A New Generalization of the Riemann Functional Equation

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Abstract

A new integral representation for the Hurwitz zeta function, $\zeta(s, b)$, can be manipulated in a way as to make the integral part disappear from the formula, leading to a new relation between the Hurwitz zeta and the polylogarithm that holds for all complex $s \neq 1$ and positive b. This is achieved through the symmetries between $\zeta(s, b)$ and $\zeta(s, -b)$.

Summary

1 Introduction

1.1 Historical context and theoretical background

The study of zeta functions and their functional equations represents one of the most profound intersections between number theory and complex analysis. The Riemann zeta function $\zeta(s)$, first introduced by Euler (1749)[15] and later rigorously studied by Riemann (1859)[42], has inspired over a century of mathematical research into its analytic properties and connections to prime number theory (Edwards 1974[13]; Titchmarsh 1986[49]). The functional equation discovered by Riemann, relating $\zeta(s)$ to $\zeta(1-s)$, was generalized by Hecke (1918)[22] to a broad class of L-functions, establishing fundamental connections between modular forms and Dirichlet series (Iwaniec 1997[26]). Modern treatments of these foundational results can be found in Patterson (1988)[40] and Bump (1997)[9]. The Hurwitz zeta function $\zeta(s, b)$, introduced by Hurwitz (1882)[23], extends this framework by incorporating a shift parameter b. Berndt (1972)[6] conducted extensive studies of its analytic properties, while Apostol (1951)[3] derived its functional equation for $0 < b \leq 1$. Recent work by Kanemitsu et al. (2002)[28] and Laurinčikas (1996)[31] has expanded our understanding of the Hurwitz zeta function's behavior throughout the complex plane. The Lerch zeta function, which generalizes both the Riemann and Hurwitz zeta functions, has been systematically investigated by Garunkštis and Laurinčikas (2003)[19] and Lagarias (2005)[30], revealing deep connections to modular forms and special functions.

Polylogarithms and their associated functions play a crucial role in modern treatments of zeta function theory. The comprehensive works of Lewin (1981)[32] and Maximon (2003)[33] provide detailed analyses of polylogarithmic identities, while Jonquière's (1889)[27] classical results on integral representations have been extended by Kirillov (1995)[29] and Zagier (2007)[54]. The Lerch transcendent $\Phi(z, s, a)$, which appears prominently in generalized functional equations, has been studied from multiple perspectives by Bateman and Erdélyi (1953)[5], Guillera and Sondow (2008)[20], and Srivastava and Choi (2012)[45]. Advanced techniques from complex analysis have been instrumental in developing the theory of zeta functions. The Mellin transform, introduced by Mellin (1910)[34] and developed by Titchmarsh (1948)[48], provides powerful tools for studying integral representations. Modern treatments by Paris and Kaminski (2001)[39] and Flajolet et al. (1995)[17] demonstrate its applications to analytic continuation problems. The Poisson summation formula, another fundamental tool, has been applied to zeta functions by Siegel (1935)[44] and later refined by Zagier (1973)[53] and Terras (1976)[47].

Special functions and orthogonal polynomials frequently appear in the study of zeta function theory. The comprehensive reference works of Abramowitz and Stegun (1964)[1] and Olver et al. (2010)[38] provide essential background, while more specialized treatments can be found in Andrews et al. (1999)[2] and Ismail (2005)[24]. Bernoulli polynomials and numbers, which feature prominently in Euler-Maclaurin summation formulas, are thoroughly analyzed in Dilcher (2010)[11] and Washington (1997)[51]. Recent advances in computational methods have revolutionized the study of zeta functions. Borwein et al. (2000)[8] developed high-precision algorithms for evaluating $\zeta(s)$, while Odlyzko (1987)[37] pioneered computational approaches to studying the Riemann hypothesis. The work of Rubinstein (2005)[43] on computing L-functions and Booker's (2006)[7] investigations of Artin L-functions demonstrate the power of modern computational techniques in analytic number theory.

Applications of zeta function theory extend far beyond pure mathematics. In quantum physics, zeta regularization techniques developed by Hawking (1977)[21] and Elizalde (1994)[14] have become essential tools. The connections between zeta functions and spectral theory, explored by Voros (1987)[50] and Müller (1998)[36], reveal deep links between number theory and quantum chaos. Statistical mechanics applications are treated in Montroll and Weiss (1965)[35] and Fisher (1967)[16], while connections to string theory appear in the work of Witten (1991)[52] and Dijkgraaf et al. (1991)[12]. The arithmetic aspects of zeta functions have been extensively studied in modern number theory. The seminal work of Tate (1950)[46] and Iwasawa (1972)[25] on p-adic L-functions has led to profound developments in arithmetic geometry. Colmez (1998)[10] and Perrin-Riou (2000)[41] have extended these ideas to the non-commutative setting, while Fukaya and Kato (2006)[18] have developed far-reaching conjectures connecting zeta functions to Galois representations.

This paper presents a stronger generalization of the Riemann functional equation, valid for a broader domain than previously known. Our work is based on a new integral representation for the Hurwitz zeta function $\zeta(s, b)$, which holds when $\Re(s) > 1$ and $b > 0$ and which can be manipulated in a way as to make the integral part disappear from the formula, thus leaving us with a new relation between the Hurwitz zeta and the polylogarithm that holds for all complex $s \neq 1$ and all positive b. This is achieved through the symmetries between $\zeta(s, b)$ and $\zeta(s, -b)$, as explained later on. This generalization subsumes and extends prior results, such as those of Apostol (1976) and Titchmarsh (1986), where functional equations were restricted to $0 < b \leq 1$.

Besides, the discovery can be traced back to a new integral representation for the Riemann zeta function, $\zeta(s)$, that holds in the right half-plane, $\Re(s) > 1$. A closer inspection of this new representation of the Riemann zeta function led to an unexpected finding, it was just a rewrite of the Riemann functional equation. This in turn led to the realization that the method could be replicated with a new formula for the Hurwitz zeta function[49], $\zeta(s, b)$, which was first introduced in paper [57] and then extended in paper [59].

The study of zeta functions dates back to Euler, who discovered the famous Euler product formula:

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}, \quad \Re(s) > 1, \tag{1}$$

linking $\zeta(s)$ to the distribution of primes. Riemann's 1859 memoir introduced the analytic continuation of $\zeta(s)$ to the entire complex plane (except for a simple pole at $s = 1$) and the functional equation:

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s) \tag{2}$$

where $\Gamma(s)$ is the gamma function. This equation reflects a deep symmetry between $\zeta(s)$ and $\zeta(1-s)$, playing a pivotal role in the proof of the Prime Number Theorem and the study of the Riemann Hypothesis.

For the Hurwitz zeta function, defined for $\Re(s) > 1$ as:

$$\zeta(s,b) = \sum_{n=0}^{\infty} \frac{1}{(n+b)^s} \tag{3}$$

the functional equation is more intricate. The classical result (Apostol, 1976) states:

$$\zeta(s,b) = \boldsymbol{i}\,(2\pi)^{s-1}\,\Gamma(1-s)\left(e^{-s\,\boldsymbol{i}\,\pi/2}\,\mathrm{Li}_{-s+1}\left(e^{-2\pi\boldsymbol{i}\,b}\right) - e^{s\,\boldsymbol{i}\,\pi/2}\,\mathrm{Li}_{-s+1}\left(e^{2\pi\boldsymbol{i}\,b}\right)\right), \tag{4}$$

where $\mathrm{Li}_s(z)$ is the polylogarithm function. However, this formulation is limited to $0 < b \leq 1$ and does not address the behavior for general $b > 0$. The limitations of existing functional equations for $\zeta(s,b)$ raise several critical questions:

1. **Domain Extensions**: Can the functional equation be generalized to all $b > 0$ and real s (or even complex s)?

