Generalizing Multidimensional Networks: A Framework of Multidimensional HyperNetworks and SuperHyperNetworks

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Abstract

Graph theory studies the mathematical structures of vertices and edges to model relationships and connectivity [1, 2]. Hypergraphs extend this framework by allowing hyperedges to connect arbitrarily many vertices at once [3], and superhypergraphs further generalize hypergraphs via iterated powerset constructions to capture hierarchical linkages among edges [4,5]. Weighted multidimensional networks model nodes connected by edges in multiple layers, assigning each edge a weight to quantify relationship strength in its specific dimension(cf. [6,7]). In this paper, we extend these ideas using hypergraphs and superhypergraphs to introduce and formalize *multidimensional hypernetworks* and *multidimensional superhypernetworks*.

Keywords: Superhypergraph, Hypergraph, Multidimensional Network, Multidimensional HyperNetwork. Multidimensional SuperHyperNetwork

1 Preliminaries

We begin by fixing notation and recalling key concepts that underlie our constructions. Unless otherwise noted, all graph-based objects in this paper are finite, simple, and undirected. For more extensive discussions of these topics, the reader is referred to the standard literature.

1.1 Hypergraphs and SuperHyperGraphs

A *hypergraph* allows edges—called *hyperedges*—to link any number of vertices at once, capturing higher-order relationships beyond pairwise connections [3, 8–11]. A *SuperHyperGraph* builds on this by applying the powerset operation repeatedly, so that edges themselves can be nested in multiple layers, representing hierarchical groupings of interactions [4, 5, 12]. In what follows, the integer $n \geq 0$ always indicates the number of times the powerset is iterated.

Definition 1.1 (Ground Set)**.** Let S be a finite *ground set* of elements under consideration. All subsequent constructions—subsets, powersets, and iterated powersets—are formed from S.

Definition 1.2 (Powerset)**.** (cf. [13–15]) Given a set S, its *powerset* $\mathcal{P}(S)$ is the collection of all subsets of S, including the empty set and S itself:

$$\mathcal{P}(S) = \{ A \mid A \subseteq S \}.$$

Definition 1.3 (Hypergraph)**.** [3, 8] A *hypergraph* $H = (V, E)$ consists of

- A finite vertex set V.
- A finite set E of nonempty subsets of V, each called a *hyperedge*.

By design, hypergraphs can model relationships that involve more than two vertices at once.

Definition 1.4 (n-th Iterated Powerset)**.** [16–18] For a set X, define its iterated powersets by

$$\mathcal{P}_0(X) = X, \qquad \mathcal{P}_{k+1}(X) = \mathcal{P}\big(\mathcal{P}_k(X)\big) \quad (k \geq 0).$$

Thus $\mathcal{P}_n(X)$ is the result of applying the powerset operation n times.

Definition 1.5 (n-SuperHyperGraph)**.** [19–21] Fix a finite ground set V_0. Let $\mathcal{P}^k(V_0)$ denote the k-th iterated powerset as above. An *n-SuperHyperGraph* is an ordered pair

$$\mathrm{SuHG}^{(n)} = (V, E), \quad \text{with } V, E \subseteq \mathcal{P}^n(V_0),$$

where elements of V are called *n-supervertices* and elements of E are *n-superedges*, each superedge being a nonempty subset of V. This structure captures interactions at up to n nested levels.

Example 1.6 (Collaborative Research 2-SuperHyperGraph)**.** Consider a research laboratory with five investigators:

$$V_0 = \{\text{Yuta}, \ \text{Hiroko}, \ \text{Shinya}, \ \text{Taka}, \ \text{Eve}\}.$$

They form three project teams (first-level groups):

$$T_1 = \{\text{Yuta}, \text{Hiroko}\}, \quad T_2 = \{\text{Shinya}, \text{Taka}\}, \quad T_3 = \{\text{Hiroko}, \text{Eve}\}.$$

These teams themselves are organized into consortia (second-level supervertices):

$$C_1 = \{T_1, T_2\}, \quad C_2 = \{T_2, T_3\}, \quad C_3 = \{T_1, T_3\}.$$

Taking $n = 2$, we have

$$V = \{\, C_1, \, C_2, \, C_3\} \ \subseteq \ \mathcal{P}^2(V_0).$$

We record collaborative initiatives as 2-superedges (each a nonempty subset of V):

$$\mathcal{E} = \big\{\{C_1\}, \ \{C_2\}, \ \{C_3\}, \ \{C_1, C_2\}, \ \{C_2, C_3\}\big\}.$$

Here

- $\{C_1\}$ represents an internal workshop run by Consortium 1.
- $\{C_1, C_2\}$ denotes a joint symposium between Consortia 1 and 2.
- $\{C_2, C_3\}$ denotes collaborative grant applications between Consortia 2 and 3.

Thus

$$\mathrm{SuHG}^{(2)} = (\, V, \ \mathcal{E}\,)$$

is a 2-SuperHyperGraph capturing collaborations at the level of researchers → teams → consortia.

1.2 Weighted Multidimensional Network

Weighted multidimensional network models nodes connected by edges in several distinct layers, assigning each edge a weight to quantify relationship strength along its specific dimension (cf. [22–24]). Related concepts include Multilayer Networks and Multiplex Networks, which are also well-studied in the literature(cf. [25–30]).

