Weak HyperFuzzy Set and Weak SuperHyperFuzzy Set

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Abstract

Set-theoretic approaches such as fuzzy sets, neutrosophic sets, plithogenic sets, rough sets, and soft sets have proven effective for modeling uncertainty. These frameworks have been enriched by hyperstructures—based on ordinary powersets—and superhyperstructures—based on n-fold iterated powersets—yielding the classes of HyperUncertain Sets and SuperHyperUncertain Sets [1, 2]. In this work, we introduce four new extensions: the *Weak HyperFuzzy Set*, the *Weak HyperNeutrosophic Set*, the *Weak SuperHyperFuzzy Set*, and the *Weak SuperHyperNeutrosophic Set*. Each is defined within the algebraic frameworks of Weak HyperStructure and Weak SuperHyperStructure, thereby generalizing existing HyperUncertain and SuperHyperUncertain Set theories. We present precise definitions and discuss how these weak variants enhance the expressive power for modeling hierarchical and component-wise uncertainty.

Keywords: HyperFuzzy Set, HyperNeutrosophic Set, SuperHyperFuzzy Set, SuperHyperNeutrosophic Set, Weak HyperStructure, Weak SuperHyperStructure

The Structure of the Paper

The structure of this paper is outlined as follows.

1 Preliminaries

In this section, we present a brief overview of the definitions and notations used throughout this paper. It should be noted that this paper does not deal with the concept of infinity; instead, we consider only finite sets.

1.1 SuperHyperstructure

Mathematical structures can be systematically extended to hyperstructures [3–6] and further to superhyperstructures by using the power set and its n-fold iteration, the so-called n-th power set [7–12]. These extensions are known to offer the advantage of modeling hierarchical and complex concepts in both mathematical theory and real-world applications. Related notions include hypergraphs [13, 14] and superhypergraphs [15–19]. Below, we give the formal definitions and illustrative examples of the n-th power set, hyperstructures, and superhyperstructures.

Definition 1.1 (Base Set)**.** A *base set* S is the foundational set from which complex structures such as powersets and hyperstructures are derived. It is formally defined as:

$$S = \{x \mid x \text{ is an element within a specified domain}\}.$$

All elements in constructs like $\mathcal{P}(S)$ or $\mathcal{P}_n(S)$ originate from the elements of S.

Definition 1.2 (Powerset)**.** Let S be any set. The *powerset* of S, denoted $\mathcal{P}(S)$, is the collection of all subsets of S, including the empty set and S itself. Formally,

$$\mathcal{P}(S) \;=\; \{\, A \mid A \subseteq S \}.$$

Example 1.3. If $S = \{a, b\}$, then

$$\mathcal{P}(S) = \bigl\{ \varnothing,\ \{a\},\ \{b\},\ \{a, b\} \bigr\}.$$

Definition 1.4 (n-th powerset)**.** [8, 9, 20] For any set H and integer $n \geq 1$, the *n-th powerset* of H, written $\mathcal{P}_n(H)$, is defined recursively by

$$\mathcal{P}_1(H) = \mathcal{P}(H), \quad \mathcal{P}_{n+1}(H) = \mathcal{P}\bigl(\mathcal{P}_n(H)\bigr).$$

Similarly, the *n-th nonempty powerset* of H, denoted $\mathcal{P}_n^*(H)$, is given by

$$\mathcal{P}_1^*(H) = \mathcal{P}(H) \setminus \{\varnothing\}, \quad \mathcal{P}_{n+1}^*(H) = \mathcal{P}\bigl(\mathcal{P}_n^*(H)\bigr) \setminus \{\varnothing\}.$$

Example 1.5. Let $H = \{0, 1\}$. Then

$$\mathcal{P}_1(H) = \{\varnothing, \{0\}, \{1\}, \{0, 1\}\},$$

and

$$\mathcal{P}_2(H) = \mathcal{P}\bigl(\mathcal{P}_1(H)\bigr) = \bigl\{ X \mid X \subseteq \{\varnothing, \{0\}, \{1\}, \{0, 1\}\} \bigr\}.$$

In particular, $\mathcal{P}_2(H)$ contains, for example, $\{\varnothing, \{0\}\}$, $\{\{1\}, \{0, 1\}\}$, and even the full set $\mathcal{P}_1(H)$ itself.

To establish a formal foundation for the concepts of Hyperstructures and Superhyperstructures, we present the following definitions and propositions.

Definition 1.6 (Classical Structure)**.** (cf. [7, 21]) A *Classical Structure* is a mathematical framework defined on a non-empty set H, equipped with one or more *Classical Operations* that satisfy specified *Classical Axioms*. Specifically:

A *Classical Operation* is a function of the form:

$$\#_0 : H^m \to H,$$

where $m \geq 1$ is a positive integer, and H^m denotes the m-fold Cartesian product of H. Common examples include addition and multiplication in algebraic structures such as groups, rings, and fields.

Definition 1.7 (Hyperoperation)**.** (cf. [22–25]) A *hyperoperation* is a generalization of a binary operation where the result of combining two elements is a set, not a single element. Formally, for a set S, a hyperoperation $\circ$ is defined as:

$$\circ : S \times S \to \mathcal{P}(S),$$

where $\mathcal{P}(S)$ is the powerset of S.

Definition 1.8 (Hyperstructure)**.** (cf. [6, 7, 21, 26–28]) A *Hyperstructure* extends the notion of a Classical Structure by operating on the powerset of a base set. Formally, it is defined as:

$$\mathcal{H} = (\mathcal{P}(S), \circ),$$

where S is the base set, $\mathcal{P}(S)$ is the powerset of S, and $\circ$ is an operation defined on subsets of $\mathcal{P}(S)$. Hyperstructures allow for generalized operations that can apply to collections of elements rather than single elements.

Example 1.9 (Chemical Reaction Hyperstructure)**.** Let the base set of chemical species be

$$S = \{\mathrm{H_2},\ \mathrm{O_2},\ \mathrm{H_2O},\ \mathrm{Na},\ \mathrm{Cl_2},\ \mathrm{NaOH},\ \mathrm{HCl}\}.$$

Define a hyperoperation $\circ : \mathcal{P}(S) \times \mathcal{P}(S) \to \mathcal{P}(S)$ by

$$A \circ B \;=\; \{\, Z \in S \mid \exists\, X \in A,\ \exists\, Y \in B \text{ such that } X + Y \to Z \text{ under standard conditions}\}.$$

For example:

$$\{H_2\} \circ \{O_2\} = \{H_2O\},$$

since hydrogen and oxygen react to form water, and

$$\{\text{Na, } Cl_2\} \circ \{H_2O\} = \{\text{NaOH, HCl}\},$$

reflecting that sodium and chlorine with water yield sodium hydroxide and hydrochloric acid. The pair $\mathcal{H} = (\mathcal{P}(S), \circ)$ therefore models a hyperstructure in which subsets of reagents combine to produce subsets of possible products.

Definition 1.10 (SuperHyperOperations)**.** (cf. [7, 29, 30]) Let H be a non-empty set, and let $\mathcal{P}(H)$ denote the powerset of H. The n-th powerset $\mathcal{P}^n(H)$ is defined recursively as follows:

$$\mathcal{P}^0(H) = H, \quad \mathcal{P}^{k+1}(H) = \mathcal{P}(\mathcal{P}^k(H)), \quad \text{for } k \geq 0.$$

A *SuperHyperOperation* of order (m, n) is an m-ary operation:

$$\circ^{(m,n)} : H^m \to \mathcal{P}^n_*(H),$$

where $\mathcal{P}^n_*(H)$ represents the n-th powerset of H, either excluding or including the empty set, depending on the type of operation:

- If the codomain is $\mathcal{P}^n_*(H)$ excluding the empty set, it is called a *classical-type* (m, n)*-SuperHyperOperation*.
- If the codomain is $\mathcal{P}^n(H)$ including the empty set, it is called a *Neutrosophic* (m, n)*-SuperHyperOperation*.

These SuperHyperOperations are higher-order generalizations of hyperoperations, capturing multi-level complexity through the construction of n-th powersets.

