

Hyperfuzzy and SuperHyperfuzzy Numbers: Generalizing Triangular, Trapezoidal, Pentagonal, Hexagonal, and Octagonal Fuzzy Numbers with Ranking Functions

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Abstract

Uncertainty modeling underpins decision-making in many domains. Over the years, numerous frameworks have been proposed to capture different facets of imprecision, including fuzzy sets [1, 2], rough sets [3, 4], hesitant fuzzy sets [5, 6], neutrosophic sets [7, 8], and plithogenic sets [9, 10]. More recently, hyperfuzzy sets and their recursive generalization, superhyperfuzzy sets, have been introduced to assign set-valued membership degrees at multiple hierarchical levels, thereby enriching the representation of uncertainty [11, 12]. In parallel, a variety of fuzzy-number extensions—triangular, trapezoidal, pentagonal, hexagonal, and octagonal fuzzy numbers—together with ranking functions on fuzzy numbers, have been studied and applied in decision-making and related fields [13, 14]. In this paper, we develop hyperfuzzy and superhyperfuzzy analogues of these fuzzy-number constructs. Our aim is to provide a unified, hierarchically structured toolkit for modeling complex uncertainty in decision support and beyond.

Keywords: Fuzzy set, HyperFuzzy Set, SuperHyperFuzzy Set, Triangular Fuzzy Numbers, Trapezoidal Fuzzy Numbers, Ranking Function on Fuzzy Numbers, Pentagonal fuzzy numbers, Octagonal Fuzzy Numbers, Hexagonal fuzzy numbers

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1 Preliminaries

We collect here the basic notions and notation used throughout the paper. Unless stated otherwise, all underlying sets are finite.

1.1 Basic Set Constructions

A *fuzzy set* assigns to each element a membership degree in the unit interval, enabling partial inclusion rather than strict binary classification [1, 15–17]. Related extensions such as bipolar fuzzy sets [18, 19] and picture fuzzy sets [20] are also well-known. A *hyperfuzzy set* refines this idea by assigning each element a nonempty set of values in the unit interval, thereby accommodating variability and imprecision in membership [21–24]. More generally, an (m, n) -*superhyperfuzzy set* assigns to each m -level subset a family of n -level membership collections, modeling multi-tiered uncertainty (cf. [12, 25]).

Definition 1.1 (Universe). Let U be a nonempty finite set, called the *universe* or *base set*. All further constructions (powersets, hyperstructures, etc.) are formed from U .

Definition 1.2 (Powerset). (cf. [26, 27]) The *powerset* of U is

$$\mathcal{P}(U) = \{A \mid A \subseteq U\}.$$

Definition 1.3 (n -fold Powerset). (cf. [28–32]) For each integer $n \geq 1$, define the n -fold iterated powerset of U by

$$\mathcal{P}^1(U) = \mathcal{P}(U), \quad \mathcal{P}^{n+1}(U) = \mathcal{P}(\mathcal{P}^n(U)).$$

If one wishes to exclude the empty set at each stage, replace \mathcal{P} by $\mathcal{P}^*(\cdot) = \mathcal{P}(\cdot) \setminus \{\emptyset\}$.

Example 1.4 (Multi-Tier Organizational Structure via Iterated Powersets). Let the base set of employees be

$$U = \{\text{Ayuna}, \text{Ichiro}, \text{Shinya}\}.$$

- $\mathcal{P}^1(U)$ (all teams):

$$\{\{\}, \{\text{Ayuna}\}, \{\text{Ichiro}\}, \{\text{Shinya}\}, \{\text{Ayuna}, \text{Ichiro}\}, \{\text{Ayuna}, \text{Shinya}\}, \{\text{Ichiro}, \text{Shinya}\}, \{\text{Ayuna}, \text{Ichiro}, \text{Shinya}\}\}.$$

Excluding the empty set gives the collection of all nonempty teams.

- $\mathcal{P}^2(U)$ (committees of teams):

$$\{\{\{\text{Ayuna}, \text{Ichiro}\}, \{\text{Ichiro}, \text{Shinya}\}\}, \{\{\text{Ayuna}\}, \{\text{Shinya}\}\}, \dots\}.$$

For example, one committee might include $\text{Team}_1 = \{\text{Ayuna}, \text{Ichiro}\}$ and $\text{Team}_2 = \{\text{Ichiro}, \text{Shinya}\}$.

- $\mathcal{P}^3(U)$ (councils of committees):

$$\{\{\{\{\text{Ayuna}, \text{Ichiro}\}, \{\text{Ichiro}, \text{Shinya}\}\}\}, \{\{\{\text{Ayuna}\}, \{\text{Shinya}\}\}, \{\{\text{Ayuna}, \text{Shinya}\}\}\}, \dots\}.$$

For instance, a council might consist of the two committees above, coordinating cross-team decisions.

In practice:

1. Level 1 (teams): small working groups drawn from the employee pool.
2. Level 2 (committees): groups of teams focusing on specific projects.
3. Level 3 (councils): assemblies of committees setting high-level strategy.

Thus the 3-fold powerset $\mathcal{P}^3(U)$ naturally models a three-tier decision-making hierarchy in an organization.

Definition 1.5 (Fuzzy Set). [1, 17] A *fuzzy set* F on U is a function

$$\mu_F: U \longrightarrow [0, 1],$$

assigning to each element $x \in U$ a *membership degree* $\mu_F(x)$.

Definition 1.6 (Fuzzy Relation). [33, 34] Given a fuzzy set F on U , a *fuzzy relation* R on U is a fuzzy subset of $U \times U$, i.e. a map

$$R: U \times U \longrightarrow [0, 1],$$

such that

$$R(x, y) \leq \min\{\mu_F(x), \mu_F(y)\} \quad \text{for all } x, y \in U.$$

Definition 1.7 (Hyperfuzzy Set). [21, 35–38] A *hyperfuzzy set* \tilde{F} on U is given by

$$\tilde{\mu}: U \longrightarrow \mathcal{P}([0, 1]) \setminus \{\emptyset\},$$

where for each $x \in U$, the set $\tilde{\mu}(x) \subseteq [0, 1]$ represents the *possible membership grades* of x .

Definition 1.8 ((m, n) -SuperHyperfuzzy Set). [11, 39] Fix nonnegative integers m, n . Let

$$\mathcal{P}_m^*(U) = \underbrace{(\mathcal{P}^* \circ \dots \circ \mathcal{P}^*)}_{m \text{ times}}(U), \quad \mathcal{P}_n^*([0, 1]) = \underbrace{(\mathcal{P}^* \circ \dots \circ \mathcal{P}^*)}_{n \text{ times}}([0, 1]),$$

where $\mathcal{P}^*(X) = \mathcal{P}(X) \setminus \{\emptyset\}$. An (m, n) -*superhyperfuzzy set* on U is a map

$$\tilde{\mu}_{m,n}: \mathcal{P}_m^*(U) \longrightarrow \mathcal{P}(\mathcal{P}_n^*([0, 1])) \setminus \{\emptyset\},$$

assigning each nonempty m -level subset of U a nonempty family of n -level membership sets, thereby modeling hierarchical uncertainty.

Example 1.9 (Hierarchical Sensor Network Reliability). Let the universe of sensors be

$$U = \{S_1, S_2, S_3\}.$$

For $m = 2$, the nonempty 1st-level subsets are

$$\mathcal{P}_1^*(U) = \{\{S_1\}, \{S_2\}, \{S_3\}, \{S_1, S_2\}, \{S_1, S_3\}, \{S_2, S_3\}, \{S_1, S_2, S_3\}\},$$

and the nonempty 2nd-level subsets are

$$\mathcal{P}_2^*(U) = \mathcal{P}^*(\mathcal{P}_1^*(U)),$$

for example

$$X = \{\{S_1, S_2\}, \{S_2, S_3\}\} \in \mathcal{P}_2^*(U).$$

Fix $n = 1$. We define a $(2, 1)$ -superhyperfuzzy membership map $\tilde{\mu}_{2,1}: \mathcal{P}_2^*(U) \rightarrow \mathcal{P}(\mathcal{P}_1([0, 1])) \setminus \{\emptyset\}$ by assigning to each 2-level subset X a nonempty family of reliability intervals. For instance:

$$\tilde{\mu}_{2,1}(\{\{S_1, S_2\}, \{S_2, S_3\}\}) = \{\{0.80, 0.85\}, \{0.75, 0.90\}\},$$

where each inner set $\{r_\ell, r_u\} \subseteq [0, 1]$ represents a plausible lower and upper bound on the aggregated cluster reliability. Thus $\tilde{\mu}_{2,1}$ captures uncertainty both within each sensor-cluster and across clusters of clusters, modeling hierarchical reliability in the network.

Example 1.10 (Sensor Cluster Reliability under Multi-Layer Uncertainty). Let the universe of sensors be $U = \{S_1, S_2, S_3\}$, and consider the 1-level subset $X = \{S_1, S_2\} \in \mathcal{P}_1^*(U)$. We wish to model the reliability of this sensor pair under three nested layers of uncertainty:

- (i) *Operational Scenario Uncertainty*: different environmental or load conditions,
- (ii) *Sensor Degradation Uncertainty*: variability in sensor aging or calibration,
- (iii) *Expert Opinion Uncertainty*: divergent assessments by multiple analysts.

Accordingly, we define the (1,3)-superhyperfuzzy membership $\tilde{\mu}_{1,3}(X) \in \mathcal{P}(\mathcal{P}(\mathcal{P}([0,1]))) \setminus \{\emptyset\}$ by two three-level membership families:

$$\tilde{\mu}_{1,3}(X) = \{W_1, W_2\},$$

where

$$W_1 = \{\{0.70, 0.75\}, \{0.80\}\}, \quad W_2 = \{\{0.65\}, \{0.85, 0.90\}\}.$$

Here:

- Each inner set (e.g. $\{0.70, 0.75\}$) is a level-1 collection of membership grades reflecting expert-opinion ranges.
- Each W_i is a level-2 collection of those grade-sets, representing variability across sensor-degradation scenarios.
- Finally, $\{W_1, W_2\}$ is the level-3 family, capturing distinct operational scenarios (e.g. “normal” vs. “harsh” environments).

Thus $\tilde{\mu}_{1,3}(X)$ models reliability with three hierarchical uncertainty layers: expert divergence, sensor aging, and environmental conditions.

1.2 HyperFuzzy Number and SuperHyperFuzzy Number

A fuzzy number is a fuzzy set on \mathbb{R} whose membership function is normal, convex, upper semi-continuous, and has compact support, modeling imprecise real values [40–45]. Several extensions of fuzzy numbers have been proposed, such as Intuitionistic Fuzzy Numbers [46,47], Complex Fuzzy Numbers [48–50], Neutrosophic Numbers [51–53], and Hesitant Fuzzy Numbers [54–56]. These concepts have been widely applied in various fields, including decision-making. A hyperfuzzy number is a hyperfuzzy set on \mathbb{R} that assigns each point a nonempty subset of $[0, 1]$, satisfying normality, hyper-convexity, upper semi-continuity, and compact support [23]. A superhyperfuzzy number of order (m, n) is a hierarchical hyperfuzzy set mapping each nonempty m -level subset of \mathbb{R} to a nonempty family of n -level membership sets, capturing multi-layer uncertainty.

Definition 1.11 (Fuzzy Number). [43, 44] A fuzzy set A on \mathbb{R} with membership function

$$\mu_A : \mathbb{R} \longrightarrow [0, 1]$$

is called a *fuzzy number* if it satisfies:

(i) **Normality:** there exists $x_0 \in \mathbb{R}$ such that $\mu_A(x_0) = 1$.

(ii) **Fuzzy convexity:** for all $x, y \in \mathbb{R}$ and $\lambda \in [0, 1]$,

$$\mu_A(\lambda x + (1 - \lambda)y) \geq \min\{\mu_A(x), \mu_A(y)\}.$$

(iii) **Upper semi-continuity:** μ_A is upper semi-continuous on \mathbb{R} .

(iv) **Compact support:** the support $\text{supp}(A) = \{x \in \mathbb{R} \mid \mu_A(x) > 0\}$ is a compact (hence bounded) subset of \mathbb{R} .

Definition 1.12 (Hyperfuzzy Number). [23] A *hyperfuzzy number* \tilde{A} is a hyperfuzzy set on \mathbb{R} (cf. Definition 1.7) whose membership map

$$\tilde{\mu}_{\tilde{A}} : \mathbb{R} \longrightarrow \mathcal{P}([0, 1]) \setminus \{\emptyset\}$$

satisfies:

(i) **Normality:** $\sup_{x \in \mathbb{R}} \tilde{\mu}_{\tilde{A}}(x) = 1$.

(ii) **Hyper-convexity:** for all $x, y \in \mathbb{R}$ and $\lambda \in [0, 1]$,

$$\min\{u, v\} \leq w \quad \text{for every } u \in \tilde{\mu}_{\tilde{A}}(x), v \in \tilde{\mu}_{\tilde{A}}(y), w \in \tilde{\mu}_{\tilde{A}}(\lambda x + (1 - \lambda)y).$$

(iii) **Upper semi-continuity:** for each $\alpha \in (0, 1]$, the α -cut $\{x \mid \exists u \geq \alpha, u \in \tilde{\mu}_{\tilde{A}}(x)\}$ is closed in \mathbb{R} .

(iv) **Compact support:** the support $\text{supp}(\tilde{A}) = \{x \mid \exists u > 0, u \in \tilde{\mu}_{\tilde{A}}(x)\}$ is a compact subset of \mathbb{R} .

Example 1.13 (Commuting Time Satisfaction as a Hyperfuzzy Number). A city planner models commuter satisfaction with travel time t (in minutes) as a hyperfuzzy number \tilde{T} . She defines

$$\underline{\mu}(t) = \begin{cases} 0, & t \leq 30, \\ \frac{t-30}{45-30}, & 30 \leq t \leq 45, \\ \frac{60-t}{60-45}, & 45 \leq t \leq 60, \\ 0, & t \geq 60, \end{cases} \quad \bar{\mu}(t) = \begin{cases} 0, & t \leq 20, \\ \frac{t-20}{40-20}, & 20 \leq t \leq 40, \\ 1, & 40 \leq t \leq 80, \\ \frac{100-t}{100-80}, & 80 \leq t \leq 100, \\ 0, & t \geq 100. \end{cases}$$

Then for each t , the hyperfuzzy membership is the interval

$$\tilde{\mu}(t) = [\underline{\mu}(t), \bar{\mu}(t)].$$

For example:

- At $t = 50$ min, $\underline{\mu}(50) = \frac{60-50}{15} \approx 0.67$ and $\bar{\mu}(50) = 1$, so satisfaction lies in $[0.67, 1]$.
- At $t = 70$ min, $\underline{\mu}(70) = 0$ and $\bar{\mu}(70) = 1$, so satisfaction lies in $[0, 1]$.
- At $t = 10$ min, $\underline{\mu}(10) = 0$ and $\bar{\mu}(10) = \frac{10-20}{20} = -0.5$ (clipped to 0), so $\tilde{\mu}(10) = [0, 0]$.

This hyperfuzzy model captures both conservative and optimistic assessments of commuting satisfaction under travel-time uncertainty.

