Completely Mixed Bi-matrix Games Without Restrictions on Payoffs at Equilibrium

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Abstract

We consider bi-matrix games of the type discussed in Raghavan (1970) and Oviedo (1996). A strategy profile is said to be completely mixed if it assigns positive probabilities to all pure strategies for both players and a bi-matrix game is said to be completely mixed if all equilibria of the game are completely mixed. Our first theorem in this note extends necessary conditions for completely mixed bi-matrix games that comprise theorem 1 in Raghavan (1970) and prove that theorem 1 in Raghavan (1970) holds without assuming zero-valued equilibrium pay-offs. Of particular significance is the result that the pay-off matrices of all completely mixed bi-matrix games are square matrices, with the rank of the matrices being at least one less than the dimension of the matrices. Results concerning ranks of matrices in bi-matrix games are in general not independent of equilibrium payoffs. An immediate corollary of our first theorem is that for completely mixed two-person zero-sum (TPZS) games with value zero, the pure-strategy pay-off matrices are square matrices with the rank of the matrices being one less than the dimension of the matrices. We apply this corollary to obtain a complete characterization of all completely mixed TPZS games that have value zero. This characterization is different from the characterization available in Kaplansky (1945). The Complementary Slackness Condition for bi-matrix games plays a very useful and important role in our analysis.

Keywords: bi-matrix games, two-person zero-sum games, equilibrium, completely mixed, unique solution

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1. Notation: For any positive integer ℓ let $\Delta^{\ell-1} = \{x \in \mathbb{R}^{\ell}_{+} | \sum_{j=1}^{\ell} x_j = 1\}$ and for $j \in \{1, \ldots, \ell\}$, let $E^{(\ell,j)}$ be the ℓ-dimensional column vector whose j^{th} coordinate is 1 and all other coordinates are equal to 0. Let $E^{(\ell)} = \sum_{j=1}^{\ell} E^{(\ell,j)}$. $E^{(\ell)}$ is said to be the ℓ- dimensional sum vector.

For positive integers m, n, and any m×n matrix P, for all $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$, the $(i, j)^{th}$ term (term at the intersection of the i^{th} row and j^{th} column) of P is denoted by p_{ij}.

For $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$, and any m×n matrix P, let P_i denote the i^{th} row of P and let P^j denote the j^{th} column of P.

For any m×n matrix P we will use P^T to denote its transpose.

2. Bi-matrix Games: For positive integers m, n, let A, B be two m×n matrices.

A **bi-matrix game** (interactive decision-making problem) with pay-off matrices A, B is a contest between two players- the **row player** and the **column player**- such that if for $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$, the row player chooses row i and the column player chooses column j, then the pay-off to the row player is a_{ij} and the pay-off to the column player is b_{ij}.

A bi-matrix game with pay-off matrices A, B is denoted by the ordered pair (A, B).

A bi-matrix game (A, B) is said to be a **two-person zero sum (TPZS) game** if B = -A.

It is customary to represent a TPZS game (A, -A) by the matrix A, and just that.

A **strategy profile** is an ordered pair $(x, y) \in \Delta^{m-1} \times \Delta^{n-1}$, where x is the vector of probabilities with which the row-player chooses each strategy and y is the vector of probabilities with which the column-player chooses each strategy.

A very "witty" justification for randomized strategies is available in (page 17, in chapter 3 of) Washburn (2014): "It is the key to resolving the '… if he thinks that I think that he thinks that…' impasse- you can't outwit someone who refuses to think".

An **equilibrium** for a bi-matrix game (A, B) is a strategy profile (x^*, y^*) that satisfies the following condition: For all $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$, $A_i y^* \leq x^{*T} A y^*$ and $x^{*T} B^j \leq x^{*T} B y^*$.

Hence, (x^*, y^*) is an equilibrium if and only if $(x^*, y^*) \in \Delta^{m-1} \times \Delta^{n-1}$ and for all $(x, y) \in \Delta^{m-1} \times \Delta^{n-1}$ it is the case that $x^T A y^* \leq x^{*T} A y^*$ and $x^{*T} B y \leq x^{*T} B y^*$.