Definition 1.11 (n-Superhyperstructure)**.** (cf. [7, 8, 21, 31]) An *n-Superhyperstructure* further generalizes a Hyperstructure by incorporating the n-th powerset of a base set. It is formally described as:

$$\mathcal{SH}_n = (\mathcal{P}_n(S), \circ),$$

where S is the base set, $\mathcal{P}_n(S)$ is the n-th powerset of S, and $\circ$ represents an operation defined on elements of $\mathcal{P}_n(S)$. This iterative framework allows for increasingly hierarchical and complex representations of relationships within the base set.

Example 1.12 (Cross-Department Project Team Formation)**.** Let the base set of employees be

$$S = \{\text{Hiroshi, Yuki, Sakura, Kenji}\}.$$

First-level groups ($\mathcal{P}_1(S)$) are project teams:

$$T_1 = \{\text{Hiroshi, Yuki}\}, \quad T_2 = \{\text{Sakura, Kenji}\}, \quad T_3 = \{\text{Yuki, Sakura}\}.$$

Second-level collections ($\mathcal{P}_2(S)$) are departments:

$$D_A = \{T_1, T_2\}, \qquad D_B = \{T_2, T_3\}.$$

Define the superhyperoperation $\circ : \mathcal{P}_2(S) \times \mathcal{P}_2(S) \to \mathcal{P}_2(S)$ by

$$A \circ B = \{ X \cup Y \mid X \in A, Y \in B \}.$$

Then

$$\begin{aligned} D_A \circ D_B &= \{T_1 \cup T_2, \ T_1 \cup T_3, \ T_2 \cup T_2, \ T_2 \cup T_3\} \\ &= \big\{\{\text{Hiroshi, Yuki, Sakura, Kenji}\}, \ \{\text{Hiroshi, Yuki, Sakura}\}, \\ &\quad \{\text{Sakura, Kenji}\}, \ \{\text{Yuki, Sakura, Kenji}\}\big\}. \end{aligned}$$

Hence $\mathcal{SH}_2 = (\mathcal{P}_2(S), \circ)$ is a concrete 2-Superhyperstructure modeling the formation of cross-department project teams with Japanese staff names.

1.2 Fuzzy Set

The Fuzzy Set is a well-known concept used to address uncertainty in set theory. These sets can be extended into Hyperfuzzy Sets and SuperHyperfuzzy Sets using hyperstructures and superhyperstructures. The definition is provided below [32–34].

Definition 1.13 (Fuzzy Set)**.** [32] Let Y be a nonempty universe. A *fuzzy set* μ on Y is a function

$$\mu : Y \longrightarrow [0,1],$$

where $\mu(y)$ denotes the membership degree of y. A *fuzzy relation* R on Y is a fuzzy subset of $Y \times Y$. If μ is a fuzzy set and R a fuzzy relation on Y, then R is said to be a *fuzzy relation on* μ if

$$R(y,z) \leq \min\{\mu(y), \mu(z)\} \quad \text{for all } y,z \in Y.$$

Example 1.14. Let $Y = \{\text{Alice}, \text{Bob}\}$. Define $\mu(\text{Alice}) = 0.8$, $\mu(\text{Bob}) = 0.6$. Then a fuzzy relation R on μ might satisfy

$$R(\text{Alice}, \text{Bob}) = 0.6 \leq \min\{0.8, 0.6\} = 0.6.$$

Definition 1.15 (HyperFuzzy Set)**.** [2,35–40] Let X be a nonempty set. A *hyperfuzzy set* on X is a function

$$\tilde{\mu} : X \longrightarrow \mathcal{P}([0,1]) \setminus \{\emptyset\},$$

assigning to each $x \in X$ a nonempty subset $\tilde{\mu}(x) \subseteq [0,1]$. This generalizes a fuzzy set by allowing each element's membership to be a range of possible values rather than a single scalar.

Example 1.16. Let $X = \{\text{R1}, \text{R2}, \text{R3}\}$ represent restaurants. A hyperfuzzy satisfaction set could be

$$\tilde{\mu}(\text{R1}) = [0.7, 0.9], \quad \tilde{\mu}(\text{R2}) = \{0.5, 0.6\}, \quad \tilde{\mu}(\text{R3}) = \{0.4, 0.8\}.$$

Here $\tilde{\mu}(\text{R1})$ indicates continuous satisfaction scores between 0.7 and 0.9.

Definition 1.17 (n-SuperHyperFuzzy Set)**.** [1,2,41] Let X be a non-empty set. The n-SuperHyperFuzzy Set is a recursive generalization of fuzzy sets, hyperfuzzy sets, and superhyperfuzzy sets. It is defined as:

$$\tilde{\mu}_n : \tilde{\mathcal{P}}_n(X) \to \tilde{\mathcal{P}}_n([0,1]),$$

where:

- $\tilde{\mathcal{P}}_1(X) = \tilde{\mathcal{P}}(X)$, and for $k \geq 2$,
$$\tilde{\mathcal{P}}_k(X) = \tilde{\mathcal{P}}(\tilde{\mathcal{P}}_{k-1}(X)),$$
represents the k-th nested family of non-empty subsets of X.
- $\tilde{\mathcal{P}}_n([0,1])$ is similarly defined for the interval $[0,1]$.
- $\tilde{\mu}_n$ assigns to each element $A \in \tilde{\mathcal{P}}_n(X)$ a non-empty subset $\tilde{\mu}_n(A) \subseteq [0,1]$, representing the degrees of membership associated with A at the n-th level.

Example 1.18 (E-commerce Bundle Satisfaction)**.** Let

$$X = \{\text{Laptop, Tablet, Smartphone}\}.$$

For $n = 2$, the set $\tilde{\mathcal{P}}_2(X)$ consists of all nonempty collections of nonempty subsets of X. Consider the bundle

$$A = \big\{\{\text{Laptop}, \text{Tablet}\}, \{\text{Smartphone}\}\big\} \in \tilde{\mathcal{P}}_2(X),$$

representing two product bundles: one combining Laptop and Tablet, and one of Smartphone alone.

We define the 2-SuperHyperFuzzy membership function $\tilde{\mu}_2 : \tilde{\mathcal{P}}_2(X) \to \tilde{\mathcal{P}}_2([0,1])$ by

$$\tilde{\mu}_2(A) = \big\{[0.75, 0.85], \{0.90, 0.95\}\big\}.$$

Here:

- $[0.75, 0.85]$ reflects survey responses indicating continuous satisfaction scores between 75% and 85% for the Laptop–Tablet bundle.
- $\{0.90, 0.95\}$ reflects two discrete high-satisfaction levels (90% and 95%) recorded for the Smartphone bundle.

This $(\tilde{\mathcal{P}}_2(X), \tilde{\mu}_2)$ thus forms a concrete 2-SuperHyperFuzzy Set, capturing hierarchical customer satisfaction for product bundles.

1.3 Neutrosophic Set

A Neutrosophic Set models uncertainty using three membership functions: truth (T), indeterminacy (I), and falsity (F) [42–47]. These sets can be extended into HyperNeutrosophic Sets and SuperHyperNeutrosophic Sets using hyperstructures and superhyperstructures.

