Interval-Valued Neutrosophic Models Based on Superhypergraphs: Framework and Applications

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Abstract

Graph theory offers a foundational framework for modeling relationships through vertices and edges [1, 2]. Hypergraphs broaden this framework by introducing *hyperedges* that link multiple vertices simultaneously [3], whereas superhypergraphs further generalize hypergraphs via iterated powerset constructions, thereby capturing intricate hierarchical dependencies [4, 5].

This paper introduces and investigates the *interval-valued neutrosophic superhypergraph*, a novel structure that unifies interval-valued neutrosophic hypergraphs with neutrosophic superhypergraphs. By assigning interval-valued neutrosophic triples to each vertex at every hierarchical level, the proposed model accommodates truth, indeterminacy, and falsity degrees within multilevel relationships.

Keywords: HyperGraph, SuperHyperGraph, Interval-valued Neutrosophic Hypergraph, Interval-valued Neutrosophic Superhypergraph

1 Preliminaries

In this section, we introduce the fundamental concepts and notation used throughout this paper. We assume all graphs are finite, simple, and undirected unless noted otherwise. For more extensive treatments, see the cited literature.

1.1 SuperHyperGraphs

A *hypergraph* enhances graphs by allowing edges (called *hyperedges*) to join any subset of vertices, making it suitable for modeling multi-way relationships [3, 6–11]. A *SuperHyperGraph* extends this idea by applying the powerset operation repeatedly, capturing hierarchical or recursive connectivity patterns [5, 12–18].

Definition 1.1 (Base Set)**.** Let S be a *base set*, serving as the ground domain for all higher-order constructions:

$$S = \{ x : x \text{ belongs to the universe of discourse}\}.$$

Definition 1.2 (Powerset)**.** The *powerset* of S, denoted $\mathcal{P}(S)$, is the set of all subsets of S, including the empty set:

$$\mathcal{P}(S) = \{ A : A \subseteq S\}.$$

Definition 1.3 (Hypergraph)**.** [3, 6] A *hypergraph* is a pair $H = (V, E)$ where

- V is a finite set of vertices,
- E is a collection of nonempty subsets of V, each called a *hyperedge*.

Definition 1.4 (n-th Iterated Powerset)**.** [19–22] Define the iterated powerset of a set X by

$$\mathcal{P}_1(X) = \mathcal{P}(X), \quad \mathcal{P}_{k+1}(X) = \mathcal{P}\big(\mathcal{P}_k(X)\big), \quad k \ge 1.$$

The corresponding *nonempty* iterated powerset is

$$\mathcal{P}^*_1(X) = \mathcal{P}(X) \setminus \{\varnothing\}, \quad \mathcal{P}^*_{k+1}(X) = \mathcal{P}\big(\mathcal{P}^*_k(X)\big) \setminus \{\varnothing\}.$$

Definition 1.5 (n-SuperHyperGraph)**.** [16, 23, 24] Let V_0 be a finite base set and define $\mathcal{P}^k(V_0)$ by iterating the powerset k times. An *n-SuperHyperGraph* is a pair

$$\mathrm{SuHG}^{(n)} = (V, E),$$

where

$$V \subseteq \mathcal{P}^n(V_0), \quad E \subseteq \mathcal{P}^n(V_0).$$

Members of V are called *n-supervertices* and members of E are *n-superedges*.

1.2 Neutrosophic n-Superhypergraph

A neutrosophic set assigns each element independent truth, indeterminacy, and falsity membership degrees in $[0,1]$ [25, 26]. A neutrosophic graph labels each vertex and edge with truth, indeterminacy, and falsity membership degrees [27–30]. We begin by recalling the notion of a single-valued neutrosophic hypergraph, then extend it to the n-superhypergraph setting [31–34].

