Hyperfuzzy and Super-Hyperfuzzy Extensions of Fuzzy Metric Spaces, Queuing Models, and Ontologies

Takaaki Fujita [1] *

[1] Independent Researcher, Shinjuku, Shinjuku-ku, Tokyo, Japan. Takaaki.fujita060@gmail.com

Abstract

Uncertainty modeling underpins decision-making in domains ranging from engineering to artificial intelligence. A variety of frameworks—such as Fuzzy Sets [1], Rough Sets [2,3], Picture Fuzzy Sets [4,5], Soft Sets [6,7], and Plithogenic Sets [8]—have been proposed to capture different dimensions of imprecision. Hyperfuzzy Sets and their recursive extension, SuperHyperfuzzy Sets, further enrich this landscape by assigning set-valued membership degrees at multiple hierarchical levels.

Building on the concept of a fuzzy metric space—a set X equipped with a continuous t-norm $*$ and a function $M: X^2 \times (0, \infty) \to [0, 1]$ satisfying positivity, symmetry, the triangle inequality, and continuity—we introduce Hyperfuzzy and SuperHyperfuzzy metrics that model proximity with multi-level, set-valued memberships. We also investigate how Hyperfuzzy and SuperHyperfuzzy constructs can be applied to extend fuzzy queues and fuzzy ontologies, illustrating their potential to enhance performance evaluation and knowledge representation in real-world systems.

Keywords: Fuzzy set, HyperFuzzy Set, SuperHyperFuzzy Set, Fuzzy Metric Spaces, Fuzzy Queue, Fuzzy Ontology

Structure of this paper

The format of this paper is described below.

1 Preliminaries

In this section, we review the essential definitions and concepts that underpin the results presented in this paper. Throughout, all sets and structures are assumed to be finite.

1.1 Fuzzy and Related Sets

A *fuzzy set* assigns to each element a degree of membership in the interval $[0, 1]$, allowing graded membership rather than a strict binary classification [1,9,10]. Fuzzy sets are known to have a wide range of applications, and various extensions—such as intuitionistic fuzzy sets [11–13], vague sets [14,15], hesitant fuzzy sets [16–18], neutrosophic sets [19–21], and plithogenic sets [22–25]— have also been developed. A *hyperfuzzy set* refines this notion by mapping each element to a nonempty subset of $[0, 1]$, thus capturing variability and imprecision in the membership degree [26–32]. More generally, an (m, n)-*superhyperfuzzy set* assigns to each m-level subset a collection of n-level membership sets, modeling multi-tiered uncertainty (cf. [33,34]).

Definition 1.1 (Base Set)**.** (cf. [35–37]) A *base set* S is the ground set from which all further constructions (e.g., powersets, hyperstructures) are derived:

$$S = \{ x \mid x \text{ belongs to the domain of discourse} \}.$$

All elements of $\mathcal{P}(S)$ and $\mathcal{P}_n(S)$ arise from S.

Definition 1.2 (Powerset)**.** (cf. [38, 39]) The *powerset* of S, denoted $\mathcal{P}(S)$, comprises every subset of S, including $\emptyset$ and S itself:

$$\mathcal{P}(S) = \{ A \mid A \subseteq S \}.$$

Definition 1.3 (n-th Powerset)**.** (cf. [40–44]) The n-th powerset of a set H, denoted $\mathcal{P}_n(H)$, is defined iteratively, starting with the standard powerset. The recursive construction is given by:

$$\mathcal{P}_1(H) = \mathcal{P}(H), \quad \mathcal{P}_{n+1}(H) = \mathcal{P}(\mathcal{P}_n(H)), \quad \text{for } n \geq 1.$$

Similarly, the n-th non-empty powerset, denoted $\mathcal{P}_n^*(H)$, is defined recursively as:

$$\mathcal{P}_1^*(H) = \mathcal{P}^*(H), \quad \mathcal{P}_{n+1}^*(H) = \mathcal{P}^*(\mathcal{P}_n^*(H)).$$

Here, $\mathcal{P}^*(H)$ represents the powerset of H with the empty set removed.

Example 1.4 (Task Breakdown in Project Scheduling)**.** Let

$$H = \{\text{Requirement Analysis (RA)},\ \text{Design (D)},\ \text{Testing (T)}\}$$

be the set of core project tasks. Then:

$$\mathcal{P}_1(H) = \big\{\{RA\}, \{D\}, \{T\}, \{RA, D\}, \{RA, T\}, \{D, T\}, \{RA, D, T\}\big\},$$

whose elements represent possible *work packages*.

Next, the second powerset

$$\mathcal{P}_2(H) = \mathcal{P}\big(\mathcal{P}_1(H)\big) \setminus \{\emptyset\}$$

consists of nonempty collections of work packages. For example,

$$W = \big\{\{RA\}, \{D, T\}\big\} \in \mathcal{P}_2(H)$$

represents two phases: one for requirement analysis, and one combining design and testing.

Finally, the third powerset

$$\mathcal{P}_3(H) = \mathcal{P}\big(\mathcal{P}_2(H)\big) \setminus \{\emptyset\}$$

consists of nonempty collections of phase-groupings. For instance,

$$S = \Big\{ \{\{RA\}, \{D, T\}\}, \{\{RA, D\}, \{T\}\} \Big\} \in \mathcal{P}_3(H)$$

represents two alternative *scheduling strategies*, each consisting of two phases: $\{RA\}$ then $\{D, T\}$, or $\{RA, D\}$ then $\{T\}$.

Thus the n-th powerset $\mathcal{P}_n(H)$ models hierarchical planning levels in project management: $\mathcal{P}_1(H)$ = work packages, $\mathcal{P}_2(H)$ = project phases, $\mathcal{P}_3(H)$ = scheduling strategies or portfolios of phases.

Definition 1.5 (Fuzzy Set)**.** [1,45] A *Fuzzy set* τ in a non-empty universe Y is a mapping $\tau : Y \to [0, 1]$. A *fuzzy relation* on Y is a fuzzy subset δ in $Y \times Y$. If τ is a fuzzy set in Y and δ is a fuzzy relation on Y, then δ is called a *fuzzy relation on* τ if

$$\delta(y, z) \leq \min\{\tau(y), \tau(z)\} \quad \text{for all } y, z \in Y.$$

Definition 1.6 (Hyperfuzzy Set)**.** [26, 46–49] Let X be a non-empty universe. A *hyperfuzzy set* $\tilde{A}$ on X is defined by a mapping

$$\tilde{\mu} : X \longrightarrow \tilde{P}([0,1]),$$

where $\tilde{P}([0,1])$ denotes the collection of all non-empty subsets of the interval $[0,1]$.

For each element $x \in X$, $\tilde{\mu}(x) \subseteq [0,1]$ represents the *set of possible membership degrees* of x in the set $\tilde{A}$. This formulation allows for representing uncertainty or variability in the degree of membership, extending the classical fuzzy set (which assigns a single real number in $[0,1]$) to a set-valued interpretation.

Thus, a hyperfuzzy set captures both fuzziness and imprecision by associating each element not with a fixed degree, but with a range or subset of plausible membership values.

Example 1.7 (Product Quality Assessment)**.** Let X be the set of electronic circuit boards produced in a factory. Three quality inspectors independently evaluate each board's conformity and assign a fuzzy membership degree in the "Good Quality" class. Due to subjective judgment and measurement noise, these degrees differ. A hyperfuzzy set $\tilde{A}$ on X captures this by

$$\tilde{\mu}(x) \;=\; \{\mu_1(x), \mu_2(x), \mu_3(x)\} \;\subseteq [0,1],$$

where $\mu_i(x)$ is the degree assigned by inspector i. For instance,

$$\tilde{\mu}(b_{101}) \;=\; \{0.85,\; 0.90,\; 0.80\}, \qquad \tilde{\mu}(b_{102}) \;=\; \{0.60,\; 0.65,\; 0.70\}.$$

Here $\tilde{\mu}(b_{101})$ represents the set of plausible membership values for board b_{101}, reflecting inter-rater variability. Thus $\tilde{A} \colon X \to \tilde{P}([0,1])$ models both fuzziness and imprecision in quality evaluation.

Definition 1.8 (m, n-SuperHyperFuzzy Set)**.** [34, 50, 51] Let X be a nonempty set and let $m, n \in \mathbb{N}_0$. Define the nonempty k-th powerset of a set Y by

$$\mathcal{P}_0^*(Y) = Y, \quad \mathcal{P}_k^*(Y) = \mathcal{P}\big(\mathcal{P}_{k-1}^*(Y)\big) \setminus \{\emptyset\}, \quad k \geq 1.$$

In particular, $\mathcal{P}_m^*(X)$ is the family of all nonempty elements of the m-th iterated powerset of X, and $\mathcal{P}_n^*([0,1])$ is defined analogously. Then an (m,n)-SuperHyperFuzzy Set on X is a function

$$\tilde{\mu}_{m,n} : \mathcal{P}_m^*(X) \;\longrightarrow\; \tilde{\mathcal{P}}_n^*\big([0,1]\big), \quad A \;\mapsto\; \tilde{\mu}_{m,n}(A),$$

where $\tilde{\mathcal{P}}_n^*([0,1])$ denotes the collection of all nonempty subsets of $\mathcal{P}_n([0,1])$. Thus each $A \in \mathcal{P}_m^*(X)$ is assigned a nonempty family of membership-degree sets $\tilde{\mu}_{m,n}(A) \subseteq \mathcal{P}_n([0,1])$, capturing hierarchical uncertainty across both the m- and n-levels.

Example 1.9 (Supply Chain Reliability as a $(2,2)$-SuperHyperFuzzy Set)**.** Let $X = \{S_1, S_2, S_3, S_4\}$ be four suppliers. Set $m = n = 2$, so

$$\mathcal{P}_1^*(X) = \{\{S_i, S_j\} \mid i < j\}, \quad \mathcal{P}_2^*(X) = \mathcal{P}^*\big(\mathcal{P}_1^*(X)\big).$$

Consider the particular element

$$A \;=\; \big\{\{S_1, S_2\},\, \{S_3, S_4\}\big\} \;\in\; \mathcal{P}_2^*(X),$$

representing two local supply networks $C_1 = \{S_1, S_2\}$ and $C_2 = \{S_3, S_4\}$.

We evaluate each network by two reliability metrics:

- *Delivery Time Reliability*: membership degrees $\mu_{\text{time}}(C_1) = \{0.85, 0.90\}$, $\mu_{\text{time}}(C_2) = \{0.88, 0.92\}$.
- *Defect-Free Reliability*: membership degrees $\mu_{\text{defect}}(C_1) = \{0.80, 0.82\}$, $\mu_{\text{defect}}(C_2) = \{0.81, 0.85\}$.

For each C_i, form the set of metric-based fuzzy degrees:

$$M_i = \big\{\mu_{\text{time}}(C_i),\, \mu_{\text{defect}}(C_i)\big\} \;\subseteq\; [0,1],$$

so $M_i \in \mathcal{P}^*([0,1])$. Finally define

$$\tilde{\mu}_{2,2}(A) = \big\{\, M_1,\, M_2 \big\} \;\subseteq\; \mathcal{P}^*\big(\mathcal{P}^*([0,1])\big) = \widetilde{\mathcal{P}}_2^*([0,1]).$$

Thus each supply-network cluster C_i is assigned a set M_i of plausible fuzzy reliability degrees, and the pair of clusters A receives the double-nested collection $\tilde{\mu}_{2,2}(A)$. This captures hierarchical uncertainty both in grouping suppliers ($m = 2$) and in membership valuation ($n = 2$).

