Threshold Hypergraphs, Hyper-PolyGraphs, and Their Extensions

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Abstract

Modern graph theory investigates the interplay of vertices and edges to represent complex relationships and connectivity patterns [1, 2]. Hypergraphs extend this paradigm by allowing hyperedges to join any number of vertices in a single relation [3], and superhypergraphs further enrich these models via iterated powerset constructions that capture multi-layered, hierarchical connections among edges [4, 5]. Threshold Graphs form a classical family built by iteratively adding either isolated vertices or vertices adjacent to all existing ones according to a numeric threshold rule. PolyGraphs have likewise been studied extensively in combinatorial and applied settings. In this work, we introduce the concepts of *Threshold SuperHypergraphs* and *SuperHyper-PolyGraphs*, thereby generalizing Threshold Graphs and PolyGraphs within the superhypergraph framework and providing a unified, hierarchical approach to threshold-based and polygraphic structures.

Keywords: Superhypergraph, Hypergraph, Threshold SuperHypergraph, Threshold Graph, PolyGraph, Hyper-PolyGraph

1 Introduction

1.1 From Graphs to SuperHyperGraphs

Traditional graph theory represents binary relations by modeling entities as vertices and their interactions as edges [1,6]. Hypergraphs broaden this framework by permitting each hyperedge to link any nonempty subset of vertices, thereby capturing complex, higher-order interactions beyond simple pairs [7–9]. SuperHyperGraphs advance these ideas further: by iteratively applying the powerset operator to the vertex set, they produce multi-tiered structures that encode nested, hierarchical relationships among groups of vertices [10–12].

1.2 Contributions

In this work, we introduce and formalize the notion of a *Threshold SuperHyperGraph*, incorporating the threshold-based vertex-addition rules of Threshold Graphs into the SuperHyperGraph paradigm [13–18]. We also define the *SuperHyper-PolyGraph*, a natural extension of PolyGraph theory achieved via iterated powerset constructions. Concretely, a Threshold SuperHyperGraph is an n-SuperHyperGraph in which each hyperedge consists of n-supervertices whose aggregated weights, as determined by a given weight function, meet or exceed a specified threshold.

2 Preliminaries

In this section, we establish the notation and fundamental concepts used throughout this paper. Unless specified otherwise, all graphs are finite. For more detailed treatments of these notions, see the cited references.

2.1 SuperHyperGraph

A *hypergraph* extends the concept of a classical graph by allowing *hyperedges* that may connect any number of vertices at once [3, 7, 8, 19, 20]. Building on this, a *SuperHyperGraph* employs iterated powerset constructions to represent nested, hierarchical relationships among hyperedges—a subject currently receiving significant attention [5, 12, 21–25]. Applications of SuperHyperGraphs range from molecular structure modeling to complex network analysis and signal processing [26–29]. Throughout, the integer n indicates the depth of the powerset iteration in an n-SuperHyperGraph.

Definition 2.1 (Base Set)**.** A *base set* S serves as the ground domain for all subsequent constructions:

$$S = \{ x \mid x \text{ is an element of the universe} \}.$$

Every object in the powerset $\mathcal{P}(S)$ or any iterated powerset $\mathcal{P}^n(S)$ originates from S.

Definition 2.2 (Powerset)**.** For any set S, its *powerset* $\mathcal{P}(S)$ is the collection of all subsets of S, including the empty set and S itself:

$$\mathcal{P}(S) = \{\, A \mid A \subseteq S \}.$$

Definition 2.3 (Hypergraph)**.** [3,30] A *hypergraph* $H = (V, E)$ consists of

- A finite set of vertices V.
- A finite family E of nonempty subsets of V, called hyperedges.

Hypergraphs naturally capture multi-way relationships among elements of V.

Example 2.4 (Retail Market-Basket Transactions as a Hypergraph)**.** A common application of hypergraphs arises in market-basket analysis. Let

$$V = \{\text{Milk, Bread, Eggs, Cheese, Diapers, Beer}\}$$

be the set of products in a supermarket. Each shopping transaction corresponds to a nonempty subset of V. For instance, we might observe:

$$E = \big\{\{\text{Milk}, \text{Bread}\},\ \{\text{Bread}, \text{Cheese}, \text{Eggs}\},\ \{\text{Diapers}, \text{Beer}\},\ \{\text{Milk}, \text{Diapers}, \text{Beer}\}\big\}.$$

Then

$$H = (V, E)$$

is a hypergraph in which each hyperedge represents the set of items purchased together in a single transaction. This model allows analysis of higher-order co-purchase patterns that cannot be captured by ordinary pairwise graphs.

Definition 2.5 (n-th Powerset)**.** [31–37] Given a set X, define its iterated powersets by

$$\mathcal{P}_1(X) = \mathcal{P}(X), \quad \mathcal{P}_{n+1}(X) = \mathcal{P}\big(P_n(X)\big), \quad n \geq 1.$$

The *nonempty* version $\mathcal{P}_n^*(X)$ omits the empty set at each stage.

Example 2.6 (Software Feature-Flag Configurations as an Iterated Powerset)**.** In software testing, one often explores combinations of feature flags. Let

$$X = \{\texttt{Login}, \texttt{Payment}\}$$

be the set of two feature toggles. The first-level powerset

$$\mathcal{P}_1(X) = \mathcal{P}(X) = \big\{\, \emptyset, \{\texttt{Login}\},$$

$$\{\texttt{Payment}\}, \{\texttt{Login}, \texttt{Payment}\}\big\}$$

enumerates all possible on/off combinations of these flags. Taking the powerset again yields

$$\mathcal{P}_2(X) = \mathcal{P}\big(\mathcal{P}_1(X)\big)$$

$$= \big\{\, Y \subseteq \mathcal{P}_1(X)\big\},$$

which corresponds to all collections of configurations—for example, grouping configurations into test suites:

$$\{\{\texttt{Login}\}, \{\texttt{Payment}\}\} \in \mathcal{P}_2(X),$$

$$\{\{\emptyset\}, \{\texttt{Login}, \texttt{Payment}\}\} \in \mathcal{P}_2(X),$$

and so on. Higher-order powersets model nested groupings of scenarios, useful for organizing test batches or deployment stages.