By the "Equivalence Theorem" of Mangasarian and Stone (1964), the set of equilibria of an m×n bi-matrix game is the projection into the m + n dimensional Euclidean space of the set of solutions of a related quadratic programming problem.

Note 1: If C and D are m×n matrix such that for some real numbers, c, d, $c_{ij} = c$ and $d_{ij} = d$ for all $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$, then it follows immediately that (x^*, y^*) is an equilibrium for (A, B) <u>if and only if</u> (x^*, y^*) is an equilibrium for (A + C, B + D). The dimensions of the matrices A and A+ C are the same and the dimensions of the matrices C and C + D are the same. However, that does not mean that the rank of the matrix A will necessarily be equal to the rank of the matrix A + C nor does it mean that the rank of the matrix B will necessarily be equal to the rank of the matrix B + C.

For the m×n bi-matrix game (A, B) and $(x, y) \in \Delta^{m-1} \times \Delta^{n-1}$ let $V(\text{row}|(A, B), (x, y)) = x^T A y$ and $V(\text{column}|(A, B), (x, y)) = x^T B y$.

Note 2: An immediate consequence of the definition of equilibrium is the following "**Complementary Slackness Condition for Bi-matrix Games**": Suppose (x^*, y^*) is an equilibrium for the bi-matrix game (A, B). (i) <u>If</u> for any $i \in \{1, \ldots, m\}$ it is the case that $A_i y^* < x^{*T} A y^*$, <u>then</u> it must be the case that $x_i^* = 0$; and (ii) <u>if</u> for any $j \in \{1, \ldots, n\}$ it is the case that $x^{*T} B^j < x^{*T} B y^*$, <u>then</u> it must be the case that $y_j^* = 0$.

It is well-known that associated with any TPZS game A, there is a real number V(A), called the **value** of A, such if (x, y) is an equilibrium of A, then $x^T A y = V(A)$.

It has been known since the seminal work of John von-Neumann and Oscar Morgenstern entitled "Theory of Games and Economic Behaviour" that every TPZS game has at least one equilibrium. A "complete characterization theorem" for the set of all equilibria in terms of the solution sets of a linear programming and its dual, for a general class of bi-matrix games that includes the set all TPZS games as well as a large class of "prisoners' dilemma" type games as a proper subset, is available as proposition 1 in Lahiri (2025). A very detailed treatment of two-person zero-sum games is available in Washburn (2014). The scope of Washburn (2014) goes way beyond the two-person zero-sum (matrix) games, that we are concerned with here.

3. Linear Systems Associated with Bi-matrix games: We now provide a short lemma for bi-matrix games. Part (iii) of the lemma will play a significant role in proving uniqueness of equilibrium for completely mixed bi-matrix games.

Lemma 1: (i) <u>If</u> (x^*, y^*) is an equilibrium for the bi-matrix game (A, B) game <u>then</u> $x^* \in \Delta^{m-1}$, $y^* \in \Delta^{n-1}$ and there exists real numbers u^*, v^* such that $Ay^* \leq u^*E^{(m)}$, $B^Tx^* \leq v^*E^{(n)}$. Further, $u^* = x^{*T}Ay^*$ and $v^* = x^{*T}By^*$.

(ii) <u>If</u> there exists $x^* \in \Delta^{m-1}$, $y^* \in \Delta^{n-1}$ and a real number u^* such that $Ay^* \leq u^*E^{(m)}$, $B^Tx^* \leq -u^*E^{(n)}$ and $x^{*T}Ay^* = - x^{*T}By^*$ then (x^*, y^*) is an equilibrium for the bi-matrix game (A, B).

(iii) <u>If</u> there exists $x^* \in \Delta^{m-1}$, $y^* \in \Delta^{n-1}$ and real numbers u^*, v^* such that $Ay^* = u^*E^{(m)}$, $B^Tx^* = v^*E^{(n)}$, then (x^*, y^*) is an equilibrium for the bi-matrix game (A, B).