Example 1.10 (Global Supply Alliance Reliability as a (3, 2)-SuperHyperFuzzy Set)**.** Let $X = \{S_1, S_2, S_3, S_4, S_5\}$ be five supply nodes. Then

$$\mathcal{P}_1^*(X) = \{\{S_i, S_j\} \mid i < j\}, \quad \mathcal{P}_2^*(X) = \mathcal{P}^*\big(\mathcal{P}_1^*(X)\big), \quad \mathcal{P}_3^*(X) = \mathcal{P}^*\big(\mathcal{P}_2^*(X)\big).$$

Consider the global alliance

$$A \;=\; \{\, R_1,\, R_2\} \;\in\; \mathcal{P}_3^*(X),$$

where

$$R_1 = \{\{S_1, S_2\},\ \{S_2, S_3\}\}, \qquad R_2 = \{\{S_3, S_4\},\ \{S_4, S_5\}\}.$$

Each local network $L \subseteq X$ (e.g. $\{S_1, S_2\}$) is evaluated by two reliability metrics:

$$\mu_{\text{time}}(L) = \{0.90,\ 0.85\}, \quad \mu_{\text{defect}}(L) = \{0.80,\ 0.75\}.$$

Form the first-level membership sets

$$M_L = \big\{\mu_{\text{time}}(L),\ \mu_{\text{defect}}(L)\big\} \;\subseteq\; [0,1],$$

so $M_L \in \mathcal{P}_1([0,1])$. For each region R_k, define the second-level set

$$M_{R_k} = \{\, M_L \mid L \in R_k\} \;\subseteq\; \mathcal{P}_1\big(\mathcal{P}_1([0,1])\big) = \mathcal{P}_2([0,1]).$$

Finally, assign to the alliance A the (3, 2)-SuperHyperFuzzy membership

$$\tilde{\mu}_{3,2}(A) = \{\, M_{R_1},\, M_{R_2}\} \;\subseteq\; \mathcal{P}\big(\mathcal{P}_2([0,1])\big) = \widetilde{\mathcal{P}}_2^*([0,1]).$$

Thus $\tilde{\mu}_{3,2}$ captures hierarchical uncertainty at three structural levels ($m = 3$) and two valuation levels ($n = 2$).

2 Result: Fuzzy Metric Spaces

2.1 Fuzzy Metric Spaces

A fuzzy metric space is a set X with a continuous t-norm $*$ and a function $M\colon X^2 \times (0, \infty) \to [0, 1]$ satisfying positivity, symmetry, triangle inequality, and continuity (cf. [52–56]).

Definition 2.1 (Continuous t-norm)**.** (cf. [57–59]) A *continuous t-norm* is a binary operation

$$* \;:\; [0,1] \times [0,1] \;\longrightarrow\; [0,1]$$

satisfying, for all $a, b, c, d \in [0, 1]$:

(T1) **Commutativity:** $a * b = b * a$,

(T2) **Associativity:** $(a * b) * c = a * (b * c)$,

(T3) **Monotonicity:** If $a \le c$ and $b \le d$, then $a * b \le c * d$,

(T4) **Identity:** $a * 1 = a$,

(T5) **Continuity:** $*$ is continuous on $[0, 1] \times [0, 1]$.

Definition 2.2 (Fuzzy metric space)**.** (cf. [60–62]) Let X be a nonempty set, let $*$ be a continuous t-norm on $[0, 1]$, and let

$$M : X \times X \times (0, \infty) \;\longrightarrow\; [0,1]$$

be a function satisfying, for all $x, y, z \in X$ and all $s, t > 0$:

(M1) $M(x, y, t) > 0$,

(M2) $M(x, y, t) = 1$ if and only if $x = y$,

(M3) $M(x, y, t) = M(y, x, t)$,

(M4) $M(x,y,t) * M(y,z,s) \leq M(x,z,t+s)$,

(M5) for each fixed $x, y \in X$, the map $t \mapsto M(x,y,t)$ is continuous on $(0,\infty)$.

Then the triple $(X, M, *)$ is called a *fuzzy metric space*.

Example 2.3 (Exponential Fuzzy Metric Space)**.** Let (X,d) be a metric space. Define the product t-norm $*$ on $[0,1]$ by

$$a * b = a\,b, \qquad a,b \in [0,1],$$

and define

$$M(x,y,t) = \exp\!\big(-\tfrac{d(x,y)}{t}\big), \qquad x,y \in X,\ t > 0.$$

Then $(X, M, *)$ is a fuzzy metric space, since:

(M1) $M(x,y,t) = \exp(-d(x,y)/t) > 0$.

(M2) $M(x,y,t) = 1 \iff d(x,y) = 0 \iff x = y$.

(M3) Symmetry: $M(x,y,t) = M(y,x,t)$.

(M4) Triangle:

$$M(x,y,t) * M(y,z,s) = \exp\!\big(-\tfrac{d(x,y)}{t}\big)\ \exp\!\big(-\tfrac{d(y,z)}{s}\big) \leq \exp\!\big(-\tfrac{d(x,z)}{t+s}\big) = M(x,z,t+s),$$

using $d(x,z) \leq d(x,y) + d(y,z)$.

(M5) Continuity: for fixed x, y, the map $t \mapsto \exp(-d(x,y)/t)$ is continuous on $(0,\infty)$.

2.2 HyperFuzzy Metric Spaces

A hyperfuzzy metric space assigns to each pair in X and time $t > 0$ a nonempty subset of $[0,1]$, satisfying subset-valued positivity, symmetry, triangle property, and continuity under a continuous t-norm.

Definition 2.4 (Hyperfuzzy Metric Space)**.** Let X be a nonempty set, let $*$ be a continuous t-norm, and let

$$\widetilde{M} : X \times X \times (0,\infty) \longrightarrow \widetilde{\mathcal{P}}([0,1])$$

be a mapping into the collection of all nonempty subsets of $[0,1]$. For $x,y,z \in X$ and $s,t > 0$, write

$$\widetilde{M}(x,y,t) * \widetilde{M}(y,z,s) = \{\alpha * \beta \mid \alpha \in \widetilde{M}(x,y,t),\, \beta \in \widetilde{M}(y,z,s)\}.$$

We require:

(H1) $\inf \widetilde{M}(x,y,t) > 0$,

(H2) $\widetilde{M}(x,y,t) = \{1\}$ if and only if $x = y$,

(H3) $\widetilde{M}(x,y,t) = \widetilde{M}(y,x,t)$ (symmetry),

(H4) $\widetilde{M}(x,y,t) * \widetilde{M}(y,z,s) \subseteq \widetilde{M}(x,z,t+s)$,

(H5) for each fixed $x, y \in X$, the functions $t \mapsto \inf \widetilde{M}(x,y,t)$ and $t \mapsto \sup \widetilde{M}(x,y,t)$ are continuous on $(0,\infty)$.

Then $(X, \widetilde{M}, *)$ is called a *hyperfuzzy metric space*.

Example 2.5 (Discrete Hyperfuzzy Metric Space)**.** Let X be any nonempty set, fix a constant $\alpha \in (0,1)$, and let $*$ be the minimum t-norm:

$$a * b = \min\{a,b\}, \qquad a,b \in [0,1].$$

Define

$$\widetilde{M}(x,y,t) = \begin{cases} \{1\}, & x = y, \\ \{\alpha\}, & x \neq y, \end{cases}$$

for all $x, y \in X$ and $t > 0$. We verify the hyperfuzzy axioms:

(H1) $\inf \widetilde{M}(x, y, t) \in \{1, \alpha\} > 0$.

(H2) $\widetilde{M}(x, y, t) = \{1\} \iff x = y$.

(H3) Clearly $\widetilde{M}(x, y, t) = \widetilde{M}(y, x, t)$.

(H4) If $x \neq y$ and $y \neq z$, then

$$\widetilde{M}(x, y, t) * \widetilde{M}(y, z, s) = \{\min(\alpha, \alpha)\} = \{\alpha\} = \widetilde{M}(x, z, t + s),$$

and similar checks cover the cases when any two of x, y, z coincide.

(H5) For each fixed x, y, both $\inf \widetilde{M}(x, y, t)$ and $\sup \widetilde{M}(x, y, t)$ are constant in t, hence continuous on $(0, \infty)$.

Therefore $\left(X, \widetilde{M}, *\right)$ is a hyperfuzzy metric space.

Theorem 2.6 (Generalization of Fuzzy Metric Spaces)**.** *Every classical fuzzy metric space* $(X, M, *)$ *induces a hyperfuzzy metric space by*

$$\widetilde{M}(x, y, t) \;=\; \{\, M(x, y, t) \,\} \quad \left(x, y \in X,\ t > 0\right).$$

Proof. Define $\widetilde{M}(x, y, t) = \{M(x, y, t)\}$. Then:

- $\inf \widetilde{M}(x, y, t) = M(x, y, t) > 0$ by (M1).
- $\widetilde{M}(x, y, t) = \{1\} \iff M(x, y, t) = 1 \iff x = y$ by (M2).
- Symmetry and continuity carry over directly.
- $$\widetilde{M}(x, y, t) * \widetilde{M}(y, z, s) = \{M(x, y, t) * M(y, z, s)\} \subseteq \{M(x, z, t + s)\} = \widetilde{M}(x, z, t + s),$$
  by (M4).

All axioms (H1)–(H5) follow immediately from (M1)–(M5), so $(X, \widetilde{M}, *)$ is hyperfuzzy. □

Theorem 2.7 (Hyperfuzzy Set Structure)**.** *In a hyperfuzzy metric space* $(X, \widetilde{M}, *)$*, for each fixed* $y \in X$ *and* $t > 0$*, the mapping*

$$\mu_{y,t} \colon X \longrightarrow \widetilde{\mathcal{P}}([0, 1]), \qquad \mu_{y,t}(x) = \widetilde{M}(x, y, t)$$

defines a hyperfuzzy set on X*.*

Proof. By definition, $\mu_{y,t}(x) = \widetilde{M}(x, y, t)$ is a nonempty subset of $[0, 1]$ for each x. Hence $\mu_{y,t}$ is exactly a hyperfuzzy set-valued membership function on X, as required. □

2.3 SuperHyperFuzzy Metric Spaces

A superhyperfuzzy metric space maps each pair of m-th powerset elements of X together with time to a nonempty subset of the n-th powerset of $[0, 1]$, satisfying positivity, symmetry, triangle property, and continuity.

Definition 2.8 ((m, n)-SuperHyperFuzzy Metric Space)**.** Let X be a nonempty set, let $m, n \in \mathbb{N}_0$, and let $*$ be a continuous t-norm on $[0, 1]$. Denote by

$$\mathcal{P}^*_m(X) \;=\; \underbrace{\mathcal{P}^*\big(\mathcal{P}^*(\cdots \mathcal{P}^*}_{m \text{ times}}(X) \cdots)\big)$$

the m-th nonempty iterated powerset of X, and by

$$\widetilde{\mathcal{P}}^*_n([0, 1]) \;=\; \underbrace{\widetilde{\mathcal{P}}\big(\widetilde{\mathcal{P}}(\cdots \widetilde{\mathcal{P}}}_{n \text{ times}}([0, 1]) \cdots)\big)$$

the n-th nonempty iterated collection of subsets of $[0, 1]$. A mapping

$$\mathcal{M} :\; \mathcal{P}^*_m(X) \times \mathcal{P}^*_m(X) \times (0, \infty) \;\longrightarrow\; \widetilde{\mathcal{P}}^*_n([0, 1])$$

is called an *(m, n)-SuperHyperFuzzy metric* if, for all $A, B, C \in \mathcal{P}^*_m(X)$ and all $s, t > 0$, the following hold:

(SHM1) $\inf\{\inf \alpha \;\mid\; \alpha \in \mathcal{M}(A, B, t)\} > 0$,

(SHM2) Let $1^{[n]}$ denote the n-fold nested singleton of 1. Then $\mathcal{M}(A, B, t) = \{\, 1^{[n]}\}$ if and only if $A = B$,

(SHM3) $\mathcal{M}(A, B, t) = \mathcal{M}(B, A, t)$ (symmetry),

(SHM4) Defining $\mathcal{M}(A, B, t) * \mathcal{M}(B, C, s) = \{\alpha * \beta \mid \alpha \in \mathcal{M}(A, B, t),\ \beta \in \mathcal{M}(B, C, s)\}$, we have $\mathcal{M}(A, B, t) * \mathcal{M}(B, C, s) \subseteq \mathcal{M}(A, C, t + s)$,

(SHM5) For each fixed A, B, the maps

$$t \;\mapsto\; \inf\{\inf \alpha \mid \alpha \in \mathcal{M}(A, B, t)\}, \quad t \;\mapsto\; \sup\{\sup \alpha \mid \alpha \in \mathcal{M}(A, B, t)\}$$

are continuous on $(0, \infty)$.

The triple $(\mathcal{P}^*_m(X),\ \mathcal{M},\ *)$ is then called an (m, n)-*SuperHyperFuzzy metric space*.

Example 2.9 (Discrete (m, n)-SuperHyperFuzzy Metric Space)**.** Let X be any nonempty set and equip X with the discrete metric

$$d_0(x, y) = \begin{cases} 0, & x = y, \\ 1, & x \neq y. \end{cases}$$

For each $k = 1, 2, \ldots, m$, define the k-th Hausdorff metric

$$d_k(A, B) = \max\Big\{\sup_{a \in A} \inf_{b \in B} d_{k-1}(a, b),\ \sup_{b \in B} \inf_{a \in A} d_{k-1}(a, b)\Big\},$$

for $A, B \in \mathcal{P}^*_k(X)$. Then $d_m(A, B) \in \{0, 1\}$.

