Linear Systems Equivalent to Two-Person Zero Sum Games

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Abstract

We provide two characterizations of the set of equilibria of a two-person zero-sum "matrix" (TPZS) game. The first is a lemma, which says that a strategy profile (pair of randomizations over pure strategies) is an equilibrium if an only if along with another real number, it satisfies a specific system of linear inequalities. The second is a proposition, which says that a strategy profile is an equilibrium if an only if along with two real numbers it solves a certain linear programming problem. The proposition is a special case of the "Equivalence Theorem" in Mangasarian and Stone (1964) and for this special case, the proof appeals to the existence result about equilibrium for a TPZS game. It is well-known that the proof of this equilibrium existence result- which is not a complete characterization by itself- requires using a linear programming problem and its dual. To the best of our knowledge, the proposition we prove, is not available anywhere that is accessible to us.

A strategy profile is said to be completely mixed if it assigns positive probabilities to all pure strategies for both players and a TPZS game is said to be completely mixed if all equilibria of the game are completely mixed. We show that a completely mixed TPZS game has a unique equilibrium strategy.

Keywords: two-person zero-sum games, equilibrium, linear programming, completely mixed, unique solution

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1. Notation: For any positive integer ℓ let $\Delta^{\ell-1} = \{x \in \mathbb{R}^{\ell}_{+} | \sum_{j=1}^{\ell} x_j = 1\}$ and for $j \in \{1, \ldots, \ell\}$, let $E^{(\ell,j)}$ be the ℓ-dimensional column vector whose j^{th} coordinate is 1 and all other coordinates are equal to 0. Let $E^{(\ell)} = \sum_{j=1}^{\ell} E^{(\ell,j)}$. $E^{(\ell)}$ is said to be the ℓ- dimensional sum vector.

2. Two-Person Zero-Sum Games: For positive integers m, n, let P be an m×n matrix such that for all $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$, the $(i, j)^{th}$ term (term at the intersection of the i^{th} row and j^{th} column) of P is p_{ij}.

A **two-person zero-sum "matrix" (TPZS) game** (interactive decision-making problem) with pay-off matrix P (or simply, a TPZS game P) is a contest between two players- the **row player** and the **column player**- such that if for $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$, the row player chooses row i and the column player chooses column j, then the pay-off to the row player is p_{ij} and the pay-off to the column player is $-p_{ij}$.

For $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$, let P_i denote the i^{th} row of P and let P^j denote the j^{th} column of P.

A **strategy profile** is an ordered pair $(x, y) \in \Delta^{m-1} \times \Delta^{n-1}$, where x is the vector of probabilities with which the row-player chooses each strategy and y is the vector of probabilities with which the column-player chooses each strategy.

A very "witty" justification for randomized strategies is available in (page 17, in chapter 3 of) Washburn (2014): "It is the key to resolving the '… if he thinks that I think that he thinks that…' impasse- you can't outwit someone who refuses to think".

An **equilibrium** for a TPZS game P is a strategy profile (x^*, y^*) that satisfies the following condition: For all $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$, $P_i y^* \leq x^{*T} P y^* \leq x^{*T} P^j$.

Hence, (x^*, y^*) is an equilibrium if and only if $(x^*, y^*) \in \Delta^{m-1} \times \Delta^{n-1}$ and for all $(x, y) \in \Delta^{m-1} \times \Delta^{n-1}$ it is the case that $xPy^* \leq x^{*T} P y^* \leq x^{*T} P y$.

It has been known since the seminal work of John von-Neumann and Oscar Morgenstern entitled "Theory of Games and Economic Behaviour" that every TPZS game has at least one equilibrium. The textbook version of this result, whose proof is based on a simple linear programming problem and its dual, is available as theorem 1 in Lahiri (2025a). A very detailed treatment of two-person zero-sum games is available in Washburn (2014). The scope of Washburn (2014) goes way beyond the two-person zero-sum (matrix) games, that we are concerned with here.

3. Linear Systems Associated with TPZS games: We now provide two characterizations of equilibrium for TPZS games in terms of linear systems, the first being a lemma.

Let P^T denote the transpose of P.