Definition 1.19 (Neutrosophic Set)**.** [43, 48] Let X be a non-empty set. A *Neutrosophic Set (NS) A* on X is characterized by three membership functions:

$$T_A : X \to [0,1], \quad I_A : X \to [0,1], \quad F_A : X \to [0,1],$$

where for each $x \in X$, the values $T_A(x)$, $I_A(x)$, and $F_A(x)$ represent the degrees of truth, indeterminacy, and falsity, respectively. These values satisfy the following condition:

$$0 \le T_A(x) + I_A(x) + F_A(x) \le 3.$$

Definition 1.20 (HyperNeutrosophic Set)**.** [1, 2, 49, 50] Let X be a non-empty set. A mapping $\tilde{\mu} : X \to \tilde{P}([0,1]^3)$ is called a *HyperNeutrosophic Set* over X, where $\tilde{P}([0,1]^3)$ denotes the family of all non-empty subsets of the unit cube $[0,1]^3$. For each $x \in X$, $\tilde{\mu}(x) \subseteq [0,1]^3$ represents a set of neutrosophic membership degrees, each consisting of truth (T), indeterminacy (I), and falsity (F) components, satisfying:

$$0 \le T + I + F \le 3.$$

Example 1.21 (Autonomous Vehicle Sensor Reliability)**.** In an autonomous driving system, let

$$X = \{\text{LiDAR, Camera, Radar}\}$$

be three sensor modalities used for obstacle detection. Define the HyperNeutrosophic membership mapping $\tilde{\mu} : X \to \mathcal{P}([0,1]^3) \setminus \{\emptyset\}$ by

$$\begin{aligned}\tilde{\mu}(\text{LiDAR}) &= \{(0.92,\ 0.04,\ 0.04),\ (0.88,\ 0.07,\ 0.05)\},\\ \tilde{\mu}(\text{Camera}) &= \{(0.85,\ 0.10,\ 0.05),\ (0.80,\ 0.15,\ 0.05)\},\\ \tilde{\mu}(\text{Radar}) &= \{(0.90,\ 0.05,\ 0.05),\ (0.87,\ 0.08,\ 0.05)\}.\end{aligned}$$

Here each neutrosophic triple (T, I, F) represents:

- T: degree of correctly detecting an obstacle,
- I: degree of indeterminacy due to sensor noise or occlusion,
- F: degree of false alarm (incorrect obstacle detection).

For example, $(0.92, 0.04, 0.04) \in \tilde{\mu}(\text{LiDAR})$ indicates very high true-positive performance with minimal indeterminacy and false alarms under optimal conditions. The pair of triples for each sensor captures performance variability across clear and adverse weather. This HyperNeutrosophic Set thus models the range of neutrosophic reliability assessments for each obstacle-detection modality.

Definition 1.22 (n-SuperHyperNeutrosophic Set)**.** [1,2] Let X be a non-empty set. An n-SuperHyperNeutrosophic Set is a recursive generalization of Neutrosophic Sets, HyperNeutrosophic Sets, and SuperHyperNeutrosophic Sets. It is defined as:

$$\tilde{A}_n : \tilde{\mathcal{P}}_n(X) \to \tilde{\mathcal{P}}_n([0,1]^3),$$

where:

- $\tilde{\mathcal{P}}_1(X) = \tilde{\mathcal{P}}(X)$, and for $k \ge 2$,
  $$\tilde{\mathcal{P}}_k(X) = \tilde{\mathcal{P}}(\tilde{\mathcal{P}}_{k-1}(X)),$$
  represents the k-th nested family of non-empty subsets of X.
- $\tilde{\mathcal{P}}_n([0,1]^3)$ is similarly defined for the unit cube $[0,1]^3$.

- The mapping $\tilde{A}_n$ assigns to each $A \in \tilde{\mathcal{P}}_n(X)$ a subset $\tilde{A}_n(A) \subseteq [0,1]^3$, representing the degrees of truth (T), indeterminacy (I), and falsity (F) for the n-th level subsets of X.

For each $A \in \tilde{\mathcal{P}}_n(X)$ and $(T, I, F) \in \tilde{A}_n(A)$, the following condition is satisfied:

$$0 \leq T + I + F \leq 3,$$

where T, I, and F represent the truth, indeterminacy, and falsity degrees, respectively.

Example 1.23 (Regional Weather Monitoring Networks)**.** In a climate-monitoring system, let

$$X = \{\text{NorthStation, EastStation, WestStation}\}$$

be three weather stations. Fix $n = 2$, so $\tilde{\mathcal{P}}_2(X)$ consists of all nonempty collections of nonempty subsets of X. Consider the cluster

$$A = \big\{\{\text{NorthStation}, \text{EastStation}\},\ \{\text{WestStation}\}\big\} \in \tilde{\mathcal{P}}_2(X),$$

representing a combined north–east network and the west station alone.

Define the reliability mapping $\tilde{A}_2 : \tilde{\mathcal{P}}_2(X) \to \tilde{\mathcal{P}}_2([0,1]^3)$ by

$$\tilde{A}_2(A) = \big\{D,\ E\big\},$$

where

$$D = \big\{(0.88,\ 0.07,\ 0.05),\ (0.85,\ 0.10,\ 0.05)\big\}, \quad E = \big\{(0.80,\ 0.12,\ 0.08),\ (0.78,\ 0.15,\ 0.07)\big\}.$$

Here each neutrosophic triple (T, I, F) denotes:

- T: confidence in correct temperature measurements,
- I: indeterminacy due to sensor drift or interference,
- F: chance of measurement failure.

Thus $(\tilde{\mathcal{P}}_2(X),\ \tilde{A}_2)$ is a concrete 2-SuperHyperNeutrosophic Set capturing hierarchical uncertainty in regional weather data collection.

1.4 Weak Hyperstructure (Hv–structure)

A weak hyperstructure (Hv–structure) is an algebraic system whose hyperoperation satisfies weak associativity: any two ways of bracketing yield intersecting product sets [51, 51–54].

Definition 1.24 (Weak Hyperstructure (Hv–structure))**.** [26] Let H be a nonempty set and let

$$\cdot\ :\ H \times H \longrightarrow \mathcal{P}^*(H)$$

be a *hyperoperation*, where $\mathcal{P}^*(H) = \mathcal{P}(H) \setminus \{\emptyset\}$ is the set of all nonempty subsets of H. We say:

1. $\cdot$ is *weakly associative* if
$$x \cdot (y \cdot z) \ \cap\ (x \cdot y) \cdot z \ \neq\ \emptyset \quad \forall x, y, z \in H.$$
2. $\cdot$ is *weakly commutative* if
$$x \cdot y \ \cap\ y \cdot x \ \neq\ \emptyset \quad \forall x, y \in H.$$
3. $(H, \cdot)$ is called an *Hv–semigroup* if $\cdot$ is weakly associative.
4. $(H, \cdot)$ is called an *Hv–group* if it is an Hv–semigroup and also satisfies the *reproduction axiom*:
$$a \cdot H = H \quad \text{and} \quad H \cdot a = H \quad \forall a \in H.$$

Example 1.25 (Collaborative Pair Formation)**.** Let

$$H = \{\text{Hiroshi, Yuki, Sakura}\}$$

be three team members. Define a hyperoperation $\cdot : H \times H \to \mathcal{P}^*(H)$ by

$$x \cdot y = \{x, y\}, \quad x, y \in H.$$

Here $x \cdot y$ represents the two-person subteam formed by x and y.

For weak associativity, take x = Hiroshi, y = Yuki, z = Sakura. First compute

$$y \cdot z = \{\text{Yuki, Sakura}\},$$

so

$$x \cdot (y \cdot z) = \bigcup_{w \in \{Yuki, Sakura\}} \{x, w\} = \{\{\text{Hiroshi, Yuki}\}, \{\text{Hiroshi, Sakura}\}\}.$$

On the other hand,

$$x \cdot y = \{\text{Hiroshi, Yuki}\}, \quad (x \cdot y) \cdot z = \bigcup_{w \in \{Hiroshi, Yuki\}} \{w, z\} = \{\{\text{Hiroshi, Sakura}\}, \{\text{Yuki, Sakura}\}\}.$$

Their intersection is nonempty:

$$x \cdot (y \cdot z) \ \cap \ (x \cdot y) \cdot z = \{\{\text{Hiroshi, Sakura}\}\} \neq \emptyset.$$

Thus $\cdot$ is weakly associative. Since $x \cdot y = \{x, y\} = y \cdot x$, it is also weakly commutative. Therefore $(H, \cdot)$ is an Hv–semigroup modeling the formation of two-person collaborations.

1.5 Weak Superhyperstructure (SHv-Structure)

A weak n-Superhyperstructure (SHv-Structure) equips the n-th powerset of a base set with a superhyperoperation satisfying weak associativity across hierarchical levels [10].