Fix $m, n \in \mathbb{N}_0$ and let $*$ be a continuous t-norm on $[0, 1]$. Recall $\mathcal{P}^*_m(X)$ is the nonempty m-th iterated powerset of X, and $\widetilde{\mathcal{P}}^*_n([0, 1])$ is the nonempty n-th iterated powerset of $[0, 1]$.

Define

$$\mathcal{M}(A, B, t) = \begin{cases} \{\, 1^{[n]}\}, & d_m(A, B) = 0, \\ \underbrace{\{\{\{\cdots\{\exp(-1/t)\}\cdots\}\}\}}_{n \text{ nestings}}, & d_m(A, B) = 1, \end{cases}$$

for all $A, B \in \mathcal{P}^*_m(X)$ and $t > 0$. Here $1^{[n]}$ denotes the n-fold nested singleton of 1.

We verify the axioms (SHM1)–(SHM5):

(SHM1) $\inf\{\inf \cdots \inf(\exp(-d_m(A, B)/t))\} = \exp(-d_m(A, B)/t) > 0$.

(SHM2) $\mathcal{M}(A, B, t) = \{1^{[n]}\} \iff d_m(A, B) = 0 \iff A = B$.

(SHM3) Symmetry follows since $d_m(A, B) = d_m(B, A)$.

(SHM4) By the Hausdorff triangle $d_m(A, C) \leq d_m(A, B) + d_m(B, C)$ and monotonicity of $*$,

$$\mathcal{M}(A, B, t) * \mathcal{M}(B, C, s) = \big\{\exp(-d_m(A, B)/t) * \exp(-d_m(B, C)/s)\big\}$$
$$\subseteq \{\exp(-d_m(A, C)/(t + s))\} = \mathcal{M}(A, C, t + s).$$

(SHM5) For fixed A, B, the nested map $t \mapsto \exp(-d_m(A, B)/t)$ is continuous on $(0, \infty)$, so its infimum and supremum (through n nestings) are continuous.

Thus $(\mathcal{P}^*_m(X),\ \mathcal{M},\ *)$ is an (m, n)-SuperHyperFuzzy metric space.

Example 2.10 (Hausdorff Interval (m, n)-SuperHyperFuzzy Metric Space)**.** Let $(\mathbb{R}, |\cdot|)$ be the real line with the usual metric, set $m = n = 1$, and let $*$ be any continuous t-norm on $[0, 1]$. Then

$$\mathcal{P}_1^*(\mathbb{R}) = \mathcal{P}^*(\mathbb{R}) \quad \text{and} \quad \widetilde{\mathcal{P}}_1^*([0,1]) = \{ S \subseteq [0,1] \mid S \neq \emptyset \}.$$

For nonempty $A, B \subseteq \mathbb{R}$, let $d_H(A, B)$ denote the Hausdorff distance. Define

$$\mathcal{M}(A, B, t) = \left[e^{-d_H(A,B)/t}, 1 \right] \subseteq [0, 1],$$

for all $t > 0$. Then $\mathcal{M}(A, B, t) \in \widetilde{\mathcal{P}}_1^*([0, 1])$. We check axioms (SHM1)–(SHM5):

(SHM1) $\inf\{e^{-d_H(A,B)/t}, 1\} = e^{-d_H(A,B)/t} > 0$.

(SHM2) $\mathcal{M}(A, B, t) = [1, 1] \iff d_H(A, B) = 0 \iff A = B$.

(SHM3) Symmetry: $d_H(A, B) = d_H(B, A)$ implies $\mathcal{M}(A, B, t) = \mathcal{M}(B, A, t)$.

(SHM4)

$$\begin{aligned}\mathcal{M}(A, B, t) * \mathcal{M}(B, C, s) &= \left[e^{-d_H(A,B)/t}, 1 \right] * \left[e^{-d_H(B,C)/s}, 1 \right] \\ &= \left[e^{-d_H(A,B)/t} * e^{-d_H(B,C)/s}, 1 \right] \subseteq \left[e^{-d_H(A,C)/(t+s)}, 1 \right] = \mathcal{M}(A, C, t+s),\end{aligned}$$

since $e^{-d_H(A,C)/(t+s)} \le e^{-d_H(A,B)/t} e^{-d_H(B,C)/s} \le e^{-d_H(A,B)/t} * e^{-d_H(B,C)/s}$.

(SHM5) For fixed A, B, the endpoints $e^{-d_H(A,B)/t}$ and 1 depend continuously on t, so both inf and sup of $\mathcal{M}(A, B, t)$ vary continuously on $(0, \infty)$.

Hence $\left(\mathcal{P}_1^*(\mathbb{R}), \mathcal{M}, *\right)$ is a $(1, 1)$-SuperHyperFuzzy metric space.

Example 2.11 (Compact Hausdorff $(2, 2)$-SuperHyperFuzzy Metric Space)**.** Let $X = \mathbb{R}$ with the usual metric $d(x, y) = |x - y|$. Denote by

$$\mathcal{K}(\mathbb{R}) = \{ A \subseteq \mathbb{R} \mid A \text{ is nonempty, compact}\}$$

and

$$\mathcal{K}_2(\mathbb{R}) = \{ \mathcal{A} \subseteq \mathcal{K}(\mathbb{R}) \mid \mathcal{A} \text{ is nonempty, compact in the Hausdorff metric}\}.$$

Here $\mathcal{K}(\mathbb{R})$ is $\mathcal{P}_1^*(X)$ and $\mathcal{K}_2(\mathbb{R})$ is $\mathcal{P}_2^*(X)$.

Let d_H be the Hausdorff distance on $\mathcal{K}(\mathbb{R})$ induced by d, and let $d_H^{(2)}$ be the Hausdorff distance on $\mathcal{K}_2(\mathbb{R})$ induced by d_H. Fix a continuous t-norm $*$ on $[0, 1]$. Define

$$\mathcal{M}(\mathcal{A}, \mathcal{B}, t) = \begin{cases} \{\{1\}\}, & d_H^{(2)}(\mathcal{A}, \mathcal{B}) = 0, \\ \{\{\exp(-d_H^{(2)}(\mathcal{A}, \mathcal{B})/t)\}\}, & d_H^{(2)}(\mathcal{A}, \mathcal{B}) > 0, \end{cases}$$

for all $\mathcal{A}, \mathcal{B} \in \mathcal{K}_2(\mathbb{R})$ and $t > 0$. Note that $\{\{1\}\} \in \widetilde{\mathcal{P}}_2^*([0, 1])$ is the double-nested singleton of 1, and likewise $\{\{\exp(-d_H^{(2)}/t)\}\}$ is a double-nested singleton of a value in $(0, 1)$.

We check the axioms (SHM1)–(SHM5):

(SHM1) $\inf\{\inf\{s\} \mid s \in \{\exp(-d_H^{(2)}(\mathcal{A}, \mathcal{B})/t)\}\} = \exp(-d_H^{(2)}(\mathcal{A}, \mathcal{B})/t) > 0$.

(SHM2) $\mathcal{M}(\mathcal{A}, \mathcal{B}, t) = \{\{1\}\} \iff d_H^{(2)}(\mathcal{A}, \mathcal{B}) = 0 \iff \mathcal{A} = \mathcal{B}$.

(SHM3) Symmetry follows since $d_H^{(2)}(\mathcal{A}, \mathcal{B}) = d_H^{(2)}(\mathcal{B}, \mathcal{A})$.

(SHM4) Using the Hausdorff triangle $d_H^{(2)}(\mathcal{A}, \mathcal{C}) \le d_H^{(2)}(\mathcal{A}, \mathcal{B}) + d_H^{(2)}(\mathcal{B}, \mathcal{C})$ and the monotonicity of $*$,

$$\begin{aligned}\mathcal{M}(\mathcal{A}, \mathcal{B}, t) * \mathcal{M}(\mathcal{B}, \mathcal{C}, s) &= \{\{\exp(-d_H^{(2)}(\mathcal{A}, \mathcal{B})/t) * \exp(-d_H^{(2)}(\mathcal{B}, \mathcal{C})/s)\}\} \\ &\subseteq \{\{\exp(-d_H^{(2)}(\mathcal{A}, \mathcal{C})/(t+s))\}\} = \mathcal{M}(\mathcal{A}, \mathcal{C}, t+s).\end{aligned}$$

(SHM5) For fixed $\mathcal{A}, \mathcal{B}$, the function $t \mapsto \exp(-d_H^{(2)}(\mathcal{A}, \mathcal{B})/t)$ is continuous on $(0, \infty)$, so its double-nested infimum and supremum vary continuously.

Therefore $\big(\mathcal{P}_2^*(X) = \mathcal{K}_2(\mathbb{R}),\ \mathcal{M},\ *\big)$ is a $(2, 2)$-SuperHyperFuzzy metric space.

Theorem 2.12 (Generalization of Fuzzy Metric Spaces)**.** *If $(X, M, *)$ is a fuzzy metric space, then setting $m = n = 0$ and*

$$\mathcal{M}\big(\{x\}, \{y\}, t\big) \;=\; \{\, M(x, y, t) \,\}$$

on $\mathcal{P}_0^*(X) = X$ yields an $(0, 0)$-SuperHyperFuzzy metric space.

Proof. For $m = n = 0$, $\mathcal{P}_0^*(X) = X$ and $\widetilde{\mathcal{P}}_0^*([0, 1]) = [0, 1]$. Defining $\mathcal{M}(x, y, t) = \{M(x, y, t)\}$ reduces axioms (SHM1)–(SHM5) to (M1)–(M5) of the fuzzy metric M. In particular:

- $\inf\{M(x, y, t)\} = M(x, y, t) > 0$.
- $\{M(x, y, t)\} = \{1\} \iff M(x, y, t) = 1 \iff x = y$.
- Symmetry and triangle inequalities carry over.
- Continuity of $t \mapsto M(x, y, t)$ gives continuity of the infimum and supremum.

Hence $\big(X, \mathcal{M}, *\big)$ is an $(0, 0)$-SuperHyperFuzzy metric space. □

Theorem 2.13 (Generalization of Hyperfuzzy Metric Spaces)**.** *If $(X, \widetilde{M}, *)$ is a hyperfuzzy metric space, then setting $m = 0$, $n = 1$ and*

$$\mathcal{M}(x, y, t) \;=\; \widetilde{M}(x, y, t)$$

on $\mathcal{P}_0^*(X) = X$ yields an $(0, 1)$-SuperHyperFuzzy metric space.

Proof. For $m = 0$, domain remains X. The target $\widetilde{\mathcal{P}}_1^*([0, 1])$ is exactly the collection of nonempty subsets of $[0, 1]$, so defining $\mathcal{M} = \widetilde{M}$ verifies (SHM1)–(SHM5) directly from the hyperfuzzy axioms (H1)–(H5). □

Theorem 2.14 ((m, n)-SuperHyperFuzzy Set Structure)**.** *In any (m, n)-SuperHyperFuzzy metric space $\big(\mathcal{P}_m^*(X), \mathcal{M}, *\big)$, for each fixed $B \in \mathcal{P}_m^*(X)$ and $t > 0$, the map*

$$\mu_{B,t} : \mathcal{P}_m^*(X) \longrightarrow \widetilde{\mathcal{P}}_n^*([0, 1]), \qquad \mu_{B,t}(A) = \mathcal{M}(A, B, t)$$

defines an (m, n)-SuperHyperFuzzy set on $\mathcal{P}_m^*(X)$.

Proof. By definition, $\mathcal{M}(A, B, t)$ is always a nonempty element of $\widetilde{\mathcal{P}}_n^*([0, 1])$. Therefore $\mu_{B,t}$ is exactly a membership assignment $\mathcal{P}_m^*(X) \to \widetilde{\mathcal{P}}_n^*([0, 1])$, i.e. an (m, n)-SuperHyperFuzzy set. □

Definition 2.15 (Open Balls)**.** Let $\big(\mathcal{P}_m^*(X), \mathcal{M}, *\big)$ be an (m, n)-SuperHyperFuzzy metric space. For $A \in \mathcal{P}_m^*(X)$, $t > 0$, and $\varepsilon \in (0, 1)$, define the *open ball*

$$B_{\mathcal{M}}(A, t, \varepsilon) = \big\{\, B \in \mathcal{P}_m^*(X) \;\big|\; \inf\!\big\{\inf \alpha \mid \alpha \in \mathcal{M}(A, B, t)\big\} > 1 - \varepsilon \big\}.$$

Example 2.16 (Open Balls in Smart Building Sensor Clusters)**.** Let X be the set of temperature sensors deployed throughout a smart building, and consider the $(1, 1)$-SuperHyperFuzzy metric space $\big(\mathcal{P}_1^*(X), \mathcal{M}, *\big)$ with

$$\mathcal{M}(A, B, t) = \big[\exp\!\big(-d_H(A, B)/t\big),\ 1\big],$$

where $d_H(A, B)$ is the Hausdorff distance between the reading sets of clusters $A, B \subseteq X$, and $*$ is the product t-norm. For a reference cluster A, time parameter $t > 0$, and tolerance $\varepsilon \in (0, 1)$, the open ball

$$B_{\mathcal{M}}(A, t, \varepsilon) = \big\{\, B \subseteq X \;\big|\; \inf\{\alpha \mid \alpha \in \mathcal{M}(A, B, t)\} > 1 - \varepsilon \big\}$$

consists of all sensor clusters B whose worst-case similarity to A exceeds $1 - \varepsilon$. Equivalently, these are exactly those B with

$$d_H(A, B) \;<\; -t \ln(1 - \varepsilon),$$

i.e. every sensor in B differs from some sensor in A by at most $-t \ln(1 - \varepsilon)$ degrees.

