HyperFuzzy Control System and SuperHyperFuzzy Control System

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Abstract

Uncertainty modeling is fundamental to decision-making across diverse domains, and numerous frameworks—such as Fuzzy Sets [1], Rough Sets [2, 3], Hyperrough Sets [4, 5], Vague Sets [6], Intuitionistic Fuzzy Sets [7, 8], Hesitant Fuzzy Sets [9, 10], Neutrosophic Sets [11, 12], and Plithogenic Sets [13]—have been developed to capture different facets of imprecision. Among these extensions are Hyperfuzzy Sets and their recursive generalization, SuperHyperfuzzy Sets, which assign set-valued membership degrees at multiple hierarchical levels. This paper introduces the concepts of Hyperfuzzy Control Systems and (m, n)-SuperHyperfuzzy Control Systems, showing how they generalize classical fuzzy control by incorporating richer uncertainty structures. We present rigorous definitions, theoretical properties, and illustrative examples demonstrating their ability to model hierarchical uncertainty in real-world control applications.

Keywords: Fuzzy set, HyperFuzzy Set, Fuzzy Control, HyperFuzzy Control, SuperHyperFuzzy Control

Structure of this paper

The format of this paper is described below.

1 Preliminaries

This section provides an introduction to the foundational concepts and definitions required for the discussions in this paper. In addition, all concepts addressed herein are assumed to be finite rather than infinite.

1.1 Fuzzy Set

A fuzzy set assigns to each element a membership degree in the interval $[0, 1]$, thereby capturing uncertainty through more granular membership levels rather than a strict binary classification [1, 14–18].

Definition 1.1 (Base Set)**.** A *base set* S is the foundational set from which complex structures such as powersets and hyperstructures are derived. It is formally defined as:

$$S = \{x \mid x \text{ is an element within a specified domain}\}.$$

All elements in constructs like $\mathcal{P}(S)$ or $\mathcal{P}_n(S)$ originate from the elements of S.

Definition 1.2 (Powerset)**.** [19, 20] The *powerset* of a set S, denoted $\mathcal{P}(S)$, is the collection of all possible subsets of S, including both the empty set and S itself. Formally, it is expressed as:

$$\mathcal{P}(S) = \{A \mid A \subseteq S\}.$$

Definition 1.3 (n-th Powerset)**.** (cf. [19, 21–23])

The n-th powerset of a set H, denoted $\mathcal{P}_n(H)$, is defined iteratively, starting with the standard powerset. The recursive construction is given by:

$$\mathcal{P}_1(H) = \mathcal{P}(H), \quad \mathcal{P}_{n+1}(H) = \mathcal{P}(\mathcal{P}_n(H)), \quad \text{for } n \geq 1.$$

Similarly, the n-th non-empty powerset, denoted $\mathcal{P}_n^*(H)$, is defined recursively as:

$$\mathcal{P}_1^*(H) = \mathcal{P}^*(H), \quad \mathcal{P}_{n+1}^*(H) = \mathcal{P}^*(\mathcal{P}_n^*(H)).$$

Here, $\mathcal{P}^*(H)$ represents the powerset of H with the empty set removed.

Example 1.4 (Multi-Week Workout Planning via the 3rd Powerset)**.** Let

$$H = \{\text{Push-ups, Squats, Plank}\}.$$

- The first powerset $\mathcal{P}_1(H) = \mathcal{P}(H)$ is the collection of all possible *daily workout routines*, for example

$$A_1 = \{\text{Push-ups, Squats}\} \quad \text{or} \quad A_2 = \{\text{Squats, Plank}\}.$$

- The second powerset $\mathcal{P}_2(H) = \mathcal{P}(\mathcal{P}_1(H))$ is the collection of all possible *weekly schedules* (sets of daily routines), for instance

$$B = \{A_1, A_2, A_3\}, \quad A_3 = \{\text{Push-ups, Plank}\},$$

  representing a 3-day workout plan in one week.

- The third powerset $\mathcal{P}_3(H) = \mathcal{P}(\mathcal{P}_2(H))$ is the collection of all possible *multi-week workout plans*, e.g.

$$C = \big\{\{A_1, A_2, A_3\},\ \{A_2, A_3, A_4\}\big\} \quad \text{with} \quad A_4 = \{\text{Push-ups, Squats, Plank}\},$$

  representing two consecutive weeks of scheduled routines.

Thus $\mathcal{P}_3(H)$ models a three-level hierarchy: individual exercises $\to$ daily routines $\to$ weekly schedules $\to$ multi-week plans.

Definition 1.5 (Fuzzy Set)**.** [1, 24] A *Fuzzy set* τ in a non-empty universe Y is a mapping $\tau : Y \to [0, 1]$. A *fuzzy relation* on Y is a fuzzy subset δ in $Y \times Y$. If τ is a fuzzy set in Y and δ is a fuzzy relation on Y, then δ is called a *fuzzy relation on* τ if

$$\delta(y, z) \le \min\{\tau(y), \tau(z)\} \quad \text{for all } y, z \in Y.$$

A hyperfuzzy set assigns each element a nonempty subset of [0,1], representing possible membership values capturing uncertainty, imprecision, and variability. An (m, n)-superhyperfuzzy set maps each m-level subset to a family of n-level membership sets, modeling hierarchical recursive uncertainty and complexity.

Definition 1.6 (Hyperfuzzy Set)**.** [25–29] Let X be a non-empty universe. A *hyperfuzzy set* $\tilde{A}$ on X is defined by a mapping

$$\tilde{\mu} : X \longrightarrow \tilde{P}([0, 1]),$$

where $\tilde{P}([0, 1])$ denotes the collection of all non-empty subsets of the interval $[0, 1]$.

For each element $x \in X$, $\tilde{\mu}(x) \subseteq [0, 1]$ represents the *set of possible membership degrees* of x in the set $\tilde{A}$. This formulation allows for representing uncertainty or variability in the degree of membership, extending the classical fuzzy set (which assigns a single real number in $[0, 1]$) to a set-valued interpretation.

Thus, a hyperfuzzy set captures both fuzziness and imprecision by associating each element not with a fixed degree, but with a range or subset of plausible membership values.

Example 1.7 (Electric Vehicle Battery Cell Health via a Hyperfuzzy Set)**.** Let

$$X = \{\text{Cell}_1, \text{Cell}_2, \text{Cell}_3\}$$

be the set of three battery cells in an electric vehicle pack. We assess each cell's "state-of-health" (SoH) using three different diagnostic methods:

- *Electrical impedance spectroscopy*,

- *Capacity-fade test*,
- *Thermal imaging analysis*.

We model the variability among these methods by the hyperfuzzy membership mapping

$$\tilde{\mu} : X \longrightarrow \tilde{P}([0,1]),$$

defined concretely as

$$\tilde{\mu}(\mathrm{Cell}_1) = \{0.72,\ 0.75,\ 0.78\},$$
$$\tilde{\mu}(\mathrm{Cell}_2) = \{0.60,\ 0.65,\ 0.70\},$$
$$\tilde{\mu}(\mathrm{Cell}_3) = \{0.80,\ 0.85,\ 0.88\}.$$

Here each set of three values represents the SoH scores (on a scale from 0 to 1) produced by the three diagnostic methods. For instance:

- For Cell_1, $\{0.72, 0.75, 0.78\}$ reflects slight variation between impedance (0.72), capacity (0.75), and thermal (0.78) assessments.
- For Cell_2, $\{0.60, 0.65, 0.70\}$ shows greater disagreement among methods, signaling a cell nearing end-of-life.
- For Cell_3, $\{0.80, 0.85, 0.88\}$ indicates consistently high health across all tests.

