Linear Systems Equivalent to Two-Person Zero Sum Games

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Abstract

We provide two characterizations of the set of equilibria of a two-person zero-sum "matrix" (TPZS) game. The first is a lemma, which says that a strategy profile (pair of randomizations over pure strategies) is an equilibrium if an only if along with another real number, it satisfies a specific system of linear inequalities. The second is a proposition, which says that a strategy profile is an equilibrium if an only if along with two real numbers it solves a certain linear programming problem. The proposition is a special case of proposition 2 in Lahiri (2025) which in turn is a special case of the "Equivalence Theorem" in Mangasarian and Stone (1964).

A strategy profile is said to be completely mixed if it assigns positive probabilities to all pure strategies for both players and a TPZS game is said to be completely mixed if all equilibria of the game are completely mixed. We show that a completely mixed TPZS game has a unique equilibrium strategy. In the proof of this result, the lemma we referred to earlier plays an important role. We then apply this result for the case of a matrix corresponding to a completely mixed TPZS game with value zero, to show (i) that a non-zero linear combination of the columns of the matrix is zero, if and only if the vector of coefficients used for the linear combination is a scalar multiple of the column player's strategy in the unique equilibrium, (ii) that a non-zero linear combination of the rows of the matrix is zero, if and only if the vector of coefficients used for the linear combination is a scalar multiple of the row player's strategy in the unique equilibrium. We apply a corollary of this proposition to obtain a complete characterization of all completely mixed TPZS games that have value zero. This characterization is different from the characterization available in Kaplansky (1945).

Keywords: two-person zero-sum games, equilibrium, linear programming, completely mixed, unique solution

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1. Notation: For any positive integer ℓ let $\Delta^{\ell-1} = \{x \in \mathbb{R}^{\ell}_{+} | \sum_{j=1}^{\ell} x_j = 1\}$ and for $j \in \{1, \dots, \ell\}$, let $E^{(\ell,j)}$ be the ℓ-dimensional column vector whose j^{th} coordinate is 1 and all other coordinates are equal to 0. Let $E^{(\ell)} = \sum_{j=1}^{\ell} E^{(\ell,j)}$. $E^{(\ell)}$ is said to be the ℓ- dimensional sum vector.

2. Two-Person Zero-Sum Games: For positive integers m, n, let P be an m×n matrix such that for all $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$, the $(i, j)^{th}$ term (term at the intersection of the i^{th} row and j^{th} column) of P is p_{ij}.

A **two-person zero-sum "matrix" (TPZS) game** (interactive decision-making problem) with pay-off matrix P (or simply, a TPZS game P) is a contest between two players- the **row player** and the **column player**- such that if for $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$, the row player chooses row i and the column player chooses column j, then the pay-off to the row player is p_{ij} and the pay-off to the column player is $-p_{ij}$.

For $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$, let P_i denote the i^{th} row of P and let P^j denote the j^{th} column of P.

A **strategy profile** is an ordered pair $(x, y) \in \Delta^{m-1} \times \Delta^{n-1}$, where x is the vector of probabilities with which the row-player chooses each strategy and y is the vector of probabilities with which the column-player chooses each strategy.

A very "witty" justification for randomized strategies is available in (page 17, in chapter 3 of) Washburn (2014): "It is the key to resolving the '… if he thinks that I think that he thinks that…' impasse- you can't outwit someone who refuses to think".

An **equilibrium** for a TPZS game P is a strategy profile (x^*, y^*) that satisfies the following condition: For all $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$, $P_i y^* \leq x^{*T} P y^* \leq x^{*T} P^j$.

Hence, (x^*, y^*) is an equilibrium if and only if $(x^*, y^*) \in \Delta^{m-1} \times \Delta^{n-1}$ and for all $(x, y) \in \Delta^{m-1} \times \Delta^{n-1}$ it is the case that $xPy^* \leq x^{*T} P y^* \leq x^{*T} P y$.

It is well-known that associated with any TPZS game P, there is a real number V(P), called the **value** of P, such if (x, y) is an equilibrium of P, then $x^T P y = V(P)$.

It has been known since the seminal work of John von-Neumann and Oscar Morgenstern entitled "Theory of Games and Economic Behaviour" that every TPZS game has at least one equilibrium. An "existence result" for a general class of two-person games that includes the set all TPZS games is available as proposition 1 in Lahiri (2025). The proof of this theorem in Lahiri (2025) is based on a simple linear programming problem and its dual. A very detailed treatment of two-person zero-sum games is available in Washburn (2014). The scope of Washburn (2014) goes way beyond the two-person zero-sum (matrix) games, that we are concerned with here.

3. Linear Systems Associated with TPZS games: We now provide two characterizations of equilibrium for TPZS games in terms of linear systems, the first being a short lemma.