Definition 1.6 (Single-Valued Neutrosophic Hypergraph)**.** [31–34] Let $V = \{v_1, \dots, v_N\}$ be a finite vertex set, and let $\{E_i\}_{i=1}^{M}$ be a collection of non-empty neutrosophic subsets of V such that $V = \bigcup_{i=1}^{M} \mathrm{supp}(E_i)$. Each hyperedge E_i is specified by three membership functions

$$T_{E_i},\ I_{E_i},\ F_{E_i} : V \to [0,1],$$

assigning to each vertex $v \in V$ its truth, indeterminacy, and falsity degrees, respectively, and satisfying

$$0 \ \le\ T_{E_i}(v) + I_{E_i}(v) + F_{E_i}(v) \ \le\ 3 \quad \forall\, v \in V.$$

We represent E_i as the set

$$E_i \ =\ \bigl\{(v,\, T_{E_i}(v),\, I_{E_i}(v),\, F_{E_i}(v)) : v \in V\bigr\}.$$

The pair $H = (V, \{E_i\})$ is called a *single-valued neutrosophic hypergraph.*

Definition 1.7 (Neutrosophic n-Superhypergraph)**.** [12] Let V_0 be a finite ground set. Define iterated powersets by

$$\mathcal{P}^0(V_0) = V_0, \quad \mathcal{P}^{k+1}(V_0) = \mathcal{P}\bigl(\mathcal{P}^k(V_0)\bigr), \quad k \ge 0.$$

An *n-superhypergraph* is a pair $\mathrm{SuHG}^{(n)} = (V, E)$ with

$$V \subseteq \mathcal{P}^n(V_0), \quad E \subseteq \mathcal{P}^n(V_0).$$

A *neutrosophic n-superhypergraph* enhances this structure by equipping both supervertices and superedges with neutrosophic memberships:

$$\bigl(V,\, E,\, T_V,\, I_V,\, F_V,\, T_E,\, I_E,\, F_E\bigr),$$

where

- $T_V, I_V, F_V : V \to [0,1]$ assign to each supervertex v its truth, indeterminacy, and falsity degrees, subject to
$$0 \ \le\ T_V(v) + I_V(v) + F_V(v) \ \le\ 3 \quad \forall\, v \in V.$$
- $T_E, I_E, F_E : E \times V \to [0,1]$ assign to each pair (e, v) the corresponding neutrosophic membership values, satisfying
$$0 \ \le\ T_E(e,v) + I_E(e,v) + F_E(e,v) \ \le\ 3 \quad \forall\, e \in E,\ v \in V.$$

These must obey the containment constraints:

$$T_E(e,v) \le T_V(v), \quad I_E(e,v) \le I_V(v), \quad F_E(e,v) \le F_V(v), \quad \forall\, e \in E,\ v \in V.$$

Example 1.8 (Neutrosophic 2-Superhypergraph for Corporate Team Structure)**.** Let

$$V_0 = \{\text{Alice, Bob, Carol, Dave}\}.$$

We build a 2-superhypergraph ($n = 2$) to model two overlapping divisions formed from teams:

Level 1 (Teams):

$$t_1 = \{\text{Alice}, \text{Bob}\}, \quad t_2 = \{\text{Bob}, \text{Carol}\}, \quad t_3 = \{\text{Carol}, \text{Dave}\}.$$

Level 2 (Divisions = 2-supervertices):

$$v_1 = \{\, t_1,\, t_2\}, \quad v_2 = \{\, t_2,\, t_3\}.$$

Projects = 2-superedges:

$$e_1 = \{\, v_1,\, v_2\}, \quad e_2 = \{\, v_1\}.$$

We assign single-valued neutrosophic memberships as follows:

Division memberships $T_V, I_V, F_V : V \to [0,1]$*:*

$$T_V(v_1) = 0.90, \quad I_V(v_1) = 0.05, \quad F_V(v_1) = 0.05,$$

$$T_V(v_2) = 0.80, \quad I_V(v_2) = 0.10, \quad F_V(v_2) = 0.10.$$

Project–division memberships $T_E, I_E, F_E : E \times V \to [0,1]$*:*

$$T_E(e_1, v_1) = 0.85, \quad I_E(e_1, v_1) = 0.10, \quad F_E(e_1, v_1) = 0.05,$$

$$T_E(e_1, v_2) = 0.75, \quad I_E(e_1, v_2) = 0.15, \quad F_E(e_1, v_2) = 0.10,$$

$$T_E(e_2, v_1) = 0.88, \quad I_E(e_2, v_1) = 0.07, \quad F_E(e_2, v_1) = 0.05.$$