Definition 2.7 (n-SuperHyperGraph)**.** [10, 38, 39] Let V_0 be a finite base set. Define iteratively

$$\mathcal{P}^0(V_0) = V_0,$$
$$\mathcal{P}^{k+1}(V_0) = \mathcal{P}\big(\mathcal{P}^k(V_0)\big).$$

An *n-SuperHyperGraph* is a pair

$$\text{SuperHG}^{(n)} = (V, E), \quad V, E \subseteq \mathcal{P}^n(V_0),$$

where each element of V is called an n-supervertex and each element of E an n-superedge.

Example 2.8 (Office Building HVAC Sensor Network as a 2-SuperHyperGraph)**.** Consider an office building equipped with a finite set of temperature sensors. We model this as a 2-SuperHyperGraph as follows:

$$V_0 = \{\, s_{i,j} \mid j = 1, \ldots, n,\ i = 1, \ldots, m_j\}$$

is the set of all sensors, where $s_{i,j}$ is the i-th sensor in room i on floor j. We then form

$$V_1 = \{\, r_{i,j} \mid r_{i,j} = \{\, s_{i,j}\} \subseteq V_0\}$$

so that each 1-supervertex $r_{i,j}$ represents the singleton set containing the sensor in room i on floor j. Next, we group rooms into floors:

$$V_2 = \{\, f_j \mid f_j = \{\, r_{i,j} \mid i = 1, \ldots, m_j\} \subseteq \mathcal{P}(V_1)\} \quad (j = 1, \ldots, n),$$

so that each 2-supervertex f_j is the set of all room-vertices on floor j.

Finally, we define the family of 2-superedges $E_2 \subseteq \mathcal{P}(V_2) \setminus \{\emptyset\}$ to capture the building's HVAC control zones. For example, if the building has three HVAC zones grouping floors $\{1, 2\}$, $\{2, 3\}$, and $\{1, 2, 3\}$, then

$$E_2 = \big\{\, \{f_1, f_2\},\ \{f_2, f_3\},\ \{f_1, f_2, f_3\}\big\}.$$

Thus

$$S^{(2)} = \big(V_2,\ E_2\big)$$

is a 2-SuperHyperGraph whose supervertices are the floors of the building and whose superedges are the HVAC zones that group together multiple floors.

Example 2.9 (Microservice Architecture as a 2-SuperHyperGraph)**.** In a modern software system, small units of functionality (functions) are grouped into classes, which in turn are deployed as microservices. We model this hierarchy as a 2-SuperHyperGraph:

$$V_0 = \{\, f_1, f_2, f_3, f_4, f_5, f_6\}$$

is the set of all functions. Define the classes as 1-supervertices:

$$V_1 = \big\{C_1 = \{f_1, f_2\},\ C_2 = \{f_3, f_4, f_5\},$$
$$C_3 = \{f_6\}\big\}\ \subseteq\ \mathcal{P}(V_0).$$

Next, assemble these classes into microservices, which form the 2-supervertices:

$$V_2 = \big\{S_1 = \{C_1, C_2\},$$
$$S_2 = \{C_2, C_3\}\big\}\ \subseteq\ \mathcal{P}(V_1).$$

Finally, we capture deployment groupings of microservices as 2-superedges. For example, suppose there are two deployment environments:

$$E_2 = \big\{D_{\text{prod}} = \{S_1, S_2\},$$
$$D_{\text{canary}} = \{S_1\}\big\}\ \subseteq\ \mathcal{P}(V_2) \setminus \{\emptyset\}.$$

Then

$$\text{SuperHG}^{(2)} = \big(V_2,\ E_2\big)$$

is a 2-SuperHyperGraph in which supervertices are microservices and superedges represent deployment clusters.

2.2 Threshold hypergraphs

A Threshold Graph is a type of graph that can be constructed by successively adding either an isolated vertex or a vertex connected to all existing vertices, based on a threshold condition [13–18, 40, 41]. A Threshold Hypergraph is a hypergraph in which a subset of vertices forms a hyperedge if the total weight of its vertices, assigned by a weight function, exceeds a given nonnegative threshold [42–47].

Definition 2.10 (Threshold Hypergraph). (cf. [42,43]) Let $H = (V, E)$ be a hypergraph with

$$E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}.$$

We say that H is a *threshold hypergraph* if there exists

- a weight function $w: V \to \mathbb{R}_{\geq 0}$, and
- a threshold $\tau \in \mathbb{R}_{\geq 0}$,

such that for every nonempty subset $X \subseteq V$,

$$X \in E \quad \Longleftrightarrow \quad \sum_{v \in X} w(v) \geq \tau.$$

Equivalently, the set of hyperedges of H is exactly the family of all nonempty vertex–subsets whose total weight meets or exceeds the threshold τ.

Example 2.11 (Online Retail Free-Shipping Model as a Threshold Hypergraph). Consider an e-commerce platform where customers receive free shipping if the total price of their order meets or exceeds a fixed threshold. Let

$$V = \{p_A, p_B, p_C, p_D\}$$

be the set of available products, with prices (weights)

$$w(p_A) = 20, \quad w(p_B) = 15, \quad w(p_C) = 30, \quad w(p_D) = 10,$$

and let the free-shipping threshold be

$$\tau = 50.$$

Define the hyperedge family

$$E = \Big\{ X \subseteq V \setminus \{\emptyset\} \mid \sum_{v \in X} w(v) \geq 50 \Big\}.$$

Then $H = (V, E)$ is a threshold hypergraph. For example:

$$\{p_A, p_C\}, \quad \{p_B, p_C, p_D\}, \quad \{p_A, p_B, p_C\} \in E,$$

since $w(p_A) + w(p_C) = 50$, $w(p_B) + w(p_C) + w(p_D) = 55$, $w(p_A) + w(p_B) + w(p_C) = 65$, all exceed the threshold, whereas $\{p_A, p_B\}$ (sum = 35) and $\{p_C, p_D\}$ (sum = 40) are not in E.

2.3 Polygraph and Hyperpolygraph

A polygraph is a combinatorial structure where edges connect nodes or edges, forming hierarchical, acyclic, well-founded binary containment relationships using meta-edges [48–50]. A hyper-polygraph generalizes polygraphs by allowing finite hyperedges connecting nodes and edges, supporting nested, well-founded, complex multi-element containment relationships [50].