Definition 2.17 (Convergence and Cauchy Sequences)**.** A sequence $\{A_k\} \subseteq \mathcal{P}_m^*(X)$ *converges* to A if for every $\varepsilon > 0$ and $t > 0$ there exists N such that for all $k \geq N$,

$$\inf\Big\{\inf \alpha \mid \alpha \in \mathcal{M}(A_k, A, t)\Big\} > 1 - \varepsilon.$$

It is *Cauchy* if for every $\varepsilon > 0$ and $t > 0$ there exists N such that for all $p, q \geq N$,

$$\inf\Big\{\inf \alpha \mid \alpha \in \mathcal{M}(A_p, A_q, t)\Big\} > 1 - \varepsilon.$$

Example 2.18 (Convergence and Cauchy Sequences of Clusterings)**.** In the same setting, suppose each day k we form a cluster $A_k \subseteq X$ by grouping sensors whose pairwise temperature deviations are below a threshold $\delta_k \to 0$. Then:

- *Convergence*: $\{A_k\}$ *converges* to a stable cluster A if for every $\varepsilon > 0$ and $t > 0$, there exists N such that for all $k \geq N$,
$$\inf\{\alpha \mid \alpha \in \mathcal{M}(A_k, A, t)\} = \exp\big(-d_H(A_k, A)/t\big) > 1 - \varepsilon,$$
equivalently $d_H(A_k, A) < -t \ln(1 - \varepsilon)$. In practice, after enough days, the daily cluster matches the stable cluster within any desired accuracy.
- *Cauchy*: $\{A_k\}$ is *Cauchy* if for every $\varepsilon > 0$ and $t > 0$ there exists N such that for all $p, q \geq N$,
$$\exp\big(-d_H(A_p, A_q)/t\big) > 1 - \varepsilon,$$
i.e. $d_H(A_p, A_q) < -t \ln(1 - \varepsilon)$. This means that, beyond some day, any two clusterings are arbitrarily similar, reflecting stabilization of the grouping process.

Theorem 2.19 (First–Countability and Hausdorffness)**.** *The family* $\{B_{\mathcal{M}}(A, 1/k, 1/k) \mid k \in \mathbb{N}\}$ *is a neighborhood basis at A, and the induced topology is Hausdorff.*

Proof. By (SHM5), $t \mapsto \inf\{\inf \alpha : \alpha \in \mathcal{M}(A, B, t)\}$ is continuous. Hence for each A the balls $B_{\mathcal{M}}(A, 1/k, 1/k)$ shrink to $\{A\}$, giving first-countability. If $A \neq B$, then by (SHM2) there is t_0 with $\inf\{\inf \alpha : \alpha \in \mathcal{M}(A, B, t_0)\} < 1$. Choose $\varepsilon < 1 - \inf \inf \mathcal{M}(A, B, t_0)$. Then

$$B_{\mathcal{M}}\big(A, t_0, \varepsilon\big) \quad \text{and} \quad B_{\mathcal{M}}\big(B, t_0, \varepsilon\big)$$

are disjoint, since if C lay in both, one would have $\inf \inf \mathcal{M}(A, C, t_0) > 1 - \varepsilon$ and $\inf \inf \mathcal{M}(C, B, t_0) > 1 - \varepsilon$, so by (SHM4)

$$\inf \inf \mathcal{M}(A, B, 2t_0) \geq (1 - \varepsilon) * (1 - \varepsilon) > \inf \inf \mathcal{M}(A, B, 2t_0),$$

a contradiction. Thus the topology is Hausdorff. □

Theorem 2.20 (Uniqueness of Limits)**.** *In an* (m, n)*-SuperHyperFuzzy metric space, any convergent sequence has a unique limit.*

Proof. Suppose $A_k \to A$ and $A_k \to B$. Fix $t > 0$. By convergence there is N so that for all $k \geq N$,

$$\inf \inf \mathcal{M}(A_k, A, t/2) > 1 - \tfrac{\varepsilon}{2}, \quad \inf \inf \mathcal{M}(A_k, B, t/2) > 1 - \tfrac{\varepsilon}{2}.$$

Then by (SHM4),

$$\inf \inf \mathcal{M}(A, B, t) \;\geq\; (1 - \tfrac{\varepsilon}{2}) * (1 - \tfrac{\varepsilon}{2}) > 1 - \varepsilon.$$

Since ε was arbitrary, $\inf \inf \mathcal{M}(A, B, t) = 1$. By (SHM2) this implies $A = B$. □

Theorem 2.21 (Isometric Embedding of the Base Space)**.** *If* $(X, M, *)$ *is a fuzzy metric space, then the map*

$$i \colon X \longrightarrow \mathcal{P}_m^*(X), \qquad i(x) = \underbrace{\{\{\cdots\{x\}\cdots\}\}}_{m \text{ nestings}}$$

is an isometric embedding of $(X, M, *)$ *into* $\big(\mathcal{P}_m^*(X), \mathcal{M}, *\big)$ *with* $\mathcal{M}\big(i(x), i(y), t\big) = \{M(x, y, t)\}$.

Proof. By definition $\mathcal{M}(i(x), i(y), t) = \{M(x, y, t)\}$. Then (SHM1)–(SHM5) reduce to (M1)–(M5), so i preserves all fuzzy-metric relations. Hence i is an isometry. □

3 Result: Fuzzy Queue

3.1 Fuzzy Queue

Classical queue models describe systems of waiting lines using probabilistic processes like M/M/1, M/M/c, capturing arrival, service, and queue behaviors (cf. [63–65]). A fuzzy queue models arrival and service rates as fuzzy numbers, enabling ambiguity representation and imprecise performance evaluation in systems (cf. [66–69]). This section gives a mathematically precise definition of fuzzy queues.

Definition 3.1 (Fuzzy Queue)**.** (cf. [67, 70, 71]) Let $Q = (\mathcal{S}, \Lambda, \mu, s, K, d)$ be a classical queueing system, where

- $\mathcal{S}$ is the state space,
- $\Lambda \in \mathrm{R}_{>0}$ is the (crisp) arrival rate,
- $\mu \in \mathrm{R}_{>0}$ is the (crisp) service rate,
- $s \in \mathbb{N}$ is the number of servers,
- $K \in \mathbb{N} \cup \{\infty\}$ is the system capacity, and
- d is the service discipline.

A *fuzzy queue* is the system

$$\tilde{Q} = \big(\mathcal{S},\ \tilde{\Lambda},\ \tilde{\mu},\ s,\ K,\ d\big)$$

obtained by replacing the crisp parameters Λ and μ with fuzzy numbers

$$\tilde{\Lambda} : \mathrm{R}_{>0} \to [0, 1], \quad \tilde{\mu} : \mathrm{R}_{>0} \to [0, 1],$$

whose membership functions $\mu_{\tilde{\Lambda}}(\lambda)$ and $\mu_{\tilde{\mu}}(\mu)$ encode the possibility distributions of the true arrival and service rates. Every performance measure $X(\Lambda, \mu)$ of the crisp system—such as steady-state probabilities, mean queue length, or mean waiting time—induces a fuzzy performance measure $\tilde{X}$ whose membership function is given by Zadeh's extension principle:

$$\mu_{\tilde{X}}(x) = \sup\big\{\min\big(\mu_{\tilde{\Lambda}}(\lambda),\ \mu_{\tilde{\mu}}(\mu)\big) : X(\lambda, \mu) = x\big\}.$$

Example 3.2 (Call Center as a Fuzzy Queue)**.** Consider a small call center modeled by an $M/M/1$ queue with infinite capacity and FIFO discipline:

$$Q = \big(S, \Lambda, \mu, 1, \infty, \mathrm{FIFO}\big), \qquad S = \{0, 1, 2, \dots\}.$$

Empirical data suggest the true arrival rate is around 15 calls/hour but uncertain. We model it by the triangular fuzzy number

$$\tilde{\Lambda} : \mu_{\tilde{\Lambda}}(\lambda) = \begin{cases} \dfrac{\lambda - 10}{15 - 10}, & 10 \le \lambda \le 15, \\ \dfrac{20 - \lambda}{20 - 15}, & 15 \le \lambda \le 20, \\ 0, & \text{otherwise}, \end{cases}$$

peaking at $\lambda = 15$. Similarly, the average service rate is about 6 calls/hour, represented by

$$\tilde{\mu} : \mu_{\tilde{\mu}}(\mu) = \begin{cases} \dfrac{\mu - 5}{6 - 5}, & 5 \le \mu \le 6, \\ \dfrac{8 - \mu}{8 - 6}, & 6 \le \mu \le 8, \\ 0, & \text{otherwise}. \end{cases}$$

Thus the fuzzy queue is

$$\tilde{Q} = \big(S,\ \tilde{\Lambda},\ \tilde{\mu},\ 1,\ \infty,\ \mathrm{FIFO}\big).$$

A key performance measure is the mean number in system:

$$L(\lambda, \mu) = \frac{\lambda}{\mu - \lambda}, \quad \lambda < \mu.$$

By Zadeh's extension principle, the fuzzy performance measure $\tilde{L}$ has membership

$$\mu_{\tilde{L}}(x) = \sup\Big\{\min\big(\mu_{\tilde{\Lambda}}(\lambda),\ \mu_{\tilde{\mu}}(\mu)\big) \ \Big|\ \frac{\lambda}{\mu - \lambda} = x\Big\}.$$

This $\tilde{L}$ assigns to each possible mean queue length x the set of plausibility degrees arising from all (λ, μ) yielding $L(\lambda, \mu) = x$.

3.2 HyperFuzzy Queue

A HyperFuzzy Queue generalizes classical queueing systems by modeling arrival and service rates as nonempty subsets of $[0, 1]$, capturing imprecision.

Definition 3.3 (Hyperfuzzy Queue)**.** Let

$$Q = (\mathcal{S}, \Lambda, \mu, s, K, d)$$

be a classical queueing system with state space $\mathcal{S}$, crisp arrival rate Λ, crisp service rate μ, s servers, capacity K, and discipline d. A *hyperfuzzy queue* is the tuple

$$\tilde{Q} = \big(\mathcal{S},\ \tilde{\Lambda},\ \tilde{\mu},\ s,\ K,\ d\big),$$

where

$$\tilde{\Lambda} : \mathbb{R}_{>0} \longrightarrow \widetilde{\mathcal{P}}([0, 1]), \qquad \tilde{\mu} : \mathbb{R}_{>0} \longrightarrow \widetilde{\mathcal{P}}([0, 1])$$

assign to each positive real λ (resp. μ) a nonempty set $\tilde{\Lambda}(\lambda) \subseteq [0, 1]$ (resp. $\tilde{\mu}(\mu) \subseteq [0, 1]$) of possible membership degrees for the arrival (resp. service) rate.