Definition 1.7 (Unweighted Multidimensional Network)**.** Let V be a finite set of *nodes* and let D be a finite set of *dimensions* (or *layers*). An *unweighted multidimensional network* is a triple

$$\mathcal{G} = (V, E, D),$$

where

- $E \subseteq V \times V \times D$ is a set of *edges* of the form (u, v, d), indicating an (undirected) connection between $u, v \in V$ in dimension $d \in D$;
- for each $d \in D$ and each unordered pair $\{u, v\} \subseteq V$, at most one edge (u, v, d) appears in E;
- if the network is *directed*, we instead allow (u, v, d) and (v, u, d) to be distinct elements of E.

We say that two nodes u, v are *adjacent in dimension d* if $(u, v, d) \in E$. The *degree* of v in dimension d is

$$k_d(v) \ = \ \big|\{\, u \in V : (u, v, d) \in E\}\big|.$$

Example 1.8 (Urban Multimodal Transport Network)**.** Let the set of *stops* in a city be

$$V = \{\mathrm{S}_1, \mathrm{S}_2, \mathrm{S}_3, \mathrm{S}_4, \mathrm{S}_5\}.$$

There are three transport modes (dimensions):

$$D = \{\text{bus}, \ \text{metro}, \ \text{tram}\}.$$

We represent direct connections by edges tagged with the mode of travel:

$$\begin{aligned} E = \{&(\mathrm{S}_1, \mathrm{S}_2, \text{bus}), \ (\mathrm{S}_2, \mathrm{S}_3, \text{bus}), \\ &(\mathrm{S}_1, \mathrm{S}_4, \text{metro}), \ (\mathrm{S}_4, \mathrm{S}_5, \text{metro}), \\ &(\mathrm{S}_2, \mathrm{S}_5, \text{tram}), \ (\mathrm{S}_3, \mathrm{S}_4, \text{tram})\}. \end{aligned}$$

No more than one connection appears for each pair and mode. Thus

$$\mathcal{G} = (V, E, D)$$

is an *unweighted multidimensional network* modeling the city's multimodal transport system:

- $(\mathrm{S}_1, \mathrm{S}_2, \text{bus})$ and $(\mathrm{S}_2, \mathrm{S}_3, \text{bus})$ are direct bus routes.
- $(\mathrm{S}_1, \mathrm{S}_4, \text{metro})$ and $(\mathrm{S}_4, \mathrm{S}_5, \text{metro})$ are metro lines.
- $(\mathrm{S}_2, \mathrm{S}_5, \text{tram})$ and $(\mathrm{S}_3, \mathrm{S}_4, \text{tram})$ are tram connections.

Passengers can analyze paths that switch modes by traversing edges in different dimensions.

Definition 1.9 (Weighted Multidimensional Network)**.** Let V, D be as in Definition 1.7, and let

$$E \ \subseteq \ V \times V \times D \times \mathbb{R}$$

be a set of *weighted edges* (u, v, d, w), where $w \in \mathbb{R}$ is the weight of the connection between u and v in dimension d. The quadruple

$$\mathcal{G} = (V, E, D, w)$$

is called a *weighted multidimensional network*. The *strength* of node v in dimension d is

$$s_d(v) \ = \sum_{(u,v,d,w)\in E} w.$$

Remark 1.10 (Adjacency Tensor)**.** In canonical tensor notation, one may encode any weighted multidimensional network $\mathcal{G} = (V, E, D)$ by a rank-4 *adjacency tensor* $M \ = \ (M^{i\alpha}_{j\beta})_{i,j\in V,\ \alpha,\beta\in D}$, where

$$M^{i\alpha}_{j\beta} = \begin{cases} w, & \text{if } (i, j, \alpha, w) \in E \text{ and } \alpha = \beta, \\ 0, & \text{otherwise.} \end{cases}$$

This representation readily extends standard matrix-based methods (e.g. centrality, spectral analysis) to the multidimensional setting.

Example 1.11 (Global Trade Weighted Multidimensional Network)**.** Consider five countries:

$$V = \{\text{US, CN, DE, JP, IN}\}.$$

We track trade in three categories (dimensions):

$$D = \{\text{oil, electronics, agriculture}\}.$$

Define weighted edges (u, v, d, w) where w is annual export volume (in billions USD):

$$\begin{aligned} E = \{&(\text{US}, \text{CN}, \text{oil}, \ 50), \ (\text{CN}, \text{US}, \text{electronics}, \ 200), \\ &(\text{DE}, \text{US}, \text{agriculture}, \ 30), \ (\text{JP}, \text{DE}, \text{electronics}, \ 120), \\ &(\text{IN}, \text{JP}, \text{agriculture}, \ 25), \ (\text{US}, \text{IN}, \text{oil}, \ 15)\}. \end{aligned}$$

Then

$$\mathcal{G} = (V, E, D, w)$$

is a *weighted multidimensional network* representing global trade:

- (US, CN, oil, 50) means the US exports \$50 billion of oil to China.
- (CN, US, electronics, 200) means China exports \$200 billion of electronics to the US.
- (DE, US, agriculture, 30) means Germany exports \$30 billion of agricultural products to the US.
- And so on for the other entries.