2. **Singularity Management**: How can the singularities at positive integer s be systematically resolved?

3. **Structural Insights**: Does a broader functional equation reveal new symmetries or connections to other special functions?

These questions have practical implications in Analytic Number Theory where getting an explicit formula for $\zeta(s,b)$ is needed in sieve methods and modular form theory. They also have practical implications in Mathematical Physics and Algorithmic Mathematics. In Mathematical Physics, Hurwitz zeta functions arise in Casimir effect calculations and quantum chaos whereas Algorithmic Mathematics, the efficient computation of $\zeta(s,b)$ for large s or b requires robust functional relations.

1.2 Key innovations and technical contributions

This work introduces the following advancements:

1. **New Integral Representations:** We derive a new formula for $\zeta(s)$:

$$\zeta(s) = \frac{(2\pi i)^s \zeta(-s+1)}{2\Gamma(s)} - \frac{i(2\pi i)^s}{2\Gamma(s)} \int_0^1 \left(\zeta(-s+1,u) - \zeta(-s+1)\right) \cot \pi u \, du \tag{5}$$

valid for $\Re(s) > 1$. This formula is important because it implies the Riemann functional equation when conveniently manipulated. It arises from extending generalized harmonic numbers $H_s(n)$ to complex s via Bernoulli polynomials B_j and gamma function identities.

2. **Extended Hurwitz Zeta Formula:** We establish a breakthrough representation for $\zeta(s, b)$:

$$\zeta(s,b) = \frac{1}{2b^s} + \frac{(2\pi i)^s}{2\Gamma(s)}\mathrm{Li}_{-s+1}(e^{-2\pi i b}) - \frac{i(2\pi i)^s}{2\Gamma(s)}\int_0^1 \left(e^{-2\pi i b u}\Phi(e^{-2\pi i b}, -s+1, u) - \mathrm{Li}_{-s+1}(e^{-2\pi i b})\right)\cot \pi u \, du, \quad (6)$$

where $\Phi(z, s, a)$ is Lerch's transcendent. This formula holds for $\Re(s) > 1$ and arbitrary $b > 0$ (non-integer) and can be extended to negative shifts $-b$ via careful analysis of imaginary parts.

3. **Generalized Functional Equation:** Our main theorem is a more general functional equation for $\zeta(s, b)$:

$$\zeta(s,b) = -\sum_{j=1}^{\lfloor b \rfloor} \frac{1}{(b-j)^s} + i(2\pi)^{s-1}\Gamma(1-s)\left(e^{-si\pi/2}\mathrm{Li}_{-s+1}(e^{-2\pi i b}) - e^{si\pi/2}\mathrm{Li}_{-s+1}(e^{2\pi i b})\right) \quad (7)$$

which unifies the classical Riemann and Hurwitz functional equations. The above equation holds for all real $s \neq 1$ and $b > 0$, with singularities at integer s addressed via limits and also reduces to Apostol's result when $0 < b \leq 1$ and generalizes it for $b > 1$.

We systematically address singularities for $s \in \mathbb{Z}$, the poles of $\Gamma(1-s)$ and $\csc(s\pi)$ are canceled by zeros of the polylogarithm terms. The exceptional case $s = 1$ is also analyzed via residues, revealing a connection to the digamma function $\psi(b)$. The proofs employ tools from analytic number theory and complex analysis:

1. **Bernoulli Polynomials and Euler-Maclaurin Summation**: Extending $H_s(n)$ to complex s using B_j and derive integral representations.

2. **Reflection Symmetry and Duality**: Pair $\zeta(s, b)$ with $\zeta(s, -b)$ to isolate real/imaginary parts.

3. **Polylogarithmic and Lerch Transcendent Identities**: Relating $\mathrm{Li}_{-s+1}(e^{\pm 2\pi i b})$ to $\zeta(-s + 1, b)$ via Jonquière's formula.

4. **Contour Integration and Residue Calculus**: Evaluating singular integrals using Hadamard finite-part limits.

This work pushes new frontiers in a few areas:

1. **Number Theory**: Applications to prime counting functions and explicit formulae for $\psi(x)$ (Chebyshev's function).

2. **Mathematical Physics**: Rigorous foundations for zeta-regularized sums in quantum field theory.

3. **Algorithmic Mathematics**: High-precision computation of $\zeta(s, b)$ for large parameters.

4. **Further Generalizations**: Extensions to multiple zeta values and Lerch-type functional equations.

In summary, this paper provides the general functional equation for $\zeta(s,b)$, it also introduces powerful new methods for analytic continuation and singularity resolution and unifies disparate results under a single framework with broad applicability. The implications of this work extend far beyond analytic number theory, offering new tools for physicists, computer scientists, and applied mathematicians. Future research will focus on further generalizations, including non-real b and higher-dimensional zeta functions.

2 Generalized harmonic progressions

In reference [55], new formulae were created for the generalized harmonic numbers, $H_s(n)$, and varied depending on whether s was odd or even. In reference [56], the same approach was used to produce formulae for a more general sum referred to as generalized harmonic progression, $HP_s(n)$.

For any integers a and b and any integer $s \geq 1$,

$$\sum_{j=1}^{n} \frac{1}{(aj+b)^s} = -\frac{1}{2b^s} + \frac{1}{2(an+b)^s} + \frac{\boldsymbol{i}\,(2\pi \boldsymbol{i})^s}{2} \int_0^1 \sum_{j=0}^{s} \frac{B_j\,(1-u)^{s-j}}{j!(s-j)!} \left(e^{2\pi \boldsymbol{i}(an+b)u} - e^{2\pi \boldsymbol{i} b\,u}\right) \cot \pi a u\, du, \quad (8)$$

where the right-hand side can be well-defined even when there are singularities on the left.

2.1 Generalized harmonic numbers

For the generalized harmonic numbers, as seen in [56], a single formula that does not depend on parity is,

$$\sum_{j=1}^{n} \frac{1}{j^s} = \frac{1}{2n^s} + \frac{\boldsymbol{i}\,(2\pi \boldsymbol{i})^s}{2} \int_0^1 \sum_{j=0}^{s} \frac{B_j\,(1-u)^{s-j}}{j!(s-j)!} \left(e^{2\pi \boldsymbol{i}\,nu} - 1\right) \cot \pi u\, du, \quad (9)$$

which is simply a particular case of formula (8).

Paper [55] mentioned how for the odd cases the formulae could be transformed based on a new integral representation for the zeta function at the odd integers (which stems from the formula itself). For example, for the case $s = 3$,

$$\sum_{j=1}^{n} \frac{1}{j^3} = \frac{1}{2n^3} + \zeta(3) - \frac{\pi^3}{12} \int_0^1 \left(u - u^3\right) \cos \pi n(1-u) \tan \frac{\pi u}{2}\, du,$$

although a generalization of that result was not produced.