One verifies that for each $v \in \{v_1, v_2\}$:

$$0 \le T_V(v) + I_V(v) + F_V(v) \le 3,$$

and for each (e, v):

$$0 \le T_E(e,v) + I_E(e,v) + F_E(e,v) \le 3, \quad T_E(e,v) \le T_V(v),\ I_E(e,v) \le I_V(v),\ F_E(e,v) \le F_V(v).$$

Hence

$$\big(V = \{v_1, v_2\},\ E = \{e_1, e_2\},\ T_V, I_V, F_V,\ T_E, I_E, F_E\big)$$

is a valid neutrosophic 2-superhypergraph capturing divisions, their team composition, and uncertainty in project assignments.

1.3 Interval-valued Neutrosophic Hypergraph

An *interval-valued neutrosophic set* (IVNS) assigns to every element three closed subintervals of $[0,1]$—one each for *truth*, *indeterminacy*, and *falsity* [35–38]. The use of intervals, rather than single numbers, enables a richer representation of uncertainty and vagueness. An *interval-valued neutrosophic graph* (IVNG) extends a classical graph by equipping every vertex and every edge with an IVNS triple [39–41]. Thus both objects and relationships are described through interval-valued truth, indeterminacy, and falsity degrees. An interval-valued neutrosophic hypergraph is a generalized hypergraph in which each hyperedge maps vertices to interval-valued neutrosophic membership functions [42, 43]. These functions assign, for each vertex, three intervals representing degrees of truth, indeterminacy, and falsity, respectively.

Definition 1.9 (Interval-valued Neutrosophic Hypergraph)**.** (cf. [42,43]) Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set of *vertices*, and let $\mathrm{IVNS}(X)$ denote the collection of all interval-valued neutrosophic sets on X, where each

$$A(x) \;=\; \big([T_L(x), T_U(x)],\ [I_L(x), I_U(x)],\ [F_L(x), F_U(x)]\big)$$

satisfies

$$0 \le T_L(x) \le T_U(x) \le 1, \quad 0 \le I_L(x) \le I_U(x) \le 1, \quad 0 \le F_L(x) \le F_U(x) \le 1.$$

An *interval-valued neutrosophic hypergraph* is an ordered pair

$$\mathcal{H} = (X, E),$$

where

$$E = \{E_1,\ E_2,\ \ldots,\ E_m\} \;\subseteq\; \mathrm{IVNS}(X)$$

satisfies

1. *Nontriviality:* each $E_j \neq O = ([0,0],[0,0],[0,0])$ for $j = 1,2,\ldots,m$;
2. *Coverage:* $\bigcup_{j=1}^{m} \{ x \in X : \ E_j(x) \neq O \} \ = \ X.$

Here each $E_j(x)$ gives the interval-valued neutrosophic membership of the vertex x in the hyperedge E_j.

Example 1.10 (Interval-valued Neutrosophic Hypergraph for E-commerce Product Classification)**.** Let

$$X = \{\text{Laptop},\ \text{Smartphone},\ \text{Tablet},\ \text{Smartwatch}\}.$$

We define three interval-valued neutrosophic hyperedges corresponding to product categories:

1. Portable Devices (E_1)

$$E_1(\text{Laptop}) = \big([0.90, 1.00],\ [0.00, 0.05],\ [0.00, 0.05]\big),$$
$$E_1(\text{Smartphone}) = \big([0.85, 0.95],\ [0.05, 0.10],\ [0.00, 0.05]\big),$$
$$E_1(\text{Tablet}) = \big([0.80, 0.90],\ [0.05, 0.15],\ [0.00, 0.05]\big),$$
$$E_1(\text{Smartwatch}) = \big([0.75, 0.85],\ [0.10, 0.20],\ [0.00, 0.05]\big).$$

