# Existence of Symmetric Equilibrium for Two-Person Symmetric Equal Row-Sums Games

**By**

**Somdeb Lahiri (Email: somdeb.lahiri@gmail.com)**

**ORCID: https://orcid.org/0000-0002-5247-3497**

**June 22, 2025.**

**This version: July 7, 2025.**

**Abstract**

A matrix is said to be an equal row-sums matrix if all row-sums are equal to one another and a two-person symmetric bi-matrix game is said to be a two-person symmetric equal row-sums (TPSERS) game if the pay-off matrix of the row player is an equal row-sums matrix. For TPSERS games, we show that the randomized strategy that assigns the same probability to all pure strategies is a symmetric equilibrium for the game and the probabilistic component of every solution of a certain quadratic programming problem is a symmetric equilibrium for the game. The main result here is motivated by the "Equivalence Theorem" in section II of Mangasarian and Stone (1964) for bi-matrix games. The version of the "Equivalence Theorem" applicable for symmetric games is available in Lahiri (2025). The proofs of both need to appeal to a prior "existence of equilibrium result" argument when it comes to establishing that every solution of the relevant quadratic programming problem yields an equilibrium for the game under consideration and hence neither proof is self-contained. For TPSERS games, we show there is a simple and self-contained proof that does not either implicitly or explicitly use any prior "existence of equilibrium result" argument and this proof is the main contribution of the paper.

**Keywords:** two-person, symmetric bi-matrix game, equilibrium, linear programming, quadratic programming

**AMS Subject Classifications:** 91A05, 91A10, 90C05, 90C20

**JEL Subject Classifications:** C72, D81

**1. Two-Person Symmetric Equal Row-Sums Games:** For a positive integer $\ell$, let $\Delta^{\ell-1} = \{x \in \mathbb{R}^{\ell}_{+} | \sum_{j=1}^{\ell} x_j = 1\}$ and for $j \in \{1, \ldots, \ell\}$, let $E^{(\ell,j)}$ be the $\ell$-dimensional column vector whose $j^{th}$ coordinate is 1 and all other coordinates are equal to 0. Let $E^{(\ell)} = \sum_{j=1}^{\ell} E^{(\ell,j)}$. $E^{(\ell,j)}$ is said to be the $\ell$-dimensional $j^{th}$ **unit coordinate vector** and $E^{(\ell)}$ is said to be the $\ell$-dimensional **sum vector**.

Given any $\ell \times \ell$ matrix B and $i, j \in \{1, \ldots, \ell\}$, we will denote the $i^{th}$ row of B by $B_i$ and the $j^{th}$ column of B by $B^j$.

The matrix B is said to be an **equal row-sums matrix** if there exists $\alpha \in \mathbb{R}$ such that for all $i \in \{1, \ldots, n\}$, $\sum_{k=1}^{n} b_{ik} = \alpha$.

For a positive integer n, let A be n×n matrix. For $i, j \in \{1, \ldots, n\}$, let $a_{ij}$ denote the $(i, j)^{th}$ entry (i.e., the entry at the intersection of the $i^{th}$ row and $j^{th}$ column) of the matrix A.

The pair $(A, A^T)$ is said to be a **symmetric bi-matrix game**.

$x^* \in \Delta^{n-1}$ is said to be a **symmetric equilibrium** of $(A, A^T)$ if $x^{*T}Ax^* \geq x^TAx^*$ for all $x \in \Delta^{n-1}$.

Thus, (and it is easily verifiable) that $x^* \in \Delta^{n-1}$ is a symmetric equilibrium of $(A, A^T)$ if and only if $(x^{*T}Ax^*)E^{(n)} \geq Ax^*$.

General results about for symmetric bi-matrix games are discussed in Lahiri (2025).

A symmetric bi-matrix game $(A, A^T)$ is said to be a **two-person symmetric equal row-sums (TPSERS) game** if A is an equal row-sums matrix.

**2. The main result:** The main result here is motivated by the "Equivalence Theorem" in section II of Mangasarian and Stone (1964) for bi-matrix games. The version of the "Equivalence Theorem" applicable for symmetric games is available in Lahiri (2025).

**Theorem 1:** Let $(A, A^T)$ be a TPSERS game. Then:

(i) There exists $x^*$, $u^*$ that solve the quadratic programming problem: Maximize $x^TAx - u$, subject to $Ax - uE^{(n)} \leq 0$, $x \in \Delta^{n-1}$, $u \in \mathbb{R}$ (the set of real numbers). Further, $x^{*T}Ax^* - u^* = 0$.

(ii) If $y \in \Delta^{n-1}$, $v \in \mathbb{R}$ solves this quadratic programming problem then y is a symmetric equilibrium for $(A, A^T)$.

**Proof:** Since A is a equal row-sums matrix, there exists there exists $\alpha \in \mathbb{R}$ such that for all $i \in \{1, \ldots, n\}$, $\sum_{k=1}^{n} a_{ik} = \alpha$.

Let $x^* = \frac{1}{n}E^{(n)}$, $u^* = \frac{\alpha}{n}$.

Thus, $Ax^* = \frac{1}{n}AE^{(n)} = \frac{\alpha}{n}E^{(n)}$. Further $x^* \in \Delta^{n-1}$ and $x^{*T}Ax^* - u^* = \frac{1}{n}E^{(n)T}(\frac{\alpha}{n}E^{(n)}) - \frac{\alpha}{n} = \frac{\alpha}{n^2}E^{(n)T}E^{(n)}) - \frac{\alpha}{n} = \frac{\alpha}{n^2}n - \frac{\alpha}{n} = \frac{\alpha}{n} - \frac{\alpha}{n} = 0$.

Let $x \in \Delta^{n-1}$ and $u \in \mathbb{R}$ satisfy $Ax - uE^{(n)} \leq 0$. Thus, $x^TAx - x^TuE^{(n)} \leq 0$.

However, $x^TuE^{(n)} = u$.

Thus, $x^TAx - u \leq 0 = x^{*T}Ax^* - u^*$, whence $x^*$, $u^*$ solves the quadratic programming problem. This proves (i).

Now suppose y, v solves the quadratic programming problem. Then, $y \in \Delta^{n-1}$, $Ay \leq vE^{(n)}$ and from (i) we know that $y^TAy - v = 0$.

Hence, $Ay \leq vE^{(n)} = (y^TAy)E^{(n)}$.

Thus, y is a symmetric equilibrium for $(A, A^T)$.

This proves (ii). Q.E.D.

## References


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