Theorem 1.14. Every (classical) fuzzy number $A: \mathbb{R} \rightarrow [0, 1]$ (cf. Definition of fuzzy number) induces a hyperfuzzy number \tilde{A} via

$$\tilde{\mu}_{\tilde{A}}(x) = \{\mu_A(x)\} \quad (x \in \mathbb{R}),$$

and conversely any hyperfuzzy number whose all $\tilde{\mu}(x)$ are singletons reduces to a fuzzy number.

Proof. Define $\tilde{\mu}_{\tilde{A}}(x) = \{\mu_A(x)\}$. Then

- *Normality:* since $\sup_x \mu_A(x) = 1$, we have $\sup \bigcup_x \{\mu_A(x)\} = 1$.
- *Hyper-convexity:* for any x, y and λ , $\mu_A(\lambda x + (1-\lambda)y) \geq \min\{\mu_A(x), \mu_A(y)\}$, so every $w = \mu_A(\lambda x + (1-\lambda)y)$ satisfies $w \geq \min\{u, v\}$ with $u = \mu_A(x)$, $v = \mu_A(y)$.
- *Upper semi-continuity* and *compact support* follow immediately from those properties of μ_A .

Conversely, if $\tilde{\mu}(x)$ is always a singleton set, define $\mu_A(x)$ to be its unique element; then A is a fuzzy number with the same normality, convexity, continuity, and support properties. \square

Definition 1.15 ((m, n)-SuperHyperfuzzy Number). An (m, n)-superhyperfuzzy number $\tilde{A}^{(m,n)}$ on \mathbb{R} is an (m, n)-superhyperfuzzy set (cf. Definition 1.8) with

$$\tilde{\mu}_{m,n}: \mathcal{P}_m^*(\mathbb{R}) \longrightarrow \mathcal{P}(\mathcal{P}_n([0, 1])) \setminus \{\emptyset\},$$

such that:

- (i) **Normality:** there exists a nonempty m -level set $X_0 \subseteq \mathbb{R}$ with $\exists V \in \tilde{\mu}_{m,n}(X_0)$ and $\sup \bigcup V = 1$.

-
- (ii) **Hierarchical convexity:** for all $X, Y \in \mathcal{P}_m^*(\mathbb{R})$ and $\lambda \in [0, 1]$, any $W \in \tilde{\mu}_{m,n}(\lambda X + (1 - \lambda)Y)$ satisfies $\min\{u, v\} \leq w$ for each $u \in U, v \in V, w \in W$, with $U \in \tilde{\mu}_{m,n}(X), V \in \tilde{\mu}_{m,n}(Y)$.
 - (iii) **Upper semi-continuity and compactness at each level** mirror the definitions of fuzzy and hyperfuzzy numbers but applied to m - and n -level cuts.

Example 1.16 (Temperature Comfort as a (1, 2)-SuperHyperfuzzy Number). We model human comfort with ambient temperature $T \in \mathbb{R}$ via a (1, 2)-superhyperfuzzy number $\tilde{C}^{(1,2)}$. Choose five nonempty 1-level subsets of \mathbb{R} :

$$\begin{aligned} X_1 &= [-10, 10] \quad (\text{“Cold”}), \\ X_2 &= [5, 20] \quad (\text{“Cool”}), \\ X_3 &= [15, 30] \quad (\text{“Comfortable”}), \\ X_4 &= [25, 40] \quad (\text{“Warm”}), \\ X_5 &= [35, 50] \quad (\text{“Hot”}). \end{aligned}$$

For each X_i , we assign two nested membership families (level-2 sets of level-1 grade-sets):

$$\tilde{\mu}_{1,2}(X_i) = \{W_{i,1}, W_{i,2}\}, \quad W_{i,j} \subseteq \mathcal{P}_1([0, 1]).$$

Concretely:

$$W_{3,1} = \{\{0.7, 0.8\}, \{0.75, 0.9\}\}, \quad W_{3,2} = \{\{0.6, 0.7\}\},$$

so for the “Comfortable” interval X_3 , one may be optimistic (0.75–0.9) or conservative (0.7–0.8), and analysts differ (0.6–0.7). Similarly, for “Cold” (X_1):

$$W_{1,1} = \{\{0.1, 0.2\}, \{0.2, 0.3\}\}, \quad W_{1,2} = \{\{0.05, 0.15\}\}.$$

Thus $\tilde{C}^{(1,2)}$ assigns each temperature range a family of possible comfort-grade intervals, capturing both intra-range uncertainty (level 1) and inter-expert variability (level 2).

Theorem 1.17. Every hyperfuzzy number \tilde{A} induces an (m, n) -superhyperfuzzy number $\tilde{A}^{(m,n)}$ by setting

$$\tilde{\mu}_{m,n}(X) = \{\tilde{\mu}_{\tilde{A}}(x) \mid x \in X\} \quad (X \in \mathcal{P}_m^*(\mathbb{R})),$$

and any (m, n) -superhyperfuzzy number whose all outputs are singletons of singleton-families reduces to a hyperfuzzy number (and hence to a fuzzy number when further all those singletons are single values).

Proof. Given \tilde{A} , for each nonempty $X \subseteq \mathbb{R}$ at level m , define $\tilde{\mu}_{m,n}(X) = \{\tilde{\mu}_{\tilde{A}}(x) : x \in X\}$, which is nonempty by normality of \tilde{A} . One checks:

- *Normality:* exists x_0 with $\sup \tilde{\mu}_{\tilde{A}}(x_0) = 1$, so $\tilde{\mu}_{m,n}(\{x_0\})$ contains a set whose supremum is 1.
- *Hierarchical convexity:* follows from hyperfuzzy convexity applied pointwise to each $x \in X$.
- *Upper semi-continuity and compactness:* inherited from \tilde{A} at each hierarchical level.

Conversely, if each $\tilde{\mu}_{m,n}(X)$ is a singleton family $\{W\}$ and each W is itself a singleton $\{w\}$, then defining $\tilde{\mu}_{\tilde{A}}(x) = w$ recovers a hyperfuzzy number, and further to a fuzzy number if w is constant for each $X = \{x\}$. \square

2 Triangular fuzzy numbers

2.1 Triangular Fuzzy numbers

Triangular fuzzy numbers have piecewise-linear membership on \mathbb{R} : rise linearly to 1 at a peak then descend linearly to 0 [57–59].

Definition 2.1 (Triangular Fuzzy Number). [60–62] A *triangular fuzzy number* \tilde{A} on the real line is a fuzzy set whose membership function $\mu_{\tilde{A}} : \mathbb{R} \rightarrow [0, 1]$ is given by

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & x \leq a, \\ \frac{x-a}{b-a}, & a \leq x \leq b, \\ \frac{c-x}{c-b}, & b \leq x \leq c, \\ 0, & x \geq c, \end{cases}$$

where $a \leq b \leq c$ are real numbers. We write $\tilde{A} = (a, b, c)$. Equivalently:

- **Support:** $\text{supp}(\tilde{A}) = [a, c]$.
- **Peak (normality):** $\mu_{\tilde{A}}(b) = 1$.
- **Convexity:** $\mu_{\tilde{A}}$ is piecewise linear and achieves its maximum at b .

The α -cut of \tilde{A} is the closed interval

$$\tilde{A}_\alpha = \{x \in \mathbb{R} \mid \mu_{\tilde{A}}(x) \geq \alpha\} = [a + \alpha(b-a), c - \alpha(c-b)], \quad \alpha \in [0, 1].$$

Example 2.2 (Project Completion Time Estimation). A project manager models the completion time T (in weeks) for a software module as a triangular fuzzy number $\tilde{T} = (4, 6, 10)$. The membership function $\mu_{\tilde{T}}(t)$ is defined by

$$\mu_{\tilde{T}}(t) = \begin{cases} 0, & t \leq 4, \\ \frac{t-4}{6-4}, & 4 \leq t \leq 6, \\ \frac{10-t}{10-6}, & 6 \leq t \leq 10, \\ 0, & t \geq 10. \end{cases}$$

- **Optimistic estimate** ($a = 4$): Completing in 4 weeks is considered impossible ($\mu = 0$).
- **Most likely** ($b = 6$): 6 weeks has full membership ($\mu = 1$).
- **Pessimistic estimate** ($c = 10$): Beyond 10 weeks is considered impossible ($\mu = 0$).

The support of \tilde{T} is $[4, 10]$, and the α -cut is

$$\tilde{T}_\alpha = [4 + 2\alpha, 10 - 4\alpha], \quad \alpha \in [0, 1].$$

For example, the 0.5-cut is $\tilde{T}_{0.5} = [5, 8]$, representing “fairly possible” completion times between 5 and 8 weeks.

2.2 Triangular HyperFuzzy numbers

Below we recall the definition of a *triangular hyperfuzzy number*.

Definition 2.3 (Triangular Hyperfuzzy Number). Let $a_u \leq a_\ell \leq b_\ell \leq c_\ell \leq c_u$ be real numbers. A *triangular hyperfuzzy number* \tilde{A} is a hyperfuzzy number (cf. Definition 1.7) whose membership map

$$\tilde{\mu}_{\tilde{A}}(x) = [\underline{\mu}(x), \bar{\mu}(x)] \subseteq [0, 1], \quad x \in \mathbb{R},$$

is the closed interval between two triangular membership functions:

$$\underline{\mu}(x) = \begin{cases} 0, & x \leq a_\ell, \\ \frac{x - a_\ell}{b_\ell - a_\ell}, & a_\ell \leq x \leq b_\ell, \\ \frac{b_\ell - a_\ell}{c_\ell - b_\ell}, & b_\ell \leq x \leq c_\ell, \\ 0, & x \geq c_\ell, \end{cases} \quad \bar{\mu}(x) = \begin{cases} 0, & x \leq a_u, \\ \frac{x - a_u}{b_\ell - a_u}, & a_u \leq x \leq b_\ell, \\ 1, & b_\ell \leq x \leq b_\ell, \\ \frac{c_u - x}{c_u - b_\ell}, & b_\ell \leq x \leq c_u, \\ 0, & x \geq c_u. \end{cases}$$

Here $\underline{\mu}$ and $\bar{\mu}$ are piecewise-linear with peak at b_ℓ , and $\underline{\mu}(x) \leq \bar{\mu}(x)$ for all x .

Example 2.4 (Commuter Travel Time Satisfaction as a Triangular Hyperfuzzy Number). A city planner models commuter satisfaction with one-way travel time t (in minutes) by a triangular hyperfuzzy number \tilde{T} with parameters

$$a_u = 10, \quad a_\ell = 20, \quad b_\ell = 30, \quad c_\ell = 50, \quad c_u = 70.$$

The lower (pessimistic) membership function is

$$\underline{\mu}(t) = \begin{cases} 0, & t \leq 20, \\ \frac{t - 20}{30 - 20}, & 20 \leq t \leq 30, \\ \frac{50 - t}{50 - 30}, & 30 \leq t \leq 50, \\ 0, & t \geq 50, \end{cases}$$

and the upper (optimistic) membership function is

$$\bar{\mu}(t) = \begin{cases} 0, & t \leq 10, \\ \frac{t - 10}{30 - 10}, & 10 \leq t \leq 30, \\ 1, & t = 30, \\ \frac{70 - t}{70 - 30}, & 30 \leq t \leq 70, \\ 0, & t \geq 70. \end{cases}$$

Thus the hyperfuzzy satisfaction interval is $\tilde{\mu}(t) = [\underline{\mu}(t), \bar{\mu}(t)]$. For example:

- At $t = 25$ min, $\underline{\mu}(25) = \frac{25-20}{10} = 0.5$, $\bar{\mu}(25) = \frac{25-10}{20} = 0.75$, so satisfaction $\in [0.5, 0.75]$.
- At $t = 40$ min, $\underline{\mu}(40) = \frac{50-40}{10} = 0.5$, $\bar{\mu}(40) = \frac{70-40}{30} = 0.75$, so satisfaction $\in [0.5, 0.75]$.
- At $t = 15$ min, $\underline{\mu}(15) = 0$, $\bar{\mu}(15) = \frac{15-10}{20} = 0.25$, so satisfaction $\in [0, 0.25]$.

This triangular hyperfuzzy model captures both conservative (lower) and optimistic (upper) assessments of commuter satisfaction under travel-time uncertainty.

Theorem 2.5. *Every triangular hyperfuzzy number \tilde{A} is a hyperfuzzy number. Moreover:*

- If $a_u = a_\ell$, $c_\ell = c_u$, then $\underline{\mu} = \bar{\mu}$ and \tilde{A} reduces to the triangular fuzzy number (a_ℓ, b_ℓ, c_ℓ) .
- If in addition $a_\ell = b_\ell = c_\ell = x_0$, then \tilde{A} further reduces to the crisp number x_0 .

Proof. We check the hyperfuzzy-number axioms:

Normality. Since $\bar{\mu}(b_\ell) = 1$, we have $\sup \bigcup_x \tilde{\mu}_{\tilde{A}}(x) = 1$.

Hyper-convexity. Let $x, y \in \mathbb{R}$ and $\lambda \in [0, 1]$. For any $u \in \tilde{\mu}_{\tilde{A}}(x)$ and $v \in \tilde{\mu}_{\tilde{A}}(y)$, $\min\{u, v\} \geq \min\{\underline{\mu}(x), \underline{\mu}(y)\}$. But triangular convexity of $\underline{\mu}$ implies $\underline{\mu}(\lambda x + (1 - \lambda)y) \geq \min\{\underline{\mu}(x), \underline{\mu}(y)\}$. Hence every $w \in \tilde{\mu}_{\tilde{A}}(\lambda x + (1 - \lambda)y)$ satisfies $w \geq \underline{\mu}(\lambda x + (1 - \lambda)y) \geq \min\{u, v\}$.

Upper semi-continuity. Each α -cut $\{x \mid \exists u \geq \alpha, u \in \tilde{\mu}_{\tilde{A}}(x)\}$ is the union of the closed α -cuts of $\underline{\mu}$ and $\overline{\mu}$, hence closed.

Compact support. $\text{supp}(\tilde{A}) = [a_u, c_u]$ is closed and bounded.

Thus \tilde{A} is a hyperfuzzy number. If $a_u = a_\ell$ and $c_\ell = c_u$, then $\underline{\mu} = \overline{\mu}$ is exactly the triangular membership of the fuzzy number (a_ℓ, b_ℓ, c_ℓ) . If further $a_\ell = b_\ell = c_\ell = x_0$, then $\underline{\mu}(x) = \overline{\mu}(x) = \delta_{x_0}(x)$, the crisp singleton at x_0 . \square

2.3 Triangular SuperHyperFuzzy numbers

Below we define the definition of a (m, n) -Triangular SuperHyperfuzzy Number.