One can compute each country's *strength* $s_d(v)$ in category d by summing outgoing volumes in that dimension.

1.3 HyperNetwork and SuperhyperNetwork

A hypernetwork is a graph generalization where hyperedges connect any number of nodes, enabling modeling of multiway relationships beyond pairwise edges [31–34]. An n-superhypernetwork uses vertices and hyperedges drawn from the n-th iterated powerset of a base node set to model nested, hierarchical groupings [34–36]. The definitions of HyperNetwork and SuperhyperNetwork are presented below [37].

Definition 1.12 (Hypernetwork). [37] A *hypernetwork* is an ordered triple

$$H = \big(V,\, \mathcal{E},\, w\big)$$

where

- V is a nonempty finite set of *nodes*;
- $\mathcal{E} \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$ is the set of *hyperedges*, each hyperedge $e \in \mathcal{E}$ being a nonempty subset of nodes (allowing multi-node interactions);
- $w\colon \mathcal{E} \to \mathbb{R}_{\geq 0}$ is a *weight or attribute function* on hyperedges (omitted if unweighted).

A *directed hypernetwork* may be defined by replacing $\mathcal{E} \subseteq \mathcal{P}(V)$ with a set of *ordered* tuples of nodes or by equipping each $e \in \mathcal{E}$ with a head-tail partition. One can further add a *node-labeling* $\ell_V : V \to L_V$ and a *hyperedge-labeling* $\ell_{\mathcal{E}} : \mathcal{E} \to L_{\mathcal{E}}$ to record types or properties.

Definition 1.13 (n-SuperHypernetwork). [31, 37, 38] Let V_0 be a finite base set of *nodes*. Define the n-th iterated powerset recursively by

$$\mathcal{P}^0(V_0) = V_0, \qquad \mathcal{P}^{k+1}(V_0) = \mathcal{P}\big(\mathcal{P}^k(V_0)\big) \quad (k \geq 0).$$

An *n-superhypernetwork* is a tuple

$$\mathcal{N}^{(n)} \;=\; \big(V,\; \mathcal{E},\; w\big)$$

where

- $V \subseteq \mathcal{P}^n(V_0)$ is a finite set of *n-supernodes*;
- $\mathcal{E} \subseteq \mathcal{P}^n(V_0)$ is a finite set of *n-superedges*, each superedge $e \in \mathcal{E}$ being a nonempty subset of V;
- $w\colon \mathcal{E} \to \mathbb{R}_{\geq 0}$ is an optional *weight function* assigning a nonnegative real weight (or confidence) to each superedge.

In other words, both vertices and hyperedges of the network are drawn from the n-th powerset of the base node set, capturing up to n levels of hierarchical grouping.

Example 1.14 (Research Consortium 2-SuperHypernetwork)**.** Consider a small research community of four investigators:

$$V_0 = \{\text{Yuta, Hiroko, Shinya, Taka}\}.$$

They form three *laboratories* (first-level teams):

$$L_1 = \{\text{Yuta}, \text{Hiroko}\}, \quad L_2 = \{\text{Shinya}, \text{Taka}\}, \quad L_3 = \{\text{Hiroko}, \text{Shinya}\}.$$

These labs themselves collaborate to form two *consortia* (second-level supernodes):

$$C_1 = \{L_1, L_2\}, \qquad C_2 = \{L_2, L_3\}.$$

Thus, taking $n = 2$, we have

$$V = \{ C_1,\ C_2\} \ \subseteq\ \mathcal{P}^2(V_0).$$

We model funded programs as *2-superedges* (each a nonempty subset of V):

$$\mathcal{E} = \big\{\{C_1\},\ \{C_2\},\ \{C_1, C_2\}\big\}.$$

Assigning each program's annual budget (in million USD) as a weight,

$$w(\{C_1\}) = 5.0, \quad w(\{C_2\}) = 3.5, \quad w(\{C_1, C_2\}) = 1.2.$$

Therefore

$$\mathcal{N}^{(2)} = \big(V,\ \mathcal{E},\ w\big)$$

is a 2-SuperHypernetwork capturing:

- the hierarchy of individual investigators $\longrightarrow$ labs $\longrightarrow$ consortia,
- the funding programs that each consortium (and joint consortia pair) secures.

2 Result: Weighted Multidimensional Hypernetwork

Weighted multidimensional hypernetwork extends networks by grouping nodes into hyperedges across distinct dimensions, assigning weights to each unique hyperedge–dimension pair to represent multiway relationship intensities.

Definition 2.1 (Weighted Multidimensional Hypernetwork)**.** Let V be a finite set of *nodes* and let D be a finite set of *dimensions*. A *weighted multidimensional hypernetwork* is a quadruple

$$\mathcal{H} \ = \ \big(V,\ \mathcal{E},\ D,\ w\big),$$

where

- $\mathcal{E} \subseteq \big(\mathcal{P}(V) \setminus \{\emptyset\}\big) \times D$ is a set of *dimension-tagged hyperedges*, each element $(e, d) \in \mathcal{E}$ consisting of a nonempty vertex subset $e \subseteq V$ and a dimension $d \in D$;
- $w\colon \mathcal{E} \to \mathbb{R}$ is a *weight function* assigning each $(e, d) \in \mathcal{E}$ a real weight $w(e, d)$.