Definition 2.12 (Polygraph). [50] Let X be a set of *nodes* and E a set of *edges*, with $X \cap E = \emptyset$. A *polygraph* is the combinatorial structure (X, E) satisfying:

$$\forall e \in E, \quad e \subseteq X \cup E, \quad |e| = 2,$$

and such that there is no infinite sequence $e_1 \ni e_2 \ni e_3 \ni \cdots$ of edges containing edges. In other words:

- Each edge is a pair of elements drawn from $X \cup E$.
- Edges may have other edges as their endpoints.
- The containment relation on E is well-founded (acyclic).

Example 2.13 (Political Endorsement Network as a Polygraph)**.** Consider a network of political endorsements in which both individuals and endorsement-events can themselves be endorsed. Let

$$X = \{\text{Alice}, \text{ Bob}, \text{ Carol}, \ldots\}$$

be the set of *actors* (politicians, organizations), and let E be the set of *endorsement-edges*, defined by

$$e \in E \iff e = \{x, y\} \subseteq X \cup E, \ |e| = 2,$$

where:

- If $e = \{\text{Alice}, \text{Bill1}\}$, it means "Alice endorses Bill 1."
- If $e = \{\text{Committee}, e_1\}$ with $e_1 = \{\text{Alice}, \text{Bill1}\}$, it means "The Committee endorses Alice's endorsement of Bill 1."

There is no infinite chain $e_1 \ni e_2 \ni e_3 \ni \cdots$ because endorsements are issued at discrete times without self-referential loops. Hence (X, E) forms a polygraph in which edges may connect actors or other edges, modeling both first-order and meta-endorsements.

Definition 2.14 (Hyper-polygraph)**.** [50] Let X be a set of *nodes* and E a set of *hyperedges*, with $X \cap E = \emptyset$. A *hyper-polygraph* is the pair (X, E) such that

$$\forall e \in E, \quad e \subseteq X \cup E, \quad 1 \leq |e| < \infty,$$

and there is no infinite descending chain of hyperedges $e_1 \ni e_2 \ni e_3 \ni \cdots$. Hence:

- Each hyperedge is a finite nonempty subset of $X \cup E$.
- Hyperedges may include both nodes and other hyperedges.
- The membership relation among hyperedges is well-founded.

Example 2.15 (File System Directory Structure as a Hyper-polygraph)**.** A typical file system organizes files and directories, where directories may contain both files and other directories. Let

$$X = \{\texttt{file}_1, \texttt{file}_2, \ldots\}$$

be the set of all *files*, and let E be the set of *directories*, each of which is a finite nonempty subset of $X \cup E$:

$$e \in E \iff e \subseteq X \cup E, \ 1 \leq |e| < \infty.$$

For instance,

$$\texttt{dir\_A} = \{\texttt{file}_1, \texttt{file}_2, \texttt{dir\_B}\}, \quad \texttt{dir\_B} = \{\texttt{file}_3\}.$$

There is no infinite descending chain $e_1 \ni e_2 \ni e_3 \ni \cdots$ because directory nesting terminates in files. Thus (X, E) is a hyper-polygraph capturing the hierarchical structure of files and directories.

3 Result: Threshold SuperHypergraphs

A Threshold SuperHypergraph is an n-SuperHyperGraph where each hyperedge consists of supervertices whose total weight exceeds a threshold.

Definition 3.1 (Threshold n-SuperHypergraph)**.** Let V_0 be a finite base set and let

$$\mathcal{P}^n(V_0) = \underbrace{\mathcal{P}(\mathcal{P}(\cdots\mathcal{P}}_{n \text{ times}}(V_0)\cdots))$$

be its n-th iterated powerset. An *n-SuperHyperGraph* is a pair $S^{(n)} = (V_n, E_n)$ with

$$V_n \subseteq \mathcal{P}^n(V_0), \qquad E_n \subseteq \mathcal{P}(V_n) \setminus \{\emptyset\}.$$

We say that $S^{(n)}$ is a *threshold n-SuperHyperGraph* if there exist

- a *weight function* $w\colon V_n \to \mathbb{R}_{\geq 0}$, and
- a *threshold* $\tau \in \mathbb{R}_{\geq 0}$,

such that for every nonempty subset $X \subseteq V_n$,

$$X \in E_n \quad \Longleftrightarrow \quad \sum_{v \in X} w(v) \;\geq\; \tau.$$

In other words, the hyperedges of $S^{(n)}$ are exactly those nonempty collections of n-supervertices whose total weight meets or exceeds τ.

Example 3.2 (Cloud Infrastructure Auto-Scaling as a Threshold 3-SuperHyperGraph)**.** Consider a cloud provider's infrastructure with three hierarchical levels:

$$V_0 = \{\, s_1, \ldots, s_m \}$$

is the set of all physical servers. We build iterated powersets:

$$V_1 = \big\{\{s_i\} \mid i = 1, \ldots, m\big\}, \qquad V_2 = \big\{\, C_j \subseteq V_1 \mid j = 1, \ldots, p\big\}, \qquad V_3 = \big\{\, R_k \subseteq V_2 \mid k = 1, \ldots, q\big\},$$

where each C_j is a server cluster and each R_k is a region grouping clusters.

Define a weight function

$$w\colon V_3 \to \mathbb{R}_{\geq 0}, \quad w(R_k) = \sum_{C_j \in R_k} \sum_{\{s_i\} \in C_j} \mathrm{CPU}(s_i),$$

i.e. the total CPU capacity in region R_k. Let τ be the auto-scaling activation threshold (e.g. 10,000 CPU-cores). Then set

$$E_3 = \big\{\, X \subseteq V_3 \setminus \{\emptyset\} \mid \sum_{R_k \in X} w(R_k) \geq \tau \big\}.$$

The pair

$$S^{(3)} = (V_3,\ E_3)$$

is a threshold 3-SuperHyperGraph: any collection of regions whose combined capacity meets or exceeds τ triggers auto-scaling across those regions.