Proof: (i) If (x^*, y^*) is an equilibrium for the bi-matrix game (A, B), then with $u^* = x^{*T}Ay^*$ and $v^* = x^{*T}By^*$ we know that for all $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$, $A_iy^* \leq x^{*T}Ay^*$ and $x^{*T}B^j \leq x^{*T}By^*$, so that $Ay^* \leq u^*E^{(m)}$ and $B^Tx^* \leq v^*E^{(n)}$. Further, $x^* \in \Delta^{m-1}$ and $y^* \in \Delta^{n-1}$.

(ii) Suppose that $x^* \in \Delta^{m-1}$, $y^* \in \Delta^{n-1}$ and there exists a real numbers u^* such that $Ay^* \leq u^*E^{(m)}$, $B^Tx^* \leq -u^*E^{(n)}$ and $x^{*T}Ay^* = - x^{*T}By^*$.

Pre-multiplying the first inequality by x^{*T} and the second by y^{*T} we get $x^{*T}Ay^* \leq u^*$ and $x^{*T}By^* = y^{*T}B^Tx^* \leq -u^*$.

Since, $x^{*T}By^* = - x^{*T}Ay^*$, we get $u^* \leq x^{*T}Ay^* \leq u^*$, so that $x^{*T}Ay^* = u^*$.

Thus, $x^{*T}By^* = -u^*$.

Since $x^* \in \Delta^{m-1}$, $y^* \in \Delta^{n-1}$, $Ay^* \leq u^*E^{(m)} = (x^{*T}Ay^*)E^{(m)}$ and $x^{*T}B \leq - u^*E^{(n)T} = (x^{*T}By^*)E^{(n)}$.

Thus, (x^*, y^*) is an equilibrium for (A, B).

(iii) Now suppose $x^* \in \Delta^{m-1}$, $y^* \in \Delta^{n-1}$ and there exists real numbers u^*, v^* such that $Ay^* = u^*E^{(m)}$, $B^Tx^* = v^*E^{(n)}$.

Thus, for any $x \in \Delta^{m-1}$ and $y \in \Delta^{n-1}$, $x^TAy^* = u^* = x^{*T}Ay^*$ and $x^{*T}By = v^* = x^{*T}By^*$.

Thus, (x^*, y^*) is an equilibrium. Q.E.D.

4. Completely Mixed Equilibrium and Completely Mixed Bi-matrix Games: We now consider some properties for equilibrium that assigns positive probabilities to all pure strategies of both players and bi-matrix games all equilibria for which satisfy such a property.

An “equilibrium” (x^*, y^*) for a bi-matrix game (A, B) is said to be **completely mixed** if for all $i \in \{1, \ldots, m\}$, $x_i^* > 0$ and for all $j \in \{1, \ldots, n\}$, $y_j^* > 0$.

An immediate consequence of note 2 in the previous section is the following lemma.

Lemma 2: If (x^*, y^*) is a completely mixed equilibrium for the bi-matrix game (A, B), then it must be the case that $Ay^* = (x^{*T}Ay^*)E^{(m)}$ and $x^{*T}B = (x^{*T}ABy^*)E^{(n)T}$.

A bi-matrix game (A, B) is said to be **completely mixed** if every equilibrium for (A, B) is completely mixed.

While the concept of completely mixed TPZS games originated in the work of Kaplansky (1945), the most recent significant results for such TPZS games as well as for completely mixed bi-matrix games can be found in Parthasarathy, Sharma and Sricharan (2020) and more recently in Parthasarathy, Gomatam and Kumar (2024). Some results in this note may overlap with results available in Oviedo (1996). We prove that if a bi-matrix game is completely mixed then it has a unique equilibrium. Unlike theorem 1 in Oviedo (1996) we do not assume that the pay-offs to the players in equilibrium is zero. We also extend several important conclusions in Raghavan (1970) and Oviedo (1996) to the set of all completely mixed bi-matrix games, without imposing any restrictions on the equilibrium pay-offs. Results, related to the ranks of the matrices of a bi-matrix game are not “obviously independent” of the kind of restrictions on payoffs invoked in Raghavan (1970) and Oviedo (1996).

Proposition 1: Suppose the bi-matrix game (A, B) is completely mixed. Then it has a unique equilibrium.