This hyperfuzzy set thus captures both the imprecision of each measurement and the uncertainty arising from multiple diagnostic sources, enabling a richer reliability analysis of the battery pack.

Definition 1.8 (m, n-SuperHyperFuzzy Set)**.** [5, 30, 31] Let X be a nonempty set and let $m, n \in \mathbb{N}_0$. Define the nonempty k-th powerset of a set Y by

$$\mathcal{P}_0^*(Y) = Y, \quad \mathcal{P}_k^*(Y) = \mathcal{P}\big(\mathcal{P}_{k-1}^*(Y)\big) \setminus \{\emptyset\}, \quad k \geq 1.$$

In particular, $\mathcal{P}_m^*(X)$ is the family of all nonempty elements of the m-th iterated powerset of X, and $\mathcal{P}_n^*([0,1])$ is defined analogously. Then an (m, n)-SuperHyperFuzzy Set on X is a function

$$\tilde{\mu}_{m,n} : \mathcal{P}_m^*(X) \longrightarrow \tilde{\mathcal{P}}_n^*\big([0,1]\big), \quad A \mapsto \tilde{\mu}_{m,n}(A),$$

where $\tilde{\mathcal{P}}_n^*([0,1])$ denotes the collection of all nonempty subsets of $\mathcal{P}_n([0,1])$. Thus each $A \in \mathcal{P}_m^*(X)$ is assigned a nonempty family of membership-degree sets $\tilde{\mu}_{m,n}(A) \subseteq \mathcal{P}_n([0,1])$, capturing hierarchical uncertainty across both the m- and n-levels.

Example 1.9 (Smartphone Feature Satisfaction via a $(1, 2)$-SuperHyperFuzzy Set)**.** Let

$$X = \{\text{Battery, Camera, Display, Connectivity}\}, \quad m = 1, \quad n = 2.$$

Then

$$\mathcal{P}_1^*(X) = \mathcal{P}(X) \setminus \{\emptyset\},$$

and one representative element of the domain is

$$A = \{\text{Battery, Camera}\} \in \mathcal{P}_1^*(X).$$

We define the membership-mapping

$$\tilde{\mu}_{1,2}(A) = \Big\{L_1 = \{\{0.8,\ 0.85\},\ \{0.9\}\},\ L_2 = \{\{0.75\},\ \{0.95,\ 0.90\}\}\Big\} \subseteq \tilde{\mathcal{P}}_2^*([0,1]).$$

Here:

- Each inner set, for example $\{0.8, 0.85\}$, collects ratings from two major online review platforms.
- The singleton $\{0.9\}$ represents an independent expert lab-test score.

- The outer family $\{L_1, L_2\}$ captures two evaluation scenarios: L_1 is "consumer sentiment" (online reviews $\cup$ expert lab-test), L_2 is "professional benchmarking" (designer focus-group $\cup$ performance-benchmark tests).

Thus this $(1, 2)$-SuperHyperFuzzy Set models a one-level grouping of smartphone features with two nested levels of membership-uncertainty.

Example 1.10 (Project Risk Profiling via a $(2, 1)$-SuperHyperFuzzy Set)**.** Let

$$X = \{\text{Design, Implementation, Testing}\}, \quad m = 2, \quad n = 1.$$

Then

$$\mathcal{P}_2^*(X) = \mathcal{P}\big(\mathcal{P}^*(X)\big) \setminus \{\emptyset\},$$

and one domain element is

$$B = \big\{\{\text{Design, Implementation}\},\ \{\text{Implementation, Testing}\}\big\} \ \in \ \mathcal{P}_2^*(X),$$

which represents overlapping project phases. We assign

$$\tilde{\mu}_{2,1}(B) = \{\,0.60,\ 0.75,\ 0.90\} \ \subseteq \ \tilde{\mathcal{P}}_1^*([0,1]),$$

where each numeric value corresponds to the overall "project-risk score" under a different assessment framework:

- 0.60: risk judged by the internal team (timeline and resource-allocation review),
- 0.75: risk judged by external consultant audit,
- 0.90: risk judged by stakeholder satisfaction surveys.

This $(2, 1)$-SuperHyperFuzzy Set thus captures two-level hierarchical grouping of project tasks with a single level of uncertainty in the risk-assessment process.

Example 1.11 (Medical Treatment Regimen Satisfaction via a $(1, 3)$-SuperHyperFuzzy Set)**.** Let

$$X = \{\text{Medication, Physiotherapy, Counseling}\}, \quad m = 1, \quad n = 3.$$

Then

$$\mathcal{P}_1^*(X) = \mathcal{P}(X) \setminus \{\emptyset\},$$

and one representative element is

$$A = \{\text{Medication, Physiotherapy}\} \ \in \ \mathcal{P}_1^*(X).$$

We define the membership mapping

$$\begin{aligned} \tilde{\mu}_{1,3}(A) = \big\{L_1 = \big\{\{\{0.80,\ 0.85\},\ \{0.90\}\},\ \{\{0.75\},\ \{0.95,\ 0.90\}\}\big\},\\ L_2 = \big\{\{\{0.70\},\ \{0.78,\ 0.82\}\},\ \{\{0.88,\ 0.92\}\}\big\}\big\}\\ \subseteq \ \tilde{\mathcal{P}}_3^*([0,1]). \end{aligned}$$

Here:

- Each innermost set—e.g. $\{0.80, 0.85\}$—collects patient self-report scores, $\{0.90\}$ the clinician's assessment, $\{0.75\}$ family-caregiver feedback, and $\{0.95, 0.90\}$ lab biomarker readings.
- The outer set L_1 groups these sources into "direct feedback" vs. "objective measures."
- The second scenario L_2 captures "daily diary entries" ($\{0.70\}, \{0.78, 0.82\}$) versus "long-term follow-up metrics" ($\{0.88, 0.92\}$).

Thus this $(1,3)$-SuperHyperFuzzy Set models a one-level grouping of treatment modalities with three nested levels of satisfaction uncertainty across multiple evaluators and measurement types.

Example 1.12 (Smart Home Energy Efficiency via a $(2,2)$-SuperHyperFuzzy Set)**.** Let

$$X = \{\text{HeatingSystem, CoolingSystem, LightingSystem}\}, \quad m = 2,\ n = 2.$$

Then

$$\mathcal{P}_2^*(X) = \mathcal{P}\big(\mathcal{P}^*(X)\big) \setminus \{\emptyset\},$$

and consider the element

$$A = \big\{\{\text{HeatingSystem, CoolingSystem}\},\ \{\text{CoolingSystem, LightingSystem}\}\big\} \in \mathcal{P}_2^*(X).$$

We assign to A a family of second-level membership sets by

$$\tilde{\mu}_{2,2}(A) = \Big\{L_1 = \{\{0.70,\ 0.75\},\ \{0.80,\ 0.85\}\},\ L_2 = \{\{0.65\},\ \{0.90,\ 0.95\}\}\Big\} \subseteq \tilde{\mathcal{P}}_2^*([0,1]).$$

Here:

- $\{0.70, 0.75\}$ and $\{0.80, 0.85\}$ are energy-efficiency scores from automated sensors during morning vs evening cycles, respectively.
- L_1 thus captures the “automated monitoring” scenario.
- $\{0.65\}$ is the occupant-reported comfort level.
- $\{0.90, 0.95\}$ is the expert energy-audit rating.
- L_2 represents the “human feedback vs professional audit” scenario.

This $(2,2)$-SuperHyperFuzzy Set models uncertainty both in the overlapping subsystem structure ($m = 2$) and in the dual-scenario efficiency assessments ($n = 2$).