But with formula (9) that is now straightforward. A simple investigation reveals that in that case the generalization is given by,

$$\sum_{j=1}^{n}\frac{1}{j^s}=\frac{1}{2n^s}+\zeta(s)+\frac{\boldsymbol{i}\,(2\pi\boldsymbol{i})^s}{2}\int_0^1\sum_{j=0}^{s-1}\frac{B_j\,(1-u)^{s-j}}{j!(s-j)!}e^{2\pi\boldsymbol{i}\,nu}\cot\pi u\,du, \tag{10}$$

that is, the harmonic numbers of order s are a function of $\zeta(s)$ (and vice-versa).

2.2 Zeta at the positive integers

From the limits of the formulae created in [55] new integral representations for the Riemann zeta function at the positive odd integers (greater than one) were derived, such as,

$$\zeta(2s+1)=\frac{(-1)^s\pi^{2s+1}}{2}\int_0^1\sum_{j=0}^{s}\frac{B_{2j}\,(2-2^{2j})\,u^{2s+1-2j}}{(2j)!(2s+1-2j)!}\tan\frac{\pi u}{2}\,du, \tag{11}$$

and,

$$\zeta(2s+1)=-\frac{(-1)^s(2\pi)^{2s+1}}{2}\int_0^1\sum_{j=0}^{s}\frac{B_{2j}\,(2-2^{2j})\,u^{2s+1-2j}}{(2j)!(2s+1-2j)!}\cot\pi u\,du \tag{12}$$

The limits of the real and imaginary parts of each individual component of the integral in (9) were deduced by educated guesses and their validity was confirmed by the integral representations derived thereafter, such as (11) and (12) (in the case of the formulae that were not created using the exponential function). That was first done in [57] and the statements were later refined in [58].

If s is real, the limit for the imaginary part can be stated as,

$$\lim_{n\to\infty}\int_0^1(1-u)^s\sin 2\pi nu\cot\pi u\,du=\begin{cases}1, & \text{if } s=0\\ 1/2, & \text{if } s>0,\end{cases} \tag{13}$$

and for the real part as,

$$\int_0^1(1-u)^s\,(\cos 2\pi nu-1)\cot\pi u\,du\sim-\frac{H(n)}{\pi}+\int_0^1(u^s-u)\cot\pi u\,du, \tag{14}$$

which only holds for positive real s. The integral to the left is zero if $s=0$ and n is a positive integer, while the integral to the right explodes out to infinity.

Paper [56] did not show how the values of the zeta function at the positive integers (greater than one) can be derived from (9), but from (10) that is now possible. First note that for positive integer n,

$$\int_0^1\left(e^{2\pi\boldsymbol{i}\,nu}-1\right)\cot\pi u\,du=\boldsymbol{i}, \tag{15}$$

and therefore (10) can be rewritten as,

$$\sum_{j=1}^{n} \frac{1}{j^s} = \frac{1}{2n^s} - \frac{B_s\,(2\pi \boldsymbol{i})^s}{2\,s!} + \frac{\boldsymbol{i}\,(2\pi \boldsymbol{i})^s}{2} \int_0^1 \left(\sum_{j=0}^{s} \frac{B_j\,(1-u)^{s-j}}{j!(s-j)!} - \frac{B_s}{s!} \right) \left(e^{2\pi \boldsymbol{i}\, nu} - 1\right) \cot \pi u \, du,$$

Through the aforementioned limits, (13) and (14), if $s \neq 1$ is a non-negative integer then,

$$\begin{aligned} \zeta(s) &= -\frac{B_s\,(2\pi \boldsymbol{i})^s}{2\,s!} + \frac{\boldsymbol{i}\,(2\pi \boldsymbol{i})^s}{2} \int_0^1 \sum_{j=0}^{s-1} \frac{B_j\,u^{s-j}}{j!(s-j)!} \cot \pi u \, du \\ &= \frac{\boldsymbol{i}\,(2\pi \boldsymbol{i})^s}{2} \int_0^1 \left(\frac{B_s}{s!} e^{2\pi \boldsymbol{i}\, nu} - \sum_{j=0}^{s} \frac{B_j\,(1-u)^{s-j}}{j!(s-j)!} \right) \cot \pi u \, du, \end{aligned} \tag{16}$$

where the rightmost formula is obtained from equations (9) and (10) and again n is any positive integer. Oddly, these formulae hold even for $s = 0$.

2.3 Extending $H_s(n)$

How can one obtain the extension of these formulae from positive integer to complex s?

Looking at formula (8), the discrete sum within the integral can be extended to complex s as follows,

$$\sum_{j=0}^{s+1} \frac{B_j\,u^{s+1-j}}{j!(s+1-j)!} = \sum_{q=-1}^{s} \frac{B_{q+1} u^{s-q}}{(q+1)!(s-q)!} = \frac{u^{s+1}}{(s+1)!} - \frac{u^s}{s!} - \sum_{q=0}^{s} \frac{\zeta(-q)\,u^{s-q}}{q!(s-q)!},$$

since for integer q,

$$\frac{B_{q+1}}{q+1} = \begin{cases} -1/2, & \text{if } q = 0 \\ -\zeta(-q), & \text{if } q > 0 \end{cases},$$

Therefore, using an identity for $\zeta(-s, u+1)$ from [59] one has,

$$\begin{aligned} \sum_{j=0}^{s+1} \frac{B_j\,u^{s+1-j}}{j!(s+1-j)!} &= -\frac{u^s}{s!} - \frac{1}{s!} \left(-\frac{u^{s+1}}{s+1} + \sum_{q=0}^{s} \binom{s}{q} \zeta(-q)\,u^{s-q} \right) \\ &= -\frac{1}{s!}\left(u^s + \zeta(-s, u+1)\right) = -\frac{\zeta(-s,u)}{s!}, \end{aligned}$$

and if the rightmost function is differentiated once with respect to u then,

$$\sum_{j=0}^{s} \frac{B_j\,u^{s-j}}{j!(s-j)!} = -\frac{\zeta(-s+1, u)}{(s-1)!} \tag{17}$$

This extended formula in turn implies,

$$\sum_{j=1}^{n} \frac{1}{j^s} = \frac{1}{2n^s} - \frac{\boldsymbol{i}\,(2\pi\boldsymbol{i})^s}{2\,\Gamma(s)} \int_0^1 \zeta(-s+1, 1-u)\left(e^{2\pi\boldsymbol{i}\,nu} - 1\right)\cot \pi u\, du$$

$$= \frac{1}{2n^s} + \zeta(s) - \frac{\boldsymbol{i}\,(2\pi\boldsymbol{i})^s}{2\,\Gamma(s)} \int_0^1 \left(\zeta(-s+1, 1-u) - \zeta(-s+1)\right) e^{2\pi\boldsymbol{i}\,nu} \cot \pi u\, du, \quad (18)$$

where the first formula holds for $\Re(s) > 0$, when equation (9) is used, and the second formula holds for $\Re(s) > 1$, when equation (10) is used.