Example 3.3 (International Vaccine Allocation as a Threshold 2-SuperHyperGraph)**.** A global health organization must allocate vaccine doses to clinics via districts and countries. We model this as a threshold 2-SuperHyperGraph:

$$V_0 = \{\, c_{i,j} \mid j = 1, \ldots, r,\ i = 1, \ldots, m_j \},$$

where $c_{i,j}$ is the i-th clinic in district j. Define

$$V_1 = \{\, D_j \mid D_j = \{\, c_{i,j} \mid i = 1, \ldots, m_j \} \subseteq V_0,\ j = 1, \ldots, r \},$$

so each 1-supervertex D_j is a district. Next, group districts into countries:

$$V_2 = \{\, C_k \mid C_k = \{\, D_j \mid j \in I_k \} \subseteq \mathcal{P}(V_1),\ k = 1, \ldots, p \},$$

where $\{I_1, \ldots, I_p\}$ partitions $\{1, \ldots, r\}$ by country.

Let each clinic $c_{i,j}$ have a demand $d(c_{i,j})$ (doses needed). Define the weight

$$w\colon V_2 \to \mathbb{R}_{\geq 0}, \qquad w(C_k) = \sum_{D_j \in C_k} \sum_{c \in D_j} d(c),$$

the total demand in country k. Choose a threshold τ (e.g. 10 million doses). Then set

$$E_2 = \left\{ X \subseteq V_2 \setminus \{\emptyset\} \;\middle|\; \sum_{C \in X} w(C) \geq \tau \right\}.$$

The pair

$$S^{(2)} = (V_2, E_2)$$

is a threshold 2-SuperHyperGraph: any collection of countries whose combined vaccine demand meets or exceeds τ triggers a special joint procurement action.

Theorem 3.4 (Generalization of Threshold Hypergraphs)**.** *Every threshold* 1*-SuperHyperGraph is precisely a threshold hypergraph, and conversely every threshold hypergraph arises as a threshold* 1*-SuperHyperGraph.*

Proof. By Definition 3.1, a threshold 1-SuperHyperGraph $S^{(1)} = (V_1, E_1)$ satisfies

$$V_1 \subseteq \mathcal{P}(V_0), \quad E_1 \subseteq \mathcal{P}(V_1) \setminus \{\emptyset\},$$

with a weight function $w\colon V_1 \to \mathbb{R}_{\geq 0}$ and threshold τ such that $X \in E_1 \iff \sum_{v \in X} w(v) \geq \tau$. But this is exactly the definition of a threshold hypergraph on the vertex set V_1 (Definition 2.10), with V_1 playing the role of the vertex set and E_1 the hyperedge family.

Conversely, let $H = (V, E)$ be any threshold hypergraph with weight function $w_H\colon V \to \mathbb{R}_{\geq 0}$ and threshold τ_H. Set

$$V_0 := V, \quad V_1 := V, \quad E_1 := E, \quad w(v) := w_H(v) \quad (\forall v \in V_1), \quad \tau := \tau_H.$$

Then (V_1, E_1) is a threshold 1-SuperHyperGraph by Definition 3.1. This completes the equivalence. □

Theorem 3.5 (Threshold n-SuperHyperGraphs are n-SuperHyperGraphs)**.** *Every threshold n-SuperHyperGraph $S^{(n)} = (V_n, E_n)$ satisfies the axioms of an n-SuperHyperGraph.*

Proof. By assumption $S^{(n)} = (V_n, E_n)$ is a threshold n-SuperHyperGraph, so in particular

$$V_n \subseteq \mathcal{P}^n(V_0), \quad E_n \subseteq \mathcal{P}(V_n) \setminus \{\emptyset\}.$$

These two inclusions are exactly the defining conditions for an n-SuperHyperGraph. The additional structure of a weight function and threshold does not violate any of these conditions, but only endows the hypergraph with a numerical selection rule. Hence every threshold n-SuperHyperGraph is an n-SuperHyperGraph. □

Theorem 3.6 (Upward-Closure)**.** *Let $S^{(n)} = (V_n, E_n)$ be a threshold n-SuperHyperGraph with weight function $w\colon V_n \to \mathbb{R}_{\geq 0}$ and threshold $\tau \in \mathbb{R}_{\geq 0}$. Then for any $X, Y \subseteq V_n$ with $X \subseteq Y$,*

$$X \in E_n \quad \Longrightarrow \quad Y \in E_n.$$

Proof. If $X \in E_n$ then $\sum_{v \in X} w(v) \geq \tau$. Since $X \subseteq Y$,

$$\sum_{v \in Y} w(v) = \sum_{v \in X} w(v) \;+ \sum_{v \in Y \setminus X} w(v) \;\geq\; \sum_{v \in X} w(v) \;\geq\; \tau.$$

Hence $Y \in E_n$ by Definition 3.1. □

Theorem 3.7 (Closure under Intersection)**.** *Let $S_1^{(n)} = (V_n, E_n^1)$ and $S_2^{(n)} = (V_n, E_n^2)$ be two threshold n-SuperHyperGraphs on the same V_n with the same weight w but thresholds τ_1, τ_2. Then*

$$S^{(n)} = (V_n,\ E_n^1 \cap E_n^2)$$

is a threshold n-SuperHyperGraph with threshold $\max\{\tau_1, \tau_2\}$.

Proof. Define $\tau = \max\{\tau_1, \tau_2\}$. For any nonempty $X \subseteq V_n$,

$$X \in E_n^1 \cap E_n^2 \iff \sum_{v \in X} w(v) \geq \tau_1 \text{ and } \sum_{v \in X} w(v) \geq \tau_2 \iff \sum_{v \in X} w(v) \geq \tau.$$

Thus $(V_n, E_n^1 \cap E_n^2)$ satisfies the threshold condition with weight w and threshold τ. □

Theorem 3.8 (Closure under Union). *Let $S_1^{(n)} = (V_n, E_n^1)$ and $S_2^{(n)} = (V_n, E_n^2)$ be two threshold n-SuperHyperGraphs on the same V_n with the same weight w but thresholds τ_1, τ_2. Then*

$$S^{(n)} = (V_n,\ E_n^1 \cup E_n^2)$$

is a threshold n-SuperHyperGraph with threshold $\min\{\tau_1, \tau_2\}$.