Proof: Suppose the bi-matrix game (A, B) is completely mixed. Thus, $A \neq 0 \neq B$.

Let (x^*, y^*) be a completely mixed equilibrium for (A, B). By note 1, we may without loss of generality assume that $x^{*T}Ay^* = 0 = x^{*T}By^*$.

Since (x^*, y^*) is completely mixed, it must be the case that $x_i^* > 0$ for all $i \in \{1, \ldots, m\}$ and $y_j^* > 0$ for all $j \in \{1, \ldots, n\}$.

By lemma 2, it follows that $Ay^* = 0$ and $B^Tx^* = 0$.

Thus, the columns of A are linearly dependent and the rows of B are linearly dependent.

Towards a contradiction suppose (x^0, y^0) is any other completely mixed equilibrium for (A, B).

Thus, $(x^0, y^0) \neq (x^*, y^*)$, $x^0 \in \Delta^{m-1}$, $y^0 \in \Delta^{n-1}$, $x_i^0 > 0$ for all $i \in \{1, \ldots, m\}$ and $y_j^0 > 0$ for all $j \in \{1, \ldots, n\}$, and by lemma 2, $Ay^0 = (x^{0T}Ay^0)E^{(m)}$ and $x^{0T}B = (x^{0T}By^0)E^{(n)T}$.

Let $u^0 = x^{0T}Ay^0$ and $v^0 = x^{0T}By^0$.

Suppose $x^0 \neq x^*$. Thus, $\{i | x_i^0 > x_i^*\} \neq \phi$ and $\{i | x_i^0 < x_i^*\}$. Thus, $t = \max\{\frac{x_i^0}{x_i^*} | i \in \{1, \ldots, m\}\} > 1 > \min\{\frac{x_i^*}{x_i^0} | i \in \{1, \ldots, m\}\} = s > 0$.

Let $x' = x^0 - sx^*$. Clearly, $B^Tx' = B^Tx^0 - sB^Tx^* = B^Tx^0 = v^0E^{(n)}$, $x' \in \mathbb{R}_+^m$.

Note that for all i∈ {1, …, m} satisfying $\frac{x_i^0}{x_i^*}$ = t, we have x_i^0 - sx_i^* = x_i^*(t – s) > 0 and for all i∈ {1, …, m} satisfying $\frac{x_i^0}{x_i^*}$ = s, we have x_i^0 - sx_i^* = x_i^*(s – s) = 0.

Thus, $x' \in \mathbb{R}_+^m \setminus \{0\}$ and the {i| x_i'> 0} is a "*strict and non-empty subset*" of {1, …, m}.

Thus, there exists α > 0 such that $\alpha x' \in \Delta^{m-1}$, $B^T(\alpha x') = 0$.

Thus, by part (iii) of lemma 1, ($\alpha x'$, y^*) is an equilibrium for (A, B).

However, ($\alpha x'$, y^*) is not completely mixed since the {i|$\alpha x_i'$ = 0} ≠ϕ, thereby contradicting (A, B) is completely mixed.

Thus, it must be that $x^0 = x^*$.

Similarly, we get $y^0 = y^*$.

Thus, $(x^0, y^0) = (x^*, y^*)$ leading to the desired contradiction and proving the proposition.

Thus, (A, B) has a unique equilibrium. Q.E.D.

The following result leads to interesting consequences for completely mixed bi-matrix games.

Proposition 2: Suppose the m×n bi-matrix game (A, B) is completely mixed. Suppose (x^*, y^*) is a completely mixed equilibrium for (A, B). Let $u^* = x^{*T}Ay^*$ and $v^* = x^{*T}By^*$. Then $\{\lambda \in \mathbb{R}^n | A\lambda = 0\} \subset \{ty^* | t \in \mathbb{R}\}$ and $\{\gamma \in \mathbb{R}^m | \gamma^T B = 0\} \subset \{tx^* | t \in \mathbb{R}\}$.