1.2 Fuzzy Control

Fuzzy control systems employ fuzzy set theory and rule-based (IF–THEN) inference to translate precise sensor measurements into smooth actuator commands (cf. [32–36]). Over the past decades, these methods have been instrumental in robotics [37–40], automated manufacturing, and hardware design [41]. Various extensions—for example, Intuitionistic Fuzzy Control [42–44]—have further enhanced their modeling flexibility and robustness.

Definition 1.13 (Fuzzy Control System)**.** [32–34] A *fuzzy control system* implements a mapping from a vector of crisp inputs to a crisp control output via fuzzy-logic reasoning. Formally, let

$$x = (x_1, \dots, x_n) \in X_1 \times \cdots \times X_n, \quad y \in Y$$

be the input and output universes. A fuzzy controller is specified by:

1. **Fuzzification:** For each input x_i one defines a finite family of fuzzy sets

$$A_{i1}, A_{i2}, \dots, A_{im_i} \subseteq X_i, \quad \mu_{A_{ij}} : X_i \to [0,1].$$

2. **Rule Base:** A collection of K IF–THEN rules

$$R_k : \quad \text{IF } x_1 \text{ is } A_{1j_1} \ \wedge\ \dots\ \wedge\ x_n \text{ is } A_{nj_n} \text{ THEN } y \text{ is } B_k,$$

   where each consequent $B_k \subseteq Y$ is a fuzzy set with membership $\mu_{B_k} : Y \to [0,1]$.

3. **Inference:** Given a specific input x, each rule fires with strength

$$\alpha_k = T\big(\mu_{A_{1j_1}}(x_1), \dots, \mu_{A_{nj_n}}(x_n)\big),$$

   where T is a chosen t-norm (e.g. $T(a,b) = \min(a,b)$).

4. **Aggregation:** The fuzzy output set $B' \subseteq Y$ is formed by

$$\mu_{B'}(y) = S_{k=1}^{K}\left[\alpha_k \otimes \mu_{B_k}(y)\right],$$

where $\otimes$ is typically the minimum and S the maximum operator.

5. **Defuzzification:** A crisp control action $y^* \in Y$ is extracted, often via the centroid formula

$$y^* = \frac{\displaystyle\int_Y y\,\mu_{B'}(y)\,dy}{\displaystyle\int_Y \mu_{B'}(y)\,dy}.$$

Example 1.14 (Home HVAC Temperature Control via a Fuzzy Control System)**.** Consider a residential heating, ventilation, and air-conditioning (HVAC) system that adjusts the heating/cooling power based on room temperature. We use two inputs and one output:

- *Inputs:*

$$x_1 = T_{\text{error}} = T_{\text{set}} - T_{\text{room}} \quad \big(\in [-10, 10]\big),$$
$$x_2 = T_{\text{rate}} = \frac{d}{dt}T_{\text{room}} \quad \big(\in [-5, 5]\big)\ (\deg \mathrm{C/min}).$$

- *Output:*

$$y = P_{\text{heat}} \quad \big(\in [-100, 100]\big),$$

where $y > 0$ means heating power (%) and $y < 0$ means cooling power (%).

1. Fuzzification Define three fuzzy sets on each input:

$$\text{Cold:} \quad \mu_{\text{Cold}}(x_1) = \begin{cases} \frac{0-x_1}{5-0}, & -5 \le x_1 \le 0, \\ 1, & x_1 \le -5, \\ 0, & x_1 \ge 0, \end{cases}$$

$$\text{Comfort:} \quad \mu_{\text{Comfort}}(x_1) = \begin{cases} \frac{x_1+2}{2}, & -2 \le x_1 \le 0, \\ \frac{2-x_1}{2}, & 0 \le x_1 \le 2, \\ 0, & \text{otherwise}, \end{cases}$$

$$\text{Hot:} \quad \mu_{\text{Hot}}(x_1) = \begin{cases} \frac{x_1}{5}, & 0 \le x_1 \le 5, \\ 1, & x_1 \ge 5, \\ 0, & x_1 \le 0. \end{cases}$$

Similarly for x_2 with sets $\{\mathit{CoolingFast}, \mathit{Stable}, \mathit{WarmingFast}\}$ defined on $[-5, 5]$ in the same triangular fashion.

On the output $y \in [-100, 100]$, define:

$$\mu_{\text{Cool}}(y) = \begin{cases} 1, & y \le -50, \\ \frac{-20-y}{-20+50}, & -50 \le y \le -20, \\ 0, & y \ge -20, \end{cases}$$

$$\mu_{\text{Off}}(y) = \begin{cases} \frac{y+10}{10}, & -10 \le y \le 0, \\ \frac{10-y}{10}, & 0 \le y \le 10, \\ 0, & |y| \ge 10, \end{cases}$$

$$\mu_{\text{Heat}}(y) = \begin{cases} 0, & y \le 20, \\ \frac{y-20}{50-20}, & 20 \le y \le 50, \\ 1, & y \ge 50. \end{cases}$$

2. Rule Base We use the following three IF–THEN rules ($T = \min$):

$$\begin{aligned}
&R_1 : \text{IF } T_{\text{error}} \text{ is Hot } \wedge\ T_{\text{rate}} \text{ is WarmingFast THEN } P_{\text{heat}} \text{ is Heat,}\\
&R_2 : \text{IF } T_{\text{error}} \text{ is Hot } \wedge\ T_{\text{rate}} \text{ is Stable THEN } P_{\text{heat}} \text{ is Heat,}\\
&R_3 : \text{IF } T_{\text{error}} \text{ is Comfort THEN } P_{\text{heat}} \text{ is Off,}\\
&R_4 : \text{IF } T_{\text{error}} \text{ is Cold THEN } P_{\text{heat}} \text{ is Cool.}
\end{aligned}$$

3. Inference For a specific measurement, say

$$T_{\text{set}} = 22\,\text{deg C}, \quad T_{\text{room}} = 18\,\text{deg C} \implies x_1 = 4, \quad x_2 = 0.5,$$

we compute:

$$\mu_{\text{Hot}}(4) = \tfrac{4}{5} = 0.8, \quad \mu_{\text{Stable}}(0.5) = \tfrac{1-0.5}{1} = 0.5, \quad \mu_{\text{WarmingFast}}(0.5) = \tfrac{0.5}{2} = 0.25.$$

Thus the rule strengths are

$$\alpha_1 = \min(0.8,\ 0.25) = 0.25, \quad \alpha_2 = \min(0.8,\ 0.5) = 0.5, \quad \alpha_3 = 0, \quad \alpha_4 = 0.$$

4. Aggregation We clip each output fuzzy set by its α and take the maximum:

$$\mu_{B'}(y) = \max\big\{0.25 \otimes \mu_{\text{Heat}}(y),\ 0.5 \otimes \mu_{\text{Heat}}(y)\big\} = 0.5\,\mu_{\text{Heat}}(y).$$

5. Defuzzification Using the centroid formula,

$$y^* = \frac{\int_{-100}^{100} y\,\mu_{B'}(y)\,dy}{\int_{-100}^{100} \mu_{B'}(y)\,dy},$$

one finds $y^* \approx 50$, i.e. the controller sets heating power to 50%.

This example illustrates how a fuzzy control system translates temperature error and its rate of change into a crisp heating/cooling command through fuzzification, rule-based inference, aggregation, and defuzzification.

2 Result: HyperFuzzy Control

A hyperfuzzy control system extends fuzzy control by assigning set-valued membership degrees that capture variability in rule firing and aggregation.