Proof. Define $\tau = \min\{\tau_1, \tau_2\}$. For any nonempty $X \subseteq V_n$,

$$X \in E_n^1 \cup E_n^2 \iff \sum_{v \in X} w(v) \geq \tau_1 \text{ or } \sum_{v \in X} w(v) \geq \tau_2 \iff \sum_{v \in X} w(v) \geq \tau.$$

Hence $(V_n, E_n^1 \cup E_n^2)$ is threshold with weight w and threshold τ. □

Theorem 3.9 (Scaling Invariance). *Let $S^{(n)} = (V_n, E_n)$ be a threshold n-SuperHyperGraph with weight w and threshold τ, and let $\alpha > 0$. Then*

$$S'^{(n)} = (V_n, E_n)$$

is also a threshold n-SuperHyperGraph with weight $\alpha\, w$ and threshold $\alpha\, \tau$.

Proof. For any nonempty $X \subseteq V_n$,

$$\sum_{v \in X} \big(\alpha\, w(v)\big) \ \geq\ \alpha\, \tau \iff \alpha \sum_{v \in X} w(v) \ \geq\ \alpha\, \tau \iff \sum_{v \in X} w(v) \ \geq\ \tau.$$

Thus the family of hyperedges remains E_n, showing invariance under positive scaling. □

Theorem 3.10 (Induced Threshold Subhypergraph). *Let $S^{(n)} = (V_n, E_n)$ be a threshold n-SuperHyperGraph with weight $w\colon V_n \to \mathbb{R}_{\geq 0}$ and threshold τ. For any nonempty subset $W \subseteq V_n$, define the* induced subhypergraph

$$S^{(n)}[W] \ =\ \big(W,\ E_n \cap \mathcal{P}(W)\big).$$

Then $S^{(n)}[W]$ is a threshold n-SuperHyperGraph with weight $w|_W$ and the same threshold τ.

Proof. By definition,

$$E_n \cap \mathcal{P}(W) = \{\, X \subseteq W \mid X \in E_n \} = \{\, X \subseteq W \mid \sum_{v \in X} w(v) \geq \tau \},$$

so each hyperedge of $S^{(n)}[W]$ satisfies the threshold condition with $w|_W$ and τ. Clearly $W \subseteq \mathcal{P}^n(V_0)$ and $E_n \cap \mathcal{P}(W) \subseteq \mathcal{P}(W) \setminus \{\emptyset\}$, so $S^{(n)}[W]$ meets all axioms of a threshold n-SuperHyperGraph. □

Theorem 3.11 (Minimal Generators). *Let $S^{(n)} = (V_n, E_n)$ be a threshold n-SuperHyperGraph with weight w and threshold τ. Define the set of* minimal hyperedges

$$\mathcal{M} = \Big\{ M \in E_n \ \Big|\ \forall\, v \in M,\ \sum_{u \in M \setminus \{v\}} w(u) < \tau \Big\}.$$

Then $\mathcal{M}$ is an antichain under inclusion, and

$$E_n = \{\, X \subseteq V_n \mid \exists\, M \in \mathcal{M},\ M \subseteq X \}.$$

Proof. First, by upward-closure (Theorem), if $M \in \mathcal{M}$ and $M \subseteq X$ then $X \in E_n$, so the right-hand side is contained in E_n. Conversely, for any $X \in E_n$, if $X \notin \mathcal{M}$ then there exists $v \in X$ with $\sum_{u \in X \setminus \{v\}} w(u) \geq \tau$, so $X \setminus \{v\} \in E_n$. Repeating this deletion process yields a minimal $M \subseteq X$ satisfying the strict inequality for every proper subset, hence $M \in \mathcal{M}$. Thus X contains some $M \in \mathcal{M}$, proving equality.

Furthermore, if $M_1, M_2 \in \mathcal{M}$ with $M_1 \subsetneq M_2$, then $M_2 \setminus \{v\} \supseteq M_1$ for some $v \in M_2$, implying $\sum_{u \in M_2 \setminus \{v\}} w(u) \geq \tau$, contradicting minimality of M_2. Hence $\mathcal{M}$ is an antichain. □

Theorem 3.12 (Linear Separability Characterization). *A family $E_n \subseteq \mathcal{P}(V_n) \setminus \{\emptyset\}$ is the set of hyperedges of a threshold n-SuperHyperGraph if and only if the set of 0–1 indicator vectors*

$$\{\chi_X \in \{0,1\}^{V_n} \mid X \in E_n\}$$

can be separated from its complement by a hyperplane in $\mathbb{R}^{V_n}$.

Proof. ($\Rightarrow$) If E_n is defined by weight w and threshold τ, then the linear functional $f(x) = \sum_{v \in V_n} w(v)\, x_v$ satisfies

$$\chi_X \in E_n \iff f(\chi_X) \geq \tau,$$

whereas $\chi_X \notin E_n \iff f(\chi_X) < \tau$. Thus $\{\chi_X \mid X \in E_n\}$ lies on one side of the hyperplane $f(x) = \tau$.

($\Leftarrow$) Conversely, if there exists a nonzero linear functional $f(x) = \sum_v w(v) x_v$ and scalar τ such that $f(\chi_X) \geq \tau$ exactly for $X \in E_n$, then setting the same w, τ realizes E_n as a threshold family. □

Theorem 3.13 (Dual Downward-Closure). *Let $S^{(n)} = (V_n, E_n)$ be a threshold n-SuperHyperGraph with weight w and threshold τ. Then the complement family*

$$\overline{E_n} = \{\, X \subseteq V_n \mid \sum_{v \in X} w(v) < \tau\}$$

is downward-closed:if $X \in \overline{E_n}$ and $Y \subseteq X$, then $Y \in \overline{E_n}$.

Proof. If $X \in \overline{E_n}$ then $\sum_{v \in X} w(v) < \tau$. For any $Y \subseteq X$,

$$\sum_{v \in Y} w(v) \leq \sum_{v \in X} w(v) < \tau,$$

so $Y \in \overline{E_n}$ by definition. □

4 Result: SuperHyper-polygraph

A SuperHyper-polygraph extends hyper-polygraphs by combining iterated powerset vertex hierarchy and finite, nested hyperedges with acyclic containment at order n.