Proof: Suppose the m×n bi-matrix game (A, B) is completely mixed. Thus, A ≠ 0 ≠ B. Since (x^*, y^*) is a completely mixed equilibrium for (A, B), by proposition 1, (x^*, y^*) is the unique equilibrium for (A, B). Since $u^* = x^{*T}Ay^*$ and $v^* = x^{*T}By^*$, (x^*, y^*) is completely mixed implies $Ay^* = u^*E^{(m)}$ and $B^Tx^* = v^*E^{(n)}$.

Suppose there exists $\lambda \in \mathbb{R}^n \setminus \{ty^* | t \in \mathbb{R}\}$ such that Aλ = 0. Since $y^* >> 0$ (i.e., all coordinates of y^* are strictly positive) for ε > 0 sufficiently small $y^* + \varepsilon\lambda >> 0$ and $A(y^* + \lambda\varepsilon) = 0$.

Thus, there exists α > 0 such that $\alpha(y^* + \lambda\varepsilon) \in \Delta^{n-1}$.

Let $y^0 = \alpha(y^* + \lambda\varepsilon)$. Then, $Ay^0 = \alpha A(y^* + \lambda\varepsilon) = \alpha Ay^* + \alpha\varepsilon A\lambda = \alpha Ay^* + 0 = \alpha u^*E^{(m)}$, since $Ay^* = u^*E^{(m)}$.

Since $x^{*T}B = (x^{*T}By^*)E^{(n)T} = v^*E^{(n)T}$, it follows from part (iii) of lemma 1 that (x^*, y^0) is an equilibrium for (A, B).

By proposition 2, we must have $y^0 = y^*$.

$y^0 = \alpha(y^* + \lambda\varepsilon) = y^*$ implies $\lambda = \frac{1-\alpha}{\varepsilon}y^*$, contradicting $\lambda \in \mathbb{R}^n \setminus \{ty^* | t \in \mathbb{R}\}$.

Thus, $\{\lambda \in \mathbb{R}^n | A\lambda = 0\} \subset \{ty^* | t \in \mathbb{R}\}$.

A similar proof works for proving $\{\gamma \in \mathbb{R}^m | \gamma^T B = 0\} \subset \{tx^* | t \in \mathbb{R}\}$. Q.E.D.

An immediate consequence of proposition 2 is the following corollary.

Corollary of proposition 2: Let (x^*, y^*) be a completely mixed equilibrium for the m×n completely mixed bi-matrix game (A, B). Then (i) rank of the matrix A is strictly less than n if and only if $Ay^* = 0$; (ii) rank of the matrix B is strictly less than m if and only if $B^Tx^* = 0$

A significant consequence of proposition 2 is the following result which includes a generalization of part (i) of theorem 1 in Oviedo (1996).

Proposition 3: Suppose the m×n bi-matrix game (A, B) is completely mixed. Then:

(i) Every collection of n-1 columns of A are linearly independent and the rank of the matrix A is at least n - 1.

(ii) Every collection of m – 1 rows of B are linearly independent and the rank of the matrix B is at least m - 1.

(iii) $n + 1 \geq m \geq n-1$.

(iv) m = n.

Proof: Since (A, B) is completely mixed, by proposition 1, it has a unique completely mixed equilibrium (x^*, y^*). Thus, $x_i^*> 0$ for all $i\in\{1, \ldots, m\}$ and $y_j^*> 0$ for all $j\in\{1, \ldots, n\}$.

(i) Towards a contradiction suppose a subset of n-1 columns of A are linearly dependent.

Without loss of generality suppose the first n-1 columns of A are linearly dependent.

Thus, there exists $\lambda = (\lambda_1, ..., \lambda_{n-1})\in \mathbb{R}^{n-1}\setminus\{0\}$ such that $\sum_{j=1}^{n-1} A^j \lambda_j = 0$.

Thus, $\sum_{j=1}^{n-1} A^j \lambda_j + 0A^n = 0$.

$\lambda = (\lambda_1, ..., \lambda_{n-1})\in \mathbb{R}^{n-1}\setminus\{0\}$ implies $\{j\in \{1, ..., n-1\} | \lambda_j > 0\} \neq \phi$.

From, proposition 2 we know that there exists $t\in \mathbb{R}$, such that $\lambda_j = ty_j^*$ for $j\in \{1, ..., n-1\}$ and $0 = ty_n^*$.