Lemma 1: (x^*, y^*) is an equilibrium for the TPZS game P <u>if and only if</u> $x^* \in \Delta^{m-1}$, $y^* \in \Delta^{n-1}$ and there exists a real number u^* such that $Py^* \leq u^* E^{(m)}$, $P^T x^* \geq u^* E^{(n)}$.

Further, such a $u^* = x^{*T} P y^*$.

Proof: If (x^*, y^*) is an equilibrium for the TPZS game P, then with $u^* = x^{*T} P y^*$, we know that for all $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$, $P_i y^* \leq x^{*T} P y^* \leq x^{*T} P^j$, so that $Py^* \leq u^* E^{(m)}$ and $P^T x^* \geq u^* E^{(n)}$. Further, $x^* \in \Delta^{m-1}$ and $y^* \in \Delta^{n-1}$.

Now suppose that $x^* \in \Delta^{m-1}$, $y^* \in \Delta^{n-1}$ and there exists a real number u^* such that $Py^* \leq u^* E^{(m)}$, $P^T x^* \geq u^* E^{(n)}$.

Pre-multiplying the first inequality by x^{*T} and the second by y^{*T} we get $x^{*T} P y^* \leq u^*$ and $x^{*T} P y^* = y^{*T} P^T x^* \geq u^*$.

Thus, $u^* = x^{*T} P y^*$.

Since $x^* \in \Delta^{m-1}$, $y^* \in \Delta^{n-1}$, $Py^* \leq u^* E^{(m)} = (x^{*T} P y^*) E^{(m)}$ and $x^{*T} P \geq u^* E^{(n)T} = (x^{*T} P y^*) E^{(n)}$.

Thus, (x^*, y^*) is an equilibrium for the TPZS game P. Q.E.D.

We now state and prove our main result for this section, which provides a second characterization of the set of equilibria as the set of solutions to a linear programming problem. While this result follows from proposition 1 of Lahiri (2025b) and hence the well-known "Equivalence Theorem" in Mangasarian and Stone (1964) for bi-matrix games, for the sake of completeness, we provide a proof of the result.

Proposition 1: (x^*, y^*) is an equilibrium for the TPZS game P <u>if and only if</u> there exist real numbers u^*, v^* such that x^*, y^*, u^*, v^* solves the following linear programming problem:

Minimize u - v, subject to $Py - uE^{(m)} \leq 0$, $-P^Tx + vE^{(n)} \leq 0$, $x \in \Delta^{m-1}$, $y \in \Delta^{n-1}$, $u, v \in \mathbb{R}$.

Further, if x^*, y^*, u^*, v^* solves the linear programming problem, then $u^* = v^* = x^{*T}Py^*$.

Proof: It is easy to see that if $Py - uE^{(m)} \leq 0$, $-P^Tx + vE^{(n)} \leq 0$, $x \in \Delta^{m-1}$, $y \in \Delta^{n-1}$, $u, v \in \mathbb{R}$, then $u - v \geq 0$.

Suppose, (x^*, y^*) is an equilibrium for the TPZS game P. Let $u^* = v^* = x^{*T}Py^*$.

Then clearly, $Py^* - u^*E^{(m)} = Py^* - (x^{*T}Py^*)E^{(m)} \leq 0$, $v^*E^{(m)} - P^Tx^* = (x^{*T}Py^*)E^{(m)} - P^Tx^* \leq 0$,

$x^* \in \Delta^{m-1}$, $y^* \in \Delta^{n-1}$, $u^*, v^* \in \mathbb{R}$.

Further, $u^* - v^* = 0$ and hence x^*, y^*, u^*, v^* solves the minimization problem.

Now suppose x^*, y^*, u^*, v^* solve the problem: Minimize u - v, subject to $Py - uE^{(m)} \leq 0$, $-P^Tx + vE^{(n)} \leq 0$, $x \in \Delta^{m-1}$, $y \in \Delta^{n-1}$, $u, v \in \mathbb{R}$.

Thus, $Py^* - u^*E^{(m)} \leq 0$, $-P^Tx^* + v^*E^{(n)} \leq 0$, $x^* \in \Delta^{m-1}$, $y^* \in \Delta^{n-1}$, $u^*, v^* \in \mathbb{R}$.