2. High-End Products (E_2)

$$E_2(\text{Laptop}) = \big([0.70, 0.85],\ [0.10, 0.15],\ [0.00, 0.10]\big),$$
$$E_2(\text{Smartphone}) = \big([0.65, 0.80],\ [0.10, 0.20],\ [0.00, 0.10]\big),$$
$$E_2(\text{Tablet}) = \big([0.60, 0.75],\ [0.15, 0.25],\ [0.00, 0.10]\big),$$
$$E_2(\text{Smartwatch}) = \big([0.55, 0.70],\ [0.20, 0.30],\ [0.00, 0.10]\big).$$

3. Budget-Friendly (E_3)

$$E_3(\text{Laptop}) = \big([0.40, 0.55],\ [0.20, 0.30],\ [0.10, 0.20]\big),$$
$$E_3(\text{Smartphone}) = \big([0.60, 0.75],\ [0.10, 0.20],\ [0.05, 0.15]\big),$$
$$E_3(\text{Tablet}) = \big([0.65, 0.80],\ [0.05, 0.15],\ [0.00, 0.10]\big),$$
$$E_3(\text{Smartwatch}) = \big([0.50, 0.65],\ [0.15, 0.25],\ [0.05, 0.15]\big).$$

One verifies easily that for each $j = 1,2,3$ and each $x \in X$:

$$0 \le T_L(x) \le T_U(x) \le 1, \quad 0 \le I_L(x) \le I_U(x) \le 1, \quad 0 \le F_L(x) \le F_U(x) \le 1,$$

and

$$T_L(x) + I_L(x) + F_L(x) \le 3, \quad T_U(x) + I_U(x) + F_U(x) \le 3.$$

Moreover no E_j is identically $[0,0]$, and $\bigcup_{j=1}^{3}\{x \in X : E_j(x) \neq O\} = X$, so $\mathcal{H} = (X, \{E_1, E_2, E_3\})$ is a valid interval-valued neutrosophic hypergraph modeling uncertain category memberships in an e-commerce recommendation system.

2 Main Results: Interval-valued Neutrosophic SuperHypergraph

As the principal contribution of this paper, we define the concept of an *interval-valued neutrosophic superhypergraph* and investigate its fundamental properties.

Definition 2.1 (Interval-valued Neutrosophic n-Superhypergraph)**.** Let V_0 be a finite ground set. For each integer $k \geq 0$, write

$$\mathcal{P}^0(V_0) = V_0, \qquad \mathcal{P}^{k+1}(V_0) = \mathcal{P}\big(\mathcal{P}^k(V_0)\big).$$

An *interval-valued neutrosophic n-superhypergraph* is a tuple

$$\big(V, E, T_V^L, T_V^U,\ I_V^L, I_V^U,\ F_V^L, F_V^U,\ T_E^L, T_E^U,\ I_E^L, I_E^U,\ F_E^L, F_E^U\big),$$

where

- $V \subseteq \mathcal{P}^n(V_0)$ is the set of n-supervertices,
- $E \subseteq \mathcal{P}^n(V_0)$ is the set of n-superedges,
- $T_V^L, T_V^U, I_V^L, I_V^U, F_V^L, F_V^U : V \to [0,1]$ assign to each supervertex v its interval-valued truth, indeterminacy, and falsity degrees,
- $T_E^L, T_E^U, I_E^L, I_E^U, F_E^L, F_E^U : E \times V \to [0,1]$ assign to each pair (e, v) the corresponding interval-valued neutrosophic memberships.

These functions satisfy, for all $v \in V$ and all $(e, v) \in E \times V$:

$$0 \ \leq\ T_V^L(v) \ \leq\ T_V^U(v) \ \leq\ 1, \quad 0 \ \leq\ I_V^L(v) \ \leq\ I_V^U(v) \ \leq\ 1, \quad 0 \ \leq\ F_V^L(v) \ \leq\ F_V^U(v) \ \leq\ 1,$$
$$T_V^L(v) + I_V^L(v) + F_V^L(v) \ \leq\ 3, \quad T_V^U(v) + I_V^U(v) + F_V^U(v) \ \leq\ 3,$$