Since, $y_j^*> 0$ for all $j\in \{1, \ldots, n\}$ and $\{j\in \{1, ..., n-1\} | \lambda_j \neq 0\} \neq \phi$, it must be the case that $t \neq 0$.

On the other hand $0 = ty_n^*$ and $y_n^* > 0$ implies, t = 0, thereby leading to a contradiction.

Hence every subset of n – 1 columns of A must be linearly independent.

This proves (i).

(ii) The proof of (ii) is exactly as in (i) with B^T playing the role of A and x^* playing the role of y^*.

(iii) From (i) it follows that the rank of the matrix A is at least n-1.

Since rank of A is equal to the rank of A^T, it must be that A^T has at least n – 1 columns and all subsets of n – 1 columns of A^T are linearly independent.

Since A is a m×n matrix, A^T is a n×m matrix.

Thus, it must be the case that $m \geq n - 1$.

From (ii) it follows that the rank of the matrix B^T is at least m-1.

Since rank of B is equal to the rank of B^T, it must be that B has at least m – 1 columns and all subsets of m – 1 columns of B are linearly independent.

Since B is a m×n matrix, it must be the case that $n \geq m - 1$.

Thus, $n + 1 \geq m \geq n - 1$.

(iv) Let (x^*, y^*) be a completely mixed equilibrium for (A, B).

Let C, D be m×n matrices such that $c_{ij} = -x^{*T}Ay^*$ and $d_{ij} = -x^{*T}By^*$ for all $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$.

By note 1 we know that the set of equilibria of the two m×n bi-matrix games (A, B), (A+C, B+D) are the same. However, it is quite possible that the rank of A is "not equal" to the rank of A + C, and/or the rank of B is "not equal" to the rank of B + D. The possibility of differences in rank does not matter here, since our purpose is to show that m = n.

Since the set of equilibria of the two m×n bi-matrix games (A, B), (A+C, B+D) are the same, (A+C, B+D) must be a completely mixed bi-matrix game with (x^*, y^*) as a completely mixed equilibrium.

$x^{*T}(A + C)y^* = x^{*T}Ay^* + x^{*T}Cy^* = x^{*T}Ay^* - x^{*T}Ay^* = 0$ and $x^{*T}(B + D)y^* = x^{*T}By^* + x^{*T}Dy^* = x^{*T}By^* - x^{*T}By^* = 0$.

We know by the Complementary Slackness Condition that $(A+C)x^* = 0$ and $(B+D)^T x^* = 0$.

Since we are concerned only with the dimension of the matrices, without loss of generality, suppose $Ay^* = 0$ and $B^T x^* = 0$.

Thus, the columns of A must be linearly dependent and the columns of B^T must be linearly dependent.

Thus, rank of A is strictly less than 'n' and the rank of B is strictly less than 'm'.

By parts (i) and (ii) of this proof, it must be the case that rank of A is n-1 and rank of B is m-1.

Towards a contradiction suppose either $m = n - 1$ or $n = m - 1$.

First suppose $m = n - 1$.

Since rank of the matrix A is n -1, without loss of generality suppose that the first n-1 columns of A are linearly independent. Let A^{-n} denote the m×m submatrix of A comprising of the first m columns of A and $y^{*(-n)}$ be the m-dimensional column vector comprising the first m coordinates of y^*.

Then, given any $\mu \in \{1, -1\}$, there exists a unique $\lambda(\mu) \in \mathbb{R}^m$ such that $A^{-n}\lambda(\mu) = \mu E^{(m)}$. Clearly, $\lambda(\mu) \neq 0$ and $\lambda(-\mu) = -\lambda(\mu)$ for $\mu \in \{1, -1\}$.

If $\{j \in \{1, \ldots, m\} | \lambda_j(1) > 0\} \neq \phi$, then set $\mu = 1$. Otherwise, set $\mu = -1$.

Let $\lambda = \lambda(\mu)$ for the chosen value of μ.

Thus, $\{j \in \{1, \ldots, m\} | \lambda_j > 0\} \neq \phi$

Thus, there exists $\alpha > 0$ such that $y_j^* - \alpha\lambda_j > 0$ for all $j\epsilon\{1, \ldots, m\}$.