Let P^T denote the transpose of P.

Lemma 1: (x^*, y^*) is an equilibrium for the TPZS game P <u>if and only if</u> $x^* \in \Delta^{m-1}$, $y^* \in \Delta^{n-1}$ and there exists a real number u^* such that $Py^* \leq u^* E^{(m)}$, $P^T x^* \geq u^* E^{(n)}$.

Further, such a $u^* = x^{*T} P y^*$.

Proof: If (x^*, y^*) is an equilibrium for the TPZS game P, then with $u^* = x^{*T} P y^*$, we know that for all $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$, $P_i y^* \leq x^{*T} P y^* \leq x^{*T} P^j$, so that $Py^* \leq u^* E^{(m)}$ and $P^T x^* \geq u^* E^{(n)}$. Further, $x^* \in \Delta^{m-1}$ and $y^* \in \Delta^{n-1}$.

Now suppose that $x^* \in \Delta^{m-1}$, $y^* \in \Delta^{n-1}$ and there exists a real number u^* such that $Py^* \leq u^* E^{(m)}$, $P^T x^* \geq u^* E^{(n)}$.

Pre-multiplying the first inequality by x^{*T} and the second by y^{*T} we get $x^{*T}Py^* \leq u^*$ and $x^{*T}Py^* = y^{*T}P^Tx^* \geq u^*$.

Thus, $u^* = x^{*T}Py^*$.

Since $x^* \in \Delta^{m-1}$, $y^* \in \Delta^{n-1}$, $Py^* \leq u^*E^{(m)} = (x^{*T}Py^*)E^{(m)}$ and $x^{*T}P \geq u^*E^{(n)T} = (x^{*T}Py^*)E^{(n)}$.

Thus, (x^*, y^*) is an equilibrium for the TPZS game P. Q.E.D.

We now state an important result, which provides a second characterization of the set of equilibria as the set of solutions to a linear programming problem. This result follows directly from proposition 2 of Lahiri (2025) and hence the well-known "Equivalence Theorem" in Mangasarian and Stone (1964) for bi-matrix games.

Proposition 1: (x^*, y^*) is an equilibrium for the TPZS game P if and only if there exist real numbers u^*, v^* such that x^*, y^*, u^*, v^* solves the following linear programming problem:

Minimize u - v, subject to $Py - uE^{(m)} \leq 0$, $-P^Tx + vE^{(n)} \leq 0$, $x \in \Delta^{m-1}$, $y \in \Delta^{n-1}$, $u, v \in \mathbb{R}$.

Further, if x^*, y^*, u^*, v^* solves the linear programming problem, then $u^* = v^* = x^{*T}Py^*$.

4. Completely Mixed Equilibrium and Completely Mixed TPZS Games: We now consider some implications of lemma 1 for equilibrium that assigns positive probabilities to all pure strategies of both players and TPZS games all equilibria for which satisfy such a property.

Lemma 1 can be equivalently stated as follows:

Lemma 1 (Alternative Statement): (x^*, y^*) is an equilibrium for a TPZS game P if and only if $x^* \in \Delta^{m-1}$, $y^* \in \Delta^{n-1,}$ $Py^* \leq (x^{*T}Py^*)E^{(m)}$, $P^Tx^* \geq (x^{*T}Py^*)E^{(n)}$.

Note 1: As a consequence of lemma 1, (i) if for any $i \in \{1, \ldots, m\}$ it is the case that $P_iy^* < u^* = x^{*T}Py^*$, then it must be the case that $x_i^* = 0$; and (ii) if for any $j \in \{1, \ldots, n\}$ it is the case that $x^{*T}P^j > u^* = x^{*T}Py^*$, then it must be the case that $y_j^* = 0$.

An "equilibrium" (x^*, y^*) for a TPZS game P is said to be **completely mixed** if for all $i \in \{1, \ldots, m\}$, $x_i^* > 0$ and for all $j \in \{1, \ldots, n\}$, $y_j^* > 0$.

A TPZS game P is said to be **completely mixed** if every equilibrium for P is completely mixed.

While the concept of completely mixed games originated in the work of Kaplansky (1945), the most recent significant results for such games as well as for completely mixed bi-matrix games can be found in Parthasarathy, Sharma and Sricharan (2020) and more recently in Parthasarathy, Gomatam and Kumar (2024). The latter mentions the work of Raghavan (1970) for completely mixed strategies for the more general class of bi-matrix games and a second significant contribution by Kaplansky (1995) for completely mixed TPZS games. Here we prove that if a TPZS game is completely mixed then it has a unique equilibrium, and hence a unique solution for the linear programming problem in proposition 1.