Example 2.2 (Corporate Team Communication Hypernetwork)**.** Consider a small engineering team with four members:

$$V = \{\text{Yuta, Hiroko, Shinya, Taka}\}.$$

They use three communication channels:

$$D = \{\text{Slack, Email, Zoom}\}.$$

We model group communications as dimension-tagged hyperedges, weighted by the average number of messages (or minutes of meeting) per week:

$$\mathcal{E} = \big\{\big(\{\text{ Yuta}, \text{Hiroko}, \text{Shinya}\}, \text{Slack}\big),\ \big(\{\text{ Yuta}, \text{Taka}\}, \text{Email}\big),\ \big(\{\text{ Hiroko}, \text{Shinya}, \text{Taka}\}, \text{Zoom}\big)\big\}.$$

Define the weight function $w\colon \mathcal{E} \to \mathbb{R}_{\geq 0}$ by

$$w\big(\{A, B, C\}, \text{Slack}\big) = 120, \quad w\big(\{A, D\}, \text{Email}\big) = 30, \quad w\big(\{B, C, D\}, \text{Zoom}\big) = 180,$$

where

$w(\{\text{Yuta, Hiroko, Shinya}\}, \text{Slack}) = 120$ (120 Slack messages/week among Yuta, Hiroko, Shinya),
$w(\{\text{Yuta, Taka}\}, \text{Email}) = 30$ (30 email exchanges/week between Yuta and Taka),
$w(\{\text{Hiroko, Shinya, Taka}\}, \text{Zoom}) = 180$ (180 minutes of Zoom meetings/week among Hiroko, Shinya, Taka).

Thus

$$\mathcal{H} = \big(V,\ \mathcal{E},\ D,\ w\big)$$

is a weighted multidimensional hypernetwork capturing the team's collaborative interactions across different channels.

Example 2.3 (Global Supply Chain Hypernetwork)**.** Consider a simplified supply chain with six locations:

$$V = \{\mathrm{F}_1, \mathrm{F}_2,\ \mathrm{W}_1, \mathrm{W}_2,\ \mathrm{R}_1, \mathrm{R}_2\},$$

where F_i are factories, W_j warehouses, and R_k retailers. There are three product categories (dimensions):

$$D = \{\text{electronics},\ \text{furniture},\ \text{food}\}.$$

We record weekly shipment groupings as dimension-tagged hyperedges, weighted by units shipped per week:

$$\begin{aligned}\mathcal{E} = \{&(\{\,\mathrm{F}_1, \mathrm{W}_1, \mathrm{R}_1\}, \text{electronics}),\\ &(\{\,\mathrm{F}_2, \mathrm{W}_2, \mathrm{R}_2\}, \text{electronics}),\\ &(\{\,\mathrm{F}_1, \mathrm{W}_1, \mathrm{W}_2, \mathrm{R}_2\}, \text{furniture}),\\ &(\{\,\mathrm{F}_2, \mathrm{W}_2, \mathrm{R}_1, \mathrm{R}_2\}, \text{food})\}.\end{aligned}$$

Assign the following weights $w\colon \mathcal{E} \to \mathbb{R}_{\geq 0}$:

$$\begin{aligned} w\big(\{\mathrm{F}_1, \mathrm{W}_1, \mathrm{R}_1\}, \text{electronics}\big) &= 1000,\\ w\big(\{\mathrm{F}_2, \mathrm{W}_2, \mathrm{R}_2\}, \text{electronics}\big) &= 800,\\ w\big(\{\mathrm{F}_1, \mathrm{W}_1, \mathrm{W}_2, \mathrm{R}_2\}, \text{furniture}\big) &= 500,\\ w\big(\{\mathrm{F}_2, \mathrm{W}_2, \mathrm{R}_1, \mathrm{R}_2\}, \text{food}\big) &= 1200.\end{aligned}$$

Thus

$$\mathcal{H} = \big(V,\ \mathcal{E},\ D,\ w\big)$$

is a weighted multidimensional hypernetwork modeling weekly shipment volumes of electronics, furniture, and food across factories, warehouses, and retailers in the supply chain.

Theorem 2.4 (Generalization of Weighted Multidimensional Networks and Hypernetworks)**.**

1. *Every weighted multidimensional network $\mathcal{N} = (V, E, D, w_{\mathcal{N}})$ (Definition 1.9) induces a weighted multidimensional hypernetwork*

$$\mathcal{H}_{\mathcal{N}} \ =\ \Big(V,\ \{\,(\{u, v\}, d) \mid (u, v, d) \in E\},\ D,\ w'_{\mathcal{N}}\Big),$$

where $w'_{\mathcal{N}}(\{u, v\}, d) = w_{\mathcal{N}}(u, v, d)$. In particular, $\mathcal{H}_{\mathcal{N}}$ is exactly the hypernetwork whose hyperedges are the size-two edges of $\mathcal{N}$.

