A Simple Proof of Existence of Symmetric Equilibrium for Symmetric Bi-matrix games: A Quadratic Programming Approach

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Abstract

We provide a proof of existence of symmetric equilibrium for symmetric bi-matrix games, a result implied by a more general result that was proved by John Nash. Our proof, unlike the original proof due to Nash, does not appeal to any fixed-point theorem. We prove that any solution to a certain specific quadratic programming problem, is a symmetric equilibrium for the associated symmetric bi-matrix game. We use no more than the continuity of real-valued multi-variable quadratic functions and the mean value theorem for real-valued quadratic functions of a single variable. This new proof can be easily understood by anyone who is familiar with a beginner's course on real analysis.

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1. Symmetric bi-matrix games: For a positive integer ℓ, let $\Delta^{\ell-1} = \{x \in \mathbb{R}^{\ell}_{+} | \sum_{j=1}^{\ell} x_j = 1\}$.

For a positive integer n, let A be an n×n matrix.

The pair (A, A^T) is said to be a **symmetric bi-matrix game**.

$x^* \in \Delta^{n-1}$ is said to be a **symmetric equilibrium** of (A, A^T) if $x^{*T}Ax^* \geq x^TAx^*$ for all $x \in \Delta^{n-1}$.

The set $\{x^* | x^*$ is a symmetric equilibrium of $(A, A^T)\}$ is **the set of all symmetric equilibria of** (A, A^T).

2. The main result: We now prove the main result presented here. The implications of the theorem we prove are discussed in Lahiri (2025).

Theorem 1 (Nash (1951)): There exists a symmetric equilibrium for (A, A^T).

Proof: Consider the function $f:\mathbb{R}^n \to \mathbb{R}$ defined as follows: for all $x \in \mathbb{R}^n$, $f(x) = x^TA^Tx$. f is continuous and twice continuously differentiable continuous on $\mathbb{R}^n$, with $Dg(x) = x^TA^T$ for all $x \in \mathbb{R}^n$. Since Δ^{n-1} is a closed and bounded subset of $\mathbb{R}^n$, and f is continuous, by Weirstrass's theorem (Corollary 5-2.1 on page 95 of Smith and Albrecht (1987)), $\underset{x \in \Delta^{n-1}}{\operatorname{argmax}} f(x) \neq \phi$.

Let $x^* \in \underset{x \in \Delta^{n-1}}{\operatorname{argmax}} f(x)$

Consider the quadratic maximization problem: Maximize f(x), subject to $x \in \Delta^{n-1}$.

Clearly x^* solves the quadratic programming problem.

We need to show that x^* solves: Maximize $x^{*T}A^Tx$, subject to $x \in \Delta^{n-1}$.

Suppose towards a contradiction x^* does not solve the linear programming problem.

Then there exists $x \in \Delta^{n-1}$ such that $Df(x^*)(x-x^*) > 0$.

Note that $x^* + t(x-x^*) \in \Delta^{n-1}$ for all $t \in [0, 1]$.

By the Mean Value Theorem (see theorem 5-6.2 on page 107 of Smith and Albrecht (1987)), for all t in (0, 1), $f(x^* + t(x-x^*)) - f(x^*) = tDf(x^* + \xi(x-x^*))(x-x^*)$ for some $\xi \in (0, t)$, possibly depending on t.

Since $Dg(x^*)(x-x^*) > 0$, by the continuity of the function $y \mapsto Df(y)(x-x^*) = y^TA^T(x-x^*)$ on $\mathbb{R}^n$, there exists $s \in (0, 1)$ such that for all $0 < r \leq s$, $Df(x^* + r(x-x^*)(x -x^*) > 0$.

Let t belong to (0, s). Then, $f(x^* + t(x-x^*)) - f(x^*) = tDf(x^* + \xi(x-x^*))(x-x^*)$ for some $\xi \in (0, t)$, possibly depending on t.

Since $0 < \xi < t < s$, it must be that $Df(x^* + \xi(x-x^*))(x-x^*) > 0$, whence $tDf(x^* + \xi(x-x^*))(x-x^*) > 0$.

Thus, $f(x^* + t(x-x^*)) - f(x^*) > 0$, contradicting x^* solves the quadratic programming problem.

Thus, it must be the case that $Dg(x^*)(x-x^*) \leq 0$ for all $x \in \Delta^{n-1}$.

This establishes the desired implication.

Thus, $x^{*T}A^Tx^* \geq x^{*T}A^Tx$ for all $x \in \Delta^{n-1}$.

Hence, $x^{*T}Ax \geq x^TAx^*$ for all $x \in \Delta^{n-1}$, i.e., x^* is a symmetric equilibrium for (A, A^T). Q.E.D.

The proof of theorem 1 is also a proof of the following result.

Theorem 2: (i) The quadratic maximization problem [Maximize x^TAx, subject to $x \in \Delta^{n-1}$] has a solution. (ii) Every solution of this problem is a symmetric equilibrium for (A, A^T).

Proof: Follows immediately from the fact that $x^TAx = x^TA^Tx$ for all $x \in \mathbb{R}^n$. Q. E. D.

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