Definition 2.6 ((m, n) -Triangular SuperHyperfuzzy Number). Fix nonnegative integers m, n . Let

$$X_{a_\ell}, X_b, X_{c_\ell} \in \mathcal{P}_m^*(\mathbb{R}), \quad X_{a_\ell} \subseteq X_b \subseteq X_{c_\ell},$$

and let

$$\underline{\tilde{A}}, \overline{\tilde{A}}: \mathcal{P}_m^*(\mathbb{R}) \longrightarrow \mathcal{P}(\mathcal{P}_n([0, 1])) \setminus \{\emptyset\}$$

be two (m, n) -superhyperfuzzy numbers satisfying for each $X \in \mathcal{P}_m^*(\mathbb{R})$,

$$\underline{\tilde{A}}(X) \subseteq \overline{\tilde{A}}(X).$$

Then the (m, n) -triangular superhyperfuzzy number $\tilde{A}^{(m, n)}$ with “vertices” $X_{a_\ell}, X_b, X_{c_\ell}$ is the (m, n) -superhyperfuzzy set

$$\tilde{\mu}_{m, n}(X) = \begin{cases} \underline{\tilde{A}}(X), & X \subseteq X_{a_\ell}, \\ (1 - \lambda(X)) \underline{\tilde{A}}(X) \cup \lambda(X) \overline{\tilde{A}}(X), & X_{a_\ell} \subseteq X \subseteq X_b, \\ \overline{\tilde{A}}(X), & X_b \subseteq X \subseteq X_{c_\ell}, \\ \{\{0\}\}, & X \supseteq X_{c_\ell}, \end{cases}$$

where $\lambda(X) = \frac{\delta(X) - \delta(X_{a_\ell})}{\delta(X_b) - \delta(X_{a_\ell})}$ and $\delta: \mathcal{P}_m^*(\mathbb{R}) \rightarrow \mathbb{R}_{\geq 0}$ is any strictly increasing “size” measure (e.g. cardinality or diameter).

Example 2.7 (Battery Pack Reliability under Load Conditions as a $(1, 2)$ -Triangular SuperHyperfuzzy Number). We model the reliability of an electric-vehicle battery pack under three load regimes: Low, Medium, High. Set

$$X_{a_\ell} = \{\text{Low}\}, \quad X_b = \{\text{Low}, \text{Medium}\}, \quad X_{c_\ell} = \{\text{Low}, \text{Medium}, \text{High}\}.$$

Take the size measure $\delta(X) = |X|$. For each X , define two nested 2-level membership families:

$$\underline{\tilde{A}}(X) = \{\{0.90, 0.95\}, \{0.85, 0.90\}\}, \quad \overline{\tilde{A}}(X) = \{\{0.92, 0.97\}, \{0.88, 0.93\}\}.$$

Then $\tilde{\mu}_{1, 2}: \mathcal{P}_1^*(\{\text{Low}, \text{Medium}, \text{High}\}) \rightarrow \mathcal{P}(\mathcal{P}_2([0, 1]))$ is given by

$$\tilde{\mu}_{1, 2}(X) = \begin{cases} \underline{\tilde{A}}(X), & \delta(X) = 1, \\ (1 - \lambda(X)) \underline{\tilde{A}}(X) \cup \lambda(X) \overline{\tilde{A}}(X), & 1 < \delta(X) < 2, \quad \lambda(X) = \frac{\delta(X) - 1}{2 - 1}. \\ \overline{\tilde{A}}(X), & \delta(X) \geq 2, \end{cases}$$

Concretely:

- $X = \{\text{Low}\}$ ($\delta = 1$): $\tilde{\mu}_{1, 2}(X) = \underline{\tilde{A}}(X) = \{\{0.90, 0.95\}, \{0.85, 0.90\}\}$.
- $X = \{\text{Low}, \text{Medium}\}$ ($\delta = 2$): $\lambda = 1$, so $\tilde{\mu}_{1, 2}(X) = \overline{\tilde{A}}(X) = \{\{0.92, 0.97\}, \{0.88, 0.93\}\}$.

- $X = \{\text{Low, Medium, High}\}$ ($\delta = 3$): $\delta \geq 2$ again yields $\tilde{\mu}_{1,2}(X) = \bar{\tilde{A}}(X)$.

This (1, 2)-triangular superhyperfuzzy number captures hierarchical uncertainty in battery reliability:

1. Level 1 ($\underline{\tilde{A}}, \bar{\tilde{A}}$) models interval forecasts from conservative and optimistic tests.
2. Level 2 (the families of intervals) represents variability across repeated trials or different expert assessments.
3. The triangular structure (via $X_{a_\ell}, X_b, X_{c_\ell}$) encodes “low-load,” “medium-load,” and “high-load” regimes with graded transitions.

Example 2.8 (Air Quality Index under Multilayer Uncertainty as a (1, 3)-Triangular SuperHyperfuzzy Number). We assess an air quality index (AQI) under nested uncertainties: sensor variability, daily fluctuations, and expert opinions. Let the region types be

$$U = \{\text{Urban, Suburban, Rural}\}.$$

Take

$$X_{a_\ell} = \{\text{Urban}\}, \quad X_b = \{\text{Urban, Suburban}\}, \quad X_{c_\ell} = U,$$

and size measure $\delta(X) = |X|$. Define two (1, 3)-superhyperfuzzy maps $\underline{\tilde{A}}, \bar{\tilde{A}}$ by:

Lower (conservative) profile:

$$\begin{aligned} \underline{\tilde{A}}(X_{a_\ell}) &= \{W_{1,1}, W_{1,2}\}, & W_{1,1} &= \{\{0.10, 0.15\}, \{0.20\}\}, & W_{1,2} &= \{\{0.05\}, \{0.25, 0.30\}\}, \\ \underline{\tilde{A}}(X_b) &= \{W_{2,1}, W_{2,2}\}, & W_{2,1} &= \{\{0.20, 0.25\}, \{0.30\}\}, & W_{2,2} &= \{\{0.15\}, \{0.35, 0.40\}\}, \\ \underline{\tilde{A}}(X_{c_\ell}) &= \{W_{3,1}, W_{3,2}\}, & W_{3,1} &= \{\{0.40, 0.45\}, \{0.50\}\}, & W_{3,2} &= \{\{0.35\}, \{0.55, 0.60\}\}. \end{aligned}$$

Upper (optimistic) profile:

$$\begin{aligned} \bar{\tilde{A}}(X_{a_\ell}) &= \{V_{1,1}, V_{1,2}\}, & V_{1,1} &= \{\{0.20, 0.25\}, \{0.30\}\}, & V_{1,2} &= \{\{0.15\}, \{0.35, 0.40\}\}, \\ \bar{\tilde{A}}(X_b) &= \{V_{2,1}, V_{2,2}\}, & V_{2,1} &= \{\{0.35, 0.40\}, \{0.45\}\}, & V_{2,2} &= \{\{0.30\}, \{0.50, 0.55\}\}, \\ \bar{\tilde{A}}(X_{c_\ell}) &= \{V_{3,1}, V_{3,2}\}, & V_{3,1} &= \{\{0.60, 0.65\}, \{0.70\}\}, & V_{3,2} &= \{\{0.55\}, \{0.75, 0.80\}\}. \end{aligned}$$

Then $\tilde{\mu}_{1,3}(X)$ is given by the triangular scheme:

$$\tilde{\mu}_{1,3}(X) = \begin{cases} \underline{\tilde{A}}(X), & X \subseteq X_{a_\ell}, \\ (1 - \lambda) \underline{\tilde{A}}(X) \cup \lambda \bar{\tilde{A}}(X), & X_{a_\ell} \subseteq X \subseteq X_b, \\ \bar{\tilde{A}}(X), & X_b \subseteq X \subseteq X_{c_\ell}, \\ \{\{0\}\}, & X \supseteq X_{c_\ell}, \end{cases} \quad \lambda = \frac{\delta(X) - 1}{2 - 1}.$$

Interpretation:

- **Level 1 (sets X):** distinguishes “Urban,” “Urban+Suburban,” and all regions.
- **Level 2 ($\underline{\tilde{A}}, \bar{\tilde{A}}$):** captures pessimistic vs. optimistic AQI interval assessments by sensors.
- **Level 3 ($W_{i,j}, V_{i,j}$):** reflects uncertainties from daily weather patterns and expert opinion variability.

Hence $\tilde{\mu}_{1,3}$ models AQI with three hierarchical uncertainty layers in a triangular transition across region types.

Theorem 2.9 (Reduction to Triangular Fuzzy, Hyperfuzzy, and SuperHyperfuzzy Numbers). *The (m, n) -triangular superhyperfuzzy number $\tilde{A}^{(m,n)}$ specializes as follows:*

- (i) *If $m = n = 0$, then $\mathcal{P}_0^*(\mathbb{R}) = \mathbb{R}$ and each of \tilde{A}, \bar{A} are singleton-valued; hence $\tilde{A}^{(0,0)}$ reduces to the classical triangular fuzzy number (a_ℓ, b, c_ℓ) where a_ℓ, b, c_ℓ are the unique points in $X_{a_\ell}, X_b, X_{c_\ell}$.*
- (ii) *If $m = 0$ but $n \geq 1$, then $\tilde{A}^{(0,n)}$ is a triangular hyperfuzzy number, since each cut $\tilde{\mu}_{0,n}(x)$ is an interval $[\underline{\mu}(x), \bar{\mu}(x)] \subseteq [0, 1]$, varying piecewise-linearly in x .*
- (iii) *For general m, n , if $\tilde{A}(X) = \bar{A}(X)$ for all X , then $\tilde{A}^{(m,n)}$ reduces to an (m, n) -superhyperfuzzy number (Definition 1.8).*

Proof. (i) When $m = n = 0$, $\mathcal{P}_0^*(\mathbb{R}) = \mathbb{R}$ and the maps \tilde{A}, \bar{A} each assign to a point $x \in \mathbb{R}$ a singleton $\{\mu_\ell(x)\}$, respectively $\{\mu_u(x)\}$. If moreover $\mu_\ell(x) = \mu_u(x)$ for all x , then the membership $\tilde{\mu}_{0,0}(x) = \{\mu(x)\}$ coincides with the piecewise-linear triangular membership of (a_ℓ, b, c_ℓ) .

- (ii) If $m = 0$, but $n \geq 1$, then each $\tilde{\mu}_{0,n}(x)$ is by construction the interval

$$[\underline{\mu}(x), \bar{\mu}(x)] \subseteq [0, 1],$$

where $\underline{\mu}, \bar{\mu}$ are piecewise-linear with peak 1 at $b \in X_b$. This matches the definition of a triangular hyperfuzzy number.

- (iii) In the general (m, n) case, if $\tilde{A}(X) = \bar{A}(X)$ for all X , then every convex combination $(1-\lambda)\tilde{A}(X) \cup \lambda\bar{A}(X)$ reduces to that same common value. Hence $\tilde{\mu}_{m,n}(X)$ is single-valued for each X , and $\tilde{A}^{(m,n)}$ coincides with the (m, n) -superhyperfuzzy number given by that value.

□

3 Ranking fuzzy numbers

3.1 Ranking Fuzzy numbers

A ranking function on fuzzy numbers maps each fuzzy number to a real score, preserving crisp consistency and membership-based ordering [63–66].

Definition 3.1 (Ranking Function on Fuzzy Numbers). [67, 68, 68, 69] Let $\mathcal{F}(\mathbb{R})$ denote the set of all fuzzy numbers on \mathbb{R} . A *ranking function* is a mapping

$$R : \mathcal{F}(\mathbb{R}) \longrightarrow \mathbb{R}$$

that induces a total order \succeq_R on $\mathcal{F}(\mathbb{R})$ via

$$A \succeq_R B \iff R(A) \geq R(B).$$

We require R to satisfy the following properties:

- (i) **Crisp consistency:** For each real x , let $\bar{x} \in \mathcal{F}(\mathbb{R})$ be the *crisp fuzzy number* defined by

$$\mu_{\bar{x}}(t) = \begin{cases} 1, & t = x, \\ 0, & t \neq x. \end{cases}$$

Then

$$R(\bar{x}) = x.$$

(ii) **Monotonicity:** If $A, B \in \mathcal{F}(\mathbb{R})$ satisfy

$$A_\alpha \subseteq B_\alpha \quad \text{for every } \alpha \in (0, 1],$$

where A_α and B_α are the usual α -cuts, then

$$R(A) \leq R(B).$$

Example 3.2 (Ranking Project Completion Estimates). A project manager has two triangular fuzzy estimates for task duration (in weeks):

$$\tilde{A} = (4, 6, 8), \quad \tilde{B} = (5, 7, 9).$$

They use the *centroid* ranking function

$$R(\tilde{X}) = \frac{a + b + c}{3}, \quad \text{for } \tilde{X} = (a, b, c).$$

Then

$$R(\tilde{A}) = \frac{4 + 6 + 8}{3} = 6, \quad R(\tilde{B}) = \frac{5 + 7 + 9}{3} = 7.$$

Crisp consistency: A “crisp” estimate $\bar{6}$ (i.e. the fuzzy number that equals 6 with membership 1) satisfies

$$R(\bar{6}) = 6.$$

Monotonicity: For each $\alpha \in (0, 1]$,

$$\tilde{A}_\alpha = [4 + 2\alpha, 8 - 2\alpha], \quad \tilde{B}_\alpha = [5 + 2\alpha, 9 - 2\alpha],$$

and indeed $\tilde{A}_\alpha \subseteq \tilde{B}_\alpha$. Consequently,

$$R(\tilde{A}) = 6 \leq 7 = R(\tilde{B}),$$

so the manager correctly ranks \tilde{B} as the longer estimate.

This example shows how a ranking function can convert fuzzy duration estimates into a precise order for project planning.

3.2 Ranking HyperFuzzy numbers

Below we present the definition of *Ranking Function on Hyperfuzzy Numbers* and a detailed illustrative example.

Definition 3.3 (Ranking Function on Hyperfuzzy Numbers). Let $\mathcal{HF}(\mathbb{R})$ denote the set of all hyperfuzzy numbers on \mathbb{R} , each given by a membership map

$$\tilde{\mu}_A : \mathbb{R} \longrightarrow \mathcal{P}([0, 1]) \setminus \{\emptyset\}.$$

For $A \in \mathcal{HF}(\mathbb{R})$, define its *lower* and *upper* membership functions by

$$\underline{\mu}_A(x) = \inf \tilde{\mu}_A(x), \quad \bar{\mu}_A(x) = \sup \tilde{\mu}_A(x), \quad x \in \mathbb{R}.$$

A *ranking function* on hyperfuzzy numbers is a map

$$R : \mathcal{HF}(\mathbb{R}) \longrightarrow \mathbb{R}$$

such that:

(i) **Crisp consistency:** For each real x_0 , let \bar{x}_0 be the *crisp hyperfuzzy number* defined by

$$\tilde{\mu}_{\bar{x}_0}(x) = \begin{cases} \{1\}, & x = x_0, \\ \{0\}, & x \neq x_0. \end{cases}$$

Then

$$R(\bar{x}_0) = x_0.$$

(ii) **Monotonicity:** If $A, B \in \mathcal{HF}(\mathbb{R})$ satisfy

$$\underline{\mu}_A(x) \leq \underline{\mu}_B(x) \quad \text{and} \quad \bar{\mu}_A(x) \leq \bar{\mu}_B(x) \quad \text{for all } x \in \mathbb{R},$$

then

$$R(A) \leq R(B).$$

Example 3.4 (Ranking House Price Estimates under Hyperfuzzy Uncertainty). A real estate analyst compares two houses A and B using hyperfuzzy price estimates (in thousands of dollars).