Definition 2.1 (Hyperfuzzy Control System)**.** Let

$$x = (x_1, \ldots, x_n) \in X_1 \times \cdots \times X_n, \quad y \in Y$$

be the crisp input and output variables. A *hyperfuzzy control system* is specified by the following components:

1. **Hyperfuzzy Fuzzification.** For each input x_i, fix a finite family of *hyperfuzzy sets*

   $$\tilde{A}_{i1}, \tilde{A}_{i2}, \ldots, \tilde{A}_{im_i} \subseteq \tilde{P}([0,1]), \quad \tilde{\mu}_{A_{ij}} : X_i \to \tilde{P}([0,1]),$$

   where each $\tilde{\mu}_{A_{ij}}(x_i) \subseteq [0,1]$ is nonempty .

2. **Rule Base.** A collection of K IF–THEN rules of the form

$$R_k: \quad \text{IF } x_1 \text{ is } \tilde{A}_{1j_1} \;\wedge\; \ldots \;\wedge\; x_n \text{ is } \tilde{A}_{nj_n} \text{ THEN } y \text{ is } \tilde{B}_k,$$

where each consequent $\tilde{B}_k \subseteq \tilde{P}([0,1])$ is a hyperfuzzy set on Y.

3. **Hyperfuzzy Inference.** Given x, each rule R_k fires with *hyperfuzzy strength*

$$\tilde{\alpha}_k = T_H\big(\tilde{\mu}_{A_{1j_1}}(x_1), \ldots, \tilde{\mu}_{A_{nj_n}}(x_n)\big),$$

where T_H is a *hyperfuzzy t-norm* (e.g. pairwise intersection of membership-sets) .

4. **Hyperfuzzy Aggregation.** The aggregated hyperfuzzy output set $\tilde{B}' \subseteq \tilde{P}([0,1])$ is

$$\tilde{B}' = \bigcup_{k=1}^{K} \tilde{\alpha}_k \,\cap\, \tilde{\mu}_{B_k}(y),$$

where each $\tilde{\mu}_{B_k}(y) \subseteq [0,1]$ and "∩" denotes set-intersection.

5. **Defuzzification via Hyperfuzzy Centroid.** A crisp control output $y^* \in Y$ is obtained by selecting the element of Y whose hyperfuzzy centroid is maximal:

$$y^* = \arg\max_{y \in Y} \ \text{Centroid}\big(\tilde{B}'(y)\big),$$

where

$$\text{Centroid}(S) = \frac{\displaystyle\int_0^1 \alpha\, \chi_S(\alpha)\, d\alpha}{\displaystyle\int_0^1 \chi_S(\alpha)\, d\alpha},$$

and χ_S is the indicator of the set $S \subseteq [0,1]$.

Example 2.2 (Autonomous Vehicle Adaptive Cruise Control via a Hyperfuzzy Control System)**.** We design a hyperfuzzy controller for adaptive cruise control (ACC) that maintains a safe following distance under uncertain sensor readings.

Inputs and Output

$$x_1 = e = d_{\text{target}} - d_{\text{actual}} \quad (d \in [-20, 20] \text{ m}), \quad x_2 = v_{\text{rel}} = v_{\text{lead}} - v_{\text{ego}} \quad (v \in [-5, 5] \text{ m/s}),$$

$$y = u \in [-100, 100]\%, \quad y > 0 = \text{throttle}, \ y < 0 = \text{brake}.$$

1. Hyperfuzzy Fuzzification Sensor fusion yields variable membership degrees from LIDAR, radar, and camera. We define two hyperfuzzy sets per input:

$$\tilde{\mu}_{\text{Close}}(e) = \{0.75, 0.80, 0.85\}, \quad \tilde{\mu}_{\text{Far}}(e) = \{0.30, 0.35, 0.40\},$$

$$\tilde{\mu}_{\text{Approach}}(v_{\text{rel}}) = \{0.65, 0.70, 0.75\}, \quad \tilde{\mu}_{\text{Recede}}(v_{\text{rel}}) = \{0.20, 0.25, 0.30\}.$$

2. Rule Base Four IF–THEN rules:

$$\begin{aligned}
R_1 &: \text{IF } e \text{ is Close } \wedge\ v_{\text{rel}} \text{ is Approach THEN } u \text{ is BrakeStrong},\\
R_2 &: \text{IF } e \text{ is Close } \wedge\ v_{\text{rel}} \text{ is Recede THEN } u \text{ is BrakeLight},\\
R_3 &: \text{IF } e \text{ is Far } \wedge\ v_{\text{rel}} \text{ is Recede THEN } u \text{ is Accelerate},\\
R_4 &: \text{IF } e \text{ is Far } \wedge\ v_{\text{rel}} \text{ is Approach THEN } u \text{ is Maintain}.
\end{aligned}$$

Each consequent is a hyperfuzzy set on u:

$$\begin{aligned}
\tilde{\mu}_{\text{BrakeStrong}}(u) &= \{0.80, 0.85\},\\
\tilde{\mu}_{\text{BrakeLight}}(u) &= \{0.40, 0.45\},\\
\tilde{\mu}_{\text{Accelerate}}(u) &= \{0.60, 0.65\},\\
\tilde{\mu}_{\text{Maintain}}(u) &= \{0.50\}.
\end{aligned}$$

3. Hyperfuzzy Inference For a measurement $e = 5$ m, $v_{\text{rel}} = -1$ m/s:

$$\tilde{\alpha}_1 = \tilde{\mu}_{\text{Close}}(5) \ \cap \ \tilde{\mu}_{\text{Approach}}(-1) = \{0.75, 0.80, 0.85\} \cap \{0.65, 0.70, 0.75\} = \{0.75\},$$

$$\tilde{\alpha}_2 = \{0.75, 0.80, 0.85\} \cap \{0.20, 0.25, 0.30\} = \emptyset, \quad \tilde{\alpha}_3 = \{0.30, 0.35, 0.40\} \cap \{0.20, 0.25, 0.30\} = \{0.30\},$$

$$\tilde{\alpha}_4 = \{0.30, 0.35, 0.40\} \cap \{0.65, 0.70, 0.75\} = \emptyset.$$

4. Hyperfuzzy Aggregation We intersect each strength with its consequent:

$$\tilde{B}'(u) = \{0.75\} \cap \{0.80, 0.85\} \ \cup \ \{0.30\} \cap \{0.60, 0.65\} = \{\,\} \ \cup \ \{\,\} = \emptyset?$$

To avoid empty result, one uses union before intersection:

$$\tilde{B}'(u) = \{0.75\} \cup \{0.30\} = \{0.30, 0.75\}.$$

5. Defuzzification Compute hyperfuzzy centroid over $[0, 1]$:

$$\text{Centroid}(\{0.30, 0.75\}) = \frac{0.30 + 0.75}{2} = 0.525.$$

Thus the controller issues $u^* \approx 52.5\%$ (mild brake).

This example demonstrates real-world ACC behavior modeled by a hyperfuzzy control system, capturing sensor variability and rule-based decision making within a hyperfuzzy framework.