By theorem 1 in Lahiri (2025a), we know that the TPZS game P has an equilibrium (x^0, y^0). Let $u^0 = v^0 = x^{0T}Py^0$.

Then, $Py^0 - u^0E^{(m)} \leq 0$, $-P^Tx^0 + v^0E^{(n)} \leq 0$, $x^0 \in \Delta^{m-1}$, $y^0 \in \Delta^{n-1}$, $u^0, v^0 \in \mathbb{R}$.

Thus, x^0, y^0, u^0, v^0 satisfy all the constraints of the minimization problem and $u^0 - v^0 = 0$.

Since, as noted right at the beginning of the proof, that $Py - uE^{(m)} \leq 0$, $-P^Tx + vE^{(n)} \leq 0$, $x \in \Delta^{m-1}$, $y \in \Delta^{n-1}$, $u, v \in \mathbb{R}$, implies $u - v \geq 0$, it must be the case that x^0, y^0, u^0, v^0 solve the minimization problem and the optimal value of the objection function is 0.

Since, x^*, y^*, u^*, v^* solve the minimization problem, it must be the case that $u^* - v^* = 0$, i.e. $u^* = v^*$.

But then, we have $Py^* - u^*E^{(m)} \leq 0$, $-P^Tx^* + v^*E^{(n)} \leq 0$, $x^* \in \Delta^{m-1}$, $y^* \in \Delta^{n-1}$, $u^*, v^* \in \mathbb{R}$ and hence by lemma 1 and $u^* = v^*$, it follows that (x^*, y^*) is an equilibrium for the TPZS game P.

Further, $[Py^* - u^*E^{(m)} \leq 0$ and $x^* \in \Delta^{m-1}]$ implies $[x^{*T}Py^* - u^*x^{*T}E^{(m)} \leq 0$, i.e., $x^{*T}Py^* \leq u^*]$.

Similarly, $[-P^Tx^* + v^*E^{(n)} \leq 0$ and $y^* \in \Delta^{n-1}]$ implies $[-y^{*T}Px^* + v^*y^{*T}E^{(n)} \leq 0$, i.e., $v^* \leq x^{*T}Py^*]$.

$[v^* \leq x^{*T}Py^* \leq u^*$ and $u^* = v^*]$ implies $[u^* = v^* = x^{*T}Py^*]$. Q.E.D.

4. Completely Mixed Equilibrium and Completely Mixed TPZS Games: We now consider some implications of lemma 1 for equilibrium that assigns positive probabilities to all pure strategies of both players and TPZS games all equilibria for which satisfy such a property.

Lemma 1 can be equivalently stated as follows:

Lemma 1 (Alternative Statement): (x^*, y^*) is an equilibrium for a TPZS game P if and only if $x^* \in \Delta^{m-1}$, $y^* \in \Delta^{n-1}$, $Py^* \leq (x^{*T}Py^*)E^{(m)}$, $P^Tx^* \geq (x^{*T}Py^*)E^{(n)}$.

Note 1: As a consequence of lemma 1, (i) if for any $i \in \{1, \ldots, m\}$ it is the case that $P_iy^* < u^* = x^{*T}Py^*$, then it must be the case that $x_i^* = 0$; and (ii) if for any $j \in \{1, \ldots, n\}$ it is the case that $x^{*T}P^j > u^* = x^{*T}Py^*$, then it must be the case that $y_j^* = 0$.

An "equilibrium" (x^*, y^*) for a TPZS game P is said to be **completely mixed** if for all $i \in \{1, \ldots, m\}$, $x_i^* > 0$ and for all $j \in \{1, \ldots, n\}$, $y_j^* > 0$.

A TPZS game P is said to be **completely mixed** if every equilibrium for P is completely mixed.