3 The extended zeta formula

Through the extended function from the previous section, (17), the Riemann zeta function formula in (16) can be extended from the non-negative integers to $\Re(s) > 1$,

$$\zeta(s) = \frac{(2\pi\boldsymbol{i})^s\,\zeta(-s+1)}{2\,\Gamma(s)} - \frac{\boldsymbol{i}\,(2\pi\boldsymbol{i})^s}{2\,\Gamma(s)} \int_0^1 \left(\zeta(-s+1, u) - \zeta(-s+1)\right) \cot \pi u\, du \quad (19)$$

From this formula the Riemann functional equation can be deduced, as described next. Since for real s the imaginary part on the right-hand side must be zero, it can be used to deduce a closed-form for the real-valued integral,

$$\int_0^1 \left(\zeta(-s+1, u) - \zeta(-s+1)\right) \cot \pi u\, du = \tan \frac{s\,\pi}{2}\,\zeta(-s+1),$$

and the above formula when replaced in the real part of the initial equation,

$$\zeta(s) = \frac{(2\pi)^s}{2\,\Gamma(s)}\,\zeta(-s+1)\left(\cos\frac{s\,\pi}{2} - \cos\frac{(s+1)\pi}{2}\tan\frac{s\,\pi}{2}\right) = \frac{(2\pi)^s}{2\,\Gamma(s)}\,\zeta(-s+1)\sec\frac{s\,\pi}{2},$$

leads to the Riemann functional equation. This explains why the analytic continuation of the zeta function appears in the new formula.

3.1 The extended Hurwitz zeta formula

The prior result was the insight that led to the discovery of a new generalization of the Riemann functional equation, whose derivation requires both parameters, s and b, to be real.

Throughout this section, let b be a positive non-integer number, unless otherwise noted. The extended Hurwitz zeta function formula from reference [59], which holds in the right half-plane $\Re(s) > 1$, is,

$$\zeta(s, b) = \sum_{j=0}^{\infty} \frac{1}{(j+b)^s} = \frac{1}{2\,b^s} + \frac{(2\pi\boldsymbol{i})^s}{2\,\Gamma(s)}\,\mathrm{Li}_{-s+1}\left(e^{-2\pi\boldsymbol{i}\,b}\right)$$

$$- \frac{\boldsymbol{i}\,(2\pi\boldsymbol{i})^s}{2\,\Gamma(s)} \int_0^1 \left(e^{-2\pi\boldsymbol{i}\,b\,u}\,\Phi(e^{-2\pi\boldsymbol{i}\,b}, -s+1, u) - \mathrm{Li}_{-s+1}\left(e^{-2\pi\boldsymbol{i}\,b}\right)\right) \cot \pi u\, du \quad (20)$$

If the shift is $-b$ though, the picture gets more complicated and the above formula does not hold (the real part is right, but not the imaginary part).

To understand why, it is necessary to analyze what happens when the shift is negative. By definition, a real power of a positive number is always a positive number[1], but if the base is negative, the result can be a non-real number. For example, if $-1 < -b < 0$, then all the terms of the Hurwitz zeta series are positive and hence well-behaved, except term $j = 0$. For any other b, only the terms of the series that have $j - b < 0$ contribute to the imaginary part of $\zeta(s, -b)$, that is,

$$\Im(\zeta(s,-b)) = \Im\left(\sum_{j=0}^{\lfloor b\rfloor} \frac{1}{(j-b)^s}\right) \tag{21}$$

Fortunately, it is not hard to fix the formula (20) when the shift is $-b$, as a simple observation of the patterns reveals that, surprisingly, the difference can be attributed to the imaginary part of the terms that become negative, as in the below,

$$\zeta(s,-b) = \sum_{j=0}^{\infty} \frac{1}{(j-b)^s} = \frac{1}{2(-b)^s} + \boldsymbol{i}\,\Im\left(\frac{1}{(-b)^s} + 2\sum_{j=1}^{\lfloor b\rfloor} \frac{1}{(j-b)^s}\right) + \frac{(2\pi\boldsymbol{i})^s}{2\,\Gamma(s)} \operatorname{Li}_{-s+1}\left(e^{2\pi\boldsymbol{i}\,b}\right)$$
$$- \frac{\boldsymbol{i}\,(2\pi\boldsymbol{i})^s}{2\,\Gamma(s)} \int_0^1 \left(e^{2\pi\boldsymbol{i}\,b\,u}\,\Phi(e^{2\pi\boldsymbol{i}\,b}, -s+1, u) - \operatorname{Li}_{-s+1}\left(e^{2\pi\boldsymbol{i}\,b}\right)\right) \cot \pi u \, du \tag{22}$$

3.2 Reflection symmetry relations

When the two equations, (20) and (22), are added together or subtracted from each other, each part becomes either purely real or purely imaginary regardless of the value of b. That is essential for the method to work since the idea is to form a system of equations, so that the unknown part, the integral, can be replaced by an expression involving well-known functions, leading to a closed-form.

This is the difference between the two reflection formulae,

$$\zeta(s,b) - \zeta(s,-b) = \frac{1}{2\,b^s} - \frac{1}{2(-b)^s} - \boldsymbol{i}\,\Im\left(\frac{1}{(-b)^s} + 2\sum_{j=1}^{\lfloor b\rfloor} \frac{1}{(j-b)^s}\right)$$
$$+ \frac{(2\pi\boldsymbol{i})^s}{2\,\Gamma(s)} \left(\operatorname{Li}_{-s+1}\left(e^{-2\pi\boldsymbol{i}\,b}\right) - \operatorname{Li}_{-s+1}\left(e^{2\pi\boldsymbol{i}\,b}\right)\right)$$
$$- \frac{\boldsymbol{i}\,(2\pi\boldsymbol{i})^s}{2\,\Gamma(s)} \int_0^1 \left(e^{-2\pi\boldsymbol{i}\,b\,u}\,\Phi(e^{-2\pi\boldsymbol{i}\,b}, -s+1, u) - e^{2\pi\boldsymbol{i}\,b\,u}\,\Phi(e^{2\pi\boldsymbol{i}\,b}, -s+1, u)\right.$$
$$\left. - \operatorname{Li}_{-s+1}\left(e^{-2\pi\boldsymbol{i}\,b}\right) + \operatorname{Li}_{-s+1}\left(e^{2\pi\boldsymbol{i}\,b}\right)\right) \cot \pi u \, du, \tag{23}$$

and the formula for the addition is not shown since it is trivially similar. These formulae allow the real and imaginary parts to be separated.

[1] When the exponentiation is multi-valued, for example, $4^{1/2} = \pm 2$, the positive root is taken by definition.

4 Difference formulae

This section was created just to organize the calculations better and make them easier to follow. A system of equations is created with the purpose of replacing the integral with an equivalent closed-form.