Theorem 2.3 (Generalization of Fuzzy Control Systems)**.** *Every classical fuzzy control system (Definition 2.1) is a special case of a hyperfuzzy control system (Definition 2.1) in which all hyperfuzzy sets reduce to singleton-valued membership sets. Conversely, if in a hyperfuzzy control system every $\tilde{A}_{ij}(x_i)$ and every $\tilde{B}_k(y)$ is a singleton $\{\mu(x_i)\}$ or $\{\mu(y)\}$, then the system reduces exactly to the corresponding fuzzy control system.*

Proof. **($\Rightarrow$) Fuzzy $\implies$ Hyperfuzzy.** Given a fuzzy control system with fuzzy sets A_{ij} and B_k, define hyperfuzzy sets by

$$\tilde{\mu}_{A_{ij}}(x) = \{\mu_{A_{ij}}(x)\}, \quad \tilde{\mu}_{B_k}(y) = \{\mu_{B_k}(y)\}.$$

Then each hyperfuzzy t-norm T_H on singletons reduces to the classical t-norm, each hyperfuzzy aggregation reduces to the classical max–min, and the hyperfuzzy centroid of a singleton set recovers the original defuzzification. Hence the hyperfuzzy control system coincides with the fuzzy one.

($\Leftarrow$) Hyperfuzzy $\implies$ Fuzzy. Conversely, if for every i, j, k and every argument the hyperfuzzy membership is a singleton, $\tilde{\mu}_{A_{ij}}(x_i) = \{\alpha_{ij}\}$ and $\tilde{\mu}_{B_k}(y) = \{\beta_k(y)\}$, then

$$\tilde{\alpha}_k = T_H(\{\alpha_{1j_1}\}, \ldots, \{\alpha_{nj_n}\}) = \{\min(\alpha_{1j_1}, \ldots, \alpha_{nj_n})\},$$

and aggregation $\bigcup_k \{\tilde{\alpha}_k\} \cap \{\beta_k(y)\}$ yields $\{\max_k \min(\tilde{\alpha}_k, \beta_k(y))\}$. The hyperfuzzy centroid of that singleton recovers the classical centroid. Thus the hyperfuzzy control system reduces exactly to the fuzzy control system. □

3 Result: (m, n)-SuperHyperFuzzy Control System

An (m, n)-superhyperfuzzy control system generalizes hyperfuzzy control using hierarchical set levels for inputs and multi-level membership uncertainties in rule inference.

Definition 3.1 ((m, n)-SuperHyperFuzzy Control System)**.** Let

$$x = (x_1, \ldots, x_n) \in X_1 \times \cdots \times X_n, \quad y \in Y$$

be the crisp input and output variables. Fix nonnegative integers m, n. An (m, n)-*superhyperfuzzy control system* consists of:

1. **SuperHyperfuzzy Fuzzification.** For each input x_i choose a finite collection of (m,n)-superhyperfuzzy sets

$$\tilde{A}_{i1},\ldots,\tilde{A}_{im_i}\subseteq \mathcal{P}^*_m(X_i),\quad \tilde{\mu}_{A_{ij}}:\mathcal{P}^*_m(X_i)\longrightarrow \tilde{\mathcal{P}}^*_n([0,1]),$$

as in Definition 1.8 .

2. **Rule Base.** A set of K IF–THEN rules

$$R_k:\quad \text{IF } x_1 \text{ is } \tilde{A}_{1j_1}\ \wedge\ \cdots\ \wedge\ x_n \text{ is } \tilde{A}_{nj_n} \text{ THEN } y \text{ is } \tilde{B}_k,$$

where each $\tilde{B}_k:\mathcal{P}^*_m(Y)\to\tilde{\mathcal{P}}^*_n([0,1])$ is an (m,n)-superhyperfuzzy set on Y.

3. **SuperHyperfuzzy Inference.** Given crisp x, lift each x_i to the singleton set $\{x_i\}\in\mathcal{P}^*_m(X_i)$. Then rule R_k fires with *strength*

$$\tilde{\alpha}_k = T_{m,n}\Big(\tilde{\mu}_{A_{1j_1}}(\{x_1\}),\ \ldots,\ \tilde{\mu}_{A_{nj_n}}(\{x_n\})\Big),$$

where $T_{m,n}$ is a chosen (m,n)-superhyperfuzzy t-norm .

4. **SuperHyperfuzzy Aggregation.** Form the aggregated output hyperfuzzy set $\tilde{B}'\subseteq\mathcal{P}^*_m(Y)$ by

$$\tilde{B}'=\bigcup_{k=1}^{K}\Big[\tilde{\alpha}_k\ \cap_{m,n}\ \tilde{\mu}_{B_k}(\{y\})\Big],$$

where $\cap_{m,n}$ is the (m,n)-superhyperfuzzy intersection.

5. **Defuzzification.** Select the crisp output

$$y^*=\arg\max_{y\in Y}\operatorname{Centroid}_{m,n}\big(\tilde{B}'(\{y\})\big),$$

where the (m,n)–superhyperfuzzy centroid is

$$\operatorname{Centroid}_{m,n}(S)=\frac{\displaystyle\int_0^1\!\!\int_0^1 \alpha\,\chi_S^{(n)}(\alpha)\,d\alpha}{\displaystyle\int_0^1\!\!\int_0^1 \chi_S^{(n)}(\alpha)\,d\alpha},$$

integrating over the n-fold membership levels.

Example 3.2 (Autonomous Vehicle Adaptive Cruise Control via a $(1,2)$-SuperHyperFuzzy Control System)**.** We design a $(1,2)$-superhyperfuzzy controller for adaptive cruise control (ACC), which must maintain a safe headway under uncertain sensor readings.

Inputs and Output

$$x_1=e=d_{\text{target}}-d_{\text{actual}}\ \big(\in[-20,20]\text{ m}\big),\quad x_2=v_{\text{rel}}=v_{\text{lead}}-v_{\text{ego}}\ \big(\in[-5,5]\text{ m/s}\big),$$

$$y=u\in[-100,100]\%,\quad y>0\text{ means throttle, } y<0\text{ means braking.}$$

1. SuperHyperfuzzy Fuzzification We lift each crisp input into the singleton $\{x_i\}\in\mathcal{P}^*_1(X_i)$ and assign two superhyperfuzzy sets per variable:

$$\tilde{\mu}_{\text{Close}}(\{e\})=\big\{\{0.70,\ 0.75\},\ \{0.85\}\big\},\quad \tilde{\mu}_{\text{Far}}(\{e\})=\big\{\{0.30,\ 0.35\},\ \{0.40\}\big\},$$

$$\tilde{\mu}_{\text{Approach}}(\{v_{\text{rel}}\})=\big\{\{0.65,\ 0.70\},\ \{0.75\}\big\},\quad \tilde{\mu}_{\text{Recede}}(\{v_{\text{rel}}\})=\big\{\{0.25\},\ \{0.30,\ 0.35\}\big\}.$$

2. Rule Base We use four IF–THEN rules:

$$\begin{aligned}
R_1 &: \text{IF } e \text{ is Close } \wedge\ v_{\text{rel}} \text{ is Approach THEN } u \text{ is BrakeStrong},\\
R_2 &: \text{IF } e \text{ is Close } \wedge\ v_{\text{rel}} \text{ is Recede THEN } u \text{ is BrakeLight},\\
R_3 &: \text{IF } e \text{ is Far } \wedge\ v_{\text{rel}} \text{ is Recede THEN } u \text{ is Accelerate},\\
R_4 &: \text{IF } e \text{ is Far } \wedge\ v_{\text{rel}} \text{ is Approach THEN } u \text{ is Maintain}.
\end{aligned}$$

Each consequent is a $(1,2)$-superhyperfuzzy set on u, for example

$$\tilde{\mu}_{\text{BrakeStrong}}(\{u\}) = \big\{\{0.80,\ 0.85\},\ \{0.90\}\big\}, \quad \tilde{\mu}_{\text{Accelerate}}(\{u\}) = \big\{\{0.60\},\ \{0.65,\ 0.70\}\big\},$$

and similarly for BrakeLight and Maintain.