House A:

$$\underline{\mu}_A(x) = \begin{cases} 0, & x \leq 100, \\ \frac{x-100}{140-x}, & 100 \leq x \leq 120, \\ \frac{120-100}{140-120}, & 120 \leq x \leq 140, \\ 0, & x \geq 140, \end{cases} \quad \bar{\mu}_A(x) = \begin{cases} 0, & x \leq 90, \\ \frac{x-90}{170-x}, & 90 \leq x \leq 125, \\ \frac{125-90}{170-125}, & 125 \leq x \leq 170, \\ 0, & x \geq 170. \end{cases}$$

House B:

$$\underline{\mu}_B(x) = \begin{cases} 0, & x \leq 110, \\ \frac{x-110}{150-x}, & 110 \leq x \leq 130, \\ \frac{130-110}{150-130}, & 130 \leq x \leq 150, \\ 0, & x \geq 150, \end{cases} \quad \bar{\mu}_B(x) = \begin{cases} 0, & x \leq 100, \\ \frac{x-100}{180-x}, & 100 \leq x \leq 135, \\ \frac{135-100}{180-135}, & 135 \leq x \leq 180, \\ 0, & x \geq 180. \end{cases}$$

Thus each hyperfuzzy membership is $\tilde{\mu}_A(x) = [\underline{\mu}_A(x), \bar{\mu}_A(x)]$ and likewise for $\tilde{\mu}_B(x)$.

We use the ranking function

$$R(\tilde{P}) = \frac{c_\ell + c_u}{2}, \quad c_\ell = \frac{a_\ell + b_\ell + c_\ell}{3}, \quad c_u = \frac{a_u + b_u + c_u}{3},$$

where (a_ℓ, b_ℓ, c_ℓ) and (a_u, b_u, c_u) are the parameters of the lower and upper triangular membership functions.

$$c_{A,\ell} = \frac{100 + 120 + 140}{3} = 120, \quad c_{A,u} = \frac{90 + 125 + 170}{3} \approx 128.33,$$

$$R(A) = \frac{120 + 128.33}{2} \approx 124.17.$$

$$c_{B,\ell} = \frac{110 + 130 + 150}{3} = 130, \quad c_{B,u} = \frac{100 + 135 + 180}{3} \approx 138.33,$$

$$R(B) = \frac{130 + 138.33}{2} \approx 134.17.$$

Crisp consistency: For the crisp hyperfuzzy number $\overline{125}$ with $\tilde{\mu}_{\overline{125}}(x) = \{1\}$ if $x = 125$, else $\{0\}$, we have $R(\overline{125}) = 125$.

Monotonicity: Since for all x ,

$$\underline{\mu}_A(x) \leq \underline{\mu}_B(x), \quad \bar{\mu}_A(x) \leq \bar{\mu}_B(x),$$

it follows that $R(A) \leq R(B)$, i.e. the ranking respects the hyperfuzzy ordering.

This example illustrates how a ranking function on hyperfuzzy numbers can convert interval-valued price estimates into a single comparable value for decision-making.

Theorem 3.5 (Reduction to Fuzzy and Hyperfuzzy Ranking). *Let R be a ranking function on $\mathcal{HF}(\mathbb{R})$ as above. Then:*

- (a) *If A is a (classical) fuzzy number, regarded as the hyperfuzzy number with $\tilde{\mu}_A(x) = \{\mu_A(x)\}$, then $R(\tilde{A})$ depends only on the single-valued membership μ_A and hence defines a valid ranking function on fuzzy numbers.*
- (b) *If all $\tilde{\mu}_A(x)$ happen to be singleton intervals $[\underline{\mu}(x), \overline{\mu}(x)]$ with $\underline{\mu}(x) = \mu(x)$, $\overline{\mu}(x) = \nu(x)$, then R restricted to such “interval hyperfuzzy numbers” yields a ranking function on hyperfuzzy numbers in the sense of [37].*

Proof. (a) Suppose A is a fuzzy number with $\mu_A : \mathbb{R} \rightarrow [0, 1]$, and form the corresponding hyperfuzzy \tilde{A} by $\tilde{\mu}_{\tilde{A}}(x) = \{\mu_A(x)\}$. Then $\underline{\mu}_{\tilde{A}}(x) = \overline{\mu}_{\tilde{A}}(x) = \mu_A(x)$. Thus any ordering or comparison of \tilde{A} with another fuzzy number \tilde{B} depends only on the single-valued μ_A, μ_B , so $R(\tilde{A})$ agrees with a ranking function on fuzzy numbers.

(b) Now suppose $A \in \mathcal{HF}(\mathbb{R})$ has $\tilde{\mu}_A(x) = [\underline{\mu}(x), \overline{\mu}(x)]$. Then the lower/upper membership functions coincide with these endpoints. Monotonicity in the hyperfuzzy sense ($\underline{\mu}_A \leq \underline{\mu}_B, \overline{\mu}_A \leq \overline{\mu}_B$) is exactly the standard interval-based ordering used in interval-valued fuzzy rankings. Hence R restricted to this subclass is a ranking function on hyperfuzzy numbers as in the existing literature. \square

3.3 Ranking SuperHyperFuzzy numbers

Below we present the definition of *Ranking Function on (m, n) -SuperHyperfuzzy Numbers* and a detailed illustrative example.

Definition 3.6 (Ranking Function on (m, n) -SuperHyperfuzzy Numbers). Let $\mathcal{SHF}^{(m,n)}(\mathbb{R})$ be the class of all (m, n) -superhyperfuzzy numbers on \mathbb{R} , each given by a membership map

$$\tilde{\mu}_{m,n} : \mathcal{P}_m^*(\mathbb{R}) \longrightarrow \mathcal{P}(\mathcal{P}_n([0, 1]) \setminus \{\emptyset\}).$$

Suppose we have a fixed ranking function

$$R_{\text{HF}} : \mathcal{HF}(\mathbb{R}) \rightarrow \mathbb{R}$$

on hyperfuzzy numbers (Definition of ranking on hyperfuzzy). Define the *aggregation map*

$$\Phi(\tilde{A}^{(m,n)}) : \mathbb{R} \longrightarrow \mathcal{P}([0, 1]) \setminus \{\emptyset\}$$

by

$$\Phi(\tilde{A}^{(m,n)})(x) = \bigcup_{\substack{X \in \mathcal{P}_m^*(\mathbb{R}) \\ x \in X}} \bigcup_{U \in \tilde{\mu}_{m,n}(X)} U, \quad x \in \mathbb{R}.$$

One checks that $\Phi(\tilde{A}^{(m,n)})$ is a hyperfuzzy number. Then the *ranking function* on (m, n) -superhyperfuzzy numbers is defined by

$$R_{m,n}(\tilde{A}^{(m,n)}) = R_{\text{HF}}(\Phi(\tilde{A}^{(m,n)})).$$

Example 3.7 (Project Lead-Time Ranking via a $(1, 1)$ -SuperHyperfuzzy Number). A project manager models the lead time t (in days) for a critical task under three overlapping time-windows:

$$X_1 = [5, 7], \quad X_2 = [7, 10], \quad X_3 = [5, 10],$$

so $X_1 \subseteq X_3$ and $X_2 \subseteq X_3$. We set $m = n = 1$ and $\delta(X) = \text{length}(X)$.

SuperHyperfuzzy membership maps: For each X_i , assign two nested membership intervals (level-1 families):

$$\tilde{A}(X_i) = \{ [a_i^\ell, b_i^\ell] \}, \quad \overline{\tilde{A}}(X_i) = \{ [a_i^u, b_i^u] \},$$

with parameters

$$(a_i^\ell, b_i^\ell, a_i^u, b_i^u) = (5, 7, 6, 8), (7, 10, 8, 9), (5, 10, 6, 9)$$

for $i = 1, 2, 3$ respectively. Thus $\tilde{\mu}_{1,1}(X_i) = \{\{[a_i^\ell, b_i^\ell]\}, \{\{[a_i^u, b_i^u]\}\}$.

Aggregation to a hyperfuzzy number: By the definition

$$\Phi(\tilde{A}^{(1,1)})(t) = \bigcup_{t \in X_i} \bigcup_{U \in \tilde{\mu}_{1,1}(X_i)} U,$$

one obtains the *triangular hyperfuzzy number* with

$$\underline{\mu}(t): (5, 7, 10), \quad \bar{\mu}(t): (6, 8, 9),$$

where $\underline{\mu}$ and $\bar{\mu}$ are the lower and upper triangular membership functions peaked at $t = 7$ and $t = 8$, respectively.

Ranking via centroid: We use the hyperfuzzy ranking function

$$R_{\text{HF}}(\tilde{T}) = \frac{\frac{5+7+10}{3} + \frac{6+8+9}{3}}{2} = \frac{22/3 + 23/3}{2} = \frac{45/3}{2} = 7.5.$$

Hence

$$R_{1,1}(\tilde{A}^{(1,1)}) = R_{\text{HF}}(\Phi(\tilde{A}^{(1,1)})) = 7.5.$$

Interpretation: Although individual windows X_1 and X_2 suggest best-case and worst-case durations, the (1, 1)-superhyperfuzzy model integrates hierarchical uncertainty and yields a single ranked lead time of 7.5 days for planning and resource allocation.

Example 3.8 (Ranking Air Quality under (1, 3)-SuperHyperfuzzy Uncertainty). Building on our (1, 3)-superhyperfuzzy AQI model (Example 4.7), we aggregate to a hyperfuzzy AQI and then rank.

Aggregation to a Triangular Hyperfuzzy Number. Applying

$$\Phi(\tilde{A}^{(1,3)})(x) = \bigcup_{x \in X} \bigcup_{U \in \tilde{\mu}_{1,3}(X)} U,$$

one obtains lower and upper triangular membership functions

$$\underline{\mu}(x): (a_\ell, b_\ell, c_\ell) = (0.2, 0.5, 0.8), \quad \bar{\mu}(x): (a_u, b_\ell, c_u) = (0.3, 0.6, 0.9).$$

Thus the equivalent triangular hyperfuzzy number has parameters (0.2, 0.5, 0.8) (pessimistic) and (0.3, 0.6, 0.9) (optimistic).

Ranking via Centroid of Hyperfuzzy Number. We use the centroid-based ranking

$$R_{\text{HF}}(\tilde{T}) = \frac{(a_\ell + b_\ell + c_\ell)/3 + (a_u + b_\ell + c_u)/3}{2}.$$

Substituting:

$$\frac{0.2 + 0.5 + 0.8}{3} = 0.50, \quad \frac{0.3 + 0.6 + 0.9}{3} = 0.60,$$

so

$$R_{1,3}(\tilde{A}^{(1,3)}) = \frac{0.50 + 0.60}{2} = 0.55.$$

Interpretation: Although experts under different conditions yielded AQI ranges from $\underline{\mu}$ to $\bar{\mu}$, the combined ranking of 0.55 (on the normalized 0–1 scale) guides decision-makers that the overall air quality is moderately good, balancing pessimistic and optimistic assessments across three hierarchical uncertainty layers.

Theorem 3.9 (Generalization of Fuzzy and Hyperfuzzy Rankings). *The ranking function $R_{m,n}$ on $\mathcal{SHF}^{(m,n)}(\mathbb{R})$ has the following reduction properties:*

- (a) **Hyperfuzzy reduction:** *If $m = 0$, then $\mathcal{P}_0^*(\mathbb{R}) = \mathbb{R}$ and $\Phi(\tilde{A}^{(0,n)}) = \tilde{A}$. Hence $R_{0,n}(\tilde{A}^{(0,n)}) = R_{\text{HF}}(\tilde{A})$, so $R_{0,n}$ coincides with the original hyperfuzzy ranking.*
- (b) **Fuzzy reduction:** *If $m = n = 0$, then $\tilde{A}^{(0,0)}$ corresponds to a classical fuzzy number A via $\tilde{\mu}(x) = \{\mu_A(x)\}$, and R_{HF} restricted to singletons is the fuzzy ranking R_F . Thus $R_{0,0}(\tilde{A}^{(0,0)}) = R_F(A)$.*

Proof. We first show $\Phi(\tilde{A}^{(m,n)})$ is a hyperfuzzy number:

- **Normality.** By the (m, n) -normality of $\tilde{A}^{(m,n)}$ there is some X_0 and $U_0 \in \tilde{\mu}_{m,n}(X_0)$ with $\sup U_0 = 1$. Since any $x_0 \in X_0$ satisfies $U_0 \subseteq \Phi(\tilde{A}^{(m,n)})(x_0)$, we get $\sup \bigcup_x \Phi(\tilde{A}^{(m,n)})(x) = 1$.
- **Hyper-convexity.** If $x, y \in \mathbb{R}$ and $\lambda \in [0, 1]$, choose $X, Y \in \mathcal{P}_m^*(\mathbb{R})$ with $x \in X$, $y \in Y$. By hierarchical convexity of $\tilde{A}^{(m,n)}$, for any $U \in \tilde{\mu}_{m,n}(X)$ and $V \in \tilde{\mu}_{m,n}(Y)$ and any $u \in U$, $v \in V$, there is $W \in \tilde{\mu}_{m,n}(\lambda X + (1 - \lambda)Y)$ and $w \in W$ with $\min\{u, v\} \leq w$. But $\lambda X + (1 - \lambda)Y$ contains the point $\lambda x + (1 - \lambda)y$, so $W \subseteq \Phi(\tilde{A}^{(m,n)})(\lambda x + (1 - \lambda)y)$. This yields hyper-convexity.
- **Upper semi-continuity and compact support.** Directly inherited from the corresponding properties at each (m, n) -level of $\tilde{A}^{(m,n)}$.

Therefore $\Phi(\tilde{A}^{(m,n)}) \in \mathcal{HF}(\mathbb{R})$, and $R_{m,n}$ is well-defined.

For part (a), if $m = 0$ then every nonempty $X \subseteq \mathbb{R}$ at level zero is of the form $X = \{x\}$, so

$$\Phi(\tilde{A}^{(0,n)})(x) = \bigcup_{X=\{x\}} \bigcup_{U \in \tilde{\mu}_{0,n}(X)} U = \tilde{\mu}(x),$$

recovering the original hyperfuzzy membership. Hence $R_{0,n} = R_{\text{HF}}$.

For part (b), when further $n = 0$, each $\tilde{\mu}(x)$ is a singleton $\{\mu_A(x)\}$, and the hyperfuzzy ranking R_{HF} reduces to the fuzzy ranking R_F by crisp consistency. Therefore $R_{0,0}$ coincides with R_F . \square

4 Trapezoidal fuzzy numbers

4.1 Trapezoidal Fuzzy numbers

Trapezoidal fuzzy numbers are fuzzy sets on \mathbb{R} with piecewise-linear membership functions that rise to full membership, plateau, and decline [70–72].

Definition 4.1 (Trapezoidal Fuzzy Number). (cf. [73–75]) A *trapezoidal fuzzy number* \tilde{A} on \mathbb{R} is a fuzzy set whose membership function $\mu_{\tilde{A}} : \mathbb{R} \rightarrow [0, 1]$ is given by

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & x \leq a, \\ \frac{x-a}{b-a}, & a \leq x \leq b, \\ 1, & b \leq x \leq c, \\ \frac{d-x}{d-c}, & c \leq x \leq d, \\ 0, & x \geq d, \end{cases}$$

where $a \leq b \leq c \leq d$ are real parameters. We denote $\tilde{A} = (a, b, c, d)$.

Equivalently:

- **Support:** $\text{supp}(\tilde{A}) = [a, d]$.
- **Core:** $\{x \mid \mu_{\tilde{A}}(x) = 1\} = [b, c]$.
- **Normality:** $\max_{x \in \mathbb{R}} \mu_{\tilde{A}}(x) = 1$.
- **Convexity:** $\mu_{\tilde{A}}$ is convex and piecewise linear.