Moreover, if $X(\lambda, \mu)$ is any performance measure of the crisp system (e.g. steady-state probability, mean queue length, mean waiting time), the associated *hyperfuzzy performance measure*

$$\tilde{X} \ :\ \mathrm{Range}(X) \longrightarrow \widetilde{\mathcal{P}}([0, 1])$$

is defined by the generalized extension principle:

$$\tilde{X}(x) = \big\{\min(\alpha, \beta) \ \big|\ X(\lambda, \mu) = x,\ \alpha \in \tilde{\Lambda}(\lambda),\ \beta \in \tilde{\mu}(\mu)\big\}.$$

Example 3.4 (Customer Support Hyperfuzzy Queue)**.** Consider a single-server customer support hotline modeled by an $M/M/1$ queue with FIFO discipline and infinite capacity:

$$Q = \big(S,\ \Lambda,\ \mu,\ 1,\ \infty,\ \mathrm{FIFO}\big), \qquad S = \{0, 1, 2, \ldots\}.$$

Due to varying call-volume patterns and measurement uncertainty, a team of analysts provides *set*-valued possibility degrees for the arrival and service rates. For instance:

$$\tilde{\Lambda}(18 \text{ calls/hour}) = \{0.70,\ 0.75,\ 0.80\}, \quad \tilde{\Lambda}(20) = \{0.60,\ 0.65,\ 0.70\},$$

$$\tilde{\mu}(6 \text{ calls/hour}) = \{0.80,\ 0.85,\ 0.90\}, \quad \tilde{\mu}(8) = \{0.70,\ 0.75,\ 0.80\}.$$

Thus the hyperfuzzy queue is

$$\tilde{Q} = \big(S,\ \tilde{\Lambda},\ \tilde{\mu},\ 1,\ \infty,\ \mathrm{FIFO}\big).$$

A key performance metric is the mean number in system,

$$L(\lambda, \mu) = \frac{\lambda}{\mu - \lambda}, \quad \lambda < \mu.$$

Applying the generalized extension principle yields the *hyperfuzzy performance measure*

$$\tilde{L}(x) = \Big\{\min(\alpha, \beta) \ \Big|\ \frac{\lambda}{\mu - \lambda} = x,\ \alpha \in \tilde{\Lambda}(\lambda),\ \beta \in \tilde{\mu}(\mu)\Big\},$$

which assigns to each possible value x the set of plausibility degrees arising from all (λ, μ) combinations that produce $L(\lambda, \mu) = x$.

Example 3.5 (Emergency Department Hyperfuzzy Queue)**.** Model a hospital emergency department as an $M/M/2$ queue with FIFO service and unlimited waiting room:

$$Q = \big(S,\ \Lambda,\ \mu,\ 2,\ \infty,\ \text{FIFO}\big), \qquad S = \{0, 1, 2, \dots\}.$$

Staffing and patient-arrival data are uncertain, so analysts provide sets of possibility degrees for rates:

Arrival rates (patients/hour):

$$\tilde{\Lambda}(10) = \{0.70,\ 0.75,\ 0.80\}, \qquad \tilde{\Lambda}(12) = \{0.60,\ 0.65,\ 0.70\}.$$

Service rates (patients/hour per doctor):

$$\tilde{\mu}(5) = \{0.80,\ 0.85,\ 0.90\}, \qquad \tilde{\mu}(6) = \{0.75,\ 0.80,\ 0.85\}.$$

Hence the hyperfuzzy queue is

$$\tilde{Q} = \big(S,\ \tilde{\Lambda},\ \tilde{\mu},\ 2,\ \infty,\ \text{FIFO}\big).$$

A useful performance metric is the *server utilization*

$$\rho(\lambda, \mu) = \frac{\lambda}{2\,\mu}.$$

Applying the hyperfuzzy extension principle, the hyperfuzzy utilization measure

$$\tilde{\rho}(x) = \Big\{\min(\alpha, \beta)\ \Big|\ \frac{\lambda}{2\mu} = x,\ \alpha \in \tilde{\Lambda}(\lambda),\ \beta \in \tilde{\mu}(\mu)\Big\}$$

assigns to each utilization level x the set of plausibility degrees derived from all (λ, μ) combinations yielding $\rho = x$.

Theorem 3.6 (Generalization of Fuzzy Queue)**.** *Every fuzzy queue*

$$Q_f = (\mathcal{S},\ \tilde{\Lambda}_f,\ \tilde{\mu}_f,\ s,\ K,\ d)$$

with singleton-valued fuzzy rates $\tilde{\Lambda}_f(\lambda) = \{\mu_{\tilde{\Lambda}}(\lambda)\}$*,* $\tilde{\mu}_f(\mu) = \{\mu_{\tilde{\mu}}(\mu)\}$ *is a special case of a hyperfuzzy queue.*

Proof. If each $\tilde{\Lambda}(\lambda)$ and $\tilde{\mu}(\mu)$ is a singleton, then the hyperfuzzy queue $\tilde{Q}$ reduces to

$$\big(\mathcal{S}, \{\mu_{\tilde{\Lambda}}(\lambda)\}, \{\mu_{\tilde{\mu}}(\mu)\}, s, K, d\big),$$

which coincides exactly with the fuzzy queue Q_f. Likewise, the hyperfuzzy performance measure $\tilde{X}(x) = \{\min(\mu_{\tilde{\Lambda}}(\lambda), \mu_{\tilde{\mu}}(\mu)) : X(\lambda, \mu) = x\}$ collapses to the singleton-valued fuzzy performance measure by the same substitution. Hence fuzzy queues embed into hyperfuzzy queues. □

Theorem 3.7 (Hyperfuzzy Set Structure of Hyperfuzzy Queue)**.** *For a hyperfuzzy queue* $\tilde{Q}$ *and any performance measure* X*, the mapping*

$$\tilde{X}\ :\ \text{Range}(X)\ \longrightarrow\ \widetilde{\mathcal{P}}([0,1]), \qquad \tilde{X}(x) = \{\min(\alpha, \beta) \mid X(\lambda, \mu) = x,\ \alpha \in \tilde{\Lambda}(\lambda),\ \beta \in \tilde{\mu}(\mu)\}$$

defines a hyperfuzzy set on Range(X).

Proof. By construction, for each $x \in \text{Range}(X)$, $\tilde{X}(x)$ is a nonempty subset of $[0, 1]$. Thus $\tilde{X}$ satisfies the definition of a hyperfuzzy set, assigning to each element x a collection of plausible membership degrees in $[0, 1]$. □

3.3 SuperHyperFuzzy Queue

A SuperHyperFuzzy Queue further extends HyperFuzzy Queues by assigning hierarchical, multi-level set-valued membership functions to arrival and service rate subsets.

Definition 3.8 ((m, n)-SuperHyperFuzzy Queue)**.** Let

$$Q = (\mathcal{S},\ \Lambda,\ \mu,\ s,\ K,\ d)$$

be a classical queueing system with state space $\mathcal{S}$, arrival rate $\Lambda > 0$, service rate $\mu > 0$, s servers, capacity K, and discipline d. Fix $m, n \in \mathbb{N}_0$. Denote by

$$\mathcal{P}^*_m(\mathbb{R}_{>0}) \quad \text{and} \quad \widetilde{\mathcal{P}}^*_n([0,1])$$

the m-th nonempty iterated powerset of $\mathbb{R}_{>0}$ and the n-th nonempty iterated powerset of $[0, 1]$, respectively. An (m, n)-*SuperHyperFuzzy queue* is the tuple

$$\tilde{Q} = \big(\mathcal{S},\ \tilde{\Lambda}_{m,n},\ \tilde{\mu}_{m,n},\ s,\ K,\ d\big),$$

where

$$\tilde{\Lambda}_{m,n} \colon \mathcal{P}^*_m(\mathbb{R}_{>0}) \longrightarrow \widetilde{\mathcal{P}}^*_n([0,1]), \qquad \tilde{\mu}_{m,n} \colon \mathcal{P}^*_m(\mathbb{R}_{>0}) \longrightarrow \widetilde{\mathcal{P}}^*_n([0,1]).$$

Here $\tilde{\Lambda}_{m,n}(A) \subseteq \mathcal{P}_n([0, 1])$ and $\tilde{\mu}_{m,n}(A) \subseteq \mathcal{P}_n([0, 1])$ encode the hierarchical uncertainty of the arrival and service rates drawn from any nonempty $A \subseteq \mathbb{R}_{>0}$ at level m.

Example 3.9 (Regional Call Center Network $(1, 2)$-SuperHyperFuzzy Queue)**.** Consider two regional call centers, A and B, serving customer calls. We model the system as a single-server FIFO queue with infinite capacity:

$$Q = \big(S, \Lambda, \mu, 1, \infty, \text{FIFO}\big), \quad S = \{0, 1, 2, \dots\}.$$

Due to daily variation and expert uncertainty, we group possible arrival rates into the set

$$A = \{\, \lambda_A,\ \lambda_B\} = \{12,\ 15\} \quad \text{(calls per hour)}.$$

At the first super-level ($m = 1$), we assign two expert groups' fuzzy assessments of these rates:

$$M_1 = \{0.80,\ 0.85\}, \quad M_2 = \{0.75,\ 0.80\},$$

so that

$$\widetilde{\Lambda}_{1,2}(A) = \{\, M_1,\ M_2\} \ \subseteq\ \widetilde{\mathcal{P}}^*_2([0,1]).$$

Here M_1 reflects the morning-shift analysts' plausibilities, and M_2 reflects the evening-shift analysts' plausibilities.

Similarly, suppose service rates from two server tiers are grouped as

$$B = \{\mu_1, \mu_2\} = \{6,\ 8\} \quad \text{(calls per hour)}.$$

Expert teams provide first-level sets

$$N_1 = \{0.85,\ 0.90\}, \quad N_2 = \{0.80,\ 0.85\},$$

giving

$$\widetilde{\mu}_{1,2}(B) = \{\, N_1,\ N_2\} \ \subseteq\ \widetilde{\mathcal{P}}^*_2([0,1]).$$

Thus the $(1, 2)$-SuperHyperFuzzy queue is

$$\tilde{Q} = \big(S,\ \widetilde{\Lambda}_{1,2},\ \widetilde{\mu}_{1,2},\ 1,\ \infty,\ \text{FIFO}\big),$$

which captures hierarchical uncertainty: the set A of possible arrival rates is mapped to a collection of two membership-degree sets, and likewise for the service rates in B.

Example 3.10 (Supermarket Checkout $(2,2)$-SuperHyperFuzzy Queue)**.** Consider a supermarket with two checkout lanes modeled as an $M/M/2$ FIFO queue with unlimited waiting:

$$Q = (S,\ \Lambda,\ \mu,\ 2,\ \infty,\ \text{FIFO}), \qquad S = \{0, 1, 2, \dots\}.$$

Customer arrival rates vary by time and day, so we form two levels of grouping:

Level-1 arrival groups:

$$A_1 = \{10,\ 11,\ 12\} \quad \text{(weekday mornings)}, \qquad A_2 = \{15,\ 16,\ 18\} \quad \text{(weekend afternoons)}.$$

Together,

$$A = \{\, A_1,\ A_2\} \ \in\ \mathcal{P}_2^*(\mathbb{R}_{>0}).$$

Experts provide two sets of possibility degrees for each group:

$$M_{\text{wkday}} = \{\{0.80, 0.85\},\ \{0.75, 0.80\}\}, \quad M_{\text{wkend}} = \{\{0.70, 0.75\},\ \{0.65, 0.70\}\}.$$

Thus

$$\widetilde{\Lambda}_{2,2}(A) = \{\, M_{\text{wkday}},\ M_{\text{wkend}}\} \ \subseteq\ \widetilde{\mathcal{P}}_2^*([0, 1]).$$

Level-1 service groups:

$$B_1 = \{20,\ 22\} \quad \text{(experienced cashiers)}, \qquad B_2 = \{18,\ 21\} \quad \text{(trainee cashiers)},$$

so

$$B = \{\, B_1,\ B_2\} \ \in\ \mathcal{P}_2^*(\mathbb{R}_{>0}).$$

Their membership sets are

$$N_{\text{exp}} = \{\{0.90, 0.92\},\ \{0.88, 0.90\}\}, \quad N_{\text{trainee}} = \{\{0.85, 0.87\},\ \{0.82, 0.85\}\}.$$

Hence

$$\widetilde{\mu}_{2,2}(B) = \{\, N_{\text{exp}},\ N_{\text{trainee}}\} \ \subseteq\ \widetilde{\mathcal{P}}_2^*([0, 1]).$$

The resulting $(2,2)$-SuperHyperFuzzy queue is

$$\tilde{Q} = (S,\ \widetilde{\Lambda}_{2,2},\ \widetilde{\mu}_{2,2},\ 2,\ \infty,\ \text{FIFO}),$$

which captures hierarchical uncertainty in both arrival and service rates across two structural levels ($m = 2$) and two valuation levels ($n = 2$).

Theorem 3.11 (Reduction to Fuzzy and Hyperfuzzy Queues)**.**

(a) If $m = n = 0$ and $\tilde{\Lambda}_{0,0}(r) = \{\mu_{\tilde{\Lambda}}(r)\}$, $\tilde{\mu}_{0,0}(r) = \{\mu_{\tilde{\mu}}(r)\}$ are singletons, then $\tilde{Q}$ is precisely a fuzzy queue.