Definition 1.26 (Weak n^{th} Superhyperstructure)**.** [10] Let S be a nonempty set and let $\mathcal{P}^n(S)$ be its n^{th} powerset (Definition **??**). Denote by $\mathcal{P}^*(\mathcal{P}^n(S)) = \mathcal{P}(\mathcal{P}^n(S)) \setminus \{\emptyset\}$ the collection of all nonempty subsets of $\mathcal{P}^n(S)$. A *weak n-Superhyperstructure* is a pair

$$\big(\mathcal{P}^n(S), \ \circ\big)$$

where

$$\circ : \ \mathcal{P}^n(S) \times \mathcal{P}^n(S) \ \longrightarrow \ \mathcal{P}^*(\mathcal{P}^n(S))$$

is a *superhyperoperation* satisfying the *weak associativity* condition:

$$X \circ \big(Y \circ Z\big) \ \cap \ \big(X \circ Y\big) \circ Z \ \neq \ \emptyset \quad \text{for all } X, Y, Z \in \mathcal{P}^n(S).$$

If moreover for each $A \in \mathcal{P}^n(S)$ one has $A \circ \mathcal{P}^n(S) = \mathcal{P}^n(S) = \mathcal{P}^n(S) \circ A$, then $(\mathcal{P}^n(S), \circ)$ is called an *Hv–group of order n*.

Example 1.27 (Nested Meal Planning in a Cafeteria)**.** Let the set of available dishes be

$$S = \{\text{Salad, Soup, Sandwich}\}.$$

For $n = 2$, the second-level powerset $\mathcal{P}^2(S)$ consists of all nonempty collections of nonempty subsets of S. For instance, consider

$$X = \big\{\{\text{Salad}\}, \{\text{Soup, Sandwich}\}\big\}, \quad Y = \big\{\{\text{Salad, Soup}\}, \{\text{Sandwich}\}\big\}, \quad Z = \big\{\{\text{Soup}\}, \{\text{Salad, Sandwich}\}\big\}.$$

Define the superhyperoperation

$$A \circ B = \Big\{ \ U \cup V \ \Big| \ U \in A, \ V \in B \Big\} \quad \text{for } A, B \in \mathcal{P}^2(S).$$

Then

$$Y \circ Z = \big\{\{\text{Salad, Soup}\}, \{\text{Sandwich}\}, \{\text{Salad, Soup, Sandwich}\}\big\},$$
$$X \circ \big(Y \circ Z\big) = \big\{\{\text{Salad, Soup}\}, \{\text{Salad, Sandwich}\}, \{\text{Salad, Soup, Sandwich}\}\big\},$$
$$X \circ Y = \big\{\{\text{Salad, Soup}\}, \{\text{Salad, Sandwich}\}, \{\text{Soup, Sandwich}\}\big\},$$
$$(X \circ Y) \circ Z = \big\{\{\text{Salad, Soup}\}, \{\text{Salad, Sandwich}\}, \{\text{Salad, Soup, Sandwich}\}\big\}.$$

Observe that {Salad, Soup, Sandwich} lies in both $X \circ (Y \circ Z)$ and $(X \circ Y) \circ Z$, so

$$X \circ \big(Y \circ Z\big) \ \cap \ \big(X \circ Y\big) \circ Z \ \neq \ \emptyset.$$

Hence $\big(\mathcal{P}^2(S), \ \circ\big)$ is a weak 2-superhyperstructure modeling hierarchical menu combinations.

2 Main Results of this paper

This section presents the main results of this paper.

2.1 Weak HyperFuzzy Set

A Weak HyperFuzzy Set assigns each element a nonempty subset of $[0, 1]$. Its hyperoperation combines membership sets by minimum and is weakly associative.

Definition 2.1 (Weak HyperFuzzy Set)**.** Let X be a nonempty set. A *Weak HyperFuzzy Set* on X is a pair $(X, \tilde{\mu})$ where

$$\tilde{\mu} : X \longrightarrow \mathcal{P}([0,1]) \setminus \{\emptyset\}$$

is a mapping into the family of all nonempty subsets of $[0, 1]$, equipped with the hyperoperation

$$\circ : \ \big(\mathcal{P}([0,1]) \setminus \{\emptyset\}\big) \times \big(\mathcal{P}([0,1]) \setminus \{\emptyset\}\big) \ \longrightarrow \ \mathcal{P}([0,1]) \setminus \{\emptyset\},$$

defined by

$$A \circ B \ = \ \big\{\min(a,b) \mid a \in A, \ b \in B\big\}.$$

We require that $(\mathcal{P}([0,1]) \setminus \{\emptyset\}, \circ)$ is an Hv–semigroup; that is, $\circ$ is *weakly associative*:

$$A \circ \big(B \circ C\big) \ \cap \ \big(A \circ B\big) \circ C \ \neq \ \emptyset \quad \forall A, B, C \subseteq [0,1], \ A, B, C \neq \emptyset.$$

Example 2.2 (Supplier Reliability Assessment)**.** In a manufacturing supply chain, let

$$X = \{S_1, \ S_2, \ S_3\}$$

denote three potential suppliers. To model the uncertain reliability of each supplier, we define a Weak HyperFuzzy Set $\tilde{\mu} \colon X \to \mathcal{P}([0,1]) \setminus \{\emptyset\}$ by

$$\tilde{\mu}(S_1) = [0.85, \ 0.95], \quad \tilde{\mu}(S_2) = \{0.70, \ 0.75, \ 0.80\}, \quad \tilde{\mu}(S_3) = [0.60, \ 0.70] \ \cup \ \{0.90\}.$$

Here:

- $\tilde{\mu}(S_1) = [0.85, 0.95]$ indicates that Supplier 1's reliability, as judged by different quality audits, ranges continuously from 85% to 95%.
- $\tilde{\mu}(S_2) = \{0.70, 0.75, 0.80\}$ reflects that Supplier 2 has been scored discretely at 70%, 75%, or 80% by various evaluators.
- $\tilde{\mu}(S_3) = [0.60, 0.70] \cup \{0.90\}$ shows that Supplier 3 generally scores between 60% and 70%, but a recent exceptional performance was recorded at 90%.

To assess the combined reliability when sourcing from two suppliers, we use the hyperoperation

$$A \circ B \ = \ \{\min(a,b) \mid a \in A, \ b \in B\}.$$

For example,

$$\tilde{\mu}(S_1) \ \circ \ \tilde{\mu}(S_2) = \big\{\min(a,b) \mid a \in [0.85, 0.95], \ b \in \{0.70, 0.75, 0.80\}\big\} = \{0.70, \ 0.75, \ 0.80\}.$$

Since min is associative, this $\circ$ makes $\mathcal{P}([0,1]) \setminus \{\emptyset\}$ into an Hv–semigroup, confirming that $(X, \tilde{\mu})$ is indeed a Weak HyperFuzzy Set.

Example 2.3 (Electric Vehicle Battery Health HyperFuzzy Set)**.** Let

$$X = \{\text{ModelA}, \ \text{ModelB}, \ \text{ModelC}\}$$

be three electric-vehicle models. A *HyperFuzzy Set* $\tilde{\mu} : X \to \mathcal{P}([0,1]) \setminus \{\emptyset\}$ capturing possible battery-health scores is

$$\tilde{\mu}(\text{ModelA}) = \{0.60, \ 0.70\}, \quad \tilde{\mu}(\text{ModelB}) = [0.80, \ 0.90], \quad \tilde{\mu}(\text{ModelC}) = [0.50, \ 0.70] \ \cup \ \{0.85\}.$$

Here each set of values represents the range of state-of-health measurements for that model under different testing conditions. No operation to combine two models' ranges is specified, so this is a pure HyperFuzzy Set without any additional algebraic structure.

Example 2.4 (Data Center Reliability Assessment)**.** In a cloud infrastructure scenario, let

$$X = \{\mathrm{DC}_A,\ \mathrm{DC}_B,\ \mathrm{DC}_C\}$$

be three geographically distinct data centers. To capture the uncertainty in their uptime reliability over a given month, we define a Weak HyperFuzzy Set $\tilde{\mu}\colon X \to \mathcal{P}([0,1]) \setminus \{\emptyset\}$ by

$$\tilde{\mu}(\mathrm{DC}_A) = [0.98,\ 1.00], \quad \tilde{\mu}(\mathrm{DC}_B) = \{0.95,\ 0.97\}, \quad \tilde{\mu}(\mathrm{DC}_C) = [0.92,\ 0.96]\ \cup\ \{0.99\}.$$

Here:

- $\tilde{\mu}(\mathrm{DC}_A) = [0.98, 1.00]$ reflects that monitoring logs report uptime between 98% and 100% in all observation periods.
- $\tilde{\mu}(\mathrm{DC}_B) = \{0.95, 0.97\}$ indicates that DC_B's uptime was recorded discretely at 95% or 97% depending on occasional maintenance windows.
- $\tilde{\mu}(\mathrm{DC}_C) = [0.92, 0.96] \cup \{0.99\}$ shows typical uptime between 92% and 96%, with one exceptional week reaching 99%.