2. *Every weighted hypernetwork $\mathcal{H} = (V, \mathcal{E}, \{d_0\}, w_{\mathcal{H}})$ with a single dimension $D = \{d_0\}$ arises as a special case of a weighted multidimensional hypernetwork by tagging all hyperedges with d_0.*

Hence the class of weighted multidimensional hypernetworks strictly generalizes both weighted multidimensional networks and weighted hypernetworks.

Proof. (1) Let $\mathcal{N} = (V, E, D, w_{\mathcal{N}})$ be a weighted multidimensional network, so

$$E \subseteq V \times V \times D, \quad w_{\mathcal{N}}\colon E \to \mathbb{R}.$$

Define

$$\mathcal{E} = \left\{ (\{u,v\}, d) \;\middle|\; (u,v,d) \in E \right\} \quad \subseteq \; (\mathcal{P}(V) \setminus \{\emptyset\}) \times D,$$

and set

$$w'_{\mathcal{N}}(\{u,v\}, d) \;=\; w_{\mathcal{N}}(u,v,d).$$

By construction $\mathcal{H}_{\mathcal{N}} = (V, \mathcal{E}, D, w'_{\mathcal{N}})$ satisfies the definition of a weighted multidimensional hypernetwork. The injective correspondence $(u,v,d) \mapsto (\{u,v\}, d)$ shows that no information is lost.

(2) Let $\mathcal{H} = (V, \mathcal{E}, \{d_0\}, w_{\mathcal{H}})$ be a weighted hypernetwork in the usual sense: $\mathcal{E} \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$ and $w_{\mathcal{H}} \colon \mathcal{E} \to \mathbb{R}$. Embed it into a weighted multidimensional hypernetwork by

$$\widetilde{\mathcal{E}} \;=\; \left\{ (e, d_0) \mid e \in \mathcal{E} \right\}, \qquad \widetilde{w}(e, d_0) = w_{\mathcal{H}}(e).$$

Again, this construction is bijective and respects all weights.

Since every weighted multidimensional network and every weighted hypernetwork injects into the class of weighted multidimensional hypernetworks, and there exist hyperedges of arbitrary cardinality or multiple dimensions that cannot be realized in the narrower settings, $\mathcal{H}$ indeed strictly generalizes both. □

Definition 2.5 (Dimension-Restricted Hypernetwork)**.** For each $d \in D$, define

$$\mathcal{E}_d \;=\; \{\, e \subseteq V : (e,d) \in \mathcal{E} \}, \quad w_d(e) = w(e,d).$$

Then $\mathcal{H}_d = (V, \mathcal{E}_d, w_d)$ is the *dimension-d restriction* of $\mathcal{H}$.

Theorem 2.6 (Restriction Yields a Weighted Hypernetwork)**.** *For each $d \in D$, the restricted structure $\mathcal{H}_d$ is itself a weighted hypernetwork in the usual sense.*

Proof. By definition $\mathcal{E}_d \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$ and w_d assigns real weights to each hyperedge $e \in \mathcal{E}_d$. Thus $\mathcal{H}_d$ meets the requirements of a weighted hypernetwork. □

Definition 2.7 (Aggregated Hypernetwork)**.** Define the *aggregation* of $\mathcal{H}$ across all dimensions by

$$\mathcal{E}_{\text{agg}} = \left\{ e \subseteq V : \exists\, d \in D,\ (e,d) \in \mathcal{E} \right\}, \quad w_{\text{agg}}(e) = \sum_{\substack{d \in D \\ (e,d) \in \mathcal{E}}} w(e,d).$$

Then $\mathcal{H}_{\text{agg}} = (V, \mathcal{E}_{\text{agg}}, w_{\text{agg}})$ is called the *aggregated hypernetwork*.

Theorem 2.8 (Aggregation Preserves Hypernetwork Structure)**.** *$\mathcal{H}_{\text{agg}}$ is a weighted hypernetwork whose edge–weight function w_{agg} is well-defined and nonnegative.*

Proof. Clearly $\mathcal{E}_{\text{agg}} \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$. Since each (e,d) contributes a real weight, the sum $w_{\text{agg}}(e) \geq 0$ is well-defined. Hence $\mathcal{H}_{\text{agg}}$ satisfies the definition of a weighted hypernetwork. □

Definition 2.9 (Primal Graph)**.** The *primal graph $G = (V, A)$* of $\mathcal{H}$ is the simple graph with

$$A \;=\; \left\{ \{u,v\} \subseteq V : \exists\, (e,d) \in \mathcal{E},\ \{u,v\} \subseteq e \right\}.$$

Theorem 2.10 (Primal Connectivity Criterion)**.** *The primal graph G is connected if and only if for every nontrivial partition $V = V_1 \cup V_2$, there exists some $(e,d) \in \mathcal{E}$ with $e \cap V_1 \neq \emptyset$ and $e \cap V_2 \neq \emptyset$.*

Proof. ($\Rightarrow$) If G is connected, no partition can isolate two sets without an edge crossing, so some hyperedge in $\mathcal{E}$ must bridge V_1 and V_2. ($\Leftarrow$) Conversely, if every partition is bridged by at least one hyperedge, the primal graph cannot split into disconnected components, hence G is connected. □