3. SuperHyperfuzzy Inference For a measurement $e = 5$ m, $v_{\text{rel}} = -1$ m/s, we compute

$$\tilde{\alpha}_1 = \tilde{\mu}_{\text{Close}}(\{5\}) \cap_{1,2} \tilde{\mu}_{\text{Approach}}(\{-1\}) = \{\{0.70, 0.75\}, \{0.85\}\} \cap \{\{0.65, 0.70\}, \{0.75\}\} = \{\{0.70\}, \{0.75\}\},$$

$$\tilde{\alpha}_2 = \{\{0.70, 0.75\}, \{0.85\}\} \cap \{\{0.25\}, \{0.30, 0.35\}\} = \emptyset,$$

$$\tilde{\alpha}_3 = \{\{0.30, 0.35\}, \{0.40\}\} \cap \{\{0.25\}, \{0.30, 0.35\}\} = \{\{0.30, 0.35\}\},$$

and $\tilde{\alpha}_4 = \emptyset$.

4. SuperHyperfuzzy Aggregation We intersect each firing strength with its consequent and take the union:

$$\begin{aligned}
\tilde{B}' &= \big(\{\{0.70\}, \{0.75\}\} \cap_{1,2} \tilde{\mu}_{\text{BrakeStrong}}\big) \ \cup\ \big(\{\{0.30, 0.35\}\} \cap_{1,2} \tilde{\mu}_{\text{Accelerate}}\big)\\
&= \{\{0.75\}\} \ \cup\ \{\{0.30\}\} = \{\{0.30\}, \{0.75\}\}.
\end{aligned}$$

5. Defuzzification Compute the $(1,2)$-superhyperfuzzy centroid by first taking the supremum of each inner set, $\sup\{0.30\} = 0.30$, $\sup\{0.75\} = 0.75$, then averaging:

$$y^* = \frac{0.30 + 0.75}{2} = 0.525,$$

so the controller applies approximately 52.5% braking effort.

This example demonstrates how a $(1,2)$-superhyperfuzzy control system captures both the hierarchical uncertainty in input fuzzification and the dual-level membership-uncertainty in rule activation and aggregation.

Example 3.3 (Greenhouse Climate Control via a $(2,2)$-SuperHyperFuzzy Control System)**.** We design a $(2,2)$-superhyperfuzzy controller to regulate both temperature and humidity in a greenhouse.

Inputs and Output

$$x_1 = e_T = T_{\text{set}} - T_{\text{room}} \quad (e_T \in [-10, 10]\,\text{deg C}), \quad x_2 = e_H = H_{\text{set}} - H_{\text{room}} \quad (e_H \in [-30, 30]\%),$$

$$y = u \in [0, 100]\%, \quad u = \text{ventilation fan speed}.$$

1. SuperHyperfuzzy Fuzzification Lift each crisp input to the singleton $\{\{x_i\}\} \in \mathcal{P}_2^*(X_i)$. Define two superhyperfuzzy sets per variable:

$$\begin{aligned}
\tilde{\mu}_{\text{Warm}}(\{\{e_T\}\}) &= \big\{L_{T1} = \{\{0.70, 0.75\}, \{0.85\}\},\ L_{T2} = \{\{0.80\}, \{0.90, 0.95\}\}\big\},\\
\tilde{\mu}_{\text{Cold}}(\{\{e_T\}\}) &= \big\{L'_{T1} = \{\{0.40, 0.45\}, \{0.50\}\},\ L'_{T2} = \{\{0.55\}, \{0.60, 0.65\}\}\big\},\\
\tilde{\mu}_{\text{Dry}}(\{\{e_H\}\}) &= \big\{L_{H1} = \{\{0.75, 0.80\}, \{0.85\}\},\ L_{H2} = \{\{0.90\}, \{0.95, 1.00\}\}\big\},\\
\tilde{\mu}_{\text{Humid}}(\{\{e_H\}\}) &= \big\{L'_{H1} = \{\{0.60\}, \{0.65, 0.70\}\},\ L'_{H2} = \{\{0.55, 0.60\}, \{0.50\}\}\big\}.
\end{aligned}$$

2. Rule Base We use four IF–THEN rules:

$$R_1 : \text{IF } T \text{ is Warm} \wedge H \text{ is Humid THEN } u \text{ is VentHigh},$$
$$R_2 : \text{IF } T \text{ is Warm} \wedge H \text{ is Dry THEN } u \text{ is VentMed},$$
$$R_3 : \text{IF } T \text{ is Cold} \wedge H \text{ is Humid THEN } u \text{ is VentLow},$$
$$R_4 : \text{IF } T \text{ is Cold} \wedge H \text{ is Dry THEN } u \text{ is VentOff}.$$

Each consequent is a $(2,2)$-superhyperfuzzy set on u:

$$\tilde{\mu}_{\text{VentHigh}}(\{\{u\}\}) = \{\{0.90\},\{0.95,1.00\}\}, \quad \tilde{\mu}_{\text{VentMed}}(\{\{u\}\}) = \{\{0.60,0.65\},\{0.70,0.75\}\},$$
$$\tilde{\mu}_{\text{VentLow}}(\{\{u\}\}) = \{\{0.30\},\{0.35,0.40\}\}, \quad \tilde{\mu}_{\text{VentOff}}(\{\{u\}\}) = \{\{0.00\},\{0.05\}\}.$$

3. SuperHyperfuzzy Inference For a measurement $e_T = 3\,\text{deg}\,\text{C}$ (warm) and $e_H = -15\%$ (dry):

$$\tilde{\alpha}_2 = \tilde{\mu}_{\text{Warm}}(\{\{3\}\}) \cap_{2,2} \tilde{\mu}_{\text{Dry}}(\{\{-15\}\}) = \big\{\{0.70,0.75\},\{0.85\}\big\} \cap \big\{\{0.75,0.80\},\{0.85\}\big\} = \{\{0.75\},\{0.85\}\},$$

while $\tilde{\alpha}_1, \tilde{\alpha}_3, \tilde{\alpha}_4$ all turn out empty.

4. SuperHyperfuzzy Aggregation Intersect the nonempty firing strength with its consequent:

$$\tilde{B}' = \{\{0.75\},\{0.85\}\} \ \cap_{2,2} \ \tilde{\mu}_{\text{VentMed}}(\{\{u\}\}) = \{\{0.75\}\} \ \cup \ \{\{0.70\}\} = \{\{0.70\},\{0.75\}\}.$$

5. Defuzzification Take the supremum of each inner set, $\sup\{0.70\} = 0.70$, $\sup\{0.75\} = 0.75$, then average:

$$u^* = \frac{0.70 + 0.75}{2} = 0.725,$$

so the fan runs at 72.5% power.

This example shows how a $(2,2)$-superhyperfuzzy control system uses two-level hierarchical uncertainty in both the input fuzzification and the dual-level membership sets to produce a robust greenhouse ventilation command.

Example 3.4 (Smart Irrigation Control via a $(1,3)$-SuperHyperFuzzy Control System)**.** We design a $(1,3)$-superhyperfuzzy controller to decide irrigation valve opening based on soil moisture and rain forecast.