Choose any such α.

Thus, $A^{-n}(\alpha\lambda) = \alpha\mu E^{(m)}$.

$\{j\epsilon\{1, \ldots, m\} | \lambda_j > 0\} \neq \phi$.

Let $\varepsilon^* = \max\{\varepsilon | y_j^* - \alpha\lambda_j \geq 0$ for all $j\epsilon\{1, \ldots, m\}\}$.

Thus, $A^{-n}(\varepsilon^*\alpha\lambda) = \alpha\varepsilon^*\mu E^{(m)}$.

Further, $\{j | y_j^* - \alpha\varepsilon^*\lambda_j > 0, j\epsilon\{1, \ldots, m\}\}$ is a proper subset of $\{1, \ldots, m\}$ and $Ay^* - A^{-n}(\varepsilon^*\alpha\lambda) = \alpha\varepsilon^* E^{(m)} = (u^* - \alpha\varepsilon^*\mu)E^{(m)}$.

Let $z^*\epsilon\mathbb{R}^n$ be such that $z_j^* = y_j^* - \alpha\varepsilon^*\lambda_j$ for $j\epsilon\{1, \ldots, n\text{-}1\}$ and $z_n^* = y_n^*$.

Thus, $z_j^* \geq 0$ for $j\epsilon\{1, \ldots, n\}$ and $z_n^* = y_n^* > 0$. Further, $\{j\epsilon\{1, \ldots, m\} | z_j^* = 0\} \neq \phi$.

Since $\sum_{j=1}^{n} z_j^* > 0$, there exists $\beta > 0$ such that $\beta z^*\epsilon\Delta^{n-1}$.

$A(\beta z^*) = \beta(u^* - \alpha\varepsilon^*\mu)E^{(m)}$.

Since, $B^T x^* = v^* E^{(n)}$, by part (iii) of lemma 1, $(x^*, \beta z^*)$ is an equilibrium for (A, B).

Since $\{j | \beta z_j^* = 0\} \neq \phi$, $(x^*, \beta z^*)$ is not completely mixed contradicting (A, B) is completely mixed.

Hence, it must be the case that $m > n - 1$, i.e., $m\epsilon\{n, n + 1\}$.

By a similar argument, it follows that $n\epsilon\{m, m + 1\}$.

Since, $n + 1 \geq m \geq n - 1$, it follows immediately from $m\epsilon\{n, n + 1\}$ and $n\epsilon\{m, m + 1\}$.

Thus, it must be the case that m = n. Q.E.D.

Note 2: The proof of part (iv) in proposition 3, is an elaboration of the proof of theorem 2 in Raghavan (1970).

The following theorem collects together the major results we have obtained so far.

Theorem 1: Let (A, B) be a m×n completely mixed bi-matrix game. Then m = n, with the rank of A being at least n-1 and the rank of B being at least n-1. Further, there exists real numbers u, v such that the linear system $[Ay = uE^{(n)}, B^T x = vE^{(n)}, x\in\Delta^{n-1}\ y\in\Delta^{n-1}]$ has a solution.

Proof: Suppose (A, B) is an m×n completely mixed bi-matrix game. By (iv) of proposition 3, we get that m = n. By proposition 1 we know that (A, B) has a unique equilibrium (x^*, y^*) that is completely mixed. By the Complementary Slackness Condition, $Ay^* = (x^{*T}Ay^*)E^{(n)}$, $B^T x^* = (x^{*T}By^*)E^{(n)}$. By (i) and (ii) of proposition 3, we know that the rank of the matrix A is at least n-1 and the rank of the matrix B is at least n – 1. Q.E.D.

5. Completely Mixed TPZS games: An immediate corollary of theorem 1 is the following result for completely mixed TPZS games.

Corollary of theorem 1: Let A be TPZS game with n columns. If A is completely mixed, then A is a square matrix, A has a unique equilibrium and rank of A is at least n - 1. If in addition V(A) = 0, then the rank of A is n – 1.