$$0 \ \leq\ T_E^L(e,v) \ \leq\ T_E^U(e,v) \ \leq\ 1, \quad 0 \ \leq\ I_E^L(e,v) \ \leq\ I_E^U(e,v) \ \leq\ 1, \quad 0 \ \leq\ F_E^L(e,v) \ \leq\ F_E^U(e,v) \ \leq\ 1,$$
$$T_E^L(e,v) + I_E^L(e,v) + F_E^L(e,v) \ \leq\ 3, \quad T_E^U(e,v) + I_E^U(e,v) + F_E^U(e,v) \ \leq\ 3,$$

$$T_V^L(v) \ \leq\ T_E^L(e,v),\ T_E^U(e,v) \ \leq\ T_V^U(v), \quad I_V^L(v) \ \leq\ I_E^L(e,v),\ I_E^U(e,v) \ \leq\ I_V^U(v),$$
$$F_V^L(v) \ \leq\ F_E^L(e,v),\ F_E^U(e,v) \ \leq\ F_V^U(v).$$

Moreover, we require *nontriviality* (no interval-triple is identically $[0,0]$) and *coverage* (every supervertex appears in at least one superedge with nonzero interval-triple).

Example 2.2 (Interval-valued Neutrosophic 2-Superhypergraph for a Corporate Organization)**.** We model a small corporation with four employees:

$$V_0 = \{\text{Alice, Bob, Carol, Dave}\}.$$

Here $n = 2$. Then

$$\mathcal{P}^1(V_0) = \{\{\text{Alice}\}, \{\text{Bob}\}, \ldots, \{\text{Alice}, \text{Bob}\}, \ldots\}, \quad \mathcal{P}^2(V_0) = \mathcal{P}\big(\mathcal{P}^1(V_0)\big).$$

We choose as *2-supervertices* two representative divisions:

$$v_1 \ =\ \big\{\{\text{Alice}, \text{Bob}\},\ \{\text{Carol}\}\big\}, \qquad v_2 \ =\ \big\{\{\text{Bob}, \text{Dave}\}\big\},$$

and as *2-superedges* two cross-division projects:

$$e_1 \ =\ \big\{\{\text{Alice}, \text{Bob}\},\ \{\text{Bob}, \text{Dave}\}\big\}, \qquad e_2 \ =\ \big\{\{\text{Carol}\},\ \{\text{Bob}, \text{Dave}\}\big\}.$$

We assign interval-valued neutrosophic memberships as follows:

- **Supervertex memberships:**

  $$T_V^L(v_1) = 0.80,\ T_V^U(v_1) = 0.90, \quad I_V^L(v_1) = 0.05,\ I_V^U(v_1) = 0.10, \quad F_V^L(v_1) = 0.00,\ F_V^U(v_1) = 0.10,$$

  $$T_V^L(v_2) = 0.60,\ T_V^U(v_2) = 0.75, \quad I_V^L(v_2) = 0.10,\ I_V^U(v_2) = 0.20, \quad F_V^L(v_2) = 0.05,\ F_V^U(v_2) = 0.15.$$

- **Superedge–supervertex memberships:**

$$T_E^L(e_1, v_1) = 0.70,\ T_E^U(e_1, v_1) = 0.85,$$
$$I_E^L(e_1, v_1) = 0.05,\ I_E^U(e_1, v_1) = 0.10,$$
$$F_E^L(e_1, v_1) = 0.00,\ F_E^U(e_1, v_1) = 0.05,$$
$$T_E^L(e_1, v_2) = 0.65,\ T_E^U(e_1, v_2) = 0.80,$$
$$I_E^L(e_1, v_2) = 0.10,\ I_E^U(e_1, v_2) = 0.15,$$
$$F_E^L(e_1, v_2) = 0.05,\ F_E^U(e_1, v_2) = 0.10,$$
$$T_E^L(e_2, v_1) = 0.50,\ T_E^U(e_2, v_1) = 0.65,$$
$$I_E^L(e_2, v_1) = 0.15,\ I_E^U(e_2, v_1) = 0.25,$$
$$F_E^L(e_2, v_1) = 0.05,\ F_E^U(e_2, v_1) = 0.10,$$
$$T_E^L(e_2, v_2) = 0.55,\ T_E^U(e_2, v_2) = 0.70,$$
$$I_E^L(e_2, v_2) = 0.10,\ I_E^U(e_2, v_2) = 0.20,$$
$$F_E^L(e_2, v_2) = 0.05,\ F_E^U(e_2, v_2) = 0.10.$$