Proposition 2: Suppose the TPZS game P is completely mixed. Then it has a unique equilibrium and there is a unique solution of the linear programming problem: Minimize u - v, subject to $Py - uE^{(m)} \leq 0$, $-P^Tx + vE^{(n)} \leq 0$, $x \in \Delta^{m-1}$, $y \in \Delta^{n-1}$, u, v$\in\mathbb{R}$.

Proof: Suppose TPZS game P is completely mixed. Thus, P ≠ 0.

Let (x^*, y^*) be a completely mixed equilibrium for P. Thus, $x_i^* > 0$ for all i∈{1, …, m} and $y_j^* > 0$ for all j∈{1, …, n}.

Let $E^{(m\times n)}$ be the m×n matrix all entries of which are equal to 1.

Then, it is easily verified that (x^*, y^*) is a completely mixed equilibrium for the TPZS game P – $(x^{*T}Py^*)E$ and $x^{*T}(P- (x^{*T}Py^*)E)y^* = 0$.

Hence, without loss of generality we may assume that the value of P, $V(P) = x^{*T}Py^* = 0$.

By note 1 (immediately after the "alternative statement of lemma 1"), it follows that $Py^* = 0$ and $P^Tx^* = 0$.

Thus, the columns of P as well the columns of P^T are linearly dependent.

Towards a contradiction suppose (x^0, y^0) is any other completely mixed equilibrium for P.

Thus, $(x^0, y^0) \neq (x^*, y^*)$, $Py^0 = 0 = P^Tx^0$, $x^0 \in \Delta^{m-1}$, $y^0 \in \Delta^{n-1}$, $x_i^0 > 0$ for all i∈{1, …, m} and $y_j^0 > 0$ for all j∈{1, …, n}.

Suppose $x^0 \neq x^*$. Thus, $\{i|x_i^0 > x_i^*\} \neq \phi$ and $\{i|x_i^0 < x_i^*\}$. Thus, $t = \max\{\frac{x_i^*}{x_i^0}| i\in\{1, …, m\}\} > 1 > \min\{\frac{x_i^*}{x_i^0}| i\in\{1, …, m\}\} = s > 0$.

Let $x' = x^* - sx^0$. Clearly, $P^Tx' = 0$, $x' \in \mathbb{R}_+^m$.

Note that for all i∈{1, …, m} satisfying $\frac{x_i^*}{x_i^0} = t$, we have $x_i^* - sx_i^0 = x_i^0(t – s) > 0$ and for all i∈{1, …, m} satisfying $\frac{x_i^*}{x_i^0} = s$, we have $x_i^* - sx_i^0 = x_i^0(s – s) = 0$.

Thus, $x' \in \mathbb{R}_+^m\backslash\{0\}$ and the $\{i|\ x_i' > 0\}$ is a "*strict and non-empty subset*" of {1, …, m}.

Thus, there exists λ > 0 such that $\lambda x' \in \Delta^{m-1}$, $P^T(\lambda x') = 0$.

Thus, $(\lambda x', y^*)$ is an equilibrium for P.

However, $(\lambda x', y^*)$ is not completely mixed since the $\{i|\lambda x_i' = 0\} \neq \phi$, thereby contradicting P is completely mixed.

Thus, it must be that $x^0 = x^*$.

Similarly, we get $y^0 = y^*$.

Thus, $(x^0, y^0) = (x^*, y^*)$ leading to the desired contradiction and proving the proposition.

Thus, P has a unique equilibrium and by proposition 1, the linear programming problem in the statement of this proposition has a unique solution. Q.E.D.

The following result leads to interesting consequences for completely mixed TPZS games.

Proposition 3: Suppose the m×n TPZS game P is completely mixed. Suppose (x^*, y^*) is a completely mixed equilibrium for P satisfying $x^{*T}Py^* = 0$. Then $\{\lambda\in\mathbb{R}^n|P\lambda = 0\} = \{ty^*|\, t\in\mathbb{R}\}$ and $\{\gamma\in\mathbb{R}^m|\gamma^T P = 0\} = \{tx^*|\, t\in\mathbb{R}\}$.

Proof: Suppose the m×n TPZS game P is completely mixed. Thus, $P \neq 0$. Since (x^*, y^*) is a completely mixed equilibrium for P, by proposition 2, (x^*, y^*) is the unique equilibrium for P. We have assumed that $x^{*T}Py^* = 0$.

Suppose there exists $\lambda\in\mathbb{R}^n\backslash\{ty^*|\, t\in\mathbb{R}\}$ such that $P\lambda = 0$. Since $y^* >> 0$ (i.e., all coordinates of y^* are strictly positive) for $\varepsilon > 0$ sufficiently small $y^* + \varepsilon\lambda >> 0$ and $P(y^* + \lambda\varepsilon) = 0$.

Thus, there exists $\alpha > 0$ such that $\alpha(y^* + \lambda\varepsilon)\in\Delta^{n-1}$.