To evaluate the reliability of a redundant dual–data-center deployment, we use the hyperoperation

$$A \circ B = \{\min(a,b) \mid a \in A,\ b \in B\}.$$

For instance,

$$\tilde{\mu}(\mathrm{DC}_A) \circ \tilde{\mu}(\mathrm{DC}_C) = \{\min(a,c) \mid a \in [0.98, 1.00],\ c \in [0.92, 0.96] \cup \{0.99\}\} = [0.92,\ 0.96].$$

Since the minimum function is associative, $\circ$ makes $\mathcal{P}([0,1]) \setminus \{\emptyset\}$ into an Hv–semigroup. Therefore $(X, \tilde{\mu})$ satisfies the axioms of a Weak HyperFuzzy Set in this real-world reliability assessment.

Example 2.5 (Electric Vehicle Combined Range Weak HyperFuzzy Set)**.** Using the same domain X and mapping $\tilde{\mu}$ above, define a hyperoperation

$$A \circ B \ = \ \{\min(a,b) \mid a \in A,\ b \in B\} \quad (A, B \subseteq [0,1],\ A, B \neq \emptyset).$$

Then $\big([0,1] \setminus \{\emptyset\}, \circ\big)$ is a weak Hv–semigroup since min is associative. For example,

$$\tilde{\mu}(\mathrm{ModelA}) \circ \tilde{\mu}(\mathrm{ModelC}) = \{\min(a,c) \mid a \in \{0.6, 0.7\},\ c \in [0.5, 0.7] \cup \{0.85\}\} = [0.5,\ 0.7].$$

Because a weakly associative hyperoperation is now available, $(X, \tilde{\mu})$ is promoted to a *Weak HyperFuzzy Set*. This contrast shows that an ordinary HyperFuzzy Set requires only a membership mapping, whereas a Weak HyperFuzzy Set additionally carries a hyperoperation satisfying the Hv–semigroup axioms.

Theorem 2.6. *The hyperoperation* $\circ$ *given by*

$$A \circ B = \{\min(a,b) \mid a \in A,\ b \in B\}$$

turns $\mathcal{P}([0,1]) \setminus \{\emptyset\}$ *into an Hv–semigroup.*

Proof. Let $A, B, C \subseteq [0,1]$ be nonempty. Since min is associative on $[0,1]$, for any $a \in A,\ b \in B,\ c \in C$ we have

$$\min\big(a, \min(b,c)\big) = \min\big(\min(a,b), c\big).$$

Hence

$$A \circ \big(B \circ C\big) = \{\min(a,d) \mid a \in A,\ d \in B \circ C\} = \{\min(a, \min(b,c))\}$$

and

$$\big(A \circ B\big) \circ C = \{\min(e,c) \mid e \in A \circ B,\ c \in C\} = \{\min(\min(a,b), c)\}.$$

Because these two sets coincide, their intersection is nonempty. Therefore $\circ$ is weakly associative, and $(\mathcal{P}([0,1]) \setminus \{\emptyset\}, \circ)$ is an Hv–semigroup. □

Theorem 2.7. *Every HyperFuzzy Set* $\tilde{\mu} : X \to \mathcal{P}([0,1]) \setminus \{\emptyset\}$ *is a Weak HyperFuzzy Set with respect to the hyperoperation* $\circ$ *above. In particular, the concept of Weak HyperFuzzy Set strictly generalizes that of HyperFuzzy Set.*

Proof. A HyperFuzzy Set is by definition just a mapping $\tilde{\mu} : X \to \mathcal{P}([0,1]) \setminus \{\emptyset\}$. By endowing the target $\mathcal{P}([0,1]) \setminus \{\emptyset\}$ with the Hv–semigroup structure given by $\circ$, the image of any hyperfuzzy set is contained in an Hv–semigroup. Thus every HyperFuzzy Set satisfies the requirements of a Weak HyperFuzzy Set, showing that our new notion indeed extends the classical one. □

2.2 Weak SuperhyperFuzzy Set

A Weak SuperHyperFuzzy Set maps each n-th level subset to a nonempty nested subset of $[0, 1]$. Its superhyperoperation uses minimum and is weakly associative.

Definition 2.8 (Weak SuperHyperFuzzy Set)**.** Let X be a nonempty set and fix $n \geq 1$. A *Weak SuperHyperFuzzy Set* on X is a pair $(\tilde{\mathcal{P}}_n(X),\ \tilde{\mu}_n)$ where

$$\tilde{\mu}_n :\ \tilde{\mathcal{P}}_n(X) \ \longrightarrow\ \tilde{\mathcal{P}}_n([0,1])$$

is a mapping into the family of all nonempty subsets of the n-th power set of $[0, 1]$, $\tilde{\mathcal{P}}_n([0, 1])$, equipped with the *superhyperoperation*

$$\circ^{(n)} :\ \tilde{\mathcal{P}}_n([0,1]) \times \tilde{\mathcal{P}}_n([0,1]) \ \longrightarrow\ \mathcal{P}^*\big(\tilde{\mathcal{P}}_n([0,1])\big),$$

defined by

$$A \ \circ^{(n)}\ B \ = \ \big\{\min(a,b) \ \big|\ a \in A,\ b \in B\big\}.$$

We require that $\big(\tilde{\mathcal{P}}_n([0,1]),\ \circ^{(n)}\big)$ is a weak n-superhyperstructure (SHv–structure), i.e. $\circ^{(n)}$ satisfies *weak associativity*:

$$A \circ^{(n)} \big(B \circ^{(n)} C\big) \ \cap\ \big(A \circ^{(n)} B\big) \circ^{(n)} C \ \neq\ \emptyset \quad \forall\, A, B, C \in \tilde{\mathcal{P}}_n([0,1]).$$

Example 2.9 (Hierarchical Sensor Cluster Reliability)**.** In a large-scale environmental monitoring system, three types of sensors—temperature (T), humidity (H), and pressure (P)—are deployed in clusters. Let

$$X \ = \ \{T,\ H,\ P\}.$$

We take $n = 2$. Thus $\tilde{\mathcal{P}}_2(X)$ consists of nonempty collections of nonempty subsets of X. For example, consider the element

$$A \ = \ \big\{\{T\},\ \{H,P\}\big\} \ \in\ \tilde{\mathcal{P}}_2(X),$$

representing a cluster composed of the single sensor T and the sub-cluster $\{H, P\}$.

We define the Weak SuperHyperFuzzy membership $\tilde{\mu}_2 : \tilde{\mathcal{P}}_2(X) \to \tilde{\mathcal{P}}_2([0, 1])$ by assigning to A the following collection of uncertainty intervals:

$$\tilde{\mu}_2(A) = \big\{\ [0.85,\ 0.95],\ \ [0.80,\ 0.90]\big\}.$$

Here:

- The interval $[0.85, 0.95]$ corresponds to the reliability of the singleton sensor T, aggregated from multiple field tests.
- The interval $[0.80, 0.90]$ corresponds to the combined reliability of the sub-cluster $\{H, P\}$, measured under varying environmental conditions.

The superhyperoperation $\circ^{(2)}$ on $\tilde{\mathcal{P}}_2([0, 1])$ is given by

$$B \ \circ^{(2)}\ C = \big\{\min(b,c) \ \big|\ b \in B,\ c \in C\big\},$$

and one checks readily—using the associativity of min—that $\big(\tilde{\mathcal{P}}_2([0, 1]),\ \circ^{(2)}\big)$ is a weak 2-superhyperstructure. Hence $(\tilde{\mathcal{P}}_2(X),\ \tilde{\mu}_2)$ is a concrete Weak SuperHyperFuzzy Set modeling hierarchical sensor-cluster reliability.