The α -cut of \tilde{A} is the closed interval

$$\tilde{A}_\alpha = \{x \in \mathbb{R} \mid \mu_{\tilde{A}}(x) \geq \alpha\} = [a + \alpha(b - a), d - \alpha(d - c)], \quad \alpha \in [0, 1].$$

Example 4.2 (Component Length Tolerance in Manufacturing). In precision machining, the acceptable length of a shaft might be modeled as a trapezoidal fuzzy number $\tilde{L} = (98, 99, 101, 102)$ mm, where:

$$\mu_{\tilde{L}}(x) = \begin{cases} 0, & x \leq 98, \\ \frac{x - 98}{99 - 98}, & 98 \leq x \leq 99, \\ 1, & 99 \leq x \leq 101, \\ \frac{102 - x}{102 - 101}, & 101 \leq x \leq 102, \\ 0, & x \geq 102. \end{cases}$$

- **Support :** [98, 102] mm is the full range of possible lengths.
- **Core :** [99, 101] mm represents the ideal tolerance band with full membership $\mu = 1$.
- **Left slope :** From 98 mm up to 99 mm, acceptability increases linearly from 0 to 1.
- **Right slope :** From 101 mm up to 102 mm, acceptability decreases linearly from 1 back to 0.

The α -cut yields

$$\tilde{L}_\alpha = [98 + \alpha(99 - 98), 102 - \alpha(102 - 101)] = [98 + \alpha, 102 - \alpha], \quad \alpha \in [0, 1],$$

so for example $\tilde{L}_{0.5} = [98.5, 101.5]$ mm is the “fairly acceptable” length interval. This trapezoidal fuzzy model captures the gradual transition from unacceptable to ideal to unacceptable lengths in a manufacturing quality-control context.

4.2 Trapezoidal HyperFuzzy numbers

Below we present the definition of *Trapezoidal HyperFuzzy numbers* and a detailed illustrative example.

Definition 4.3 (Trapezoidal Hyperfuzzy Number). Let real parameters satisfy

$$a_U \leq a_L < b_L \leq c_L < d_L \leq d_U, \quad a_U < b_U \leq b_L, \quad c_L \leq c_U < d_U.$$

Define two piecewise-linear membership functions on \mathbb{R} :

$$\underline{\mu}(x) = \begin{cases} 0, & x \leq a_L, \\ \frac{x - a_L}{b_L - a_L}, & a_L \leq x \leq b_L, \\ 1, & b_L \leq x \leq c_L, \\ \frac{d_L - x}{d_L - c_L}, & c_L \leq x \leq d_L, \\ 0, & x \geq d_L, \end{cases} \quad \bar{\mu}(x) = \begin{cases} 0, & x \leq a_U, \\ \frac{x - a_U}{b_U - a_U}, & a_U \leq x \leq b_U, \\ 1, & b_U \leq x \leq c_U, \\ \frac{d_U - x}{d_U - c_U}, & c_U \leq x \leq d_U, \\ 0, & x \geq d_U. \end{cases}$$

Because $a_U \leq a_L, d_L \leq d_U$, and $b_U \leq b_L \leq c_L \leq c_U$, one has $\underline{\mu}(x) \leq \bar{\mu}(x)$ for all x . The map

$$\tilde{\mu}(x) = [\underline{\mu}(x), \bar{\mu}(x)] \subseteq [0, 1] \quad (x \in \mathbb{R})$$

then defines a *trapezoidal hyperfuzzy number*.

Example 4.4 (Machining Tolerance with Measurement Uncertainty). A factory inspects steel shafts with nominal diameter 100 mm and tolerance ± 2 mm, but measurement gauges introduce additional uncertainty. We model the acceptable diameter x (in mm) as a trapezoidal hyperfuzzy number with

$$a_U = 97, \quad a_L = 98, \quad b_U = 99, \quad b_L = 100, \quad c_L = 102, \quad c_U = 103, \quad d_L = 104, \quad d_U = 105.$$

Lower (strict) membership $\underline{\mu}(x)$:

$$\underline{\mu}(x) = \begin{cases} 0, & x \leq 98, \\ \frac{x-98}{100-98}, & 98 \leq x \leq 100, \\ 1, & 100 \leq x \leq 102, \\ \frac{104-x}{104-102}, & 102 \leq x \leq 104, \\ 0, & x \geq 104. \end{cases}$$

Upper (lenient) membership $\bar{\mu}(x)$:

$$\bar{\mu}(x) = \begin{cases} 0, & x \leq 97, \\ \frac{x-97}{99-97}, & 97 \leq x \leq 99, \\ 1, & 99 \leq x \leq 103, \\ \frac{105-x}{105-103}, & 103 \leq x \leq 105, \\ 0, & x \geq 105. \end{cases}$$

Because $97 \leq 98$, $100 \leq 99$, and $102 \leq 103$, we have $\underline{\mu}(x) \leq \bar{\mu}(x)$. Hence the hyperfuzzy membership is

$$\tilde{\mu}(x) = [\underline{\mu}(x), \bar{\mu}(x)].$$

Interpretation:

- At $x = 100$ mm, $\underline{\mu} = 1$ and $\bar{\mu} = 1$, so full acceptability.
- At $x = 98$ mm, $\underline{\mu}(98) = 0$, $\bar{\mu}(98) = \frac{1}{2}$, so acceptability $[0, 0.5]$.
- At $x = 104$ mm, $\underline{\mu}(104) = 0$, $\bar{\mu}(104) = \frac{1}{2}$, so acceptability $[0, 0.5]$.
- At $x = 97$ mm or $x = 105$ mm, both memberships are 0, so rejection.

This trapezoidal hyperfuzzy model captures both the strict design tolerance (inner plateau $[100,102]$) and the looser gauge uncertainty ($[99,103]$), yielding interval-valued acceptability grades for quality control.

Theorem 4.5. *Every trapezoidal hyperfuzzy number \tilde{A} is a hyperfuzzy number. Moreover:*

- If $a_U = a_L$, $b_U = b_L$, $c_U = c_L$, and $d_U = d_L$, then $\underline{\mu} = \bar{\mu}$ and \tilde{A} reduces to the trapezoidal fuzzy number (a_L, b_L, c_L, d_L) .
- If in addition $b_L = c_L$, $b_U = c_U$, then \tilde{A} reduces to the triangular hyperfuzzy number with parameters $(a_U, a_L, b_L, c_L, d_L, d_U)$.

Proof. We verify the hyperfuzzy-number axioms for \tilde{A} :

Normality. Since $\bar{\mu}(x) = 1$ for all $x \in [b_U, c_U]$, we have $\sup \bigcup_{x \in \mathbb{R}} \tilde{\mu}(x) = 1$.

Hyper-convexity. Let $x, y \in \mathbb{R}$ and $\lambda \in [0, 1]$. Choose any $u \in [\underline{\mu}(x), \bar{\mu}(x)]$ and $v \in [\underline{\mu}(y), \bar{\mu}(y)]$. By convexity of each piecewise-linear function,

$$\underline{\mu}(\lambda x + (1 - \lambda)y) \geq \min\{\underline{\mu}(x), \underline{\mu}(y)\}, \quad \bar{\mu}(\lambda x + (1 - \lambda)y) \geq \min\{\bar{\mu}(x), \bar{\mu}(y)\}.$$

Therefore every $w \in [\underline{\mu}(\lambda x + (1 - \lambda)y), \bar{\mu}(\lambda x + (1 - \lambda)y)]$ satisfies $w \geq \min\{u, v\}$. This establishes hyper-convexity.

Upper semi-continuity. Each α -cut $\{x \mid \exists u \geq \alpha, u \in \tilde{\mu}(x)\}$ is the union of the closed α -cuts of $\underline{\mu}$ and $\bar{\mu}$, hence closed.

Compact support. $\text{supp}(\tilde{A}) = [a_U, d_U]$ is a compact interval.

Thus \tilde{A} satisfies all requirements of a hyperfuzzy number.

For (a), if $a_U = a_L, b_U = b_L, c_U = c_L, d_U = d_L$, then $\underline{\mu}(x) = \bar{\mu}(x)$ equals the standard trapezoidal membership of (a_L, b_L, c_L, d_L) , so $\tilde{\mu}(x) = \{\mu(x)\}$ and \tilde{A} is exactly that trapezoidal fuzzy number.

For (b), further setting $b_L = c_L$ and $b_U = c_U$ collapses the ‘‘core’’ plateau to a point, recovering the triangular hyperfuzzy profile as per the triangular hyperfuzzy definition. \square

4.3 Trapezoidal SuperHyperFuzzy numbers

Below we present the definition of (m, n) -Trapezoidal SuperHyperfuzzy Numbers and a detailed illustrative example.

Definition 4.6 ((m, n) -Trapezoidal SuperHyperfuzzy Number). Let $m, n \geq 0$ be fixed. Choose four nonempty m -level sets

$$X_a, X_b, X_c, X_d \in \mathcal{P}_m^*(\mathbb{R}), \quad X_a \subseteq X_b \subseteq X_c \subseteq X_d,$$

and let

$$\tilde{A}, \bar{\tilde{A}} : \mathcal{P}_m^*(\mathbb{R}) \longrightarrow \mathcal{P}(\mathcal{P}_n([0, 1])) \setminus \{\emptyset\}$$

be two (m, n) -superhyperfuzzy numbers satisfying $\tilde{A}(X) \subseteq \bar{\tilde{A}}(X)$ for all X . Fix a strictly increasing ‘‘size’’ function $\delta : \mathcal{P}_m^*(\mathbb{R}) \rightarrow \mathbb{R}_{\geq 0}$. Define

$$\lambda_1(X) = \frac{\delta(X) - \delta(X_a)}{\delta(X_b) - \delta(X_a)}, \quad \lambda_2(X) = \frac{\delta(X_d) - \delta(X)}{\delta(X_d) - \delta(X_c)}.$$

Then the (m, n) -trapezoidal superhyperfuzzy number $\tilde{A}^{(m, n)}$ is the map $\tilde{\mu}_{m, n} : \mathcal{P}_m^*(\mathbb{R}) \rightarrow \mathcal{P}(\mathcal{P}_n([0, 1]))$ given by

$$\tilde{\mu}_{m, n}(X) = \begin{cases} \tilde{A}(X), & \delta(X) \leq \delta(X_a), \\ (1 - \lambda_1(X)) \tilde{A}(X) \cup \lambda_1(X) \bar{\tilde{A}}(X), & \delta(X_a) \leq \delta(X) \leq \delta(X_b), \\ \bar{\tilde{A}}(X), & \delta(X_b) \leq \delta(X) \leq \delta(X_c), \\ (1 - \lambda_2(X)) \bar{\tilde{A}}(X) \cup \lambda_2(X) \tilde{A}(X), & \delta(X_c) \leq \delta(X) \leq \delta(X_d), \\ \{\emptyset\}, & \delta(X_d) \leq \delta(X). \end{cases}$$

Example 4.7 (Customer Support Response Time under Trapezoidal SuperHyperfuzzy Uncertainty). We model a help-desk's promised response time t (in hours) under hierarchical uncertainty using a (1, 2)-trapezoidal superhyperfuzzy number. Let

$$X_a = [0, 1], \quad X_b = [0, 2], \quad X_c = [0, 4], \quad X_d = [0, 8] \subseteq \mathcal{P}_1^*(\mathbb{R}),$$

with $\delta(X) = \text{length}(X)$.

Lower (conservative) membership families:

$$\underline{\tilde{A}}(X) = \begin{cases} \{\{0.95, 1.00\}, \{0.90, 0.98\}\}, & X = X_a, \\ \{\{0.85, 0.92\}, \{0.80, 0.90\}\}, & X = X_b, \\ \{\{0.70, 0.85\}, \{0.65, 0.80\}\}, & X = X_c, \\ \{\{0.50, 0.70\}, \{0.45, 0.65\}\}, & X = X_d. \end{cases}$$

Upper (optimistic) membership families:

$$\overline{\tilde{A}}(X) = \begin{cases} \{\{0.98, 1.00\}, \{0.93, 0.99\}\}, & X = X_a, \\ \{\{0.90, 0.95\}, \{0.85, 0.93\}\}, & X = X_b, \\ \{\{0.75, 0.88\}, \{0.70, 0.85\}\}, & X = X_c, \\ \{\{0.55, 0.75\}, \{0.50, 0.72\}\}, & X = X_d. \end{cases}$$

By the definition, for any $X \in \mathcal{P}_1^*(\mathbb{R})$:

$$\tilde{\mu}_{1,2}(X) = \begin{cases} \underline{\tilde{A}}(X), & \delta(X) \leq 1, \\ (1 - \lambda_1) \underline{\tilde{A}}(X) \cup \lambda_1 \overline{\tilde{A}}(X), & 1 \leq \delta(X) \leq 2, \\ \overline{\tilde{A}}(X), & 2 \leq \delta(X) \leq 4, \\ (1 - \lambda_2) \overline{\tilde{A}}(X) \cup \lambda_2 \underline{\tilde{A}}(X), & 4 \leq \delta(X) \leq 8, \\ \{\{0\}\}, & \delta(X) \geq 8, \end{cases}$$

where $\lambda_1 = (\delta(X) - 1)/(2 - 1)$, $\lambda_2 = (8 - \delta(X))/(8 - 4)$.

Concrete evaluation: For $X = [0, 1.5]$ ($\delta = 1.5$), $\lambda_1 = 0.5$, so

$$\tilde{\mu}_{1,2}(X) = \frac{1}{2} \underline{\tilde{A}}(X_b) \cup \frac{1}{2} \overline{\tilde{A}}(X_b).$$

Thus the support team's "fair-case" response is an average of conservative $\{0.85, 0.92\}$, $\{0.80, 0.90\}$ and optimistic $\{0.90, 0.95\}$, $\{0.85, 0.93\}$ grades.

Interpretation:

- X_a (urgent): high, tightly-clustered satisfaction intervals (~ 0.95 – 1.00).
- X_b (fast): moderate intervals (~ 0.80 – 0.95).
- X_c (standard): wider intervals (~ 0.65 – 0.88).
- X_d (slow): lower intervals (~ 0.45 – 0.75).
- The trapezoidal structure captures a plateau of optimistic grades for $\delta \in [2, 4]$, blending toward conservative as δ grows beyond 4.

This (1,2)-trapezoidal superhyperfuzzy model provides a nuanced, hierarchical depiction of response-time satisfaction under multiple layers of uncertainty.

Example 4.8 (Cloud Service Uptime under Nested Cluster Uncertainty). Consider a cloud provider with three servers S_1, S_2, S_3 . We examine redundancy clusters at two hierarchical levels ($m = 2$) and model uptime reliability under measurement uncertainty ($n = 2$).

Level-2 sets:

$$\begin{aligned} X_a &= \{\{S_1\}\}, \\ X_b &= \{\{S_1\}, \{S_2\}\}, \\ X_c &= \{\{S_1\}, \{S_2\}, \{S_3\}\}, \\ X_d &= \{\{S_1\}, \{S_2\}, \{S_3\}, \{S_1, S_2\}\}. \end{aligned}$$

Define the size measure $\delta(X) = |X|$, so $\delta(X_a) = 1, \dots, \delta(X_d) = 4$.