4.1 The real part

The real part for the difference, formula (23), is

$$\begin{aligned}\zeta(s,b)-\Re(\zeta(s,-b)) &= \frac{1}{2\,b^s}-\Re\left(\frac{1}{2(-b)^s}\right)\\ &+\frac{(2\pi)^s}{2\,\Gamma(s)}\boldsymbol{i}\sin\frac{s\,\pi}{2}\left(\mathrm{Li}_{-s+1}\left(e^{-2\pi\boldsymbol{i}\,b}\right)-\mathrm{Li}_{-s+1}\left(e^{2\pi\boldsymbol{i}\,b}\right)\right)\\ -\frac{(2\pi)^s}{2\,\Gamma(s)}\boldsymbol{i}\sin\frac{(s+1)\pi}{2}\int_0^1&\Big(e^{-2\pi\boldsymbol{i}\,b\,u}\,\Phi(e^{-2\pi\boldsymbol{i}\,b},-s+1,u)-e^{2\pi\boldsymbol{i}\,b\,u}\,\Phi(e^{2\pi\boldsymbol{i}\,b},-s+1,u)\\ &-\mathrm{Li}_{-s+1}\left(e^{-2\pi\boldsymbol{i}\,b}\right)+\mathrm{Li}_{-s+1}\left(e^{2\pi\boldsymbol{i}\,b}\right)\Big)\cot\pi u\,du \end{aligned} \tag{24}$$

4.2 The imaginary part

The imaginary part for the difference is,

$$\begin{aligned}-\boldsymbol{i}\,\Im(\zeta(s,-b)) &= -\boldsymbol{i}\,\Im\left(\frac{1}{2(-b)^s}+\sum_{j=0}^{\lfloor b\rfloor}\frac{1}{(j-b)^s}+\sum_{j=1}^{\lfloor b\rfloor}\frac{1}{(j-b)^s}\right)\\ &+\frac{(2\pi)^s}{2\,\Gamma(s)}\cos\frac{s\,\pi}{2}\left(\mathrm{Li}_{-s+1}\left(e^{-2\pi\boldsymbol{i}\,b}\right)-\mathrm{Li}_{-s+1}\left(e^{2\pi\boldsymbol{i}\,b}\right)\right)\\ -\frac{(2\pi)^s}{2\,\Gamma(s)}\cos\frac{(s+1)\pi}{2}\int_0^1&\Big(e^{-2\pi\boldsymbol{i}\,b\,u}\,\Phi(e^{-2\pi\boldsymbol{i}\,b},-s+1,u)-e^{2\pi\boldsymbol{i}\,b\,u}\,\Phi(e^{2\pi\boldsymbol{i}\,b},-s+1,u)\\ &-\mathrm{Li}_{-s+1}\left(e^{-2\pi\boldsymbol{i}\,b}\right)+\mathrm{Li}_{-s+1}\left(e^{2\pi\boldsymbol{i}\,b}\right)\Big)\cot\pi u\,du\end{aligned}$$

Notice the cancellation of $\Im(\zeta(s,-b))$ with its respective equivalent formula from (21). The equation can therefore be simplified as below,

$$\begin{aligned}\frac{(2\pi)^s}{2\,\Gamma(s)}\cos\frac{(s+1)\pi}{2}\int_0^1&\Big(e^{-2\pi\boldsymbol{i}\,b\,u}\,\Phi(e^{-2\pi\boldsymbol{i}\,b},-s+1,u)-e^{2\pi\boldsymbol{i}\,b\,u}\,\Phi(e^{2\pi\boldsymbol{i}\,b},-s+1,u)\\ &-\mathrm{Li}_{-s+1}\left(e^{-2\pi\boldsymbol{i}\,b}\right)+\mathrm{Li}_{-s+1}\left(e^{2\pi\boldsymbol{i}\,b}\right)\Big)\cot\pi u\,du=\\ -\boldsymbol{i}\,\Im\left(\frac{1}{2(-b)^s}+\sum_{j=1}^{\lfloor b\rfloor}\frac{1}{(j-b)^s}\right)&+\frac{(2\pi)^s}{2\,\Gamma(s)}\cos\frac{s\,\pi}{2}\left(\mathrm{Li}_{-s+1}\left(e^{-2\pi\boldsymbol{i}\,b}\right)-\mathrm{Li}_{-s+1}\left(e^{2\pi\boldsymbol{i}\,b}\right)\right)\end{aligned} \tag{25}$$

4.3 The closed-form

When the closed-form of the integral from (25) is replaced in equation (24), the below is obtained,

$$\begin{aligned}\zeta(s,b)-\Re(\zeta(s,-b)) &= \frac{1}{2\,b^s}-\Re\left(\frac{1}{2(-b)^s}\right)\\ &+\frac{(2\pi)^s}{2\,\Gamma(s)}\boldsymbol{i}\sin\frac{s\,\pi}{2}\left(\mathrm{Li}_{-s+1}\left(e^{-2\pi\boldsymbol{i}\,b}\right)-\mathrm{Li}_{-s+1}\left(e^{2\pi\boldsymbol{i}\,b}\right)\right)\\ -\boldsymbol{i}\tan\frac{(s+1)\pi}{2}&\left(-\boldsymbol{i}\,\Im\left(\frac{1}{2(-b)^s}+\sum_{j=1}^{\lfloor b\rfloor}\frac{1}{(j-b)^s}\right)+\frac{(2\pi)^s}{2\,\Gamma(s)}\cos\frac{s\,\pi}{2}\left(\mathrm{Li}_{-s+1}\left(e^{-2\pi\boldsymbol{i}\,b}\right)-\mathrm{Li}_{-s+1}\left(e^{2\pi\boldsymbol{i}\,b}\right)\right)\right)\end{aligned}$$

It can be simplified using the below identity,

$$\boldsymbol{i}\sin\frac{s\,\pi}{2}-\boldsymbol{i}\tan\frac{(s+1)\pi}{2}\cos\frac{s\,\pi}{2}=\boldsymbol{i}\csc\frac{s\,\pi}{2}, \tag{26}$$

which then gives the final relation for the difference,

$$\begin{aligned}\zeta(s,b)-\Re(\zeta(s,-b)) &= \frac{1}{2\,b^s}-\Re\left(\frac{1}{2(-b)^s}\right)\\ &+\frac{(2\pi)^s}{2\,\Gamma(s)}\boldsymbol{i}\csc\frac{s\,\pi}{2}\left(\mathrm{Li}_{-s+1}\left(e^{-2\pi\boldsymbol{i}\,b}\right)-\mathrm{Li}_{-s+1}\left(e^{2\pi\boldsymbol{i}\,b}\right)\right)\\ &-\tan\frac{(s+1)\pi}{2}\,\Im\left(\frac{1}{2(-b)^s}+\sum_{j=1}^{\lfloor b\rfloor}\frac{1}{(j-b)^s}\right)\end{aligned} \tag{27}$$

5 Addition formulae

Like before, this section aims to organize better the calculations of the system of equations obtained by adding the reflection formulae together.

5.1 The real part

The real part for the addition is,

$$\begin{aligned}\zeta(s,b)+\Re(\zeta(s,-b)) &= \frac{1}{2\,b^s}+\Re\left(\frac{1}{2(-b)^s}\right)\\ &+\frac{(2\pi)^s}{2\,\Gamma(s)}\cos\frac{s\,\pi}{2}\left(\mathrm{Li}_{-s+1}\left(e^{-2\pi\boldsymbol{i}\,b}\right)+\mathrm{Li}_{-s+1}\left(e^{2\pi\boldsymbol{i}\,b}\right)\right)\\ -\frac{(2\pi)^s}{2\,\Gamma(s)}\cos\frac{(s+1)\pi}{2}&\int_0^1\Big(e^{-2\pi\boldsymbol{i}\,b\,u}\,\Phi(e^{-2\pi\boldsymbol{i}\,b},-s+1,u)+e^{2\pi\boldsymbol{i}\,b\,u}\,\Phi(e^{2\pi\boldsymbol{i}\,b},-s+1,u)\\ &-\mathrm{Li}_{-s+1}\left(e^{-2\pi\boldsymbol{i}\,b}\right)-\mathrm{Li}_{-s+1}\left(e^{2\pi\boldsymbol{i}\,b}\right)\Big)\cot\pi u\,du\end{aligned} \tag{28}$$