Inputs and Output

$$x_1 = e_S = M_{\text{set}} - M_{\text{soil}} \quad (e_S \in [-50,50]\%), \qquad x_2 = f_R = R_{\text{forecast}} - R_{\text{threshold}} \quad (f_R \in [-20,20] \text{ mm}),$$
$$y = u \in [0,100]\%, \quad u = \text{valve opening percentage}.$$

1. SuperHyperfuzzy Fuzzification Lift each crisp input to $\{\{x_i\}\} \in \mathcal{P}_1^*(X_i)$. Define two $(1,3)$-superhyperfuzzy sets per input:

$$\tilde{\mu}_{\text{Dry}}\big(\{\{e_S\}\}\big) = \big\{L_1 = \{\{0.70,0.75\},\ \{0.80\}\},\ L_2 = \{\{0.85\},\ \{0.90,0.95\}\}\big\},$$
$$\tilde{\mu}_{\text{Wet}}\big(\{\{e_S\}\}\big) = \big\{L_1' = \{\{0.40,0.45\},\ \{0.50\}\},\ L_2' = \{\{0.55\},\ \{0.60,0.65\}\}\big\},$$
$$\tilde{\mu}_{\text{NoRain}}\big(\{\{f_R\}\}\big) = \big\{R_1 = \{\{0.60\},\ \{0.65,0.70\}\},\ R_2 = \{\{0.75\},\ \{0.80,0.85\}\}\big\},$$
$$\tilde{\mu}_{\text{RainExpected}}\big(\{\{f_R\}\}\big) = \big\{R_1' = \{\{0.30,0.35\},\ \{0.40\}\},\ R_2' = \{\{0.45\},\ \{0.50,0.55\}\}\big\}.$$

2. Rule Base Four IF–THEN rules:

$$R_1 : \text{IF } e_S \text{ is Dry } \wedge\ f_R \text{ is NoRain THEN } u \text{ is IrrigateHigh},$$
$$R_2 : \text{IF } e_S \text{ is Dry } \wedge\ f_R \text{ is RainExpected THEN } u \text{ is IrrigateMedium},$$
$$R_3 : \text{IF } e_S \text{ is Wet } \wedge\ f_R \text{ is NoRain THEN } u \text{ is IrrigateLow},$$
$$R_4 : \text{IF } e_S \text{ is Wet } \wedge\ f_R \text{ is RainExpected THEN } u \text{ is IrrigateOff}.$$

Each consequent is a $(1,3)$-superhyperfuzzy set on $\{\{u\}\}$, for instance:

$$\tilde{\mu}_{\text{IrrigateHigh}}(\{\{u\}\}) = \big\{H_1 = \{\{0.85, 0.90\},\ \{0.95\}\},\ H_2 = \{\{1.00\},\ \{0.98, 0.99\}\}\big\},$$

$$\tilde{\mu}_{\text{IrrigateMedium}}(\{\{u\}\}) = \big\{M_1 = \{\{0.60, 0.65\},\ \{0.70\}\},\ M_2 = \{\{0.75\},\ \{0.80, 0.82\}\}\big\},$$

and similarly for IrrigateLow and IrrigateOff.

3. SuperHyperfuzzy Inference For a measurement $e_S = 30\%$ (dry) and $f_R = -5$ mm (no rain):

$$\tilde{\alpha}_1 = \tilde{\mu}_{\text{Dry}}(\{\{30\}\}) \ \cap_{1,3}\ \tilde{\mu}_{\text{NoRain}}(\{\{-5\}\}) = \{L_1, L_2\} \cap \{R_1, R_2\} = \{\, L_1 \cap R_1,\ L_2 \cap R_1 \,\},$$

where

$$L_1 \cap R_1 = \{\{0.70\}, \{0.65\}\}, \quad L_2 \cap R_1 = \{\{0.85\}, \{0.70\}\}.$$

All other $\tilde{\alpha}_k$ are empty.

4. SuperHyperfuzzy Aggregation Intersect the firing strengths with their consequents and union:

$$\tilde{B}' = \big\{L_1 \cap R_1,\ L_2 \cap R_1\big\} \ \cap_{1,3}\ \tilde{\mu}_{\text{IrrigateHigh}}(\{\{u\}\}) = \big\{\{0.85\}, \{0.70\}\big\}.$$

5. Defuzzification Take the triple-supremum:

$$\overset{(3)}{\sup}\{0.85\} = 0.85, \quad \overset{(3)}{\sup}\{0.70\} = 0.70,$$

then average for crisp output:

$$u^* = \frac{0.85 + 0.70}{2} = 0.775,$$

so the irrigation valve opens to 77.5%.

This example shows how a $(1,3)$-superhyperfuzzy control system captures one level of set-valued domain uncertainty and three nested levels of membership uncertainty to make robust irrigation decisions.

Theorem 3.5 (Generalization of Fuzzy and Hyperfuzzy Control Systems)**.** *Let $m, n \geq 0$. The (m,n)-superhyperfuzzy control system of Definition 4.1 satisfies:*

(i) ***Fuzzy Control as** $(0,1)$**-Case.*** *If $m = 0$, $n = 1$, then $\mathcal{P}_0^*(X_i) = X_i$ and $\tilde{\mathcal{P}}_1^*([0,1])$ is all nonempty subsets of $[0,1]$. All superhyperfuzzy operations reduce to classical fuzzy operations, recovering exactly a fuzzy control system (Definition 2.1).*

(ii) ***Hyperfuzzy Control as** $(0,n)$**-Case.*** *If $m = 0$ but $n \geq 1$, then one obtains a hyperfuzzy control system (Definition 3.1) in which each membership is a level-n set of membership degrees.*

(iii) ***Singleton Reduction to Crisp Control.*** *If every (m,n)-membership $\tilde{\mu}_{A_{ij}}(A)$ and $\tilde{\mu}_{B_k}(B)$ is a singleton $\{\mu\}$, then the system reduces to the classical crisp controller with Boolean rules.*

Proof. (i) Set $m = 0$. Then $\mathcal{P}_0^*(X_i) = X_i$, so each $\tilde{\mu}_{A_{ij}} \colon X_i \to \tilde{\mathcal{P}}_1^*([0,1])$ is exactly a fuzzy-membership map $X_i \to [0,1]$. The $(0,1)$-superhyperfuzzy t-norm, aggregation, and centroid specialize to the classical min–max inference and centroid defuzzification. Hence the classical fuzzy control system is recovered.

(ii) With $m = 0$ but general n, each $\tilde{\mu}_{A_{ij}} \colon X_i \to \tilde{\mathcal{P}}_n^*([0,1])$ is a hyperfuzzy membership of level n. The inference and aggregation become those of a hyperfuzzy control system.

(iii) If for every rule and every argument $\tilde{\mu}(A) \subseteq [0,1]$ is a singleton $\{\alpha\}$, then every superhyperfuzzy t-norm and intersection yields a singleton $\{\min\}$, and aggregation a singleton $\{\max\}$. The centroid of a singleton is that value. Thus all computations collapse to Boolean rule inference with crisp outputs. □

Theorem 3.6 (Reduction to Classical Fuzzy Control)**.** *Let $m = 0$ and $n = 1$. Then any (m,n)-superhyperfuzzy control system (Definition 4.1) reduces exactly to a classical fuzzy control system (Definition 2.1).*

Proof. With $m = 0$, we have $\mathcal{P}_0^*(X_i) = X_i$, so each membership map

$$\tilde{\mu}_{A_{ij}} \colon X_i \to \tilde{\mathcal{P}}_1^*([0,1])$$

assigns to each $x_i \in X_i$ a nonempty subset of $[0,1]$. Since $n = 1$, each such subset is simply a singleton $\{\mu_{A_{ij}}(x_i)\}$. Thus

$$\tilde{\mu}_{A_{ij}}(x_i) = \{\mu_{A_{ij}}(x_i)\},$$

and likewise for each consequent $\tilde{B}_k$. The $(0,1)$-superhyperfuzzy t-norm $T_{0,1}$ on singletons coincides with the classical t-norm, and the $(0,1)$-centroid reduces to the usual fuzzy centroid. Therefore every step—fuzzification, inference, aggregation, defuzzification—recovers the fuzzy-logic operations of Definition 2.1, proving the reduction. □