Example 2.10 (Smart Home Device Cluster Reliability)**.** Consider a smart home system with three critical device types:

$$X = \{\text{Thermostat, Camera, SmartLock}\}.$$

We set $n = 2$, so $\tilde{\mathcal{P}}_2(X)$ consists of nonempty collections of nonempty subsets of X. For instance, take

$$A \ = \ \big\{\{\text{Thermostat}, \text{Camera}\},\ \{\text{SmartLock}\}\big\} \ \in\ \tilde{\mathcal{P}}_2(X),$$

representing the cluster of environmental sensors $\{\text{Thermostat}, \text{Camera}\}$ and the standalone access controller $\{\text{SmartLock}\}$.

Define the Weak SuperHyperFuzzy membership $\tilde{\mu}_2 : \tilde{\mathcal{P}}_2(X) \to \tilde{\mathcal{P}}_2([0, 1])$ by

$$\tilde{\mu}_2(A) = \big\{\ [0.88,\ 0.94],\ \ [0.76,\ 0.86]\big\}.$$

Here:

- $[0.88, 0.94]$ quantifies the combined reliability of the Thermostat–Camera sensor pair, based on multiple uptime logs.
- $[0.76, 0.86]$ quantifies the reliability of the standalone SmartLock, reflecting variability due to firmware updates.

The superhyperoperation $\circ^{(2)}$ on $\tilde{\mathcal{P}}_2([0,1])$ is

$$B \circ^{(2)} C = \{\min(b, c) \mid b \in B,\ c \in C\},$$

which, by the associativity of min, makes $(\tilde{\mathcal{P}}_2([0,1]),\ \circ^{(2)})$ a weak 2-superhyperstructure. Therefore, $(\tilde{\mathcal{P}}_2(X),\ \tilde{\mu}_2)$ is a concrete Weak SuperHyperFuzzy Set modeling hierarchical reliability in a smart home device network.

Theorem 2.11. *$(\tilde{\mathcal{P}}_n([0,1]),\ \circ^{(n)})$ is an SHv–structure of order n.*

Proof. Let $A, B, C \subseteq \tilde{\mathcal{P}}_n([0,1])$ be nonempty. By definition,

$$B \circ^{(n)} C = \{\min(b, c) \mid b \in B,\ c \in C\}, \quad A \circ^{(n)} B = \{\min(a, b) \mid a \in A,\ b \in B\}.$$

Since min is associative on $[0, 1]$, $\min(a, \min(b, c)) = \min(\min(a, b), c)$ for all $a \in A,\ b \in B,\ c \in C$. Hence

$$A \circ^{(n)} \left(B \circ^{(n)} C\right) = \{\min(a, \min(b, c))\}, \quad \left(A \circ^{(n)} B\right) \circ^{(n)} C = \{\min(\min(a, b), c)\}.$$

These two sets coincide, so their intersection is nonempty. Therefore $\circ^{(n)}$ is weakly associative, and $(\tilde{\mathcal{P}}_n([0,1]),\ \circ^{(n)})$ is indeed an SHv–structure. □

Theorem 2.12. *Every SuperHyperFuzzy Set $\tilde{\mu}_n : \tilde{\mathcal{P}}_n(X) \to \tilde{\mathcal{P}}_n([0,1])$ is a Weak SuperHyperFuzzy Set with respect to $\circ^{(n)}$. In particular, this notion strictly generalizes both the classical SuperHyperFuzzy Set and, when $n = 1$, the Weak HyperFuzzy Set.*

Proof. A SuperHyperFuzzy Set is, by definition, any mapping $\tilde{\mu}_n : \tilde{\mathcal{P}}_n(X) \to \tilde{\mathcal{P}}_n([0,1])$. Endowing the target $\tilde{\mathcal{P}}_n([0,1])$ with the weak superhyperoperation $\circ^{(n)}$ shown above makes its image lie in an SHv–structure. Hence every SuperHyperFuzzy Set satisfies the axioms of a Weak SuperHyperFuzzy Set. When $n = 1$, one recovers precisely the structure of a Weak HyperFuzzy Set, showing that our new concept generalizes both SuperHyperFuzzy and Weak HyperFuzzy Sets. □

2.3 Weak HyperNeutrosophic Set

A Weak HyperNeutrosophic Set maps each element to a nonempty subset of $[0,1]^3$. Its hyperoperation merges neutrosophic triples by min/max componentwise and is weakly associative.

Definition 2.13 (Weak HyperNeutrosophic Set)**.** Let X be a nonempty set. Denote

$$\tilde{\mathcal{P}}\left([0,1]^3\right) = \mathcal{P}\left([0,1]^3\right) \setminus \{\emptyset\}.$$

A *Weak HyperNeutrosophic Set* on X is a pair $(X, \tilde{\mu})$ where

$$\tilde{\mu} : X \longrightarrow \tilde{\mathcal{P}}\left([0,1]^3\right)$$

is a mapping into the family of all nonempty subsets of the neutrosophic cube $[0,1]^3$, equipped with the hyperoperation

$$\star : \ \tilde{\mathcal{P}}\left([0,1]^3\right) \times \tilde{\mathcal{P}}\left([0,1]^3\right) \longrightarrow \mathcal{P}^*\Big(\tilde{\mathcal{P}}\left([0,1]^3\right)\Big),$$

defined by

$$A \star B \ = \ \left\{ \left(\min(T_a, T_b),\ \max(I_a, I_b),\ \max(F_a, F_b)\right) \ \Big| \ (T_a, I_a, F_a) \in A,\ (T_b, I_b, F_b) \in B \right\}.$$

We require that $(\tilde{\mathcal{P}}([0,1]^3), \star)$ is an Hv–semigroup; that is, $\star$ is *weakly associative*:

$$A \star \left(B \star C\right) \ \cap \ \left(A \star B\right) \star C \ \neq \ \emptyset \quad \forall\, A, B, C \subseteq [0,1]^3,\ A, B, C \neq \emptyset.$$

Example 2.14 (Medical Diagnostic Test Uncertainty)**.** In a clinical decision-making context, let

$$X = \{\text{BloodTest, Imaging, ClinicalExam}\}$$

represent three diagnostic modalities for detecting a particular disease. To capture the neutrosophic uncertainty in each modality's outcome, we define a Weak HyperNeutrosophic Set $\tilde{\mu} : X \to \mathcal{P}\big([0,1]^3\big) \setminus \{\emptyset\}$ by

$$\begin{aligned}
\tilde{\mu}(\text{BloodTest}) &= \big\{\,(0.90,\ 0.05,\ 0.05),\ \ (0.85,\ 0.10,\ 0.05)\big\},\\
\tilde{\mu}(\text{Imaging}) &= \big\{\,(0.80,\ 0.15,\ 0.05),\ \ (0.75,\ 0.20,\ 0.05)\big\},\\
\tilde{\mu}(\text{ClinicalExam}) &= \big\{\,(0.70,\ 0.20,\ 0.10),\ \ (0.65,\ 0.25,\ 0.10)\big\}.
\end{aligned}$$

Here each triple (T, I, F) corresponds to:

- T: degree of positively detecting the disease,
- I: degree of indeterminacy due to ambiguous or conflicting evidence,
- F: degree of negatively detecting the disease.

For example, BloodTest yields two neutrosophic assessments: one very confident positive $(0.90, 0.05, 0.05)$ and one slightly less confident $(0.85, 0.10, 0.05)$. Imaging and ClinicalExam are interpreted similarly.

To illustrate the hyperoperation $\star$, consider combining BloodTest and Imaging:

$$\tilde{\mu}(\text{BloodTest}) \ \star\ \tilde{\mu}(\text{Imaging})$$

$$= \big\{\ (\min(T_a, T_b),\ \max(I_a, I_b),\ \max(F_a, F_b))$$

$$\big|\ (T_a, I_a, F_a) \in \tilde{\mu}(\text{BloodTest}),\ (T_b, I_b, F_b) \in \tilde{\mu}(\text{Imaging})\big\}.$$

One representative element of this set is

$$\big(\min(0.90, 0.80),\ \max(0.05, 0.15),\ \max(0.05, 0.05)\big) = (0.80,\ 0.15,\ 0.05).$$

Since the componentwise min/max operations are associative in this way, $\star$ endows $\mathcal{P}([0,1]^3) \setminus \{\emptyset\}$ with a weakly associative Hv–semigroup structure, confirming that $(X, \tilde{\mu})$ is indeed a Weak HyperNeutrosophic Set for modeling hierarchical diagnostic uncertainty.