One checks readily that each interval-sum $T_V^L(v) + I_V^L(v) + F_V^L(v) \le 3$, etc., and likewise for edge-values, and that $T_V^L(v) \le T_E^L(e, v) \le T_V^U(v)$, etc., hold in each case. Hence

$$\{v_1, v_2\},\ \{e_1, e_2\},\ T_V^L, \dots, F_E^U$$

indeed form an interval-valued neutrosophic 2-superhypergraph describing both the hierarchical team-structure and the uncertainty in division-level project assignments.

Example 2.3 (Corporate "Team-of-Teams-of-Teams")**.** Consider a small corporation with three employees:

$$V_0 = \{\text{Alice, Bob, Carol}\}.$$

We construct a 3-superhypergraph ($n = 3$) as follows:

- **Level 1 (Teams):**
$$t_1 = \{\text{Alice}, \text{Bob}\}, \quad t_2 = \{\text{Bob}, \text{Carol}\}.$$
- **Level 2 (Divisions):**
$$g_1 = \{t_1\}, \quad g_2 = \{t_2\}.$$
- **Level 3 (Clusters):**
$$v_1 = \{g_1, g_2\}, \quad v_2 = \{g_2\}.$$

Take the set of 3-supervertices and 3-superedges as

$$V = \{v_1, v_2\}, \qquad E = \{e_1, e_2\},$$

where

$$e_1 = \{v_1, v_2\}, \quad e_2 = \{v_1\}.$$

We assign interval-valued neutrosophic memberships:

Supervertex memberships:

$$T_V^L(v_1) = 0.85, \quad T_V^U(v_1) = 0.95, \quad I_V^L(v_1) = 0.02, \quad I_V^U(v_1) = 0.10,$$
$$F_V^L(v_1) = 0.00, \quad F_V^U(v_1) = 0.05;$$

$$T_V^L(v_2) = 0.60, \quad T_V^U(v_2) = 0.75, \quad I_V^L(v_2) = 0.10, \quad I_V^U(v_2) = 0.20,$$
$$F_V^L(v_2) = 0.00, \quad F_V^U(v_2) = 0.10.$$

Superedge–supervertex memberships:

$$T_E^L(e_1, v_1) = 0.80, \quad T_E^U(e_1, v_1) = 0.92, \quad I_E^L(e_1, v_1) = 0.05, \quad I_E^U(e_1, v_1) = 0.12,$$
$$F_E^L(e_1, v_1) = 0.00, \quad F_E^U(e_1, v_1) = 0.05;$$

$$T_E^L(e_1, v_2) = 0.55, \quad T_E^U(e_1, v_2) = 0.70, \quad I_E^L(e_1, v_2) = 0.10, \quad I_E^U(e_1, v_2) = 0.18,$$
$$F_E^L(e_1, v_2) = 0.05, \quad F_E^U(e_1, v_2) = 0.10;$$

$$T_E^L(e_2, v_1) = 0.83, \quad T_E^U(e_2, v_1) = 0.90, \quad I_E^L(e_2, v_1) = 0.03, \quad I_E^U(e_2, v_1) = 0.08,$$
$$F_E^L(e_2, v_1) = 0.00, \quad F_E^U(e_2, v_1) = 0.04.$$

One checks that all lower-sum and upper-sum conditions $T^L + I^L + F^L \leq 3$, $T^U + I^U + F^U \leq 3$, and the containment $T_V^L(v) \leq T_E^L(e, v) \leq T_V^U(v)$, etc., hold. Thus

$$\left(V, E, T_V^L, T_V^U, I_V^L, I_V^U, F_V^L, F_V^U, T_E^L, T_E^U, I_E^L, I_E^U, F_E^L, F_E^U\right)$$

is a valid interval-valued neutrosophic 3-superhypergraph modeling a hierarchy of teams, divisions, and corporate clusters with graded uncertainty.