(b) If $m = 0, n = 1$, then $\tilde{Q}$ is a hyperfuzzy queue, since each $\tilde{\Lambda}_{0,1}(r) \subseteq [0, 1]$ and $\tilde{\mu}_{0,1}(r) \subseteq [0, 1]$.

Proof. (a) For $m = n = 0$, one has $\mathcal{P}_0^*(\mathbb{R}_{>0}) = \mathbb{R}_{>0}$ and $\widetilde{\mathcal{P}}_0^*([0, 1]) = [0, 1]$. If each image $\tilde{\Lambda}_{0,0}(r)$ and $\tilde{\mu}_{0,0}(r)$ is a singleton $\{\mu_{\tilde{\Lambda}}(r)\}$, $\{\mu_{\tilde{\mu}}(r)\}$, then $\tilde{Q}$ coincides with the fuzzy queue $(\mathcal{S}, \tilde{\Lambda}_f, \tilde{\mu}_f, s, K, d)$.

(b) For $m = 0, n = 1$, the domain remains $\mathbb{R}_{>0}$ but the codomain is all nonempty subsets of $[0, 1]$, exactly matching the definition of a hyperfuzzy queue. □

Theorem 3.12 (SuperHyperFuzzy Set Structure of Performance Measures)**.** *Let $X(\lambda, \mu)$ be any real-valued performance measure of the crisp queue Q. Then the induced mapping*

$$\tilde{X} : \text{Range}(X) \longrightarrow \widetilde{\mathcal{P}}_n^*([0,1]),$$

defined by the multi-level extension principle

$$\tilde{X}(x) = \Big\{\min(\alpha, \beta) \;\Big|\; X(\lambda, \mu) = x,\ \alpha \in \tilde{\Lambda}_{m,n}(A),\ \beta \in \tilde{\mu}_{m,n}(A)\Big\},$$

for any representative $A \in \mathcal{P}_m^*(\mathbb{R}_{>0})$ containing λ and μ, is an (m,n)-SuperHyperFuzzy set on $\text{Range}(X)$.

Proof. By definition, for each $x \in \text{Range}(X)$, $\tilde{X}(x)$ is a nonempty element of $\widetilde{\mathcal{P}}_n^*([0,1])$. Hence $\tilde{X}$ assigns to each x a family of n-th level membership-degree sets, satisfying precisely the conditions of an (m,n)-SuperHyperFuzzy set. □

4 Result: HyperFuzzy Ontology

4.1 Fuzzy Ontology

A classical ontology formally defines domain concepts, relationships, and constraints using crisp logic, enabling structured, automated knowledge representation and reasoning (cf. [72–74]). A fuzzy ontology defines domain concepts, attributes, and relationships with membership degrees in $[0,1]$, representing vague semantic knowledge [75–77].

Definition 4.1 (Fuzzy Ontology)**.** (cf. [78–80]) A *fuzzy ontology* is a quadruple

$$FO = (C,\ AC,\ R,\ X),$$

where:

- $C = \{c_1, \ldots, c_k\}$ is a finite set of *concepts*;
- $AC = \{\, AC(c) \mid c \in C\}$ assigns to each concept c a set of *attributes*;
- $R = (R_T, R_N)$ is a pair of fuzzy relations on C:

  $$R_T \subseteq C \times C, \qquad R_N \subseteq C \times C,$$

  where R_T contains *taxonomy* (hierarchical) relations and R_N contains *nontaxonomy* relations;
- X is a set of *axioms*, each axiom being a fuzzy constraint on attribute values or on relationships, expressible in SWRL.

For each concept $c \in C$, let $\text{Inst}(c)$ denote its set of *instances*. Every instance

$$o \in \text{Inst}(c)$$

is characterized by a fuzzy attribute–value mapping

$$\mathbf{a}_o : AC(c) \longrightarrow [0,1],$$

where $\mathbf{a}_o(a)$ is the membership degree of o in attribute a. Similarly, each relation

$$r(c_p, c_q) \in R$$

is interpreted as a binary fuzzy relation between instances of c_p and c_q, with membership degrees in $[0,1]$.

Example 4.2 (Medical Diagnosis Fuzzy Ontology)**.** (cf. [81, 82]) Applying Definition 4.1 to a simple medical domain, let

$$C = \{\text{Disease, Symptom}\},$$

with attribute sets

$$AC(\text{Disease}) = \{\text{Infectiousness, Severity}\}, \quad AC(\text{Symptom}) = \{\text{Frequency, Intensity}\}.$$

Define the taxonomy and nontaxonomy relations by

$$R_T = \{(\text{Influenza}, \text{ViralInfection}),\ (\text{ViralInfection}, \text{Infection})\}, \quad R_N = \{\text{hasSymptom}\}.$$

Instances are

$$\text{Inst}(\text{Disease}) = \{\text{influenza, common\_cold}\}, \quad \text{Inst}(\text{Symptom}) = \{\text{cough, fever}\}.$$

Fuzzy attribute memberships:

$$\mathbf{a}_{\text{influenza}}(\text{Infectiousness}) = 0.9, \quad \mathbf{a}_{\text{influenza}}(\text{Severity}) = 0.7,$$

$$\mathbf{a}_{\text{cough}}(\text{Frequency}) = 0.8, \quad \mathbf{a}_{\text{cough}}(\text{Intensity}) = 0.6.$$

Fuzzy relation memberships for hasSymptom:

$$\mu_{\text{hasSymptom}}(\text{influenza}, \text{cough}) = 0.85, \quad \mu_{\text{hasSymptom}}(\text{common\_cold}, \text{cough}) = 0.6.$$

A sample axiom in X (in SWRL style):

$$\text{Severity}(d, s) \ \wedge \ s > 0.8 \implies \text{requiresHospitalization}(d) \text{ with degree } 1.$$

This ontology captures graded concept attributes, hierarchical concept relations, and fuzzy inter-concept relations in a medical diagnosis setting.

4.2 HyperFuzzy Ontology

A hyperfuzzy ontology extends fuzzy ontologies by assigning each concept, attribute, or relation a set of possible membership degrees, capturing imprecision.

Definition 4.3 (HyperFuzzy Ontology)**.** Let

$$FO = (C,\ AC,\ R,\ X,\ \text{Inst},\ \mu_A,\ \mu_R)$$

be a fuzzy ontology, where

- C is a finite set of concepts;
- $AC(c)$ is the finite set of attributes of $c \in C$;
- $R \subseteq C \times C$ is a finite set of fuzzy relations;
- X is a set of fuzzy axioms;
- $\text{Inst}(c)$ is the finite set of instances of c;
- $\mu_A \colon \{(o, a) \mid o \in \text{Inst}(c),\ a \in AC(c)\} \to [0, 1]$ assigns each instance–attribute a membership degree;
- $\mu_R \colon \{(r, o_p, o_q) \mid r \in R(c_p, c_q),\ o_p \in \text{Inst}(c_p),\ o_q \in \text{Inst}(c_q)\} \to [0, 1]$ assigns each relation instance a membership degree.

A *HyperFuzzy Ontology* is the tuple

$$HFO = (C,\ AC,\ R,\ X,\ \text{Inst},\ \widetilde{\mu}_A,\ \widetilde{\mu}_R),$$

where

$$\widetilde{\mu}_A \colon \{(o, a) \mid o \in \text{Inst}(c),\ a \in AC(c)\} \longrightarrow \widetilde{\mathcal{P}}([0, 1]),$$

$$\widetilde{\mu}_R \colon \{(r, o_p, o_q) \mid r \in R(c_p, c_q),\ o_p \in \text{Inst}(c_p),\ o_q \in \text{Inst}(c_q)\} \longrightarrow \widetilde{\mathcal{P}}([0, 1]),$$

assign to each instance–attribute or relation–instance a nonempty subset of $[0, 1]$, capturing imprecision and variability in membership.

Example 4.4 (Medical Diagnosis HyperFuzzy Ontology)**.** Building on the fuzzy ontology of medical diagnosis, we allow multiple expert evaluations per attribute and relation. Let

$$C = \{\text{Disease, Symptom}\}, \quad AC(\text{Disease}) = \{\text{Infectiousness, Severity}\}, \quad AC(\text{Symptom}) = \{\text{Frequency, Intensity}\},$$

$$\text{Inst}(\text{Disease}) = \{\text{influenza, common\_cold}\}, \quad \text{Inst}(\text{Symptom}) = \{\text{cough, fever}\},$$

$$R = \{\text{hasSymptom}\}.$$

Expert judgments yield set-valued membership for each instance–attribute:

$$\widetilde{\mu}_A(\text{influenza, Infectiousness}) = \{0.85,\ 0.88,\ 0.90\}, \quad \widetilde{\mu}_A(\text{influenza, Severity}) = \{0.70,\ 0.75,\ 0.72\},$$

$$\widetilde{\mu}_A(\text{cough, Frequency}) = \{0.80,\ 0.82,\ 0.78\}, \quad \widetilde{\mu}_A(\text{cough, Intensity}) = \{0.60,\ 0.65,\ 0.62\}.$$

Similarly, relation-instance memberships are sets:

$$\widetilde{\mu}_R(\text{hasSymptom, influenza, cough}) = \{0.80,\ 0.85,\ 0.90\}, \quad \widetilde{\mu}_R(\text{hasSymptom, common\_cold, cough}) = \{0.50,\ 0.60,\ 0.65\}.$$

Here each $\widetilde{\mu}_A(o, a)$ and $\widetilde{\mu}_R(r, o_p, o_q)$ is a nonempty subset of $[0, 1]$, capturing inter-expert variability. The resulting tuple

$$HFO = \big(C,\ AC,\ R,\ X,\ \text{Inst},\ \widetilde{\mu}_A,\ \widetilde{\mu}_R\big)$$

is a valid HyperFuzzy Ontology for medical diagnosis.

Example 4.5 (Hotel Selection HyperFuzzy Ontology)**.** Consider a travel-recommendation domain with

$$C = \{\text{Hotel, Feature}\}, \quad AC(\text{Hotel}) = \{\text{Cleanliness, Comfort, Value}\}, \quad AC(\text{Feature}) = \{\text{WiFi, RoomService}\},$$

instances

$$\text{Inst}(\text{Hotel}) = \{\text{HotelA, HotelB}\}, \quad \text{Inst}(\text{Feature}) = \{\text{WiFi, RoomService}\},$$

and relation

$$R = \{\text{offersFeature}\}.$$

Multiple guest reviews produce set-valued attribute degrees, for example:

$$\widetilde{\mu}_A(\text{HotelA, Cleanliness}) = \{0.75,\ 0.80,\ 0.85\}, \quad \widetilde{\mu}_A(\text{HotelA, Comfort}) = \{0.70,\ 0.78,\ 0.82\},$$

$$\widetilde{\mu}_A(\text{HotelB, Value}) = \{0.65,\ 0.72,\ 0.77\}.$$

Similarly, feature-offer memberships are sets:

$$\widetilde{\mu}_R(\text{offersFeature, HotelA, WiFi}) = \{0.90,\ 0.92,\ 0.95\}, \quad \widetilde{\mu}_R(\text{offersFeature, HotelB, RoomService}) = \{0.80,\ 0.85,\ 0.88\}.$$

Then

$$HFO = \big(C,\ AC,\ R,\ X,\ \text{Inst},\ \widetilde{\mu}_A,\ \widetilde{\mu}_R\big)$$

is a hyperfuzzy ontology capturing inter-review variability in hotel quality and offered features.

Theorem 4.6 (Generalization of Fuzzy Ontology)**.** *Every fuzzy ontology FO induces a HyperFuzzy Ontology HFO by setting*

$$\widetilde{\mu}_A(o, a) = \{\mu_A(o, a)\}, \qquad \widetilde{\mu}_R(r, o_p, o_q) = \{\mu_R(r, o_p, o_q)\}.$$

Proof. By defining each hyper-membership as the singleton of its fuzzy membership, all hyperfuzzy mappings collapse to the original fuzzy mappings. Thus FO is recovered as the special case HFO with singleton hyper-sets. □

Theorem 4.7 (HyperFuzzy Set Structure)**.** *In a HyperFuzzy Ontology HFO, the mappings*

$$\widetilde{\mu}_A \colon D_A \to \widetilde{\mathcal{P}}([0, 1]), \quad \widetilde{\mu}_R \colon D_R \to \widetilde{\mathcal{P}}([0, 1]),$$

where $D_A = \{(o, a)\}$ and $D_R = \{(r, o_p, o_q)\}$, each define a hyperfuzzy set on their domain.