Example 2.15 (Credit Approval Risk Analysis)**.** In a consumer lending scenario, let

$$X = \{\text{CreditScore, IncomeCheck, RepaymentHistory}\}$$

be three independent checks used to decide on loan approval. We define a Weak HyperNeutrosophic Set $\tilde{\mu} : X \to \mathcal{P}([0,1]^3) \setminus \{\emptyset\}$ by

$$\begin{aligned}
\tilde{\mu}(\text{CreditScore}) &= \big\{\,(0.80,\ 0.10,\ 0.10),\ \ (0.75,\ 0.15,\ 0.10)\big\},\\
\tilde{\mu}(\text{IncomeCheck}) &= \big\{\,(0.85,\ 0.05,\ 0.10),\ \ (0.82,\ 0.08,\ 0.10)\big\},\\
\tilde{\mu}(\text{RepaymentHistory}) &= \big\{\,(0.90,\ 0.05,\ 0.05),\ \ (0.88,\ 0.07,\ 0.05)\big\}.
\end{aligned}$$

Here each triple (T, I, F) represents:

- T: degree of "approve" (positive evidence),
- I: degree of indeterminacy (conflicting or missing information),
- F: degree of "reject" (negative evidence).

To aggregate two checks, e.g. CreditScore and IncomeCheck, we use the hyperoperation

$$A \star B = \left\{ (\min(T_a, T_b),\ \max(I_a, I_b),\ \max(F_a, F_b)) \;\middle|\; (T_a, I_a, F_a) \in A,\ (T_b, I_b, F_b) \in B \right\}.$$

For instance,

$$\tilde{\mu}(\text{CreditScore}) \star \tilde{\mu}(\text{IncomeCheck}) \ni (\min(0.80, 0.85),\ \max(0.10, 0.05),\ \max(0.10, 0.10)) = (0.80,\ 0.10,\ 0.10).$$

Since the component-wise min and max operations are associative and commutative in the Hv–sense, $\star$ makes $\mathcal{P}([0,1]^3) \setminus \{\emptyset\}$ into an Hv–semigroup. Therefore $(X, \tilde{\mu})$ models the neutrosophic uncertainty in credit approval and constitutes a concrete Weak HyperNeutrosophic Set.

Theorem 2.16. *$(\tilde{\mathcal{P}}([0,1]^3), \star)$ is an Hv–semigroup.*

Proof. Let $A, B, C \subseteq [0,1]^3$ be nonempty. For each $(T_a, I_a, F_a) \in A,\ (T_b, I_b, F_b) \in B,\ (T_c, I_c, F_c) \in C$, the componentwise operations min on truth and max on indeterminacy and falsity are associative:

$$\min\big(T_a, \min(T_b, T_c)\big) = \min\big(\min(T_a, T_b), T_c\big), \quad \max\big(I_a, \max(I_b, I_c)\big) = \max\big(\max(I_a, I_b), I_c\big),$$

and similarly for F. Hence

$$A \star (B \star C) = \big\{(\min(T_a, \min(T_b, T_c)), \dots)\big\}, \quad (A \star B) \star C = \big\{(\min(\min(T_a, T_b), T_c), \dots)\big\},$$

and these two sets coincide. Their intersection is therefore nonempty, establishing weak associativity. □

Theorem 2.17. *Every HyperNeutrosophic Set $\tilde{\mu} : X \to \tilde{\mathcal{P}}([0,1]^3)$ is a Weak HyperNeutrosophic Set. In particular, this notion generalizes both classical HyperNeutrosophic Sets and, when restricted to the first coordinate only, Weak HyperFuzzy Sets.*

Proof. A HyperNeutrosophic Set is by definition any mapping $\tilde{\mu} : X \to \tilde{\mathcal{P}}([0,1]^3)$. Equipping the codomain with the Hv–semigroup structure $(\tilde{\mathcal{P}}([0,1]^3), \star)$ makes $\tilde{\mu}(X)$ lie in an Hv–semigroup. Thus the pair $(X, \tilde{\mu})$ satisfies the axioms of a Weak HyperNeutrosophic Set. Restricting attention to truth–degrees alone recovers exactly the Weak HyperFuzzy Set structure, showing that both previously known notions embed into our new framework. □

2.4 Weak SuperhyperNeutrosophic Set

A Weak n-SuperhyperNeutrosophic Set maps n-th level sets to nonempty nested subsets of $[0,1]^3$. Its superhyperoperation merges triples via min/max and is weakly associative.

Definition 2.18 (Weak n-SuperhyperNeutrosophic Set)**.** Let X be a nonempty set and fix $n \geq 1$. Denote

$$\tilde{\mathcal{P}}_n\big([0,1]^3\big) \;=\; \mathcal{P}\big(\tilde{\mathcal{P}}_{n-1}([0,1]^3)\big) \setminus \{\emptyset\},$$

with $\tilde{\mathcal{P}}_1([0,1]^3) = \mathcal{P}([0,1]^3) \setminus \{\emptyset\}$. A *Weak n-SuperhyperNeutrosophic Set* on X is a pair $\big(\tilde{\mathcal{P}}_n(X),\ \tilde{A}_n\big)$ where

$$\tilde{A}_n :\ \tilde{\mathcal{P}}_n(X) \;\longrightarrow\; \tilde{\mathcal{P}}_n\big([0,1]^3\big)$$

is a mapping into nonempty n-th superhyperpowersets of the neutrosophic cube, equipped with the *superhyperoperation*

$$\star^{(n)} : \tilde{\mathcal{P}}_n\big([0,1]^3\big) \times \tilde{\mathcal{P}}_n\big([0,1]^3\big) \;\longrightarrow\; \mathcal{P}^*\big(\tilde{\mathcal{P}}_n([0,1]^3)\big),$$

defined by

$$A \;\star^{(n)}\; B \;=\; \left\{ \big(\min(T_a, T_b),\ \max(I_a, I_b),\ \max(F_a, F_b)\big) \;\middle|\; (T_a, I_a, F_a) \in A,\ (T_b, I_b, F_b) \in B \right\}.$$

We require that $\big(\tilde{\mathcal{P}}_n([0,1]^3),\ \star^{(n)}\big)$ is a *weak n-superhyperstructure* (SHv–structure), i.e. for all nonempty $A, B, C \subseteq \tilde{\mathcal{P}}_n([0,1]^3)$,

$$A \star^{(n)} \big(B \star^{(n)} C\big) \;\cap\; \big(A \star^{(n)} B\big) \star^{(n)} C \;\neq\; \emptyset.$$

Example 2.19 (Financial Portfolio Cluster Neutrosophic Risk)**.** Consider a hierarchical financial portfolio with three asset classes:

$$X = \{\text{Equity, Debt, RealEstate}\}.$$

Fix $n = 2$. Then $\tilde{\mathcal{P}}_2(X)$ consists of nonempty collections of nonempty subsets of X. For instance, let

$$A = \big\{\{\text{Equity}\}, \{\text{Debt, RealEstate}\}\big\} \in \tilde{\mathcal{P}}_2(X),$$

representing two portfolio clusters: a pure equity tranche and a combined debt–real-estate tranche.

We define the Weak 2-SuperhyperNeutrosophic membership $\tilde{A}_2 : \tilde{\mathcal{P}}_2(X) \to \tilde{\mathcal{P}}_2([0,1]^3)$ by

$$\tilde{A}_2(A) = \big\{D, E\big\},$$

where

$$D = \big\{(0.75, 0.15, 0.10), (0.70, 0.20, 0.10)\big\}, \quad E = \big\{(0.65, 0.25, 0.10), (0.60, 0.30, 0.10)\big\}.$$

Here each neutrosophic triple (T, I, F) encodes:

- T: belief in achieving target return,
- I: indeterminacy due to market volatility,
- F: belief in experiencing a loss.

Thus D captures two scenarios for the equity cluster, while E captures two scenarios for the debt–real-estate cluster.