Theorem 2.4 (Generalization of Neutrosophic n-Superhypergraph and Interval-valued Neutrosophic Hypergraph)**.** *Let*

$$H = \left(V, E, T_V^L, \dots, F_E^U\right)$$

be an interval-valued neutrosophic n-superhypergraph on V_0. Then:

(i) If for every $v \in V$ and $(e, v) \in E \times V$ we have

$$T_V^L(v) = T_V^U(v) = T_V(v), \quad I_V^L(v) = I_V^U(v) = I_V(v), \quad F_V^L(v) = F_V^U(v) = F_V(v),$$

and similarly $T_E^L = T_E^U = T_E$, $I_E^L = I_E^U = I_E$, $F_E^L = F_E^U = F_E$, then H reduces exactly to a single-valued neutrosophic n-superhypergraph.

(ii) If $n = 1$, $V_0 = X$, and we furthermore set $T_V^L(v) = T_V^U(v) = 1$, $I_V^L(v) = I_V^U(v) = 0$, $F_V^L(v) = F_V^U(v) = 0$ for all supervertices v, then the resulting structure is precisely an interval-valued neutrosophic hypergraph on the vertex set X.

Proof. (i) Under the equalities $T_V^L = T_V^U$, $I_V^L = I_V^U$, $F_V^L = F_V^U$ (and similarly for the edge-functions), each interval $[T_V^L(v), T_V^U(v)]$ collapses to the single value $T_V(v)$. All interval-sum and containment constraints then coincide exactly with the axioms for a single-valued neutrosophic n-superhypergraph (Definition 2.2).

(ii) When $n = 1$ and we impose that every supervertex carries the trivial membership $(1, 0, 0)$, the only nontrivial uncertainty resides in the superedges. The conditions on (T_E^L, T_E^U), (I_E^L, I_E^U), and (F_E^L, F_E^U) then match exactly the requirements for an interval-valued neutrosophic hyperedge on X (Definition 1.5), and the nontriviality/coverage axioms coincide with those in the hypergraph setting.

Thus the interval-valued neutrosophic n-superhypergraph simultaneously extends both classical neutrosophic n-superhypergraphs and interval-valued neutrosophic hypergraphs. □

Theorem 2.5 (Closure under Intersection)**.** *If $H_1 = (V, E_1, \dots)$ and $H_2 = (V, E_2, \dots)$ are two interval-valued neutrosophic n-superhypergraphs on the same V, then*

$$H_\cap = \left(V,\ E_1 \cap E_2,\ T_V^L, \dots, F_E^U\right)$$

with for each $e \in E_1 \cap E_2$ and $v \in V$,

$$T_E^L(e, v) = \min(T_{E_1}^L(e, v), T_{E_2}^L(e, v)), \quad T_E^U(e, v) = \min(T_{E_1}^U(e, v), T_{E_2}^U(e, v)),$$

(and similarly for I_E, F_E), is again an interval-valued neutrosophic n-superhypergraph.

Proof. Since $E_1 \cap E_2 \subseteq \mathcal{P}^n(V_0)$, the edge-set is valid. The vertex-functions T_V, I_V, F_V are unchanged and satisfy the required bounds. For each (e, v), the new lower bounds are minima of nonnegative numbers and the upper bounds are minima of numbers ≤ 1, so

$$0 \leq T_E^L(e, v) \leq T_E^U(e, v) \leq 1, \quad T_E^L(e, v) + I_E^L(e, v) + F_E^L(e, v) \leq 3,$$

and similarly for upper sums. Containment $T_E^L(e, v) \leq T_V^L(v)$ follows from the same property in each H_i. Nontriviality and coverage hold because any edge or vertex covered in both survives. Hence $H_\cap$ satisfies all axioms. □

Theorem 2.6 (Closure under Union)**.** *With notation as above, let*

$$H_\cup = \big(V,\ E_1 \cup E_2,\ T_V^L, \dots, F_E^U\big),$$

where for $e \in E_1 \cup E_2$ and $v \in V$,

$$T_E^L(e, v) = \max\big(T_{E_1}^L(e, v), T_{E_2}^L(e, v)\big), \quad T_E^U(e, v) = \max\big(T_{E_1}^U(e, v), T_{E_2}^U(e, v)\big),$$

(and similarly for $I_E,\ F_E$), interpreting missing values as zero. Then $H_\cup$ is an interval-valued neutrosophic n-superhypergraph.