Theorem 3.7 (Reduction to Hyperfuzzy Control)**.** *Let $m = 0$ but $n \geq 1$. Then an (m,n)-superhyperfuzzy control system reduces to a hyperfuzzy control system as in Definition 3.1.*

Proof. Again with $m = 0$, the domain collapse $\mathcal{P}_0^*(X_i) = X_i$ holds, so each input membership $\tilde{\mu}_{A_{ij}} \colon X_i \to \tilde{\mathcal{P}}_n^*([0,1])$ is exactly a hyperfuzzy membership of level n. The $(0,n)$-superhyperfuzzy t-norm and intersection coincide with the hyperfuzzy operations $\cap_h$ and $\otimes_h$, and the $(0,n)$-centroid is the hyperfuzzy centroid. Hence the system specializes to a hyperfuzzy control system. □

Theorem 3.8 (Compactness and Existence of Defuzzified Output)**.** *Assume each membership map $\tilde{\mu}_{A_{ij}}$ and $\tilde{\mu}_{B_k}$ takes values in nonempty, compact subsets of $\mathcal{P}_n([0,1])$. Then for any input x, the aggregated output $\tilde{B}' \subseteq \mathcal{P}_m^*(Y)$ is nonempty and compact at each membership level, and the (m,n)-superhyperfuzzy centroid* $\mathrm{Centroid}_{m,n}(\tilde{B}'(\{y\}))$ *exists for all $y \in Y$, ensuring a well-defined crisp output.*

Proof. Each hyperlevel set $\tilde{\mu}_{A_{ij}}(A)$ is compact by hypothesis. Finite intersections ($\cap_{m,n}$) and finite unions preserve compactness in the powerset topology. Thus each rule strength $\tilde{\alpha}_k$ is compact, and each clipped consequent $\tilde{\alpha}_k \cap_{m,n} \tilde{\mu}_{B_k}(\{y\})$ remains compact. Their union over $k = 1, \ldots, K$ yields $\tilde{B}'(\{y\})$ compact and nonempty. A compact subset of $[0,1]$ has a well-defined supremum, so the nested integrals in the centroid formula converge, guaranteeing existence of $\mathrm{Centroid}_{m,n}(\tilde{B}'(\{y\}))$ for each y. □

Theorem 3.9 (Monotonicity in Rule Strength)**.** *Let the (m,n)-superhyperfuzzy t-norm $T_{m,n}$ and intersection $\cap_{m,n}$ be monotone in each argument. Then increasing any input membership sets $\tilde{\mu}_{A_{ij}}(\{x_i\})$ (in the sense of set-inclusion) cannot decrease the final defuzzified output y^*.*

Proof. Suppose one input's membership map $\tilde{\mu}_{A_{ij}}$ is replaced by a larger set $S' \supseteq S$. By monotonicity of $T_{m,n}$, every rule strength $\tilde{\alpha}_k$ can only increase (set-inclusion) when one argument increases. Consequently each clipped consequent $\tilde{\alpha}_k \cap_{m,n} \tilde{\mu}_{B_k}$ also increases. The union of these sets, $\tilde{B}'$, thus grows by inclusion. A larger hyperfuzzy output set at each membership level yields a larger supremum in the centroid integrals, so the defuzzified y^* cannot decrease. □

Theorem 3.10 (Commutativity and Associativity of Aggregation)**.** *In any (m, n)-superhyperfuzzy control system, the (m, n)-aggregation operator*

$$\bigcup_{k=1}^{K}\left[\tilde{\alpha}_k \ \cap_{m,n} \ \tilde{\mu}_{B_k}(y)\right]$$

is both commutative and associative in the index k. Hence the final aggregated output $\tilde{B}'(y)$ is independent of rule ordering.

Proof. The aggregation is defined as a finite union of terms of the form $\tilde{\alpha}_k \cap_{m,n} \tilde{\mu}_{B_k}(y)$. In the lattice of subsets of $\mathcal{P}_m^*(Y)$, union is commutative and associative. Moreover, by construction $\cap_{m,n}$ is commutative and associative on $\tilde{\mathcal{P}}_n^*([0, 1])$. Therefore

$$\left(A \cap_{m,n} B\right) \cup_{m,n} C = A \cap_{m,n} \left(B \cup_{m,n} C\right) = \left(B \cup_{m,n} A\right) \cap_{m,n} C,$$

etc., showing that neither the grouping nor the order of the union affects $\tilde{B}'(y)$. Hence aggregation is rule-order independent. □

Theorem 3.11 (Monotonicity under Rule Base Expansion)**.** *Let a (m, n)-superhyperfuzzy control system have rule base $\{R_1, \ldots, R_K\}$, and let $\tilde{B}'_K$ denote its aggregated output. If we add an extra rule R_{K+1}, yielding $\tilde{B}'_{K+1}$, then for every $y \in Y$,*

$$\tilde{B}'_K(y) \ \subseteq \ \tilde{B}'_{K+1}(y),$$

and consequently the defuzzified output $y^*_{K+1} \geq y^*_K$ in the sense of the superhyperfuzzy centroid order.

Proof. Adding R_{K+1} contributes one more term $\tilde{\alpha}_{K+1} \cap_{m,n} \tilde{\mu}_{B_{K+1}}(y)$ to the union defining $\tilde{B}'$. Since

$$\tilde{B}'_{K+1}(y) = \tilde{B}'_K(y) \ \cup \ \left[\tilde{\alpha}_{K+1} \cap_{m,n} \tilde{\mu}_{B_{K+1}}(y)\right],$$

we have $\tilde{B}'_K(y) \subseteq \tilde{B}'_{K+1}(y)$. A larger hyperfuzzy set at each y yields a greater or equal (m, n)-centroid, so $y^*_{K+1} \geq y^*_K$. □

Theorem 3.12 (Continuity of the Defuzzified Control Law)**.** *Suppose each membership map $\tilde{\mu}_{A_{ij}} : \mathcal{P}_m^*(X_i) \to \tilde{\mathcal{P}}_n^*([0, 1])$ is continuous in the Vietoris topology, and the (m, n)-t-norm $T_{m,n}$ and intersection $\cap_{m,n}$ are continuous operations on compact sets. Then the mapping*

$$F \colon X_1 \times \cdots \times X_n \ \longrightarrow \ Y, \quad F(x) = y^*$$

that assigns each input x its defuzzified output y^* is continuous.

Proof. Continuity of each $\tilde{\mu}_{A_{ij}}$ means small changes in x_i produce small changes (in Hausdorff metric) of the superhyperfuzzy membership sets. By joint continuity of $T_{m,n}$ and $\cap_{m,n}$, the rule strengths $\tilde{\alpha}_k$ vary continuously in x. Finite unions and intersections of compact sets vary continuously in the Vietoris topology, so the aggregated output $\tilde{B}'$ is a continuous function of x. The (m, n)-centroid, defined by integrals of supremum operations over compact sets, is continuous in the Hausdorff sense. Composing these yields that $F(x) = y^*$ is continuous. □

4 Conclusion

This paper introduced the concepts of Hyperfuzzy Control Systems and (m, n)-SuperHyperfuzzy Control Systems, demonstrating how they extend classical fuzzy control by embedding richer, hierarchical uncertainty structures. In future work, we plan to investigate extensions based on Plithogenic Sets [45–47], develop and analyze efficient solution algorithms, and perform empirical validation on physical hardware to assess real-world performance.

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Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

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