Definition 2.11 (Node Strength and Dimension Relevance)**.** For each node $v \in V$ define its *dimension-d strength*

$$s_d(v) = \sum_{\substack{(e,d)\in\mathcal{E} \\ v\in e}} w(e,d),$$

and its *total strength*

$$s(v) = \sum_{d\in D} s_d(v).$$

The *relevance* of dimension d at v is

$$R(v,d) = \frac{s_d(v)}{s(v)}.$$

Theorem 2.12 (Relevance Forms a Probability Distribution)**.** *For each $v \in V$ with $s(v) > 0$, the values $\{R(v,d)\}_{d\in D}$ satisfy $0 \le R(v,d) \le 1$ and $\sum_{d\in D} R(v,d) = 1$.*

Proof. By construction $s_d(v) \ge 0$ and $s(v) > 0$. Thus $0 \le R(v,d) \le 1$. Moreover

$$\sum_{d\in D} R(v,d) = \sum_{d\in D} \frac{s_d(v)}{s(v)} = \frac{1}{s(v)} \sum_{d\in D} s_d(v) = 1,$$

as required. □

3 Result: Weighted Multidimensional SuperHypernetwork

Weighted multidimensional superhypernetwork generalizes hypernetworks by using iterated powersets to create distinct nested supernodes and superedges across dimensions, with weights capturing hierarchical multiway relationship strengths.

Definition 3.1 (Weighted Multidimensional SuperHypernetwork)**.** Let V_0 be a finite *base set* and let D be a finite set of *dimensions*. Fix a nonnegative integer n. Denote by

$$\mathcal{P}^0(V_0) = V_0, \quad \mathcal{P}^{k+1}(V_0) = \mathcal{P}\big(\mathcal{P}^k(V_0)\big) \quad (k \ge 0).$$

A *weighted multidimensional n-superhypernetwork* is a quadruple

$$\mathcal{S}^{(n)} \;=\; \big(V,\; \mathcal{E},\; D,\; w\big),$$

where

- $V \subseteq \mathcal{P}^n(V_0)$ is the set of *n-supernodes*;
- $\mathcal{E} \subseteq \big(\mathcal{P}^n(V_0) \setminus \{\emptyset\}\big) \times D$ is the set of *dimension-tagged n-superedges*, each $(e,d) \in \mathcal{E}$ consisting of a nonempty n-supernode subset $e \subseteq V$ in dimension $d \in D$;
- $w\colon \mathcal{E} \to \mathbb{R}$ assigns each (e,d) a real *weight* $w(e,d)$.

Example 3.2 (Regional Multimodal Logistics SuperHypernetwork)**.** Consider a logistics company serving five cities:

$$V_0 = \{A,B,C,D,E\}.$$

There are three transportation modes (dimensions):

$$D = \{\text{road},\ \text{rail},\ \text{air}\}.$$

We choose $n = 1$, so

$$\mathcal{P}^1(V_0) = \mathcal{P}(V_0) = \{\,\emptyset, \{A\}, \{B\}, \{C\}, \{D\}, \{E\}, \{A,B\}, \dots, \{A,B,C,D,E\}\}.$$

Define the set of *1–supernodes* as two key regions:

$$V \;=\; \big\{\{A,B\},\ \{C,D,E\}\big\}.$$

Each region groups nearby cities into a single operational cluster.

Next, we model both intra- and inter-region shipping via dimension-tagged superedges:

$$\mathcal{E} = \big\{\big(\{A,B\}, \mathit{road}\big),\ \big(\{A,B\}, \mathit{rail}\big),\ \big(\{C,D,E\}, \mathit{air}\big),\ \big(\{\{A,B\},\{C,D,E\}\}, \mathit{rail}\big)\big\}.$$

- $(\{A, B\}, \text{road})$: shipments by truck within Region $\{A, B\}$.
- $(\{A, B\}, \text{rail})$: rail shipments within Region $\{A, B\}$.
- $(\{C, D, E\}, \text{air})$: air shipments within Region $\{C, D, E\}$.
- $(\{\{A, B\}, \{C, D, E\}\}, \text{rail})$: inter-region rail shipments connecting the two regions.

We assign daily shipment volumes (in tons) as weights:

$$\begin{aligned} w\big(\{A, B\}, \text{road}\big) &= 500, \\ w\big(\{A, B\}, \text{rail}\big) &= 300, \\ w\big(\{C, D, E\}, \text{air}\big) &= 200, \\ w\big(\{\{A, B\}, \{C, D, E\}\}, \text{rail}\big) &= 150. \end{aligned}$$

Thus

$$\mathcal{S}^{(1)} = \big(V,\ \mathcal{E},\ D,\ w\big)$$

is a weighted multidimensional 1-superhypernetwork capturing both hierarchical clustering of cities into regions and the multimodal shipment volumes within and between those regions.