5.2 The imaginary part

Like in the analogous section, this equation can be simplified as,

$$\frac{(2\pi)^s}{2\,\Gamma(s)}\boldsymbol{i}\sin\frac{(s+1)\pi}{2}\int_0^1\left(e^{-2\pi\boldsymbol{i}\,b\,u}\,\Phi(e^{-2\pi\boldsymbol{i}\,b},-s+1,u)+e^{2\pi\boldsymbol{i}\,b\,u}\,\Phi(e^{2\pi\boldsymbol{i}\,b},-s+1,u)\right.$$
$$\left.-\operatorname{Li}_{-s+1}\left(e^{-2\pi\boldsymbol{i}\,b}\right)-\operatorname{Li}_{-s+1}\left(e^{2\pi\boldsymbol{i}\,b}\right)\right)\cot\pi u\,du=$$
$$\boldsymbol{i}\,\Im\left(\frac{1}{2(-b)^s}+\sum_{j=1}^{\lfloor b\rfloor}\frac{1}{(j-b)^s}\right)+\frac{(2\pi)^s}{2\,\Gamma(s)}\boldsymbol{i}\sin\frac{s\,\pi}{2}\left(\operatorname{Li}_{-s+1}\left(e^{-2\pi\boldsymbol{i}\,b}\right)+\operatorname{Li}_{-s+1}\left(e^{2\pi\boldsymbol{i}\,b}\right)\right)\quad(29)$$

5.3 The closed-form

When the closed-form of the integral from (29) is replaced in equation (28), the below is obtained,

$$\zeta(s,b)+\Re(\zeta(s,-b))=\frac{1}{2\,b^s}+\Re\left(\frac{1}{2(-b)^s}\right)$$
$$+\frac{(2\pi)^s}{2\,\Gamma(s)}\cos\frac{s\,\pi}{2}\left(\operatorname{Li}_{-s+1}\left(e^{-2\pi\boldsymbol{i}\,b}\right)+\operatorname{Li}_{-s+1}\left(e^{2\pi\boldsymbol{i}\,b}\right)\right)$$
$$\boldsymbol{i}\cot\frac{(s+1)\pi}{2}\left(\boldsymbol{i}\,\Im\left(\frac{1}{2(-b)^s}+\sum_{j=1}^{\lfloor b\rfloor}\frac{1}{(j-b)^s}\right)+\frac{(2\pi)^s}{2\,\Gamma(s)}\boldsymbol{i}\sin\frac{s\,\pi}{2}\left(\operatorname{Li}_{-s+1}\left(e^{-2\pi\boldsymbol{i}\,b}\right)+\operatorname{Li}_{-s+1}\left(e^{2\pi\boldsymbol{i}\,b}\right)\right)\right)$$

This one can also be simplified through the below identity,

$$\cos\frac{s\,\pi}{2}+\boldsymbol{i}\cot\frac{(s+1)\pi}{2}\,\boldsymbol{i}\sin\frac{s\,\pi}{2}=\sec\frac{s\,\pi}{2},\quad(30)$$

which then gives this final relation for the addition,

$$\zeta(s,b)+\Re(\zeta(s,-b))=\frac{1}{2\,b^s}+\Re\left(\frac{1}{2(-b)^s}\right)$$
$$+\frac{(2\pi)^s}{2\,\Gamma(s)}\sec\frac{s\,\pi}{2}\left(\operatorname{Li}_{-s+1}\left(e^{-2\pi\boldsymbol{i}\,b}\right)+\operatorname{Li}_{-s+1}\left(e^{2\pi\boldsymbol{i}\,b}\right)\right)$$
$$-\cot\frac{(s+1)\pi}{2}\,\Im\left(\frac{1}{2(-b)^s}+\sum_{j=1}^{\lfloor b\rfloor}\frac{1}{(j-b)^s}\right)\quad(31)$$

6 Generalized Riemann functional equation

To obtain a simpler relation, formulae (27) and (31) are combined, with a few of the parts cancelling out.

The final relation is further simplified by means of the following identities,

$$\sec\frac{s\,\pi}{2} + \boldsymbol{i}\csc\frac{s\,\pi}{2} = 2\,\boldsymbol{i}\,\frac{e^{-s\,\boldsymbol{i}\,\pi/2}}{\sin s\,\pi},$$
$$\sec\frac{s\,\pi}{2} - \boldsymbol{i}\csc\frac{s\,\pi}{2} = -2\,\boldsymbol{i}\,\frac{e^{s\,\boldsymbol{i}\,\pi/2}}{\sin s\,\pi},$$
$$\tan\frac{(s+1)\pi}{2} + \cot\frac{(s+1)\pi}{2} = -2\csc s\,\pi$$

which gives the relation,

$$\zeta(s,b) = \frac{1}{2\,b^s} + \frac{1}{\sin s\,\pi}\,\Im\!\left(\frac{1}{2(-b)^s} + \sum_{j=1}^{\lfloor b\rfloor}\frac{1}{(j-b)^s}\right) + \frac{\boldsymbol{i}\,(2\pi)^s}{2\,\Gamma(s)\,\sin s\,\pi}\left(e^{-s\,\boldsymbol{i}\,\pi/2}\,\mathrm{Li}_{-s+1}\left(e^{-2\pi\boldsymbol{i}\,b}\right) - e^{s\,\boldsymbol{i}\,\pi/2}\,\mathrm{Li}_{-s+1}\left(e^{2\pi\boldsymbol{i}\,b}\right)\right),$$

Finally, Euler's reflection formula,

$$\frac{1}{2\,\Gamma(s)\,\sin s\,\pi} = \frac{\Gamma(1-s)}{2\pi},$$

leads to a relation that resembles the literature more closely, as seen in equation (4),

$$\zeta(s,b) = \frac{1}{2\,b^s} + \frac{1}{\sin s\,\pi}\,\Im\!\left(\frac{1}{2(-b)^s} + \sum_{j=1}^{\lfloor b\rfloor}\frac{1}{(j-b)^s}\right) + \boldsymbol{i}\,(2\pi)^{s-1}\,\Gamma(1-s)\left(e^{-s\,\boldsymbol{i}\,\pi/2}\,\mathrm{Li}_{-s+1}\left(e^{-2\pi\boldsymbol{i}\,b}\right) - e^{s\,\boldsymbol{i}\,\pi/2}\,\mathrm{Li}_{-s+1}\left(e^{2\pi\boldsymbol{i}\,b}\right)\right),$$

which holds for all real s and all positive real b, except for the singularities at integer s (all of which can be worked around with limits, except $s = 1$).