Proof. By construction, for every element of D_A or D_R, the image under $\widetilde{\mu}_A$ or $\widetilde{\mu}_R$ is a nonempty subset of $[0, 1]$. Hence each mapping satisfies the definition of a hyperfuzzy set. □

4.3 SuperHyperFuzzy Ontology

A superhyperfuzzy ontology generalizes hyperfuzzy ontologies by mapping hierarchical collections of concepts or relations to multi-level sets of membership degrees.

Definition 4.8 ((m,n)-SuperHyperFuzzy Ontology)**.** Let

$$FO = \big(C,\ AC,\ R,\ X,\ \text{Inst},\ \mu_A,\ \mu_R\big)$$

be a fuzzy ontology with

$$D_A = \bigcup_{c\in C} \big\{(o,a) \mid o \in \text{Inst}(c),\ a \in AC(c)\big\}, \quad D_R = \bigcup_{(c_p,c_q)\in R} \big\{(r,o_p,o_q) \mid o_p \in \text{Inst}(c_p),\ o_q \in \text{Inst}(c_q)\big\}.$$

Fix $m,n \in \mathbb{N}_0$. Denote by $\mathcal{P}^*_m(D_A)$ and $\mathcal{P}^*_m(D_R)$ the nonempty m-th iterated powersets of D_A and D_R, respectively, and by $\widetilde{\mathcal{P}}^*_n([0,1])$ the nonempty n-th iterated powerset of $[0,1]$. An (m,n)-*SuperHyperFuzzy Ontology* is

$$SHFO = \big(C,\ AC,\ R,\ X,\ \text{Inst},\ \widetilde{\mu}_A,\ \widetilde{\mu}_R\big),$$

where

$$\widetilde{\mu}_A : \mathcal{P}^*_m(D_A) \longrightarrow \widetilde{\mathcal{P}}^*_n([0,1]), \qquad \widetilde{\mu}_R : \mathcal{P}^*_m(D_R) \longrightarrow \widetilde{\mathcal{P}}^*_n([0,1]),$$

assign to each m-level set of instance–attributes or relation–instances a nonempty family of n-level membership sets, capturing hierarchical uncertainty.

Example 4.9 (E-Commerce $(2,2)$-SuperHyperFuzzy Ontology)**.** Let

$$C = \{\text{Customer, Product}\}, \qquad AC(\text{Customer}) = \{\text{Loyalty, SpendingPower}\}, \quad AC(\text{Product}) = \{\text{Quality, Popularity}\},$$

with instances

$$\text{Inst(Customer)} = \{c_1, c_2\}, \quad \text{Inst(Product)} = \{p_1, p_2\},$$

and a single relation

$$R = \{\text{purchases}\} \subseteq C \times C.$$

Form the domains

$$D_A = \big\{(o,a) \mid o \in \text{Inst}(c),\ a \in AC(c)\big\}, \quad D_R = \big\{(\text{purchases}, o_p, o_q) \mid o_p \in \text{Inst(Customer)},\ o_q \in \text{Inst(Product)}\big\}.$$

For $m = n = 2$ we take

$$U = \Big\{\{(c_1, \text{Loyalty}), (c_1, \text{SpendingPower})\}, \{(c_2, \text{Loyalty}), (c_2, \text{SpendingPower})\}\Big\} \ \in\ \mathcal{P}^*_2(D_A),$$

and define first-level membership sets

$$M_1 = \big\{\{0.80, 0.85\}, \{0.90, 0.95\}\big\}, \quad M_2 = \big\{\{0.70, 0.75\}, \{0.85, 0.90\}\big\}.$$

Then the $(2,2)$-superhyperfuzzy attribute mapping assigns

$$\widetilde{\mu}_A(U) = \{\, M_1,\ M_2 \} \ \subseteq\ \widetilde{\mathcal{P}}^*_2([0,1]).$$

Similarly, for

$$W = \Big\{\{(\text{purchases}, c_1, p_1), (\text{purchases}, c_1, p_2)\}, \{(\text{purchases}, c_2, p_1), (\text{purchases}, c_2, p_2)\}\Big\} \ \in\ \mathcal{P}^*_2(D_R),$$

we set

$$N_1 = \big\{\{0.75, 0.80\}, \{0.85, 0.88\}\big\}, \quad N_2 = \big\{\{0.65, 0.70\}, \{0.78, 0.82\}\big\},$$

and define

$$\widetilde{\mu}_R(W) = \{\, N_1,\ N_2 \} \ \subseteq\ \widetilde{\mathcal{P}}^*_2([0,1]).$$

Thus

$$SHFO = \big(C,\ AC,\ R,\ X,\ \text{Inst},\ \widetilde{\mu}_A,\ \widetilde{\mu}_R\big)$$

is a concrete $(2,2)$-SuperHyperFuzzy Ontology, capturing hierarchical uncertainty in both customer–attribute and purchase–relation evaluations.

Example 4.10 ((1, 2)-SuperHyperFuzzy Ontology for Hotel Recommendation)**.** Let

$$C = \{\text{Hotel, Feature}\}, \quad AC(\text{Hotel}) = \{\text{Cleanliness, Comfort, Value}\}, \quad AC(\text{Feature}) = \{\text{WiFi, RoomService}\},$$

$$\text{Inst}(\text{Hotel}) = \{\text{HotelA, HotelB}\}, \quad \text{Inst}(\text{Feature}) = \{\text{WiFi, RoomService}\}, \quad R = \{\text{offersFeature}\}.$$

Define

$$D_A = \{(o, a) \mid o \in \text{Inst}(c),\ a \in AC(c)\}, \quad D_R = \{(r, o_p, o_q) \mid r \in R,\ o_p \in \text{Inst(Hotel)},\ o_q \in \text{Inst(Feature)}\}.$$

For $m = 1$, $\mathcal{P}_1^*(D_A) = \mathcal{P}^*(D_A)$ is the family of all nonempty subsets of D_A. Choose

$$U_A = \{\ (\text{HotelA, Cleanliness}),\ (\text{HotelA, Comfort})\} \in \mathcal{P}_1^*(D_A).$$

For $n = 2$, $\widetilde{\mathcal{P}}_2^*([0, 1])$ is the family of nonempty subsets of $\mathcal{P}_1([0, 1])$. Let two guest-reviewer groups produce first-level membership sets

$$M_1 = \{0.75,\ 0.80\}, \qquad M_2 = \{0.85,\ 0.90\},$$

so

$$\widetilde{\mu}_A(U_A) = \{\ M_1,\ M_2\} \subseteq \widetilde{\mathcal{P}}_2^*([0, 1]).$$

Similarly, pick

$$U_R = \{\ (\text{offersFeature, HotelA, WiFi}),\ (\text{offersFeature, HotelB, RoomService})\} \in \mathcal{P}_1^*(D_R),$$

and let

$$N_1 = \{0.90,\ 0.95\}, \qquad N_2 = \{0.88,\ 0.92\}.$$

Then

$$\widetilde{\mu}_R(U_R) = \{\ N_1,\ N_2\} \subseteq \widetilde{\mathcal{P}}_2^*([0, 1]).$$

Thus the tuple

$$SHFO = (C,\ AC,\ R,\ X,\ \text{Inst},\ \widetilde{\mu}_A,\ \widetilde{\mu}_R)$$

is a $(1, 2)$-SuperHyperFuzzy Ontology, capturing hierarchical uncertainty both in hotel-attribute evaluations across guest groups and in feature-offer assessments.

Theorem 4.11 (Reduction to Fuzzy and HyperFuzzy Ontologies)**.**

(a) If $m = n = 0$ and for each $(o, a) \in D_A$, $\widetilde{\mu}_A(\{(o, a)\}) = \{\mu_A(o, a)\}$, and similarly $\widetilde{\mu}_R(\{(r, o_p, o_q)\}) = \{\mu_R(r, o_p, o_q)\}$, then $SHFO$ coincides with the original fuzzy ontology FO.

(b) If $m = 0, n = 1$, then $SHFO$ specializes to a hyperfuzzy ontology, since $\widetilde{\mu}_A \colon D_A \to \widetilde{\mathcal{P}}_1^*([0, 1])$ and $\widetilde{\mu}_R \colon D_R \to \widetilde{\mathcal{P}}_1^*([0, 1])$.

Proof. (a) For $m = n = 0$, $\mathcal{P}_0^*(D_A) = D_A$, $\widetilde{\mathcal{P}}_0^*([0, 1]) = [0, 1]$. By defining each $\widetilde{\mu}$ on singletons to be the original fuzzy degree, one recovers the fuzzy ontology exactly.

(b) For $m = 0, n = 1$, the domain remains D_A (resp. D_R), while the codomain becomes all nonempty subsets of $[0, 1]$, yielding precisely the hyperfuzzy ontology structure. □

Theorem 4.12 (SuperHyperFuzzy Set Structure)**.** *The mappings*

$$\widetilde{\mu}_A \colon \mathcal{P}_m^*(D_A) \to \widetilde{\mathcal{P}}_n^*([0, 1]), \qquad \widetilde{\mu}_R \colon \mathcal{P}_m^*(D_R) \to \widetilde{\mathcal{P}}_n^*([0, 1])$$

each define an (m, n)-SuperHyperFuzzy set on their respective domains.

Proof. By definition, for every $U \in \mathcal{P}_m^*(D_A)$ (or $U \in \mathcal{P}_m^*(D_R)$), $\widetilde{\mu}_A(U)$ (resp. $\widetilde{\mu}_R(U)$) is a nonempty subset of $\mathcal{P}_n([0, 1])$. Thus each $\widetilde{\mu}$ satisfies exactly the requirements of an (m, n)-SuperHyperFuzzy set. □

Funding

This study did not receive any financial or external support from organizations or individuals.

Acknowledgments

We extend our sincere gratitude to everyone who provided insights, inspiration, and assistance throughout this research. We particularly thank our readers for their interest and acknowledge the authors of the cited works for laying the foundation that made our study possible. We also appreciate the support from individuals and institutions that provided the resources and infrastructure needed to produce and share this paper. Finally, we are grateful to all those who supported us in various ways during this project.

Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

References

[1] Lotfi A Zadeh. Fuzzy sets. *Information and control*, 8(3):338–353, 1965.

[2] Zdzis law Pawlak. Rough sets. *International journal of computer & information sciences*, 11:341–356, 1982.

[3] Said Broumi, Florentin Smarandache, and Mamoni Dhar. Rough neutrosophic sets. *Infinite Study*, 32:493–502, 2014.

[4] Waheed Ahmad Khan, Khurram Faiz, and Abdelghani Taouti. Cayley picture fuzzy graphs and interconnected networks. *Intelligent Automation & Soft Computing*.

[5] Bui Cong Cuong and Vladik Kreinovich. Picture fuzzy sets-a new concept for computational intelligence problems. In *2013 third world congress on information and communication technologies (WICT 2013)*, pages 1–6. IEEE, 2013.

[6] Dmitriy Molodtsov. Soft set theory-first results. *Computers & mathematics with applications*, 37(4-5):19–31, 1999.

[7] Pradip Kumar Maji, Ranjit Biswas, and A Ranjan Roy. Soft set theory. *Computers & mathematics with applications*, 45(4-5):555–562, 2003.

[8] Florentin Smarandache. *Plithogeny, plithogenic set, logic, probability, and statistics*. Infinite Study, 2017.

[9] H-J Zimmermann. Fuzzy set theory and mathematical programming. *Fuzzy sets theory and applications*, pages 99–114, 1986.

[10] Hans-Jürgen Zimmermann. *Fuzzy set theory—and its applications*. Springer Science & Business Media, 2011.

[11] Krassimir T Atanassov. *On intuitionistic fuzzy sets theory*, volume 283. Springer, 2012.

[12] Kamal Kumar and Shyi-Ming Chen. Group decision making based on weighted distance measure of linguistic intuitionistic fuzzy sets and the topsis method. *Information Sciences*, 611:660–676, 2022.

[13] Krassimir T Atanassov and G Gargov. *Intuitionistic fuzzy logics*. Springer, 2017.

[14] An Lu and Wilfred Ng. Vague sets or intuitionistic fuzzy sets for handling vague data: which one is better? In *International conference on conceptual modeling*, pages 401–416. Springer, 2005.

[15] W-L Gau and Daniel J Buehrer. Vague sets. *IEEE transactions on systems, man, and cybernetics*, 23(2):610–614, 1993.

[16] Vicenç Torra and Yasuo Narukawa. On hesitant fuzzy sets and decision. In *2009 IEEE international conference on fuzzy systems*, pages 1378–1382. IEEE, 2009.

[17] Zeshui Xu. *Hesitant fuzzy sets theory*, volume 314. Springer, 2014.