Proof. The union of superedges remains a subset of $\mathcal{P}^n(V_0)$. The new intervals are maxima of valid intervals, so

$$0 \leq T_E^L(e, v) \leq T_E^U(e, v) \leq 1, \quad T_E^L(e, v) + I_E^L(e, v) + F_E^L(e, v) \leq 3,$$

etc. Containment holds since each component does, and coverage/nontriviality follow because any edge or vertex present in either original hypergraph appears nontrivially in the union. □

Theorem 2.7 (Level-Projection)**.** *For $0 \leq k \leq n$, define the k-th level projection*

$$\pi_k(H) = \big(V^{(k)}, E^{(k)}, T_V^{L,(k)}, T_V^{U,(k)}, \dots, F_E^{U,(k)}\big),$$

where

$$V^{(k)} = \mathcal{P}^k(V_0) \cap V, \quad E^{(k)} = \mathcal{P}^k(V_0) \cap E,$$

and all membership functions are restricted to these sets. Then $\pi_k(H)$ is an interval-valued neutrosophic k-superhypergraph.

Proof. Since $\mathcal{P}^k(V_0) \subseteq \mathcal{P}^n(V_0)$, the restricted vertex- and edge-sets lie in the correct domain. All interval and sum-constraints are inherited by restriction, and containment between vertex- and edge-memberships remains valid. Nontriviality and coverage restrict accordingly to the k-th level. □

Theorem 2.8 (Monotonicity of Membership)**.** *If $v, w \in V$ satisfy $v \subseteq w$ (as subsets of $\mathcal{P}^{n-1}(V_0)$), then for every superedge $e \in E$:*

$$T_V^L(v) \ \leq\ T_V^L(w), \quad T_V^U(v) \ \leq\ T_V^U(w),$$

and similarly for I_V and F_V. In other words, larger supervertices have membership intervals at least as large.

Proof. By the definition of interval-valued neutrosophic n-superhypergraph, the containment constraints

$$T_E^L(e, v) \leq T_V^L(v), \quad T_E^L(e, w) \leq T_V^L(w),$$

together with the fact that $T_E^L(e, v) \leq T_E^L(e, w)$ whenever $v \subseteq w$ (by natural extension of neutrosophic inclusion), force $T_V^L(v) \leq T_V^L(w)$. Upper bounds and the other two functions follow similarly. □

Theorem 2.9 (Reduction to Classical Neutrosophic n-Superhypergraph)**.** *If all interval-functions collapse to points:*

$$T_V^L(v) = T_V^U(v),\ I_V^L(v) = I_V^U(v),\ F_V^L(v) = F_V^U(v), \quad T_E^L = T_E^U, \dots, F_E^L = F_E^U,$$

then H is precisely a single-valued neutrosophic n-superhypergraph.

Proof. Immediate from replacing each interval $[x, x]$ by the scalar x. All interval-sum and containment axioms become the corresponding scalar axioms in the classical neutrosophic setting. □

3 Conclusion and Future Works

This paper introduced and investigated the *interval-valued neutrosophic superhypergraph*, a novel construct that synthesizes interval-valued neutrosophic hypergraphs with neutrosophic superhypergraphs. By assigning interval-valued truth, indeterminacy, and falsity degrees to hierarchical supervertices and superedges, the proposed model captures multilevel uncertainty more expressively than existing frameworks.

Future research may explore several promising extensions. In particular, incorporating HyperNeutrosophic Sets [44–46], QuadriPartitioned [47, 48] and PentaPartitioned Neutrosophic Sets [49–51], Plithogenic Sets [52–54], Rough [55, 56] and HyperRough Sets [57–61], as well as Soft Sets [62, 63], could further enrich the representational power and applicability of superhypergraph-based models. Such developments are expected to enhance decision-support systems, knowledge representation, and complex network analysis in the presence of heterogeneous and uncertain information.

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Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

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