Proof: We only need to show that if A is an n×n completely mixed TPZS game and rank of A is at least n – 1, then [V(A) = 0 implies rank of A is n – 1].

Let (x^*, y^*) be a completely mixed equilibrium for A. Then, by the Complementary Slackness Condition, $Ay^* = (x^{*T}Ay^*)E^{(n)}$.

Since V(A) = 0, it follows that $x^{*T}Ay^* = 0$. Thus, $Ay^* = 0$ and hence the columns of A are linearly dependent. Hence rank of A is strictly less than 'n'. Since rank of A is at least n-1, it follows that rank of A is n-1. Q.E.D.

Using the corollary, we can establish the following alternative version of the main result in Kaplansky (1945).

Theorem 2: Let A be a TPZS game with 'n' columns and suppose V(A) = 0. Then A is completely mixed <u>if and only if</u> A is a square matrix with the rank of A being equal to n-1, and the linear system $[Ay = 0, y \in \Delta^{n-1}]$ has a solution.

Proof: From corollary of theorem 2, we know that if A is a completely mixed TPZS game with n columns and V(A) = 0, then A is a square matrix with the rank of A equal to n-1, and the linear system $[Ay = 0, y \in \Delta^{n-1}]$ has a solution.

Thus, in view of the corollary of theorem 2, all that we need to show for a TPZS game A with 'n' columns satisfying V(A) = 0, is that <u>if</u> A is a square matrix of size 'n' (i.e., n×n matrix), the rank of A is n-1, and the linear system $[Ay = 0, y \in \Delta^{n-1}]$ has a solution, <u>then</u> A is completely mixed.

Let (x^*, y^*) be an equilibrium of the TPZS game A. Thus, $x^* \in \Delta^{m-1}$, $y^* \in \Delta^{n-1,}$ $Ay^* \leq (x^{*T}Ay^*)E^{(m)}$, $A^Tx^* \geq (x^{*T}Ay^*)E^{(n)} = 0$, since V(A) = 0.

$Ay = 0$ implies $x^TAy = x^{*T}Ay = 0$ for all $x \in \Delta^{n-1}$.

$A^Tx^* \geq 0$ implies $x^{*T}Az \geq 0 = x^{*T}Ay$ for all $z \in \Delta^{n-1}$.

Since $y \in \Delta^{n-1}$, we get (x^*, y) is also an equilibrium of the TPZS game A.

Hence, without loss of generality suppose, $y = y^*$.

If $\{j| y_j^* = 0\} \neq \phi$, then $\sum_{\{j|y_j^* > 0\}} A^j y_j^* = 0$, with the cardinality of $\{j| y_j^* > 0\}$ being at least 1 and at most n-1, contradicts our assumption that the rank of A is n-1.

Hence $\{j| y_j^* = 0\} = \phi$.

Thus, by the Complementary Slackness Condition, it must be the case that $x^{*T}A = 0$.

If $\{i| x_i^* = 0\} \neq \phi$, then $\sum_{\{i|x_i^* > 0\}} A_i x_i^* = 0$, with the cardinality of $\{i| x_i^* > 0\}$ being at least 1 and at most n-1, contradicting our assumption that the rank of A is n-1.

Thus, it must be the case that $\{i| x_i^* = 0\} = \phi$.

Suppose, (x^0, y^0) is an equilibrium for A. Then (x^0, y^*) and (x^*, y^0) are also equilibria of A.

Thus, $\{j | y_j^* = 0\} = \phi$ implies by the Complementary Slackness Condition that $x^{0T}A = 0$ and $\{i | x_i^* = 0\} = \phi$ implies by the Complementary Slackness Condition $Ay^0 = 0$.

Since rank of A is n-1 it follows by an argument similar to that for x^* and y^* that $\{i | x_i^0 = 0\} = \phi$ and $\{j | y_j^0 = 0\} = \phi$.

Thus, (x^0, y^0) is a completely mixed equilibrium, there by leading to the conclusion that A is a completely mixed TPZS game. Q.E.D.

Note: This is a revised version of an earlier paper entitled "Linear Systems Equivalent to Two-Person Zero Sum Games". This version generalizes and extends several results in the earlier version.

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