Let $y^0 = \alpha(y^* + \lambda\varepsilon)$. Then, $Py^0 = 0$.

Since $x^{*T}P = (x^{*T}Py^*)E^{(n)T} = 0$, we get $Py^0 = 0 = (x^{*T}Py^0)E^{(m)}$.

Thus, (x^*, y^0) is an equilibrium for P. By proposition 2, we must have $y^0 = y^*$.

$y^0 = \alpha(y^* + \lambda\varepsilon) = y^*$ implies $\lambda = \frac{1-\alpha}{\varepsilon}y^*$, contradicting $\lambda\in\mathbb{R}^n\backslash\{ty^*|\, t\in\mathbb{R}\}$.

Thus, $\{\lambda\in\mathbb{R}^n|P\lambda = 0\} \subset \{ty^*|\, t\in\mathbb{R}\}$.

Since $Py^* = 0$, it must be the case that $\{ty^*|\, t\in\mathbb{R}\} \subset \{\lambda\in\mathbb{R}^n|P\lambda = 0\}$.

Hence, $\{\lambda\in\mathbb{R}^n|P\lambda = 0\} = \{ty^*|\, t\in\mathbb{R}\}$.

A similar proof works for proving $\{\gamma\in\mathbb{R}^m|\gamma^T P = 0\} = \{tx^*|\, t\in\mathbb{R}\}$. Q.E.D.

An immediate consequence of proposition 3 is the following result implied in the version of the main result of Kaplansky (1945) which appears as theorem 2 (without proof) in Parthasarathy et al. (2020).

Corollary of Proposition 3: Suppose the m×n TPZS game P is completely mixed with V(P) = 0. Then:

(i) Every collection of n-1 columns of P are linearly independent.

(ii) Every collection of m – 1 rows of P are linearly independent. Hence,

(iii) m = n and the rank of the matrix P is n – 1.

Using the corollary, we can establish the following alternative version of the main result in Kaplansky (1945).

Theorem 1: Let P be a TPZS game with 'n' columns and suppose V(P) = 0. Then P is completely mixed <u>if and only if</u> P is a square matrix with the rank of P being equal to n-1, and the linear system $[Py = 0, y\in\Delta^{n-1}]$ has a solution.

Proof: It is easy to see that if P is completely mixed, then $Py = 0$, $y\in\Delta^{n-1}$ has a solution.

Thus, in view of the corollary of proposition 3, all that we need to show for a TPZS game P with 'n' columns satisfying V(P) = 0, is that <u>if</u> P is a square matrix of size 'n' (i.e., n×n matrix), the rank of P is n-1, and the linear system [Py = 0, $y \in \Delta^{n-1}$] has a solution, then P is completely mixed.

Let (x^*, y^*) be an equilibrium of the TPZS game P. Then by the alternative statement of lemma 1, $x^* \in \Delta^{m-1}$, $y^* \in \Delta^{n-1}$, $Py^* \leq (x^{*T}Py^*)E^{(m)}$, $P^Tx^* \geq (x^{*T}Py^*)E^{(n)}$.

Since, Py = 0, $x^{*T}Py = 0 = V(P) = (x^{*T}Py^*)$.

Thus, (x^*, y) is also an equilibrium of the TPZS P. Hence, without loss of generality suppose, $y = y^*$.

If $\{j|\ y_j^* = 0\} \neq \phi$, then $\sum_{\{j|y_j^* > 0\}} P^j y_j^* = 0$, with the cardinality of $\{j|\ y_j^* > 0\}$ being at least 1 and at most n-1, contradicting our assumption that the rank of P is n-1.

Hence $\{j|\ y_j^* = 0\} = \phi$.

Thus, it must be the case that $x^{*T}P = 0$.

If $\{i|\ x_i^* = 0\} \neq \phi$, then $\sum_{\{i|x_i^* > 0\}} P_i x_i^* = 0$, with the cardinality of $\{i|\ x_i^* > 0\}$ being at least 1 and at most n-1, contradicting our assumption that the rank of P is n-1.

Thus, it must be the case that $\{i|\ x_i^* = 0\} = \phi$.

Suppose, (x^0, y^0) is an equilibrium for P. Then (x^0, y^*) and (x^*, y^0) are also equilibria of P.

Thus, $\{j|\ y_j^* = 0\} = \phi$ implies $x^{0T}P = 0$ and $\{i|\ x_i^* = 0\} = \phi$ implies $Py^0 = 0$.

Since rank of P is n-1 it follows by an argument similar to that for x^* and y^* that $\{i|\ x_i^0 = 0\} = \phi$ and $\{j|\ y_j^0 = 0\} = \phi$.

Thus, (x^0, y^0) is a completely mixed equilibrium, there by leading to the conclusion that P is a completely mixed TPZS game. Q.E.D.

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