While the concept of completely mixed games originated in the work of Kaplansky (1945), the most recent significant results for such games as well as for completely mixed bi-matrix games can be found in Parthasarathy, Sharma and Sricharan (2020) and more recently in Parthasarathy, Gomatam and Kumar (2024). The latter mentions the work of Raghavan (1970) for completely mixed strategies for the more general class of bi-matrix games and a second significant contribution by Kaplansky (1995) for completely mixed TPZS games. Here we prove that if a TPZS game is completely mixed then it has a unique equilibrium, and hence a unique solution for the linear programming problem in proposition 1.

Proposition 2: Suppose the TPZS game P is completely mixed. Then it has a unique equilibrium and there is a unique solution of the linear programming problem: Minimize u - v, subject to $Py - uE^{(m)} \leq 0$, $-P^Tx + vE^{(n)} \leq 0$, $x \in \Delta^{m-1}$, $y \in \Delta^{n-1}$, $u, v \in \mathbb{R}$.

Proof: Suppose TPZS game P is completely mixed. Thus, $P \neq 0$.

Let (x^*, y^*) be a completely mixed equilibrium for P. Thus, $x_i^* > 0$ for all $i \in \{1, \ldots, m\}$ and $y_j^* > 0$ for all $j \in \{1, \ldots, n\}$.

Let $E^{(m\times n)}$ be the m×n matrix all entries of which are equal to 1.

Then, it is easily verified that (x^*, y^*) is a completely mixed equilibrium for the TPZS game $P - (x^{*T}Py^*)E$ and $x^{*T}(P - (x^{*T}Py^*)E)y^* = 0$.

Hence, without loss of generality we may assume that $x^{*T}Py^* = 0$.

By note 1 (immediately after the "alternative statement of lemma 1"), it follows that $Py^* = 0$ and $P^Tx^* = 0$.

Thus, the columns of P as well the columns of P^T are linearly dependent.

Towards a contradiction suppose (x^0, y^0) is any other completely mixed equilibrium for P.

Thus, $(x^0, y^0) \neq (x^*, y^*)$, $Py^0 = 0 = P^Tx^0$, $x^0 \in \Delta^{m-1}$, $y^0 \in \Delta^{n-1}$, $x_i^0 > 0$ for all $i \in \{1, \ldots, m\}$ and $y_j^0 > 0$ for all $j \in \{1, \ldots, n\}$.

Suppose $x^0 \neq x^*$. Thus, $\{i|x_i^0 > x_i^*\} \neq \phi$ and $\{i|x_i^0 < x_i^*\}$. Thus, $t = \max\{\frac{x_i^*}{x_i^0}|\ i \in \{1, \ldots, m\} > 1 > \min\{\frac{x_i^*}{x_i^0}|\ i \in \{1, \ldots, m\} = s > 0$.

Let $x' = x^* - sx^0$. Clearly, $P^T x' = 0$, $x' \in \mathbb{R}_+^m$.

Note that for all $i \in \{1, \ldots, m\}$ satisfying $\frac{x_i^*}{x_i^0} = t$, we have $x_i^* - sx_i^0 = x_i^0(t - s) > 0$ and for all $i \in \{1, \ldots, m\}$ satisfying $\frac{x_i^*}{x_i^0} = s$, we have $x_i^* - sx_i^0 = x_i^0(s - s) = 0$.

Thus, $x' \in \mathbb{R}_+^m \backslash \{0\}$ and the $\{i|\ x_i' > 0\}$ is a "*strict and non-empty subset*" of $\{1, \ldots, m\}$.

Thus, there exists $\lambda > 0$ such that $\lambda x' \in \Delta^{m-1}$, $P^T(\lambda x') = 0$.

Thus, $(\lambda x', y^*)$ is an equilibrium for P.

However, $(\lambda x', y^*)$ is not completely mixed since the $\{i|\lambda x_i' = 0\} \neq \phi$, thereby contradicting P is completely mixed.

Thus, it must be that $x^0 = x^*$.

Similarly, we get $y^0 = y^*$.

Thus, $(x^0, y^0) = (x^*, y^*)$ leading to the desired contradiction and proving the proposition.

Thus, P has a unique equilibrium and by proposition 1, the linear programming in the statement of this proposition has a unique solution. Q.E.D.

Note 2: We follow the policy that citations in scientific report should reflect relevance, without sacrificing precision.

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