Lower membership families $\underline{\tilde{A}}$:

$$\underline{\tilde{A}}(X) = \begin{cases} \{\{0.98, 0.95\}, \{0.95, 0.99\}\}, & X = X_a, \\ \{\{0.95, 0.92\}, \{0.92, 0.96\}\}, & X = X_b, \\ \{\{0.90, 0.88\}, \{0.88, 0.92\}\}, & X = X_c, \\ \{\{0.85, 0.80\}, \{0.80, 0.88\}\}, & X = X_d. \end{cases}$$

Upper membership families $\overline{\tilde{A}}$:

$$\overline{\tilde{A}}(X) = \begin{cases} \{\{1.00, 0.99\}, \{0.97, 0.995\}\}, & X = X_a, \\ \{\{0.99, 0.97\}, \{0.94, 0.97\}\}, & X = X_b, \\ \{\{0.96, 0.93\}, \{0.90, 0.94\}\}, & X = X_c, \\ \{\{0.92, 0.88\}, \{0.85, 0.90\}\}, & X = X_d. \end{cases}$$

Trapezoidal super-combination:

$$\tilde{\mu}_{2,2}(X) = \begin{cases} \underline{\tilde{A}}(X), & \delta(X) \leq 1, \\ (1 - \lambda_1) \underline{\tilde{A}}(X) \cup \lambda_1 \overline{\tilde{A}}(X), & 1 \leq \delta(X) \leq 2, \\ \overline{\tilde{A}}(X), & 2 \leq \delta(X) \leq 3, \\ (1 - \lambda_2) \overline{\tilde{A}}(X) \cup \lambda_2 \underline{\tilde{A}}(X), & 3 \leq \delta(X) \leq 4, \\ \{\{0\}\}, & \delta(X) \geq 4, \end{cases}$$

where $\lambda_1 = (\delta(X) - 1)/(2 - 1)$ and $\lambda_2 = (4 - \delta(X))/(4 - 3)$.

Concrete evaluation:

- For $X = X_a$ ($\delta = 1$), $\tilde{\mu}_{2,2}(X_a) = \underline{\tilde{A}}(X_a)$, reflecting two conservative uptime intervals $\{0.98, 0.95\}$ and $\{0.95, 0.99\}$.
- For $X = X_b$ ($\delta = 2$), $\tilde{\mu}_{2,2}(X_b) = \overline{\tilde{A}}(X_b)$, giving optimistic intervals $\{0.99, 0.97\}$ and $\{0.94, 0.97\}$.
- For $X = X_d$ ($\delta = 4$), $\tilde{\mu}_{2,2}(X_d) = \{\{0\}\}$, indicating no guaranteed reliability under overly broad cluster uncertainty.

This (2, 2)-trapezoidal superhyperfuzzy number captures hierarchical server-cluster reliability with nested conservative and optimistic estimates, blending them linearly as cluster complexity grows.

Theorem 4.9 (Reductions of (m, n) -Trapezoidal SuperHyperfuzzy Numbers). *The (m, n) -trapezoidal superhyperfuzzy number $\tilde{A}^{(m,n)}$ generalizes:*

- (i) Trapezoidal fuzzy numbers: *If $m = n = 0$, then each $X \subseteq \mathbb{R}$ is a singleton $\{x\}$, and $\underline{\tilde{A}}(X) = \{\mu_\ell(x)\}$, $\overline{\tilde{A}}(X) = \{\mu_u(x)\}$ with μ_ℓ, μ_u piecewise-linear as in the trapezoidal fuzzy definition. Moreover $\mu_\ell = \mu_u$ precisely when $\tilde{A}^{(0,0)}$ is the classical trapezoidal fuzzy number (a, b, c, d) .*
- (ii) Trapezoidal hyperfuzzy numbers: *If $m = 0$ but $n \geq 1$, then $\tilde{\mu}_{0,n}(x) = [\underline{\mu}(x), \overline{\mu}(x)] \subseteq [0, 1]$ recovers the trapezoidal hyperfuzzy number of Definition 7.1.*
- (iii) (m, n) -SuperHyperfuzzy numbers: *If $\underline{\tilde{A}}(X) = \overline{\tilde{A}}(X)$ for all X , then every convex combination $(1 - \lambda)\underline{\tilde{A}}(X) \cup \lambda\overline{\tilde{A}}(X)$ reduces to that single value, and $\tilde{A}^{(m,n)}$ is exactly the underlying (m, n) -superhyperfuzzy number.*

Proof. (i) When $m = n = 0$, one has $\mathcal{P}_0^*(\mathbb{R}) = \mathbb{R}$. Writing $X = \{x\}$, define $\mu_\ell(x) = u$ for the unique $u \in \underline{\tilde{A}}(X)$ and similarly $\mu_u(x)$ from $\overline{\tilde{A}}(X)$. The piecewise definition of $\tilde{\mu}_{0,0}(x)$ then matches the classical trapezoidal form. If $\mu_\ell = \mu_u$, the lower and upper maps coincide, yielding exactly one membership value $\mu(x)$, i.e. the crisp trapezoidal fuzzy number (a, b, c, d) .

(ii) If $m = 0$, then each $X = \{x\}$ and $\tilde{\mu}_{0,n}(x) = (\underline{\mu}(x), \overline{\mu}(x))$ is the interval of possible membership grades, with $\underline{\mu}, \overline{\mu}$ piecewise-linear and plateauing on $[b, c]$. This recovers precisely the trapezoidal hyperfuzzy number.

(iii) In the general (m, n) case, if $\underline{\tilde{A}}(X) = \overline{\tilde{A}}(X)$ for all X , then each convex combination in the definition collapses to that common set. Hence $\tilde{\mu}_{m,n}(X)$ agrees with the single-valued (m, n) -superhyperfuzzy number $\tilde{A}(X)$, completing the reduction. \square

5 Pentagonal fuzzy numbers

5.1 Pentagonal Fuzzy numbers

A pentagonal fuzzy number employs five parameters to define a membership function with two increasing, one constant, and two decreasing segments [76–78].

Definition 5.1 (Pentagonal Fuzzy Number). [79–81] A *pentagonal fuzzy number* on \mathbb{R} is a quintuple

$$\tilde{A} = (a_1, a_2, a_3, a_4, a_5), \quad a_1 \leq a_2 \leq a_3 \leq a_4 \leq a_5,$$

equipped with a continuous membership function $\mu_{\tilde{A}} : \mathbb{R} \rightarrow [0, 1]$ of the form

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & x \leq a_1, \\ f_1(x), & a_1 \leq x \leq a_2, \\ f_2(x), & a_2 \leq x \leq a_3, \\ f_3(x), & a_3 \leq x \leq a_4, \\ f_4(x), & a_4 \leq x \leq a_5, \\ 0, & x \geq a_5, \end{cases}$$

where

$$0 = f_1(a_1) < f_1(a_2) \leq f_2(a_2) < f_2(a_3) = 1, \quad 1 = f_3(a_3) > f_3(a_4) \geq f_4(a_4) > f_4(a_5) = 0,$$

with f_1, f_2 strictly increasing and f_3, f_4 strictly decreasing on their respective intervals.

Example 5.2 (Customer Delivery Satisfaction Model). Consider modeling customer delivery satisfaction as the pentagonal fuzzy number $\tilde{A} = (1, 3, 5, 7, 10)$ (in days). Its membership function $\mu_{\tilde{A}}(x)$ is

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & x \leq 1, \\ \frac{x-1}{3-1} \cdot 0.5, & 1 < x < 3, \\ \frac{x-3}{5-3} \cdot 0.5 + 0.5, & 3 \leq x \leq 5, \\ \frac{7-x}{7-5} \cdot 0.5 + 0.5, & 5 < x < 7, \\ \frac{10-x}{10-7} \cdot 0.5, & 7 \leq x < 10, \\ 0, & x \geq 10. \end{cases}$$

Here:

- $x \leq 1$: satisfaction is 0 (too early).
- $1 < x \leq 3$: satisfaction increases to 0.5.
- $3 < x \leq 5$: satisfaction increases to 1.
- $5 < x \leq 7$: satisfaction decreases to 0.5.
- $7 < x \leq 10$: satisfaction decreases to 0.
- $x \geq 10$: satisfaction is 0 (too late).

This pentagonal fuzzy model captures customers' graded satisfaction for early, on-time, and late deliveries.

5.2 Pentagonal Hyperfuzzy numbers

Below we present the definition of *Pentagonal Hyperfuzzy numbers* and a detailed illustrative example.

Definition 5.3 (Pentagonal Hyperfuzzy Number). Let

$$a_1 \leq a_2 \leq a_3 \leq a_4 \leq a_5, \quad b_1 \leq b_2 \leq b_3 \leq b_4 \leq b_5,$$

be two nondecreasing sequences of real numbers satisfying $a_i \leq b_i$ for $i = 1, \dots, 5$. Define two pentagonal membership functions $\underline{\mu}, \bar{\mu} : \mathbb{R} \rightarrow [0, 1]$ by

$$\underline{\mu}(x) = \begin{cases} 0, & x \leq a_1, \\ \frac{x-a_1}{a_2-a_1}, & a_1 < x < a_2, \\ \frac{a_2-a_1}{x-a_2}, & a_2 \leq x \leq a_3, \\ \frac{a_3-a_2}{a_4-x}, & a_3 < x < a_4, \\ \frac{a_4-a_3}{a_5-x}, & a_4 \leq x < a_5, \\ \frac{a_5-a_4}{a_5-x}, & a_4 < x < a_5, \\ 0, & x \geq a_5, \end{cases} \quad \bar{\mu}(x) = \begin{cases} 0, & x \leq b_1, \\ \frac{x-b_1}{b_2-b_1}, & b_1 < x < b_2, \\ \frac{b_2-b_1}{x-b_2}, & b_2 \leq x \leq b_3, \\ \frac{b_3-b_2}{b_4-x}, & b_3 < x < b_4, \\ \frac{b_4-b_3}{b_5-x}, & b_4 \leq x < b_5, \\ \frac{b_5-b_4}{b_5-x}, & b_4 < x < b_5, \\ 0, & x \geq b_5. \end{cases}$$

Since $a_i \leq b_i$ implies $\underline{\mu}(x) \leq \bar{\mu}(x)$ for all x , the map

$$\tilde{\mu}(x) = [\underline{\mu}(x), \bar{\mu}(x)] \subseteq [0, 1]$$

defines a hyperfuzzy membership. The pair $\tilde{A} = (\tilde{\mu})$ is called a *pentagonal hyperfuzzy number*.

Example 5.4 (Hyperfuzzy Customer Delivery Satisfaction). Suppose a logistics company models customer satisfaction with shipment time x (in days) as a pentagonal hyperfuzzy number. Let the *lower* satisfaction profile be determined by

$$(a_1, a_2, a_3, a_4, a_5) = (1, 2, 4, 6, 8),$$

and the *upper* (optimistic) profile by

$$(b_1, b_2, b_3, b_4, b_5) = (2, 3, 5, 7, 9).$$

Then the lower membership function is

$$\underline{\mu}(x) = \begin{cases} 0, & x \leq 1, \\ \frac{x-1}{2-1}, & 1 \leq x \leq 2, \\ \frac{x-2}{4-2}, & 2 \leq x \leq 4, \\ \frac{4-x}{6-4}, & 4 \leq x \leq 6, \\ \frac{6-x}{8-6}, & 6 \leq x \leq 8, \\ 0, & x \geq 8, \end{cases}$$

and the upper membership function is

$$\bar{\mu}(x) = \begin{cases} 0, & x \leq 2, \\ \frac{x-2}{3-2}, & 2 \leq x \leq 3, \\ \frac{x-3}{5-3}, & 3 \leq x \leq 5, \\ \frac{5-x}{7-5}, & 5 \leq x \leq 7, \\ \frac{7-x}{9-7}, & 7 \leq x \leq 9, \\ 0, & x \geq 9. \end{cases}$$

Thus the hyperfuzzy satisfaction is the interval

$$\tilde{\mu}(x) = [\underline{\mu}(x), \bar{\mu}(x)].$$

For example:

- If $x = 3$ days, $\underline{\mu}(3) = \frac{3-2}{4-2} = 0.5$ and $\bar{\mu}(3) = \frac{3-2}{3-2} = 1$, so satisfaction lies in $[0.5, 1]$.
- If $x = 6$ days, $\underline{\mu}(6) = 0$ and $\bar{\mu}(6) = \frac{7-6}{7-5} = 0.5$, so satisfaction lies in $[0, 0.5]$.

This pentagonal hyperfuzzy model captures both conservative and optimistic customer satisfaction evaluations under delivery-time uncertainty.

Theorem 5.5 (Generalization of Pentagonal Fuzzy and Hyperfuzzy Numbers). *Let \mathcal{PF} be the class of pentagonal fuzzy numbers and \mathcal{HF} the class of hyperfuzzy numbers. Denote by \mathcal{PHF} the class of pentagonal hyperfuzzy numbers. Then*

$$\mathcal{PF} \subseteq \mathcal{PHF} \subseteq \mathcal{HF}.$$

Proof. (i) $\mathcal{PF} \subseteq \mathcal{PHF}$. Let $A = (a_1, \dots, a_5)$ be a pentagonal fuzzy number with membership $\mu_A(x)$. Define $\underline{\mu}(x) = \bar{\mu}(x) = \mu_A(x)$. Then $\tilde{\mu}(x) = [\mu_A(x), \mu_A(x)]$ satisfies the definition of a pentagonal hyperfuzzy number with $b_i = a_i$. Thus every pentagonal fuzzy number yields a pentagonal hyperfuzzy number.

(ii) $\mathcal{PHF} \subseteq \mathcal{HF}$. By construction, a pentagonal hyperfuzzy number \tilde{A} is a hyperfuzzy set on \mathbb{R} (its membership at each x is the nonempty interval $[\underline{\mu}(x), \bar{\mu}(x)]$). It therefore satisfies the axioms of a hyperfuzzy number (normality, hyper-convexity, upper semi-continuity, compact support). Hence $\tilde{A} \in \mathcal{HF}$. \square

5.3 Pentagonal SuperHyperFuzzy numbers

Below we present the definition of (m, n) -Pentagonal SuperHyperfuzzy Numbers and a detailed illustrative example.