Example 3.3 (Corporate Communication SuperHypernetwork)**.** Consider a small company with four employees:

$$V_0 = \{\text{ Yuta, Hiroko, Shinya, Taka}\}.$$

There are three communication modes (dimensions):

$$D = \{\text{email, meeting, codeReview}\}.$$

We take $n = 2$. Then

$$\mathcal{P}^1(V_0) = \big\{\{\text{Yuta}, \text{Hiroko}\},\ \{\text{Shinya}, \text{Taka}\},\ \{\text{Hiroko}, \text{Shinya}\}, \dots\big\},$$

and

$$\mathcal{P}^2(V_0) = \mathcal{P}\big(\mathcal{P}^1(V_0)\big),$$

whose elements are *sets of teams*. Choose three key teams:

$$T_1 = \{\text{Yuta}, \text{Hiroko}\}, \quad T_2 = \{\text{Shinya}, \text{Taka}\}, \quad T_3 = \{\text{Hiroko}, \text{Shinya}\}.$$

Form two departments as 2-supernodes:

$$V = \big\{D_A = \{T_1, T_2\}, \quad D_B = \{T_2, T_3\}\big\} \ \subseteq\ \mathcal{P}^2(V_0).$$

Here D_A groups Team 1 and Team 2; D_B groups Team 2 and Team 3.

Next, define dimension-tagged 2-superedges (e, d) to capture both intra- and inter-department communications:

$$\mathcal{E} = \big\{(\ \{D_A\},\ \text{email}\ ), (\ \{D_A\},\ \text{meeting}\ ), (\ \{D_B\},\ \text{codeReview}\ ), (\ \{D_A, D_B\},\ \text{meeting}\ ), (\ \{D_A, D_B\},\ \text{email}\ )\big\}.$$

We assign monthly communication volumes as weights $w\colon \mathcal{E} \to \mathbb{R}_{\geq 0}$:

$$\begin{aligned} w\big(\{D_A\}, \text{email}\big) &= 1200, \\ w\big(\{D_A\}, \text{meeting}\big) &= 150, \\ w\big(\{D_B\}, \text{codeReview}\big) &= 300, \\ w\big(\{D_A, D_B\}, \text{meeting}\big) &= 40, \\ w\big(\{D_A, D_B\}, \text{email}\big) &= 800. \end{aligned}$$

Thus

$$\mathcal{S}^{(2)} = \big(V,\ \mathcal{E},\ D,\ w\big)$$

is a weighted multidimensional 2-superhypernetwork modeling:

- intra-department email and meetings within D_A,
- code-review activity within D_B,
- cross-department email and joint meetings between D_A and D_B.

Theorem 3.4 (Generalization of Earlier Models)**.** *The class of weighted multidimensional n-superhypernetworks (Definition 3.1) strictly generalizes each of the following:*

1. Weighted multidimensional networks $(V, E, D, w_{\mathcal{N}})$*, by taking* $n = 0$ *and restricting all hyperedges to cardinality 2.*
2. Weighted multidimensional hypernetworks $(V, \mathcal{E}, D, w_{\mathcal{H}})$*, by taking* $n = 0$*.*
3. Superhypernetworks $(V, \mathcal{E}, w_{\mathcal{S}})$ *(with a single trivial dimension), by taking* $D = \{\star\}$*.*

Proof. We exhibit embedding constructions for each case:

(1) From weighted multidimensional networks. Let $\mathcal{N} = (V, E, D, w_{\mathcal{N}})$ be a weighted multidimensional network, so

$$E \subseteq V \times V \times D, \quad w_{\mathcal{N}} : E \to \mathbb{R}.$$

Choose $n = 0$. Define

$$V' = V \subseteq \mathcal{P}^0(V_0), \qquad \mathcal{E}' = \big\{ (\{u, v\}, d) \bigm| (u, v, d) \in E \big\} \subseteq \big(\mathcal{P}^0(V_0) \setminus \{\emptyset\}\big) \times D,$$

and set $w'(\{u, v\}, d) = w_{\mathcal{N}}(u, v, d)$. Then $\mathcal{S}' = \big(V', \mathcal{E}', D, w'\big)$ is a weighted multidimensional 0-superhypernetwork whose superedges are exactly the ordinary edges of $\mathcal{N}$.

(2) From weighted multidimensional hypernetworks. Let $\mathcal{H} = (V, \mathcal{E}, D, w_{\mathcal{H}})$ be a weighted multidimensional hypernetwork with $\mathcal{E} \subseteq (\mathcal{P}(V) \setminus \{\emptyset\}) \times D$. Again take $n = 0$ and set

$$V' = V \subseteq \mathcal{P}^0(V_0), \qquad \mathcal{E}' = \mathcal{E}, \quad w' = w_{\mathcal{H}}.$$

Then $\mathcal{S}' = (V', \mathcal{E}', D, w')$ is a weighted multidimensional 0-superhypernetwork identical to $\mathcal{H}$.

(3) From superhypernetworks. Let $\mathcal{S} = (V, \mathcal{E}, w_{\mathcal{S}})$ be an (unweighted or weighted) n-superhypernetwork without multiple dimensions. Embed it by setting

$$D' = \{\star\}, \quad V' = V \subseteq \mathcal{P}^n(V_0), \quad \mathcal{E}' = \{ (e, \star) \mid e \in \mathcal{E}\}, \quad w'(e, \star) = w_{\mathcal{S}}(e).$$

Then $\big(V', \mathcal{E}', D', w'\big)$ is a weighted multidimensional n-superhypernetwork reproducing $\mathcal{S}$.