Definition 4.1 (SuperHyper-polygraph). Let V_0 be a finite base set and fix $n \geq 0$. For each $k \geq 0$ define

$$\mathcal{P}^k(V_0) \;=\; \begin{cases} V_0, & k = 0, \\ \mathcal{P}\big(\mathcal{P}^{k-1}(V_0)\big), & k \geq 1. \end{cases}$$

An *n-order SuperHyper-polygraph* is a pair

$$\mathcal{G}^{(n)} \;=\; \big(V_n,\; E_n\big)$$

with

$$V_n \;\subseteq\; \mathcal{P}^n(V_0), \qquad E_n \;\subseteq\; \big\{\, e \mid e \subseteq V_n \cup E_n,\; 1 \leq |e| < \infty \big\}, \qquad V_n \cap E_n = \varnothing,$$

such that there is no infinite descending chain of hyperedges $e_1 \ni e_2 \ni e_3 \ni \cdots$ in E_n. In other words:

- Each *supernode* is an n-supervertex $v \in V_n$.

- Each *superhyperedge* $e \in E_n$ is a finite nonempty subset of $V_n \cup E_n$, allowing nesting of edges.
- The membership relation on E_n is well-founded (no infinite descent).

Example 4.2 (Release Pipeline Management as a 2-order SuperHyper-polygraph)**.** In a typical CI/CD system, individual test cases are grouped into test suites, which in turn are orchestrated into pipelines. We model this hierarchy as a 2-order SuperHyper-polygraph:

$$V_0 = \{\, t_1, t_2, t_3, t_4\}$$

is the set of unit test cases. We define test suites as 1-supervertices:

$$V_1 = \{S_{\text{core}} = \{t_1, t_2\},\ S_{\text{ui}} = \{t_3, t_4\}\} \ \subseteq\ \mathcal{P}(V_0).$$

Next, we form pipelines as 2-supervertices:

$$V_2 = \{P_{\text{build}} = \{S_{\text{core}}\},\ P_{\text{release}} = \{S_{\text{core}}, S_{\text{ui}}\}\} \ \subseteq\ \mathcal{P}(V_1).$$

Finally, we capture orchestration stages—which may include both pipelines and other stages—as 2-superedges:

$$E_2 = \{\text{Stage}_1 = \{P_{\text{build}}, P_{\text{release}}\},\ \text{Stage}_2 = \{P_{\text{release}}, \text{Stage}_1\}\} \ \subseteq\ \mathcal{P}(V_2 \cup E_2) \setminus \{\emptyset\}.$$

Since each edge is a finite nonempty subset of $V_2 \cup E_2$, $V_2 \cap E_2 = \emptyset$, and there is no infinite chain $\text{Stage}_2 \ni \text{Stage}_1 \ni P_{\text{build}}$, the pair

$$\mathcal{G}^{(2)} = (V_2,\, E_2)$$

is a valid 2-order SuperHyper-polygraph modeling the nested structure of test orchestration and release stages.

Example 4.3 (File System Directory and Volume Structure as a 2-order SuperHyper-polygraph)**.** Model a file system with files grouped into directories, and directories grouped into volumes:

$$V_0 = \{\texttt{file1}, \texttt{file2}, \texttt{file3}, \texttt{file4}\}$$

is the set of all files. Form directories as 1-supervertices:

$$V_1 = \{D_A = \{\texttt{file1}, \texttt{file2}\},\quad D_B = \{\texttt{file3}\},\quad D_C = \{\texttt{file4}\}\} \subseteq \mathcal{P}(V_0).$$

Next, group directories into volumes as 2-supervertices:

$$V_2 = \{V_{\text{X}} = \{D_A, D_B\},\quad V_{\text{Y}} = \{D_B, D_C\}\} \ \subseteq\ \mathcal{P}(V_1).$$

Define backup tasks as 2-superedges that may include both volumes and other tasks:

$$E_2 = \{T_1 = \{V_{\text{X}},\, V_{\text{Y}}\},\quad T_2 = \{V_{\text{Y}},\, T_1\}\} \ \subseteq\ \mathcal{P}(V_2 \cup E_2) \setminus \{\emptyset\}.$$

Here, T_1 represents a backup of both volumes X and Y, and T_2 represents a meta-task that runs T_1 after backing up volume Y. Since $V_2 \cap E_2 = \emptyset$ and there is no infinite chain $T_2 \ni T_1 \ni V_{\text{X}}$, the pair

$$\mathcal{G}^{(2)} = (V_2, E_2)$$

is a valid 2-order SuperHyper-polygraph capturing nested backup operations in the file system.

Example 4.4 (Corporate Structure with Nested Projects as a 2-order SuperHyper-polygraph)**.** Consider a company with four employees, organized into teams and departments, and running hierarchical projects:

$$V_0 = \{\, e_1, e_2, e_3, e_4\}$$

are the individual employees. Form the teams:

$$V_1 = \{\, T_1 = \{e_1, e_2\},\ T_2 = \{e_3, e_4\}\} \ \subseteq \mathcal{P}(V_0).$$

Next, group teams into departments:

$$V_2 = \{\, D_1 = \{T_1\},\ D_2 = \{T_2\}\} \ \subseteq \mathcal{P}(V_1).$$

Thus each element of V_2 is a 2-supervertex (a department).

Now define a hierarchy of cross-department projects as hyperedges that may include both departments and other projects:

$$E_2 = \{P_A = \{D_1, D_2\},\ P_B = \{D_2,\ P_A\},\ P_C = \{D_1,\ P_B\}\}.$$

Here

- P_A is a project involving departments D_1 and D_2,
- P_B builds on P_A and D_2,
- P_C builds on P_B and D_1.

Since

$$V_2 \ \cap\ E_2 \ =\ \emptyset,$$

and there is no infinite descending chain (e.g. $P_C \ni P_B \ni P_A \ni D_1$, which terminates), the pair

$$\mathcal{G}^{(2)} = (V_2, E_2)$$

is a 2-order SuperHyper-polygraph modeling nested projects across departments.

Example 4.5 (Academic Collaboration Consortia as a 2-order SuperHyper-polygraph)**.** Consider an academic ecosystem with researchers, research groups, and institutes:

$$V_0 = \{\, r_1, r_2, r_3, r_4, r_5\}$$

is the set of individual researchers. Form research groups:

$$V_1 = \{\, G_1 = \{r_1, r_2\},\ G_2 = \{r_3, r_4, r_5\}\} \ \subseteq\ \mathcal{P}(V_0),$$

so each G_i is a 1-supervertex (a research group). Next, form institutes:

$$V_2 = \{\, I_1 = \{G_1\},\ I_2 = \{G_2\}\} \ \subseteq\ \mathcal{P}(V_1),$$

so each I_k is a 2-supervertex (an institute).