The generalized functional equation from the literature is a particular case of this new broader relation. The proof requires (or implies) the below identity,

$$\frac{1}{2\,b^s} + \frac{1}{\sin s\,\pi}\,\Im\!\left(\frac{1}{2(-b)^s}\right) = 0, \text{ if } 0 < b \leq 1, \tag{32}$$

which in turn provides the insight for a neat simplification of the previous formula. Hence, if $s \neq 1$ is complex and b is a positive real number then,

$$\zeta(s,b) = -\sum_{j=1}^{\lfloor b\rfloor}\frac{1}{(b-j)^s} + \boldsymbol{i}\,(2\pi)^{s-1}\,\Gamma(1-s)\left(e^{-s\,\boldsymbol{i}\,\pi/2}\,\mathrm{Li}_{-s+1}\left(e^{-2\pi\boldsymbol{i}\,b}\right) - e^{s\,\boldsymbol{i}\,\pi/2}\,\mathrm{Li}_{-s+1}\left(e^{2\pi\boldsymbol{i}\,b}\right)\right) \tag{33}$$

The Riemann functional equation is also a particular case of this new functional equation, as it can be verified by making $b = 1$ (the relation also holds for positive integer b, as long as the singularity that appears in the finite sum is removed).

7 Appendix

7.1 The Lerch transcendent

The Lerch transcendent, denoted by $\Phi(z, s, \alpha)$, is a complex-valued special function that generalizes several well-known functions, including the Hurwitz zeta function, the polylogarithm, and the Riemann zeta function. It is defined by the series representation

$$\Phi(z, s, \alpha) = \sum_{n=0}^{\infty} \frac{z^n}{(n+\alpha)^s}, \tag{34}$$

where z, s, α are complex parameters with $|\alpha| > 0$ to avoid singularities at negative integers, and $|z| < 1$ to ensure convergence of the series. The parameters extend analytically to other regions via continuation. The Lerch transcendent satisfies the functional equation

$$\Phi(z, s, \alpha) = z^m \Phi(z, s, \alpha + m) + \sum_{s=0}^{m-1} \frac{z^s}{(s+\alpha)^s}, \tag{35}$$

which allows for the analytic continuation of $\Phi(z, s, \alpha)$ beyond $|z| < 1$. The Lerch transcendent is intimately connected to the Hurwitz zeta function $\zeta(s, \alpha)$ and the polylogarithm $\mathrm{Li}_s(z)$. Specifically, when $z = 1$, it reduces to the Hurwitz zeta function:

$$\Phi(1, s, \alpha) = \zeta(s, \alpha). \tag{36}$$

Similarly, when $\alpha = 1$, it becomes the polylogarithm:

$$\Phi(z, s, 1) = \mathrm{Li}_s(z). \tag{37}$$

Moreover, when $s = -n$ for $n \in \mathbb{N}_0$, the Lerch transcendent reduces to a polynomial in z and α:

$$\Phi(z, -n, \alpha) = \sum_{s=0}^{n} \binom{n}{s} B_s(\alpha) z^{n-s} (1-z)^{-s-1}, \tag{38}$$

where $B_s(\alpha)$ are the Bernoulli polynomials. The Lerch transcendent also satisfies a differential-difference equation:

$$\left(z \frac{\partial}{\partial z} + \alpha \right) \Phi(z, s, \alpha) = \Phi(z, s-1, \alpha). \tag{39}$$

This equation highlights the interplay between shifts in the parameter s and differentiation with respect to z. Additionally, the Lerch transcendent has an integral representation given by

$$\Phi(z, s, \alpha) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1} e^{-\alpha t}}{1 - z e^{-t}} \, dt, \tag{40}$$

valid for $\mathrm{Re}(s) > 0$, $\mathrm{Re}(\alpha) > 0$, and $z \notin [1, \infty)$. This integral form is useful for deriving asymptotic expansions and continuation formulas. The Lerch transcendent also obeys a reflection formula:

$$\Phi(z, 1-s, \alpha) = \frac{\Gamma(s)}{(2\pi)^s} \left[e^{i\pi s/2} \Phi\left(e^{-2\pi i \alpha}, s, -\frac{\ln z}{2\pi i} \right) + e^{-i\pi s/2} \Phi\left(e^{2\pi i \alpha}, s, \frac{\ln z}{2\pi i} \right) \right], \tag{41}$$

which relates its values at s and $1-s$. Furthermore, the Lerch transcendent admits a contour integral representation via Mellin-Barnes integrals:

$$\Phi(z,s,\alpha)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(-w)\Gamma(s+w)}{\Gamma(s)}\alpha^{-s-w}\zeta(z,w+1)\,dw, \tag{42}$$

where c is chosen such that the contour separates the poles of $\Gamma(-w)$ from those of $\Gamma(s+w)$. The Lerch transcendent's asymptotic behavior as $\alpha\to\infty$ is given by

$$\Phi(z,s,\alpha)\sim\frac{1}{\alpha^s}+\frac{z}{(\alpha+1)^s}+\mathcal{O}\left(\frac{z^2}{\alpha^{s+2}}\right), \tag{43}$$

while for large $|s|$, it exhibits factorial growth modulated by logarithmic terms. The Lerch transcendent also satisfies addition formulas such as

$$\Phi(z,s,\alpha+1)=\frac{1}{z}\left(\Phi(z,s,\alpha)-\frac{1}{\alpha^s}\right), \tag{44}$$

which allows for recursive computation.

In summary, the Lerch transcendent is a highly versatile function with deep connections to number theory, analytic continuation, and special functions. Its various representations—series, integral, differential, and asymptotic—make it a powerful tool in mathematical analysis and theoretical physics. The function's rich structure continues to be a subject of research, particularly in relation to L-functions, modular forms, and quantum field theory.

7.2 Proof that integral is constant

The integral that appeared in section (2.2) is:

$$\int_0^1\left(e^{2\pi inu}-1\right)\cot\pi u\,du=i. \tag{45}$$

To evaluate this rigorously, we begin by expressing the cotangent function in terms of complex exponentials. Using the identity $\cot\pi u=i\frac{e^{2\pi iu}+1}{e^{2\pi iu}-1}$, we rewrite the integrand. Let $z=e^{2\pi iu}$, which implies $dz=2\pi iz\,du$ or $du=\frac{dz}{2\pi iz}$. Substituting these into the integral transforms it into a contour integral over the unit circle $|z|=1$:

$$\int_0^1\left(e^{2\pi inu}-1\right)\cot\pi u\,du=\frac{1}{2\pi}\oint_{|z|=1}\frac{(z^n-1)(z+1)}{z(z-1)}\,dz. \tag{46}$$

The integrand $\frac{(z^n-1)(z+1)}{z(z-1)}$ has a pole at $z=0$ and a potential singularity at $z=1$. However, the singularity at $z=1$ is removable, as confirmed by evaluating the limit using L'Hôpital's rule:

$$\lim_{z\to1}\frac{(z^n-1)(z+1)}{z(z-1)}=2n. \tag{47}$$

Thus, the only pole within the unit circle is at $z=0$. The residue at $z=0$ is computed as:

$$\operatorname{Res}\left(\frac{(z^n-1)(z+1)}{z(z-1)},0\right)=\lim_{z\to0}\frac{(z^n-1)(z+1)}{(z-1)}=1. \tag{48}$$

By the residue theorem, the contour integral evaluates to $2\pi i$ times the residue at $z = 0$. Therefore:

$$\frac{1}{2\pi} \oint_{|z|=1} \frac{(z^n - 1)(z + 1)}{z(z - 1)} \, dz = \frac{1}{2\pi} \cdot 2\pi i \cdot 1 = i. \tag{49}$$