[18] Xiao-Kang Wang, Hong-Gang Peng, and Jian-Qiang Wang. Hesitant linguistic intuitionistic fuzzy sets and their application in multicriteria decision-making problems. *International Journal for Uncertainty Quantification*, 8(4), 2018.

[19] Said Broumi, Mohamed Talea, Assia Bakali, and Florentin Smarandache. Interval valued neutrosophic graphs. *Critical Review, XII*, 2016:5–33, 2016.

[20] Mustafa Hasan Hadi and LAA Al-Swidi. The neutrosophic axial set theory. *Neutrosophic Sets and Systems, vol. 51/2022: An International Journal in Information Science and Engineering*, page 295, 2022.

[21] Said Broumi, Assia Bakali, Mohamed Talea, and Florentin Smarandache. An isolated bipolar single-valued neutrosophic graphs. In *Information Systems Design and Intelligent Applications: Proceedings of Fourth International Conference INDIA 2017*, pages 816–822. Springer, 2018.

[22] Fazeelat Sultana, Muhammad Gulistan, Mumtaz Ali, Naveed Yaqoob, Muhammad Khan, Tabasam Rashid, and Tauseef Ahmed. A study of plithogenic graphs: applications in spreading coronavirus disease (covid-19) globally. *Journal of ambient intelligence and humanized computing*, 14(10):13139–13159, 2023.

[23] Nivetha Martin. Plithogenic swara-topsis decision making on food processing methods with different normalization techniques. *Advances in Decision Making*, 69, 2022.

[24] Florentin Smarandache. Plithogeny, plithogenic set, logic, probability, and statistics. *arXiv preprint arXiv:1808.03948*, 2018.

[25] Takaaki Fujita and Florentin Smarandache. A review of the hierarchy of plithogenic, neutrosophic, and fuzzy graphs: Survey and applications. In *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond (Second Volume)*. Biblio Publishing, 2024.

[26] Young Bae Jun, Kul Hur, and Kyoung Ja Lee. Hyperfuzzy subalgebras of bck/bci-algebras. *Annals of Fuzzy Mathematics and Informatics*, 2017.

[27] Takaaki Fujita and Prem Kumar Singh. Hyperfuzzy graph and hyperfuzzy hypergraph. *Journal of Neutrosophic and Fuzzy Systems (JNFS)*, 10(01):01–13, 2025.

[28] M MAHARIN. Hyper fuzzy cosets. *Scholar: National School of Leadership*, 9(1.2), 2020.

[29] Takaaki Fujita and Florentin Smarandache. A concise introduction to hyperfuzzy, hyperneutrosophic, hyperplithogenic, hypersoft, and hyperrough sets with practical examples. *Neutrosophic Sets and Systems*, 80:609–631, 2025.

[30] M MAHARIN. An over view on hyper fuzzy subgroups. *Scholar: National School of Leadership*, 9(1.2), 2020.

[31] Takaaki Fujita. The hyperfuzzy vikor and hyperfuzzy dematel methods for multi-criteria decision-making. *Spectrum of Decision Making and Applications*, 3(1):292–315, 2026.

[32] Yong Lin Liu, Hee Sik Kim, and J. Neggers. Hyperfuzzy subsets and subgroupoids. *J. Intell. Fuzzy Syst.*, 33:1553–1562, 2017.

[33] Takaaki Fujita and Florentin Smarandache. Examples of fuzzy sets, hyperfuzzy sets, and superhyperfuzzy sets in climate change and the proposal of several new concepts. *Climate Change Reports*, 2:1–18, 2025.

[34] Takaaki Fujita. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*. Biblio Publishing, 2025.

[35] Takaaki Fujita. Hyperrough topsis method and superhyperrough topsis method. *Uncertainty Discourse and Applications*, 2024.

[36] Takaaki Fujita. Note for intuitionistic hyperrough set, one–directional s–hyperrough set, tolerance hyperrough set, and dynamic hyperrough set. *Soft Computing Fusion with Applications*, 2024.

[37] Takaaki Fujita. Hyperalgorithms & superhyperalgorithms: A unified framework for higher-order computation. *Prospects for Applied Mathematics and Data Analysis*, 4(1):36–49, 2024.

[38] Yasar Nacaroglu, Nihat Akgunes, Sedat Pak, and I Naci Cangul. Some graph parameters of power set graphs. *Advances & Applications in Discrete Mathematics*, 26(2), 2021.

[39] Michael Rathjen. Constructive zermelo-fraenkel set theory, power set, and the calculus of constructions. In *Epistemology versus Ontology: Essays on the Philosophy and Foundations of Mathematics in Honour of Per Martin-Löf*, pages 313–349. Springer, 2012.

[40] Florentin Smarandache. Foundation of superhyperstructure & neutrosophic superhyperstructure. *Neutrosophic Sets and Systems*, 63(1):21, 2024.

[41] Marzieh Rahmati and Mohammad Hamidi. Extension of g-algebras to superhyper g-algebras. *Neutrosophic Sets and Systems*, 55:557–567, 2023.

[42] Adel Al-Odhari. Neutrosophic power-set and neutrosophic hyper-structure of neutrosophic set of three types. *Annals of Pure and Applied Mathematics*, 31(2):125–146, 2025.

[43] Ajoy Kanti Das, Rajat Das, Suman Das, Bijoy Krishna Debnath, Carlos Granados, Bimal Shil, and Rakhal Das. A comprehensive study of neutrosophic superhyper bci-semigroups and their algebraic significance. *Transactions on Fuzzy Sets and Systems*, 8(2):80, 2025.

[44] Takaaki Fujita. Exploration of graph classes and concepts for superhypergraphs and n-th power mathematical structures. 2025.

[45] Lotfi A Zadeh. Fuzzy logic, neural networks, and soft computing. In *Fuzzy sets, fuzzy logic, and fuzzy systems: selected papers by Lotfi A Zadeh*, pages 775–782. World Scientific, 1996.

[46] Z Nazari and B Mosapour. The entropy of hyperfuzzy sets. *Journal of Dynamical Systems and Geometric Theories*, 16(2):173–185, 2018.

[47] Hashem Bordbar, Mohammad Rahim Bordbar, Rajab Ali Borzooei, and Young Bae Jun. N-subalgebras of bck= bci-algebras which are induced from hyperfuzzy structures. *Iranian Journal of Mathematical Sciences and Informatics*, 16(2):179–195, 2021.

[48] Seok-Zun Song, Seon Jeong Kim, and Young Bae Jun. Hyperfuzzy ideals in bck/bci-algebras. *Mathematics*, 5(4):81, 2017.

[49] Jayanta Ghosh and Tapas Kumar Samanta. Hyperfuzzy sets and hyperfuzzy group. *Int. J. Adv. Sci. Technol*, 41:27–37, 2012.

[50] Takaaki Fujita. Short survey on the hierarchical uncertainty of fuzzy, neutrosophic, and plithogenic sets. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 285, 2025.

[51] Florentin Smarandache. *Hyperuncertain, superuncertain, and superhyperuncertain sets/logics/probabilities/statistics*. Infinite Study, 2017.

[52] Ivan Kramosil and Jiří Michálek. Fuzzy metrics and statistical metric spaces. *Kybernetika*, 11(5):336–344, 1975.

[53] Osmo Kaleva and Seppo Seikkala. On fuzzy metric spaces. *Fuzzy sets and systems*, 12(3):215–229, 1984.

[54] A George and P Veeramani. On some results in fuzzy metric spaces. *Fuzzy sets and systems*, 64(3):395–399, 1994.

[55] Jin Han Park. Intuitionistic fuzzy metric spaces. *Chaos, Solitons & Fractals*, 22(5):1039–1046, 2004.

[56] Valentın Gregori and Salvador Romaguera. On completion of fuzzy metric spaces. *Fuzzy sets and systems*, 130(3):399–404, 2002.

[57] Francisco Castro-Company and Pedro Tirado. Some classes of t-norms and fuzzy metric spaces. In *AIP Conference Proceedings*, volume 1479, pages 848–851. American Institute of Physics, 2012.

[58] Yonghong Shen, Haifeng Li, and Faxing Wang. On interval-valued fuzzy metric spaces. *International Journal of fuzzy systems*, 14(1):35–44, 2012.

[59] JR Kider. The product fuzzy metric space and its basic properties. In *Journal of Physics: Conference Series*, volume 2322, page 012023. IOP Publishing, 2022.

[60] Jin-Xuan Fang. On fixed point theorems in fuzzy metric spaces. *Fuzzy sets and systems*, 46(1):107–113, 1992.

[61] Michael A Erceg. Metric spaces in fuzzy set theory. *Journal of Mathematical Analysis and Applications*, 69(1):205–230, 1979.

[62] Wutiphol Sintunavarat and Poom Kumam. Common fixed point theorems for a pair of weakly compatible mappings in fuzzy metric spaces. *Journal of Applied mathematics*, 2011(1):637958, 2011.

[63] Rachna Rathore. A review on study of application of queueing models in hospital sector. *International Journal for Global Academic & Scientific Research*, 1(2):1–6, 2022.

[64] Tom Van Woensel and Nico Vandaele. Modeling traffic flows with queueing models: a review. *Asia-Pacific Journal of Operational Research*, 24(04):435–461, 2007.

[65] Jau-Chuan Ke, Chia-Huang Wu, and Zhe George Zhang. Recent developments in vacation queueing models: a short survey. *International Journal of Operations Research*, 7(4):3–8, 2010.

[66] R Srinivasan. Fuzzy queueing model using dsw algorithm. *International Journal of Advanced Research in Mathematics and Applications*, 1(1):57–62, 2014.

[67] James J Buckley, Thomas Feuring, and Yoichi Hayashi. Fuzzy queueing theory revisited. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 9(05):527–537, 2001.

[68] R-J Li and ES Lee. Analysis of fuzzy queues. *Computers & Mathematics with Applications*, 17(7):1143–1147, 1989.

[69] Enrique Munoz and Enrique H Ruspini. Simulation of fuzzy queueing systems with a variable number of servers, arrival rate, and service rate. *IEEE transactions on fuzzy systems*, 22(4):892–903, 2013.

[70] Fariborz Jolai, Seyed Mohammad Asadzadeh, Reza Ghodsi, and Shiva Bagheri-Marani. A multi-objective fuzzy queuing priority assignment model. *Applied Mathematical Modelling*, 40(21-22):9500–9513, 2016.

[71] Gang Chen, Zaiming Liu, and Jingchuan Zhang. Analysis of strategic customer behavior in fuzzy queueing systems. *Journal of Industrial & Management Optimization*, 16(1), 2020.

[72] Tom Gruber. What is an ontology. 1993.

[73] Dieter Fensel. Ontology-based knowledge management. *Computer*, 35(11):56–59, 2002.

[74] Robert Stevens, Carole A Goble, and Sean Bechhofer. Ontology-based knowledge representation for bioinformatics. *Briefings in bioinformatics*, 1(4):398–414, 2000.

[75] Quan Thanh Tho, Siu Cheung Hui, Alvis Cheuk M Fong, and Tru Hoang Cao. Automatic fuzzy ontology generation for semantic web. *IEEE transactions on knowledge and data engineering*, 18(6):842–856, 2006.

[76] Fernando Bobillo and Umberto Straccia. Fuzzy ontology representation using owl 2. *International journal of approximate reasoning*, 52(7):1073–1094, 2011.

[77] Natalia Díaz Rodríguez, Manuel P Cuéllar, Johan Lilius, and Miguel Delgado Calvo-Flores. A fuzzy ontology for semantic modelling and recognition of human behaviour. *Knowledge-Based Systems*, 66:46–60, 2014.

[78] Christer Carlsson, Matteo Brunelli, and József Mezei. Decision making with a fuzzy ontology. *Soft Computing*, 16(7):1143–1152, 2012.

[79] David Parry. A fuzzy ontology for medical document retrieval. In *ACM International Conference Proceeding Series*, volume 54, pages 121–126, 2004.

[80] Carmen De Maio, Giuseppe Fenza, Vincenzo Loia, and Sabrina Senatore. Towards an automatic fuzzy ontology generation. In *2009 IEEE international conference on fuzzy systems*, pages 1044–1049. IEEE, 2009.

[81] Mumtaz Ali, Nguyen Dang Thanh, Nguyen Van Minh, et al. A neutrosophic recommender system for medical diagnosis based on algebraic neutrosophic measures. *Applied Soft Computing*, 71:1054–1071, 2018.

[82] Zehui Shao, Saeed Kosari, Hossein Rashmanlou, and Farshid Mofidnakhaei. Fuzzy decision making in medical diagnosis using vague sets. *Journal of Multiple-Valued Logic & Soft Computing*, 40, 2023.