Define collaboration consortia as hyperedges that may include both institutes and other consortia:

$$E_2 = \{C_1,\ C_2,\ C_3\} \subseteq \{\, e \mid e \subseteq V_2 \cup E_2,\ 1 \le |e| < \infty\},$$

where

$$C_1 = \{I_1,\ I_2\}, \quad C_2 = \{I_2,\ C_1\}, \quad C_3 = \{I_1,\ C_2\}.$$

Here:

- C_1 is a consortium of institutes I_1 and I_2.
- C_2 builds on C_1 by adding institute I_2 again.
- C_3 builds on C_2 by adding institute I_1.

Since $V_2 \cap E_2 = \emptyset$ and there is no infinite chain $C_3 \ni C_2 \ni C_1 \ni I_1$, the pair

$$\mathcal{G}^{(2)} = (V_2,\ E_2)$$

is a 2-order SuperHyper-polygraph modeling nested consortia in academic collaborations.

Theorem 4.6 (Every SuperHyper-polygraph induces a SuperHyperGraph)**.** *Let $\mathcal{G}^{(n)} = (V_n, E_n)$ be an n-order SuperHyper-polygraph. Define*

$$E'_n \;=\; \{\, e \cap V_n \mid e \in E_n \,\} \;\subseteq\; \mathcal{P}(V_n) \setminus \{\emptyset\}.$$

Then $S^{(n)} = (V_n, E'_n)$ is an n-SuperHyperGraph.

Proof. By construction $E'_n \subseteq \mathcal{P}(V_n) \setminus \{\emptyset\}$ and $V_n \subseteq \mathcal{P}^n(V_0)$. These two conditions exactly match the definition of an n-SuperHyperGraph. The well-foundedness of E_n ensures no pathological infinite nesting arises when restricting to vertex-only subsets, but this property is not required in the SuperHyperGraph axioms. Hence (V_n, E'_n) satisfies all requirements for an n-SuperHyperGraph. □

Theorem 4.7 (Hyper-polygraphs as 0-order SuperHyper-polygraphs)**.** *A pair (X, E) is a hyper-polygraph (Definition) if and only if it is a SuperHyper-polygraph of order $n = 0$.*

Proof. When $n = 0$, $\mathcal{P}^0(V_0) = V_0$. Setting $X = V_0$ and $E_0 = E$, the SuperHyper-polygraph conditions become:

$$E_0 \subseteq \{\, e \mid e \subseteq X \cup E_0,\ 1 \le |e| < \infty \},\quad X \cap E_0 = \emptyset,$$

and no infinite chain $e_1 \ni e_2 \ni \cdots$. These are exactly the axioms of a hyper-polygraph. Conversely, any hyper-polygraph (X, E) satisfies these conditions for $n = 0$, so it is a 0-order SuperHyper-polygraph. □

Theorem 4.8 (Closure under Disjoint Union)**.** *Let $\mathcal{G}_1^{(n)} = (V_n^1, E_n^1)$ and $\mathcal{G}_2^{(n)} = (V_n^2, E_n^2)$ be two n-order SuperHyper-polygraphs with $V_n^1 \cap V_n^2 = \emptyset$ and $E_n^1 \cap E_n^2 = \emptyset$. Then their disjoint union*

$$\mathcal{G}^{(n)} = \big(V_n^1 \cup V_n^2,\ E_n^1 \cup E_n^2\big)$$

is again an n-order SuperHyper-polygraph.

Proof. Since $V_n^1 \cap V_n^2 = \emptyset$, clearly $V_n^1 \cup V_n^2 \subseteq \mathcal{P}^n(V_0)$ and $(E_n^1 \cup E_n^2) \subseteq \{\, e \mid e \subseteq (V_n^1 \cup V_n^2) \cup (E_n^1 \cup E_n^2),\ 1 \le |e| < \infty \}$. Disjointness ensures no new identifications arise, and any descending chain in $E_n^1 \cup E_n^2$ must lie entirely in either E_n^1 or E_n^2, both of which are well-founded. Hence no infinite chain exists in the union, and $V_n \cap E_n = \emptyset$ remains valid. Therefore the union satisfies all axioms of an n-order SuperHyper-polygraph. □

Theorem 4.9 (Order-Lifting to $(n+1)$-order)**.** *Let $\mathcal{G}^{(n)} = (V_n, E_n)$ be an n-order SuperHyper-polygraph. Define*

$$V_{n+1} = \{\, \{v\} \mid v \in V_n \,\}, \qquad E_{n+1} = \{\, \{e\} \mid e \in E_n \,\}.$$

Then $\mathcal{G}^{(n+1)} = (V_{n+1}, E_{n+1})$ is an $(n+1)$-order SuperHyper-polygraph.

Proof. By construction $V_{n+1} \subseteq \mathcal{P}(V_n) \subseteq \mathcal{P}^{n+1}(V_0)$. Each superhyperedge $\{e\} \in E_{n+1}$ is a singleton subset of $V_{n+1} \cup E_{n+1}$ of finite size, and clearly $V_{n+1} \cap E_{n+1} = \emptyset$. Any descending chain in E_{n+1} would induce a descending chain in E_n by unwrapping the singletons; since E_n is well-founded, no infinite descent can occur in E_{n+1}. Hence the axioms for an $(n+1)$-order SuperHyper-polygraph are satisfied. □

Theorem 4.10 (Acyclicity of the Incidence Graph)**.** *Let $\mathcal{G}^{(n)} = (V_n, E_n)$ be an n-order SuperHyper-polygraph. Construct its incidence directed graph I with vertex set $V_n \cup E_n$ and an arc $x \to e$ whenever $x \in e$. Then I contains no directed cycles.*

Proof. Suppose for contradiction that there is a directed cycle

$$x_1 \to e_1 \to x_2 \to e_2 \to \cdots \to x_k \to e_k \to x_1$$

in I, where each $x_i \in V_n \cup E_n$ and each $e_i \in E_n$. By construction edges only point from nodes or edges into hyperedges and from hyperedges into their members, so such a cycle would yield an infinite descending membership chain

$$e_1 \ni x_2,\ e_2 \ni x_3,\ \ldots,\ e_k \ni x_1,\ e_1 \ni x_2,\ \ldots$$

repeating indefinitely. This contradicts the well-foundedness axiom of E_n. Therefore I must be acyclic. □

Theorem 4.11 (Hereditary Induced Sub-SuperHyper-polygraphs)**.** *Let $\mathcal{G}^{(n)} = (V_n, E_n)$ be an n-order SuperHyper-polygraph. Suppose*

$$A \subseteq V_n \cup E_n \quad \text{and} \quad \forall\, e \in E_n \cap A,\ e \subseteq A.$$

Define

$$V_n' = V_n \cap A, \qquad E_n' = E_n \cap A.$$

Then $\mathcal{G}'^{(n)} = (V_n', E_n')$ is an n-order SuperHyper-polygraph.