Definition 5.6 ((m, n) -Pentagonal SuperHyperfuzzy Number). Let $m, n \geq 0$. Choose five nonempty m -level sets

$$X_1 \subseteq X_2 \subseteq X_3 \subseteq X_4 \subseteq X_5 \in \mathcal{P}_m^*(\mathbb{R}),$$

and two (m, n) -superhyperfuzzy maps

$$\underline{\tilde{A}}, \overline{\tilde{A}}: \mathcal{P}_m^*(\mathbb{R}) \longrightarrow \mathcal{P}(\mathcal{P}_n([0, 1])) \setminus \{\emptyset\}$$

satisfying $\underline{\tilde{A}}(X) \subseteq \overline{\tilde{A}}(X)$ for every X . Let $\delta: \mathcal{P}_m^*(\mathbb{R}) \rightarrow \mathbb{R}_{\geq 0}$ be a strictly increasing size measure, and define four weight functions

$$\lambda_i(X) = \frac{\delta(X) - \delta(X_i)}{\delta(X_{i+1}) - \delta(X_i)}, \quad i = 1, 2, 3, 4,$$

whenever $\delta(X_i) \leq \delta(X) \leq \delta(X_{i+1})$. Then the (m, n) -pentagonal superhyperfuzzy number $\tilde{A}^{(m, n)}$ is the map

$$\tilde{\mu}_{m, n}(X) = \begin{cases} \{\{0\}\}, & \delta(X) \leq \delta(X_1), \\ \lambda_1(X) \underline{\tilde{A}}(X), & \delta(X_1) \leq \delta(X) \leq \delta(X_2), \\ (1 - \lambda_2(X)) \underline{\tilde{A}}(X) \cup \lambda_2(X) \overline{\tilde{A}}(X), & \delta(X_2) \leq \delta(X) \leq \delta(X_3), \\ (1 - \lambda_3(X)) \overline{\tilde{A}}(X) \cup \lambda_3(X) \underline{\tilde{A}}(X), & \delta(X_3) \leq \delta(X) \leq \delta(X_4), \\ (1 - \lambda_4(X)) \underline{\tilde{A}}(X), & \delta(X_4) \leq \delta(X) \leq \delta(X_5), \\ \{\{0\}\}, & \delta(X) \geq \delta(X_5). \end{cases}$$

Example 5.7 (Battery Reliability in Usage Contexts). Fix $m = n = 1$. Let the universe of usage scenarios be

$$U = \{\text{Idle, Light, Moderate, Heavy, Emergency}\},$$

and choose the following nonempty 1-level sets:

$$\begin{aligned} X_1 &= \{\text{Idle}\}, \\ X_2 &= \{\text{Idle, Light}\}, \\ X_3 &= \{\text{Idle, Light, Moderate}\}, \\ X_4 &= \{\text{Idle, Light, Moderate, Heavy}\}, \\ X_5 &= U. \end{aligned}$$

Define the size measure $\delta(X) = |X|$. For each X_i set

$$\underline{\tilde{A}}(X_i) = \{\{r_i^\ell\}\}, \quad \overline{\tilde{A}}(X_i) = \{\{r_i^u\}\},$$

where

$$\begin{aligned} (r_1^\ell, r_1^u) &= (0.90, 1.00), \\ (r_2^\ell, r_2^u) &= (0.80, 0.95), \\ (r_3^\ell, r_3^u) &= (0.70, 0.85), \\ (r_4^\ell, r_4^u) &= (0.60, 0.75), \\ (r_5^\ell, r_5^u) &= (0.50, 0.65). \end{aligned}$$

Then for any $X \in \mathcal{P}_1^*(U)$, the pentagonal-style supercombination $\tilde{\mu}_{1, 1}(X)$ interpolates between $\underline{\tilde{A}}(X)$ and $\overline{\tilde{A}}(X)$ according to Definition 4.1. For instance, if $\delta(X_2) = 2 \leq \delta(X) = 3 \leq \delta(X_3) = 3$, then

$$\tilde{\mu}_{1, 1}(X) = (1 - \lambda_2) \underline{\tilde{A}}(X_2) \cup \lambda_2 \overline{\tilde{A}}(X_3), \quad \lambda_2 = \frac{\delta(X) - \delta(X_2)}{\delta(X_3) - \delta(X_2)} = 0.$$

Hence $\tilde{\mu}_{1, 1}(X) = \underline{\tilde{A}}(X_2) = \{\{0.80\}\}$, reflecting conservative reliability when grouping two scenarios. This example illustrates how $(1, 1)$ -pentagonal superhyperfuzzy numbers capture hierarchical, context-dependent uncertainty in battery performance.

Example 5.8 (EV Battery Reliability under Nested Driving Modes). Consider modeling the reliability of an electric-vehicle battery under combinations of driving modes. Let the base set of modes be

$$U = \{\text{City, Highway, Mixed}\}.$$

For $m = 2$, we work in $\mathcal{P}_2^*(U)$, the nonempty collections of nonempty mode-subsets. Choose five nested 2-level sets:

$$\begin{aligned} X_1 &= \{\{\text{City}\}\}, \\ X_2 &= \{\{\text{City}\}, \{\text{Highway}\}\}, \\ X_3 &= \{\{\text{City}\}, \{\text{Highway}\}, \{\text{Mixed}\}\}, \\ X_4 &= \{\{\text{City}\}, \{\text{Highway}\}, \{\text{Mixed}\}, \{\text{City, Highway}\}\}, \\ X_5 &= \mathcal{P}(U) \setminus \{\emptyset\}. \end{aligned}$$

Define the size measure $\delta(X) = |X|$. For each $i = 1, \dots, 5$, let the lower and upper super-membership maps be singleton families of two possible reliability grades:

$$\underline{\tilde{A}}(X_i) = \{\{\ell_i, \ell_i + 0.05\}\}, \quad \overline{\tilde{A}}(X_i) = \{\{u_i, u_i - 0.05\}\},$$

with

$$(\ell_i, u_i) = (0.90, 1.00), (0.80, 0.95), (0.70, 0.85), (0.60, 0.80), (0.50, 0.75)$$

for $i = 1, \dots, 5$. Then for any $X \in \mathcal{P}_2^*(U)$, the $(2, 2)$ -pentagonal combination $\tilde{\mu}_{2,2}(X)$ interpolates between $\underline{\tilde{A}}(X)$ and $\overline{\tilde{A}}(X)$ as in Definition 4.1.

For instance, if $X = X_2$ (so $\delta(X) = 2$), then

$$\lambda_1(X) = \frac{\delta(X) - \delta(X_1)}{\delta(X_2) - \delta(X_1)} = \frac{2 - 1}{2 - 1} = 1,$$

and thus

$$\tilde{\mu}_{2,2}(X) = \lambda_1(X) \underline{\tilde{A}}(X_1) = \underline{\tilde{A}}(X_1) = \{\{0.90, 0.95\}\}.$$

This means that when considering only City and Highway modes as separate subsets, the battery's reliability is estimated conservatively between 0.90 and 0.95. In contrast, for $X = X_5$ (all mode-subsets), $\tilde{\mu}_{2,2}(X_5) = \{\{0.50, 0.75\}\}$, reflecting the widest uncertainty under the most complex mode combinations.

Theorem 5.9 (Reduction to Pentagonal Fuzzy, Hyperfuzzy, and SuperHyperfuzzy). *The above (m, n) -pentagonal superhyperfuzzy number $\tilde{A}^{(m,n)}$ specializes as follows:*

- (i) *If $m = n = 0$, then each $X \subseteq \mathbb{R}$ is a singleton $\{x\}$, and $\underline{\tilde{A}}(\{x\}) = \{\mu_\ell(x)\}$, $\overline{\tilde{A}}(\{x\}) = \{\mu_u(x)\}$ define a classical pentagonal fuzzy membership via μ_ℓ, μ_u . Moreover $\mu_\ell = \mu_u$ yields exactly the standard pentagonal fuzzy number (a_1, \dots, a_5) .*
- (ii) *If $m = 0$ and $n \geq 1$, then for each x the set $\tilde{\mu}_{0,n}(x) \subseteq [0, 1]$ is exactly the interval $[\underline{\mu}(x), \overline{\mu}(x)]$ of a pentagonal hyperfuzzy number.*
- (iii) *If $\underline{\tilde{A}}(X) = \overline{\tilde{A}}(X)$ for all X , then every convex combination in the definition collapses, and $\tilde{A}^{(m,n)}$ reduces to the underlying (m, n) -superhyperfuzzy number \tilde{A} .*

Proof. (i) When $m = n = 0$, one has $\mathcal{P}_0^*(\mathbb{R}) = \mathbb{R}$. Writing $X = \{x\}$, set $\mu_\ell(x)$ and $\mu_u(x)$ as the unique elements of $\underline{\tilde{A}}(X)$ and $\overline{\tilde{A}}(X)$. The piecewise definition of $\tilde{\mu}_{0,0}(x)$ then recovers the five-segment pentagonal shape, and if $\mu_\ell = \mu_u$ this coincides with the classical pentagonal fuzzy number.

(ii) If $m = 0$, then each $X = \{x\}$ and $\tilde{\mu}_{0,n}(x)$ is the union of all $U \in \underline{\tilde{A}}(X) \cup \overline{\tilde{A}}(X)$ according to the piecewise weights. One checks that this yields exactly the interval $[\underline{\mu}(x), \overline{\mu}(x)]$ of a pentagonal hyperfuzzy number as in the previous definition.

(iii) If for every X we have $\underline{\tilde{A}}(X) = \overline{\tilde{A}}(X)$, then each convex-combination step $(1 - \lambda_i)\underline{\tilde{A}}(X) \cup \lambda_i\overline{\tilde{A}}(X)$ collapses to that single set. Hence $\tilde{\mu}_{m,n}(X) = \underline{\tilde{A}}(X)$ for all X , showing that $\tilde{A}^{(m,n)}$ is exactly the given (m, n) -superhyperfuzzy number. \square

6 Hexagonal fuzzy numbers

6.1 Hexagonal Fuzzy numbers

A hexagonal fuzzy number uses six parameters to define two distinct full-membership plateaux and three linear transition segments for membership [82–84].

Definition 6.1 (Hexagonal Fuzzy Number). (cf. [85–87]) A *hexagonal fuzzy number* \tilde{H} on \mathbb{R} is a fuzzy set whose membership function $\mu_{\tilde{H}} : \mathbb{R} \rightarrow [0, 1]$ is determined by six real parameters

$$a_1 \leq a_2 \leq a_3 \leq a_4 \leq a_5 \leq a_6$$

as follows:

$$\mu_{\tilde{H}}(x) = \begin{cases} 0, & x \leq a_1, \\ \frac{x - a_1}{a_2 - a_1}, & a_1 < x \leq a_2, \\ 1, & a_2 < x \leq a_3, \\ \frac{a_4 - x}{a_4 - a_3}, & a_3 < x \leq a_4, \\ 1, & a_4 < x \leq a_5, \\ \frac{a_6 - x}{a_6 - a_5}, & a_5 < x \leq a_6, \\ 0, & x > a_6. \end{cases}$$

Here the function rises linearly from 0 to 1 on $[a_1, a_2]$, remains at 1 on $[a_2, a_3]$, decreases linearly on $[a_3, a_4]$, again stays at 1 on $[a_4, a_5]$, and finally decreases to 0 on $[a_5, a_6]$.

Example 6.2 (Optimal Relative Humidity for Archival Preservation). A museum conservator models the “suitability” of relative humidity (RH, in %) for preserving ancient manuscripts. Experts agree there are two distinct optimal ranges: moderate (40 – 55 %) and high (65 – 80 %). A hexagonal fuzzy number \tilde{H} captures this:

$$a_1 = 30, \quad a_2 = 40, \quad a_3 = 55, \quad a_4 = 60, \quad a_5 = 65, \quad a_6 = 80.$$

The membership function $\mu_{\tilde{H}}(x)$ is

$$\mu_{\tilde{H}}(x) = \begin{cases} 0, & x \leq 30, \\ \frac{x - 30}{40 - 30}, & 30 < x \leq 40, \\ 1, & 40 < x \leq 55, \\ \frac{60 - x}{60 - 55}, & 55 < x \leq 60, \\ 1, & 60 < x \leq 65, \\ \frac{80 - x}{80 - 65}, & 65 < x \leq 80, \\ 0, & x > 80. \end{cases}$$

Interpretation:

- RH 30 % or below: unsuitable ($\mu = 0$).
- RH increases from 30 to 40 %: suitability rises linearly $0 \rightarrow 1$.
- RH 40–55 %: fully suitable plateau ($\mu = 1$).
- RH 55–60 %: suitability drops linearly $1 \rightarrow 0$.
- RH 60–65 %: second plateau of full suitability ($\mu = 1$).

- RH 65–80 %: suitability decreases linearly $1 \rightarrow 0$.
- RH 80 % or above: unsuitable ($\mu = 0$).

This hexagonal fuzzy model reflects two expert-identified humidity windows that maximize preservation quality, with rapid loss of suitability between them.

6.2 Hexagonal Hyperfuzzy numbers

Below we present the definition of *hexagonal hyperfuzzy numbers* and a detailed illustrative example.

Definition 6.3 (Hexagonal Hyperfuzzy Number). Let

$$a_1 \leq a_2 \leq a_3 \leq a_4 \leq a_5 \leq a_6, \quad b_1 \leq b_2 \leq b_3 \leq b_4 \leq b_5 \leq b_6,$$

be real parameters satisfying $a_i \leq b_i$ for $i = 1, \dots, 6$. Define two piecewise-linear membership functions

$$\underline{\mu}(x) = \begin{cases} 0, & x \leq a_1, \\ \frac{x - a_1}{a_2 - a_1}, & a_1 \leq x \leq a_2, \\ 1, & a_2 \leq x \leq a_3, \\ \frac{a_4 - x}{a_4 - a_3}, & a_3 \leq x \leq a_4, \\ 1, & a_4 \leq x \leq a_5, \\ \frac{a_6 - x}{a_6 - a_5}, & a_5 \leq x \leq a_6, \\ 0, & x \geq a_6, \end{cases} \quad \bar{\mu}(x) = \begin{cases} 0, & x \leq b_1, \\ \frac{x - b_1}{b_2 - b_1}, & b_1 \leq x \leq b_2, \\ 1, & b_2 \leq x \leq b_3, \\ \frac{b_4 - x}{b_4 - b_3}, & b_3 \leq x \leq b_4, \\ 1, & b_4 \leq x \leq b_5, \\ \frac{b_6 - x}{b_6 - b_5}, & b_5 \leq x \leq b_6, \\ 0, & x \geq b_6. \end{cases}$$

Since $a_i \leq b_i$ implies $\underline{\mu}(x) \leq \bar{\mu}(x)$ for all x , the interval

$$\tilde{\mu}(x) = [\underline{\mu}(x), \bar{\mu}(x)] \subseteq [0, 1]$$

defines a *hexagonal hyperfuzzy number*.

Example 6.4 (Archival Humidity Suitability as a Hexagonal Hyperfuzzy Number). A museum conservator assesses relative humidity (RH, in %) for preserving rare manuscripts. Expert studies define two optimal RH plateaux: moderate (40–55 %) and high (65–80 %). Measurement and modeling uncertainties widen these ranges. We model suitability by a hexagonal hyperfuzzy number \tilde{H} with

$$a_1 = 30, \quad a_2 = 40, \quad a_3 = 55, \quad a_4 = 60, \quad a_5 = 65, \quad a_6 = 80, \\ b_1 = 32, \quad b_2 = 42, \quad b_3 = 57, \quad b_4 = 62, \quad b_5 = 67, \quad b_6 = 82,$$

so $a_i \leq b_i$ for $i = 1, \dots, 6$. The lower (pessimistic) membership is

$$\underline{\mu}(x) = \begin{cases} 0, & x \leq 30, \\ \frac{x - 30}{40 - 30}, & 30 \leq x \leq 40, \\ 1, & 40 \leq x \leq 55, \\ \frac{60 - x}{60 - 55}, & 55 \leq x \leq 60, \\ 1, & 60 \leq x \leq 65, \\ \frac{80 - x}{80 - 65}, & 65 \leq x \leq 80, \\ 0, & x \geq 80, \end{cases}$$

and the upper (optimistic) membership is

$$\bar{\mu}(x) = \begin{cases} 0, & x \leq 32, \\ \frac{x-32}{42-32}, & 32 \leq x \leq 42, \\ 1, & 42 \leq x \leq 57, \\ \frac{62-x}{62-57}, & 57 \leq x \leq 62, \\ 1, & 62 \leq x \leq 67, \\ \frac{82-x}{82-67}, & 67 \leq x \leq 82, \\ 0, & x \geq 82. \end{cases}$$

Since $a_i \leq b_i$, we have $\underline{\mu}(x) \leq \bar{\mu}(x)$ for all x . Therefore

$$\tilde{\mu}(x) = [\underline{\mu}(x), \bar{\mu}(x)] \subseteq [0, 1]$$

defines a *hexagonal hyperfuzzy number* capturing both strict and lenient suitability assessments.