The superhyperoperation $\star^{(2)}$ on $\tilde{\mathcal{P}}_2([0,1]^3)$ is defined by

$$A' \star^{(2)} B' = \big\{(\min(T_a, T_b), \max(I_a, I_b), \max(F_a, F_b)) \;\big|\; (T_a, I_a, F_a) \in A', (T_b, I_b, F_b) \in B'\big\}.$$

For example, combining the equity scenario $(0.75, 0.15, 0.10) \in D$ with the debt–real-estate scenario $(0.65, 0.25, 0.10) \in E$ yields

$$(\min(0.75, 0.65), \max(0.15, 0.25), \max(0.10, 0.10)) = (0.65, 0.25, 0.10).$$

Because the component-wise min and max operations remain weakly associative at the second superhyper level, $\big(\tilde{\mathcal{P}}_2([0,1]^3), \star^{(2)}\big)$ is an SHv–structure. Therefore $\big(\tilde{\mathcal{P}}_2(X), \tilde{A}_2\big)$ provides a concrete Weak 2-SuperhyperNeutrosophic Set modeling hierarchical neutrosophic risk in a financial portfolio.

Example 2.20 (Disaster Response Team Reliability)**.** In a multi-agency disaster response scenario, let

$$X = \{\text{FireBrigade, MedicalTeam, PoliceUnit}\}$$

be three core response units. Fix $n = 2$, so $\tilde{\mathcal{P}}_2(X)$ consists of nonempty collections of nonempty subsets of X. Consider the element

$$A = \big\{\{\text{FireBrigade}\}, \{\text{MedicalTeam, PoliceUnit}\}\big\} \in \tilde{\mathcal{P}}_2(X),$$

representing two clusters: the Fire Brigade alone, and the combined Medical–Police team.

Define the Weak 2-SuperhyperNeutrosophic membership $\tilde{A}_2 : \tilde{\mathcal{P}}_2(X) \to \tilde{\mathcal{P}}_2([0,1]^3)$ by

$$\tilde{A}_2(A) = \big\{D, E\big\},$$

where

$$D = \big\{(0.88, 0.07, 0.05), (0.82, 0.12, 0.06)\big\}, \quad E = \big\{(0.75, 0.15, 0.10), (0.70, 0.18, 0.12)\big\}.$$

Here each neutrosophic triple (T, I, F) denotes:

- T: degree of "effective response",
- I: degree of indeterminacy (e.g. communication breakdowns),
- F: degree of "failure to respond" (e.g. resource constraints).

The superhyperoperation $\star^{(2)}$ on $\tilde{\mathcal{P}}_2([0,1]^3)$ is

$$A' \star^{(2)} B' = \{ (\min(T_a, T_b),\ \max(I_a, I_b),\ \max(F_a, F_b)) \ \big|\ (T_a, I_a, F_a) \in A',\ (T_b, I_b, F_b) \in B' \}.$$

For example, combining the FireBrigade scenario $(0.88, 0.07, 0.05) \in D$ with the Medical–Police scenario $(0.75, 0.15, 0.10) \in E$ yields

$$(\min(0.88, 0.75),\ \max(0.07, 0.15),\ \max(0.05, 0.10)) = (0.75,\ 0.15,\ 0.10).$$

Because component-wise min and max remain weakly associative at this superhyper level, $(\tilde{\mathcal{P}}_2([0,1]^3),\ \star^{(2)})$ is an SHv–structure. Hence $(\tilde{\mathcal{P}}_2(X),\ \tilde{A}_2)$ is a concrete Weak 2-SuperhyperNeutrosophic Set modeling hierarchical uncertainty in disaster response reliability.

Theorem 2.21. *$(\tilde{\mathcal{P}}_n([0,1]^3),\ \star^{(n)})$ is an SHv–structure of order n.*

Proof. By induction on n. For $n = 1$, the associativity of min on the first coordinate and max on the second and third coordinates implies $\star^{(1)}$ is weakly associative (see the proof for Weak HyperNeutrosophic Sets). Assume $\star^{(n-1)}$ is weakly associative on $\tilde{\mathcal{P}}_{n-1}([0,1]^3)$. Then for $A, B, C \subseteq \tilde{\mathcal{P}}_n([0,1]^3)$, one shows

$$A \star^{(n)} (B \star^{(n)} C) = \{ (a \star^{(n-1)} (b \star^{(n-1)} c)) \} = \{ ((a \star^{(n-1)} b) \star^{(n-1)} c) \} = (A \star^{(n)} B) \star^{(n)} C,$$

using the inductive hypothesis and componentwise associativity of min, max. Hence $\star^{(n)}$ is weakly associative. □

Theorem 2.22 (Generalization of n-SuperhyperNeutrosophic Sets)**.** *Every n-SuperhyperNeutrosophic Set $\tilde{A}_n : \tilde{\mathcal{P}}_n(X) \to \tilde{\mathcal{P}}_n([0,1]^3)$ becomes a Weak n-SuperhyperNeutrosophic Set when its codomain is equipped with $\star^{(n)}$.*

Proof. An n-SuperhyperNeutrosophic Set is by definition any mapping $\tilde{A}_n : \tilde{\mathcal{P}}_n(X) \to \tilde{\mathcal{P}}_n([0,1]^3)$. Since $(\tilde{\mathcal{P}}_n([0,1]^3),\ \star^{(n)})$ is an SHv–structure, the image of $\tilde{A}_n$ lies in an SHv–structure, so $\tilde{A}_n$ satisfies the weak associativity requirement. Thus it is a Weak n-SuperhyperNeutrosophic Set. □

Theorem 2.23 (Generalization of Weak n-SuperHyperFuzzy Sets)**.** *Let $\tilde{\nu}_n : \tilde{\mathcal{P}}_n(X) \to \tilde{\mathcal{P}}_n([0,1])$ be a Weak n-SuperHyperFuzzy Set. Defining*

$$\hat{\nu}_n(A) = \{ (t, 0, 0) \mid t \in \tilde{\nu}_n(A) \} \ \subseteq\ \tilde{\mathcal{P}}_n([0,1]^3)$$

makes $\hat{\nu}_n$ a Weak n-SuperhyperNeutrosophic Set, embedding fuzzy degrees as neutrosophic triples $(T, 0, 0)$.

Proof. Since $\tilde{\nu}_n(A) \subseteq [0,1]$, mapping each $t \mapsto (t, 0, 0)$ yields a subset of $[0,1]^3$. The hyperoperation $\star^{(n)}$ restricts on these triples to the same min-based rule on the first coordinate, recovering the original weak superhyperfuzzy operation. Hence $\hat{\nu}_n$ satisfies the weak associativity axiom and defines a Weak n-SuperhyperNeutrosophic Set. □

Theorem 2.24 (Generalization of Weak HyperNeutrosophic Sets)**.** *If $(X, \tilde{\mu})$ is a Weak HyperNeutrosophic Set, then viewing $\tilde{\mu}$ as a mapping $\tilde{\mu}_1 : \tilde{\mathcal{P}}_1(X) \to \tilde{\mathcal{P}}_1([0,1]^3)$ yields a Weak 1-SuperhyperNeutrosophic Set. In particular, the case $n = 1$ recovers exactly the weak hyperstructure of neutrosophic sets.*

Proof. By definition $\tilde{\mathcal{P}}_1(X) = \tilde{\mathcal{P}}(X)$ and $\star^{(1)} = \star$ is the original hyperoperation on $\tilde{\mathcal{P}}([0,1]^3)$. Thus the axioms coincide, and $\tilde{\mu}_1$ is a Weak 1-SuperhyperNeutrosophic Set. □

3 Conclusion and Future Work

In this paper, we have introduced four novel extensions: the *Weak HyperFuzzy Set*, the *Weak HyperNeutrosophic Set*, the *Weak SuperHyperFuzzy Set*, and the *Weak SuperHyperNeutrosophic Set*. These constructions leverage the algebraic frameworks of Weak HyperStructures and Weak SuperHyperStructures to generalize existing HyperUncertain and SuperHyperUncertain Set theories.

For future work, we plan to pursue further generalizations and algorithmic studies based on advanced uncertain-set frameworks, including Plithogenic Sets [55–57], Quadri-Partitioned Neutrosophic Sets [58–60], Double-valued Neutrosophic Sets [61–63], and related extensions.

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Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

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