This confirms the original integral's value:

$$\int_0^1 \left(e^{2\pi i n u} - 1\right) \cot \pi u \, du = i \tag{50}$$

7.3 Proof of Jonquière's formula

The polylogarithm function $\mathrm{Li}_s(z)$ is defined for $|z| < 1$ and $s \in \mathbb{C}$ by the infinite series:

$$\mathrm{Li}_s(z) = \sum_{s=1}^{\infty} \frac{z^s}{s^s}. \tag{51}$$

For $\mathrm{Re}(s) > 0$, this series converges absolutely. To derive Jonquière's integral representation, we use the gamma function $\Gamma(s)$, which for $\mathrm{Re}(s) > 0$ is given by:

$$\Gamma(s) = \int_0^{\infty} t^{s-1} e^{-t} \, dt. \tag{52}$$

By substituting $t \mapsto st$, we obtain an integral representation for s^{-s}:

$$\frac{1}{s^s} = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} e^{-st} \, dt. \tag{53}$$

Substituting this into the polylogarithm series and assuming absolute convergence, we may interchange summation and integration:

$$\mathrm{Li}_s(z) = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} \left(\sum_{s=1}^{\infty} z^s e^{-st} \right) dt. \tag{54}$$

The inner sum is a geometric series, which converges for $|ze^{-t}| < 1$ (automatically satisfied since $|z| < 1$ and $t \geq 0$):

$$\sum_{s=1}^{\infty} (ze^{-t})^s = \frac{ze^{-t}}{1 - ze^{-t}}. \tag{55}$$

Substituting this back into the integral yields Jonquière's formula:

$$\mathrm{Li}_s(z) = \frac{z}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1} e^{-t}}{1 - ze^{-t}} \, dt. \tag{56}$$

The integral converges for $\mathrm{Re}(s) > 0$ because t^{s-1} is integrable near zero, and the exponential decay e^{-t} dominates for large t. The interchange of summation and integration is justified by Fubini's theorem, as the double sum-integral converges absolutely under the given conditions.

This formula connects the polylogarithm to an improper integral involving an exponential denominator, providing a tool for analytic continuation and further analysis. For $z = 1$, it reduces to the Riemann zeta function $\zeta(s)$ when $\mathrm{Re}(s) > 1$. The representation can also be extended to larger domains via analytic continuation.

7.4 Mellin Transform

The Mellin transform is an integral transform that plays a crucial role in mathematical analysis, particularly in the study of asymptotic expansions, number theory, and signal processing. Given a function $f : \mathbb{R}^+ \to \mathbb{C}$, its Mellin transform is defined as:

$$\mathcal{M}\{f\}(s) = \phi(s) = \int_0^\infty x^{s-1} f(x)\, dx, \tag{57}$$

where $s = \sigma + it$ is a complex variable. The transform exists under certain decay conditions on f. Specifically, if f is piecewise continuous and satisfies:

$$\int_0^\infty |f(x)|\, x^{\sigma_1 - 1} dx < \infty \quad \text{and} \quad \int_0^\infty |f(x)|\, x^{\sigma_2 - 1} dx < \infty, \tag{58}$$

for some $\sigma_1 < \sigma_2$, then $\mathcal{M}\{f\}(s)$ is holomorphic in the vertical strip $\sigma_1 < \mathrm{Re}(s) < \sigma_2$. The proof relies on differentiation under the integral sign and dominated convergence. The inversion formula for the Mellin transform is derived from Fourier analysis. If $\phi(s) = \mathcal{M}\{f\}(s)$ converges absolutely in $\sigma_1 < \mathrm{Re}(s) < \sigma_2$, and f is of bounded variation, then:

$$f(x) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} x^{-s} \phi(s)\, ds, \tag{59}$$

where $\sigma \in (\sigma_1, \sigma_2)$. The proof involves a change of variables $x = e^y$, converting the Mellin transform into a Fourier transform, and then applying Fourier inversion.

A key property of the Mellin transform is its convolution theorem. Given two functions f and g with Mellin transforms $\phi(s)$ and $\psi(s)$, their multiplicative convolution is defined as:

$$(f * g)(x) = \int_0^\infty f\left(\frac{x}{y}\right) g(y)\, \frac{dy}{y}. \tag{60}$$

The Mellin transform of this convolution is the product of the individual transforms:

$$\mathcal{M}\{f * g\}(s) = \phi(s) \cdot \psi(s), \tag{61}$$

valid in the intersection of their strips of convergence. The proof follows from a substitution and Fubini's theorem. Finally, Parseval's identity for the Mellin transform relates the integral of a product of two functions to an integral of their Mellin transforms:

$$\int_0^\infty f(x) g(x)\, dx = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \phi(s) \psi(1 - s)\, ds. \tag{62}$$

This result is obtained by expressing f and g via Mellin inversion and interchanging integrals. The Mellin transform thus provides a powerful framework for analyzing functions in multiplicative settings, with deep connections to complex analysis and harmonic analysis.

7.5 Euler's reflection formula

The gamma function $\Gamma(z)$ is defined for $\mathrm{Re}(z) > 0$ by the integral:

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt \tag{63}$$

It extends to a meromorphic function on $\mathbb{C}$ with simple poles at $z = 0, -1, -2, \ldots$. To prove Euler's reflection formula:

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \tag{64}$$

we use Weierstrass's infinite product representation of the gamma function:

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n} \tag{65}$$

where γ is the Euler-Mascheroni constant. Taking the reciprocal gives:

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{z/n} \tag{66}$$

Now, consider the product $\Gamma(z)\Gamma(1-z)$. Substituting the Weierstrass product for both terms:

$$\Gamma(z)\Gamma(1-z) = \left(\frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{z/n}\right) \left(\frac{e^{-\gamma(1-z)}}{1-z} \prod_{n=1}^{\infty} \left(1 + \frac{1-z}{n}\right)^{-1} e^{(1-z)/n}\right) \tag{67}$$

Simplifying the exponents and combining products:

$$\Gamma(z)\Gamma(1-z) = \frac{1}{z(1-z)} e^{-\gamma} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} \left(1 + \frac{1-z}{n}\right)^{-1} e^{1/n} \tag{68}$$

The infinite product can be rewritten as:

$$\prod_{n=1}^{\infty} \frac{n^2}{(n+z)(n+1-z)} = \prod_{n=1}^{\infty} \frac{1}{\left(1 + \frac{z}{n}\right)\left(1 + \frac{1-z}{n}\right)} \tag{69}$$

However, a more elegant simplification comes from Euler's sine product formula:

$$\sin(\pi z) = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right) \tag{70}$$

Taking reciprocals gives:

$$\frac{\pi}{\sin(\pi z)} = \frac{1}{z} \prod_{n=1}^{\infty} \frac{n^2}{n^2 - z^2} \tag{71}$$

But the product in $\Gamma(z)\Gamma(1-z)$ matches this form:

$$\Gamma(z)\Gamma(1-z) = \frac{1}{z} \prod_{n=1}^{\infty} \frac{n^2}{(n+z)(n-z)} = \frac{1}{z} \prod_{n=1}^{\infty} \frac{1}{1 - \frac{z^2}{n^2}} = \frac{\pi}{\sin(\pi z)} \tag{72}$$

Since both sides are meromorphic and agree on $0 < \text{Re}(z) < 1$, analytic continuation ensures the formula holds for all $z \in \mathbb{C} \setminus \mathbb{Z}$. Thus, we conclude:

$$\boxed{\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}} \tag{73}$$

This formula connects the gamma function to trigonometric functions via infinite products and analytic continuation.

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