Proof. First, $V_n' \subseteq V_n \subseteq \mathcal{P}^n(V_0)$ by hypothesis. Next,

$$E_n' \subseteq E_n \subseteq \{\, e \mid e \subseteq V_n \cup E_n,\ 1 \le |e| < \infty\},$$

so each $e \in E_n'$ is a finite nonempty subset of $V_n' \cup E_n'$. Since $V_n \cap E_n = \emptyset$, we also have $V_n' \cap E_n' = \emptyset$. Finally, any infinite descending chain $e_1 \ni e_2 \ni e_3 \ni \cdots$ in E_n' would also lie in E_n, contradicting its well-foundedness. Hence $\mathcal{G}'^{(n)}$ satisfies all axioms of an n-order SuperHyper-polygraph. □

Theorem 4.12 (Intersection Closure)**.** *Let $\mathcal{G}_1^{(n)} = (V_n, E_n^1)$ and $\mathcal{G}_2^{(n)} = (V_n, E_n^2)$ be two n-order SuperHyper-polygraphs on the same supervertex set V_n. Then*

$$\mathcal{G}^{(n)} = \left(V_n,\ E_n^1 \cap E_n^2\right)$$

is also an n-order SuperHyper-polygraph.

Proof. Clearly $E_n^1 \cap E_n^2 \subseteq \mathcal{P}(V_n) \setminus \{\emptyset\}$ since each $E_n^i \subseteq \mathcal{P}(V_n) \setminus \{\emptyset\}$. Also $V_n \cap (E_n^1 \cap E_n^2) = \emptyset$. If there were an infinite descending chain in $E_n^1 \cap E_n^2$, it would be an infinite chain in both E_n^1 and E_n^2, contradicting their well-foundedness. Thus $\mathcal{G}^{(n)}$ meets all requirements of an n-order SuperHyper-polygraph. □

Theorem 4.13 (Generalized Order-Lifting)**.** *Let $\mathcal{G}^{(n)} = (V_n, E_n)$ be an n-order SuperHyper-polygraph. For any integer $m \ge n$, define recursively for $k = n, n+1, \ldots, m-1$:*

$$V_{k+1} = \big\{\{v\} \mid v \in V_k\big\}, \qquad E_{k+1} = \big\{\{e\} \mid e \in E_k\big\}.$$

Then $\mathcal{G}^{(m)} = (V_m, E_m)$ is an m-order SuperHyper-polygraph, and the mapping $\iota\colon V_n \cup E_n \to V_m \cup E_m$ sending $\iota(v) = \{\cdots\{\{v\}\}\cdots\}$ (with $m-n$ nestings) and similarly for edges is an isomorphic embedding of $\mathcal{G}^{(n)}$ into $\mathcal{G}^{(m)}$.

Proof. We proceed by induction on k. The base case $k = n$ holds by hypothesis. Assume $\mathcal{G}^{(k)} = (V_k, E_k)$ is a k-order SuperHyper-polygraph. Then $V_{k+1} \subseteq \mathcal{P}(V_k) \subseteq \mathcal{P}^{k+1}(V_0)$, each $\{e\} \in E_{k+1}$ is a finite nonempty subset of $V_{k+1} \cup E_{k+1}$, and $V_{k+1} \cap E_{k+1} = \emptyset$. Any infinite chain in E_{k+1} would project to an infinite chain in E_k, violating well-foundedness. Thus $\mathcal{G}^{(k+1)}$ is a $(k+1)$-order SuperHyper-polygraph. Iterating up to m yields $\mathcal{G}^{(m)}$.

The embedding ι is bijective onto its image and preserves membership: for $x \in V_n \cup E_n$, $\iota(x) \in E_{n+1}$ if and only if $x \in E_n$, and so on, ensuring that incidence and acyclicity are carried over. Hence ι is an isomorphic embedding. □

5 Conclusion and Future Work

In this work, we have introduced and rigorously formalized the concept of a *Threshold SuperHypergraph*, thereby extending the classical notion of a Threshold Graph into the SuperHypergraph framework. We have also defined the *SuperHyper-PolyGraph* and examined its fundamental mathematical properties.

Looking ahead, we plan to investigate further generalizations of Threshold SuperHypergraphs by integrating theories of Fuzzy Sets [51–54] HyperFuzzy Sets [55–60], Superhyperfuzzy Sets [61, 62], Picture Fuzzy Sets [63, 64], Intuitionistic Fuzzy Sets [65, 66], Rough Sets [67–69], Vague Sets [70, 71], Neutrosophic Sets [72–74], and Plithogenic Sets [75–79]. Furthermore, we aim to conduct computational experiments, develop efficient graph-theoretic algorithms, and explore applications of these hierarchical and threshold-based structures across diverse scientific and engineering domains.

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Data Availability

No datasets were generated or analyzed during this study, as it is purely theoretical in nature. We welcome future empirical investigations to test and extend the concepts presented here.

Ethical Approval

Not applicable. This research does not involve human participants or animal subjects.

Conflicts of Interest

The authors declare that they have no conflicts of interest with respect to the research or its publication.

Disclaimer

The ideas and models introduced in this paper are theoretical and have not yet been validated in practical settings. Readers are encouraged to verify results independently and to explore real-world applications. Any errors or omissions remain the responsibility of the authors, and the views expressed herein do not necessarily reflect those of any affiliated organizations.

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