In each case, no information beyond simple relabeling is lost, and the restrictions (to $n = 0$, to edge-cardinality 2, or to a singleton D) exhibit that the general definition indeed encompasses all earlier models. Moreover, allowing arbitrary n, arbitrary dimension set D, and arbitrary superedge cardinalities strictly enlarges the modeling power, so the generalization is proper. □

Definition 3.5 (Dimension-Restricted SuperHypernetwork)**.** Let $\mathcal{S}^{(n)} = (V, \mathcal{E}, D, w)$ be a weighted multidimensional n-superhypernetwork. For each $d \in D$, define

$$\mathcal{E}_d \;=\; \{\, e \subseteq V \mid (e, d) \in \mathcal{E}\}, \quad w_d(e) \;=\; w(e, d).$$

Then $\mathcal{S}^{(n)}_d = (V, \mathcal{E}_d, w_d)$ is called the *dimension-d restriction* of $\mathcal{S}^{(n)}$.

Example 3.6 (Dimension-Restricted Email SuperHypernetwork)**.** Starting from the corporate communication superhypernetwork of Example 3.3,

$$\mathcal{S}^{(2)} = \big(V,\ \mathcal{E},\ D,\ w\big),$$

with

$$V = \{D_A, D_B\}, \quad D = \{\text{email, meeting, codeReview}\},$$

we form the *email* restriction by selecting

$$\mathcal{E}_{\text{email}} = \big\{\, e \subseteq V \mid (e, \text{email}) \in \mathcal{E} \big\} = \big\{\{D_A\},\ \{D_A, D_B\}\big\},$$

and setting

$$w_{\text{email}}\big(\{D_A\}\big) = 1200, \qquad w_{\text{email}}\big(\{D_A, D_B\}\big) = 800.$$

Thus the dimension-restricted superhypernetwork

$$\mathcal{S}^{(2)}_{\text{email}} = \big(V,\ \mathcal{E}_{\text{email}},\ w_{\text{email}}\big)$$

captures *only* the email traffic:

- $\{D_A\}$ with weight 1200 represents department D_A's internal emails;
- $\{D_A, D_B\}$ with weight 800 represents cross-department emails between D_A and D_B.

Theorem 3.7 (Dimension Restriction)**.** *For each $d \in D$, the triple $\mathcal{S}^{(n)}_d = (V, \mathcal{E}_d, w_d)$ is itself a weighted n-superhypernetwork (with a single dimension), inheriting all structural and weight properties from $\mathcal{S}^{(n)}$.*

Proof. By construction, $\mathcal{E}_d \subseteq \mathcal{P}^n(V_0)$ and $w_d : \mathcal{E}_d \to \mathbb{R}$. All supernodes remain in $V \subseteq \mathcal{P}^n(V_0)$, and each superedge $e \in \mathcal{E}_d$ is nonempty. Therefore $\mathcal{S}^{(n)}_d$ satisfies Definition 3.1 with $D = \{d\}$. □

Definition 3.8 (Primal Graph)**.** The *primal graph* $G(V, A)$ of $\mathcal{S}^{(n)} = (V, \mathcal{E}, D, w)$ is the simple graph whose vertex set is V and whose edge set

$$A \;=\; \big\{\{u, v\} \subseteq V : \exists\,(e, d) \in \mathcal{E},\ u \neq v,\ \{u, v\} \subseteq e\big\}.$$

Theorem 3.9 (Primal Connectivity)**.** *The primal graph G of $\mathcal{S}^{(n)}$ is connected if and only if for every nontrivial partition $V = V_1 \cup V_2$, there exists $(e, d) \in \mathcal{E}$ with $e \cap V_1 \neq \emptyset$ and $e \cap V_2 \neq \emptyset$.*

Proof. $\Rightarrow$: If G is connected, then no nontrivial partition can isolate vertices without an edge between them. Any separating partition would contradict connectivity. $\Leftarrow$: Conversely, if every bipartition is bridged by some superedge, then the primal graph cannot decompose into disconnected components, hence must be connected. □

Definition 3.10 (Supernode Strength and Relevance)**.** For $v \in V$, define its *total strength*

$$s(v) = \sum_{\substack{(e,d)\in\mathcal{E} \\ v\in e}} w(e, d),$$

and its *dimension-d strength*

$$s_d(v) = \sum_{\substack{(e,d)\in\mathcal{E} \\ v\in e}} w(e, d).$$

The *dimension relevance* of d at v is

$$R(v, d) = \frac{s_d(v)}{s(v)}, \quad d \in D.$$

Theorem 3.11 (Relevance Distribution)**.** *For each $v \in V$, the relevance values $\{R(v, d)\}_{d\in D}$ form a probability distribution:*

$$R(v, d) \in [0, 1], \quad \sum_{d\in D} R(v, d) = 1.$$

Proof. By definition $s_d(v) \geq 0$ and $s(v) = \sum_d s_d(v) > 0$ whenever v lies in at least one superedge. Hence $0 \leq R(v, d) \leq 1$. Moreover,

$$\sum_{d \in D} R(v, d) = \sum_{d \in D} \frac{s_d(v)}{s(v)} = \frac{1}{s(v)} \sum_{d \in D} s_d(v) = 1.$$

□

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Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

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