Sample evaluations:

- At $x = 50\%$, $\underline{\mu}(50) = 1$, $\bar{\mu}(50) = 1$, so suitability is $[1, 1]$.
- At $x = 57\%$, $\underline{\mu}(57) = \frac{60-57}{5} = 0.6$, $\bar{\mu}(57) = 1$, so suitability is $[0.6, 1]$.
- At $x = 35\%$, $\underline{\mu}(35) = \frac{35-30}{10} = 0.5$, $\bar{\mu}(35) = \frac{35-32}{10} = 0.3$, so suitability is $[0.5, 0.3]$ clipped to $[0.3, 0.5]$.
- At $x = 82\%$, $\underline{\mu}(82) = 0$, $\bar{\mu}(82) = 0$, so suitability is $[0, 0]$.

This hexagonal hyperfuzzy model flexibly represents two expert-identified preservation windows and quantifies both conservative and optimistic membership under measurement uncertainty.

Theorem 6.5 (Generalization of Hexagonal Fuzzy and Hyperfuzzy Numbers). *Let $\mathcal{HF}(\mathbb{R})$ be the class of hyperfuzzy numbers and $\mathcal{HF}_{\text{hex}}(\mathbb{R})$ its hexagonal subclass. Then*

$$\mathcal{HF}_{\text{hex}}(\mathbb{R}) \subseteq \mathcal{HF}(\mathbb{R}),$$

and if in addition $a_i = b_i$ for all i , each hexagonal hyperfuzzy number reduces to a hexagonal fuzzy number.

Proof. **1. Hyperfuzzy axioms.** Let $\tilde{\mu}$ be defined as above.

- *Normality:* On $[b_2, b_3]$ we have $\bar{\mu}(x) = 1$, so $\sup \bigcup_x \tilde{\mu}(x) = 1$.
- *Hyper-convexity:* For any $x, y \in \mathbb{R}$ and $\lambda \in [0, 1]$, pick $u \in [\underline{\mu}(x), \bar{\mu}(x)]$ and $v \in [\underline{\mu}(y), \bar{\mu}(y)]$. By piecewise convexity of $\underline{\mu}$ and $\bar{\mu}$, $\underline{\mu}(\lambda x + (1-\lambda)y) \geq \min\{\underline{\mu}(x), \underline{\mu}(y)\}$ and similarly for $\bar{\mu}$. Hence any $w \in \tilde{\mu}(\lambda x + (1-\lambda)y)$ satisfies $w \geq \min\{u, v\}$.
- *Upper semi-continuity:* Each α -cut $\{x \mid \exists u \geq \alpha, u \in \tilde{\mu}(x)\}$ is the union of closed α -cuts of $\underline{\mu}$ and $\bar{\mu}$, thus closed.
- *Compact support:* $\text{supp}(\tilde{\mu}) = [a_1, b_6]$ is compact.

2. Reduction to hexagonal fuzzy. If $a_i = b_i$ for all i , then $\underline{\mu} = \bar{\mu}$ coincides with the classical hexagonal fuzzy membership, so $\tilde{\mu}(x) = \{\mu(x)\}$ defines a hexagonal fuzzy number. \square

6.3 Hexagonal SuperHyperFuzzy numbers

Below we present the definition of *hexagonal superhyperfuzzy numbers* and a detailed illustrative example.

Definition 6.6 ((m, n) -Hexagonal SuperHyperfuzzy Number). Fix nonnegative integers m, n . Let

$$X_{a_1} \subseteq X_{a_2} \subseteq X_{a_3} \subseteq X_{a_4} \subseteq X_{a_5} \subseteq X_{a_6} \in \mathcal{P}_m^*(\mathbb{R}),$$

and let

$$\underline{\tilde{A}}, \bar{\tilde{A}} : \mathcal{P}_m^*(\mathbb{R}) \longrightarrow \mathcal{P}(\mathcal{P}_n([0, 1])) \setminus \{\emptyset\}$$

be two (m, n) -superhyperfuzzy membership maps satisfying $\underline{\tilde{A}}(X) \subseteq \bar{\tilde{A}}(X)$ for every X . Choose a strictly increasing size measure $\delta : \mathcal{P}_m^*(\mathbb{R}) \rightarrow \mathbb{R}_{\geq 0}$. Define weight functions

$$\lambda_1(X) = \frac{\delta(X) - \delta(X_{a_1})}{\delta(X_{a_2}) - \delta(X_{a_1})}, \quad \lambda_2(X) = \frac{\delta(X) - \delta(X_{a_3})}{\delta(X_{a_4}) - \delta(X_{a_3})}, \quad \lambda_3(X) = \frac{\delta(X) - \delta(X_{a_5})}{\delta(X_{a_6}) - \delta(X_{a_5})}.$$

Then the (m, n) -hexagonal superhyperfuzzy number $\tilde{A}^{(m,n)}$ is the map $\tilde{\mu}_{m,n} : \mathcal{P}_m^*(\mathbb{R}) \rightarrow \mathcal{P}(\mathcal{P}_n([0, 1]))$ given by

$$\tilde{\mu}_{m,n}(X) = \begin{cases} \{\{0\}\}, & \delta(X) \leq \delta(X_{a_1}), \\ (1 - \lambda_1(X)) \underline{\tilde{A}}(X) \cup \lambda_1(X) \bar{\tilde{A}}(X), & \delta(X_{a_1}) \leq \delta(X) \leq \delta(X_{a_2}), \\ \bar{\tilde{A}}(X), & \delta(X_{a_2}) \leq \delta(X) \leq \delta(X_{a_3}), \\ (1 - \lambda_2(X)) \bar{\tilde{A}}(X) \cup \lambda_2(X) \underline{\tilde{A}}(X), & \delta(X_{a_3}) \leq \delta(X) \leq \delta(X_{a_4}), \\ \bar{\tilde{A}}(X), & \delta(X_{a_4}) \leq \delta(X) \leq \delta(X_{a_5}), \\ (1 - \lambda_3(X)) \bar{\tilde{A}}(X) \cup \lambda_3(X) \underline{\tilde{A}}(X), & \delta(X_{a_5}) \leq \delta(X) \leq \delta(X_{a_6}), \\ \{\{0\}\}, & \delta(X) \geq \delta(X_{a_6}). \end{cases}$$

Example 6.7 (Network Latency Quality under Hierarchical Uncertainty). We model end-to-end network latency t (in milliseconds) as a $(1, 2)$ -hexagonal superhyperfuzzy number, capturing six nested latency regimes and two levels of uncertainty (sensor vs. expert).

Level-1 sets (latency regimes):

$$\begin{aligned} X_{a_1} &= [0, 50] \quad (\text{“Ultra-Low”}), \\ X_{a_2} &= [0, 100] \quad (\text{“Low”}), \\ X_{a_3} &= [0, 200] \quad (\text{“Moderate”}), \\ X_{a_4} &= [0, 500] \quad (\text{“High”}), \\ X_{a_5} &= [0, 1000] \quad (\text{“Severe”}), \\ X_{a_6} &= [0, 2000] \quad (\text{“Extreme”}). \end{aligned}$$

Clearly $X_{a_1} \subseteq X_{a_2} \subseteq \dots \subseteq X_{a_6}$. Let $\delta(X) = \text{length}(X)$.

Lower-level membership families $\underline{\tilde{A}}$:

$$\underline{\tilde{A}}(X_{a_i}) = \{U_{i,1}, U_{i,2}\}, \quad U_{i,1}, U_{i,2} \subseteq [0, 1],$$

with

$$\begin{aligned} U_{1,1} &= \{0.90, 0.95\}, & U_{1,2} &= \{0.85, 0.92\}, \\ U_{2,1} &= \{0.80, 0.88\}, & U_{2,2} &= \{0.75, 0.85\}, \\ U_{3,1} &= \{0.60, 0.75\}, & U_{3,2} &= \{0.55, 0.70\}, \\ U_{4,1} &= \{0.40, 0.60\}, & U_{4,2} &= \{0.35, 0.55\}, \\ U_{5,1} &= \{0.20, 0.40\}, & U_{5,2} &= \{0.15, 0.35\}, \\ U_{6,1} &= \{0.00, 0.20\}, & U_{6,2} &= \{0.00, 0.10\}. \end{aligned}$$

Upper-level membership families \bar{A} :

$$\bar{A}(X_{a_i}) = \{V_{i,1}, V_{i,2}\},$$

with

$$\begin{aligned} V_{1,1} &= \{0.95, 1.00\}, & V_{1,2} &= \{0.90, 0.98\}, \\ V_{2,1} &= \{0.88, 0.95\}, & V_{2,2} &= \{0.82, 0.92\}, \\ V_{3,1} &= \{0.70, 0.82\}, & V_{3,2} &= \{0.65, 0.78\}, \\ V_{4,1} &= \{0.55, 0.68\}, & V_{4,2} &= \{0.50, 0.65\}, \\ V_{5,1} &= \{0.30, 0.50\}, & V_{5,2} &= \{0.25, 0.45\}, \\ V_{6,1} &= \{0.10, 0.30\}, & V_{6,2} &= \{0.05, 0.25\}. \end{aligned}$$

Hexagonal superhyperfuzzy membership $\tilde{\mu}_{1,2}(X)$: For any $X \subseteq \mathbb{R}$,

$$\tilde{\mu}_{1,2}(X) = \begin{cases} \{\{0\}\}, & \delta(X) \leq 50, \\ (1 - \lambda_1) \underline{\tilde{A}}(X) \cup \lambda_1 \bar{\tilde{A}}(X), & 50 \leq \delta(X) \leq 100, \\ \bar{\tilde{A}}(X), & 100 \leq \delta(X) \leq 200, \\ (1 - \lambda_2) \bar{\tilde{A}}(X) \cup \lambda_2 \underline{\tilde{A}}(X), & 200 \leq \delta(X) \leq 500, \\ \underline{\tilde{A}}(X), & 500 \leq \delta(X) \leq 1000, \\ (1 - \lambda_3) \underline{\tilde{A}}(X) \cup \lambda_3 \bar{\tilde{A}}(X), & 1000 \leq \delta(X) \leq 2000, \\ \{\{0\}\}, & \delta(X) \geq 2000, \end{cases}$$

where

$$\lambda_1 = \frac{\delta(X) - 50}{50}, \quad \lambda_2 = \frac{\delta(X) - 200}{300}, \quad \lambda_3 = \frac{\delta(X) - 1000}{1000}.$$

Interpretation:

- $\delta \leq 50$ ms: “Ultra-Low” regime, no guaranteed quality ($\mu = \{0\}$).
- $50 \leq \delta \leq 100$: blend conservative vs. optimistic intervals for “Low” latency.
- $100 \leq \delta \leq 200$: optimistic “Moderate” regime.
- $200 \leq \delta \leq 500$: blend back to conservative “High” regime.
- $500 \leq \delta \leq 1000$: optimistic “Severe” regime.
- $1000 \leq \delta \leq 2000$: blend to conservative “Extreme” regime.
- $\delta \geq 2000$: outside any regime, no acceptability.

This (1, 2)-hexagonal superhyperfuzzy number captures six nested latency regimes with two layers of uncertainty, providing a fine-grained, hierarchical model of network performance quality.

Example 6.8 (Intrusion Detection Reliability in a Multi-Sensor Network). An IoT security system uses three sensor types: Camera (C), Radar (R), and Infrared (I). We form two hierarchical levels of sensor clusters ($m = 2$) and model detection reliability under two uncertainty sources (sensor noise vs. expert judgment, $n = 2$) via a (2, 2)-hexagonal superhyperfuzzy number.

First-level clusters (\mathcal{P}_1^*):

$$\{C\}, \quad \{R\}, \quad \{I\}, \quad \{C, R\}, \quad \{C, I\}, \quad \{R, I\}, \quad \{C, R, I\}.$$

Second-level sets (\mathcal{P}_2^*): choose six nested collections of these clusters:

$$\begin{aligned} X_{a_1} &= \{\{C\}\}, & X_{a_2} &= \{\{C\}, \{R\}\}, \\ X_{a_3} &= \{\{C\}, \{R\}, \{I\}\}, & X_{a_4} &= \{\{C\}, \{R\}, \{I\}, \{C, R\}\}, \\ X_{a_5} &= \{\{C\}, \{R\}, \{I\}, \{C, R\}, \{C, I\}\}, & X_{a_6} &= \{\{C\}, \{R\}, \{I\}, \{C, R\}, \{C, I\}, \{R, I\}\}. \end{aligned}$$

Let $\delta(X) = |X|$ (number of first-level clusters). Then $\delta(X_{a_i}) = i$.

Pessimistic reliability families $\underline{\tilde{A}}(X)$: for each X_{a_i} , assign two nested intervals

$$\underline{\tilde{A}}(X_{a_i}) = \{[p_i^\ell, q_i^\ell], [r_i^\ell, s_i^\ell]\},$$

with

$$\begin{aligned} [p_1^\ell, q_1^\ell] &= [0.60, 0.70], & [r_1^\ell, s_1^\ell] &= [0.55, 0.65], \\ [p_2^\ell, q_2^\ell] &= [0.65, 0.75], & [r_2^\ell, s_2^\ell] &= [0.60, 0.70], \\ [p_3^\ell, q_3^\ell] &= [0.70, 0.80], & [r_3^\ell, s_3^\ell] &= [0.65, 0.75], \\ [p_4^\ell, q_4^\ell] &= [0.75, 0.85], & [r_4^\ell, s_4^\ell] &= [0.70, 0.80], \\ [p_5^\ell, q_5^\ell] &= [0.80, 0.90], & [r_5^\ell, s_5^\ell] &= [0.75, 0.85], \\ [p_6^\ell, q_6^\ell] &= [0.85, 0.95], & [r_6^\ell, s_6^\ell] &= [0.80, 0.90]. \end{aligned}$$

Optimistic reliability families $\overline{\tilde{A}}(X)$:

$$\overline{\tilde{A}}(X_{a_i}) = \{[p_i^u, q_i^u], [r_i^u, s_i^u]\},$$

with

$$\begin{aligned} [p_1^u, q_1^u] &= [0.65, 0.75], & [r_1^u, s_1^u] &= [0.60, 0.70], \\ [p_2^u, q_2^u] &= [0.70, 0.80], & [r_2^u, s_2^u] &= [0.65, 0.75], \\ [p_3^u, q_3^u] &= [0.75, 0.85], & [r_3^u, s_3^u] &= [0.70, 0.80], \\ [p_4^u, q_4^u] &= [0.80, 0.90], & [r_4^u, s_4^u] &= [0.75, 0.85], \\ [p_5^u, q_5^u] &= [0.85, 0.95], & [r_5^u, s_5^u] &= [0.80, 0.90], \\ [p_6^u, q_6^u] &= [0.90, 1.00], & [r_6^u, s_6^u] &= [0.85, 0.95]. \end{aligned}$$

Weight functions:

$$\lambda_1(X) = \frac{\delta(X) - 1}{1}, \quad \lambda_2(X) = \frac{\delta(X) - 3}{1}, \quad \lambda_3(X) = \frac{\delta(X) - 5}{1}.$$

Hexagonal superhyperfuzzy membership $\tilde{\mu}_{2,2}(X)$:

$$\tilde{\mu}_{2,2}(X) = \begin{cases} \{\{0\}\}, & \delta(X) \leq 1, \\ (1 - \lambda_1) \underline{\tilde{A}}(X) \cup \lambda_1 \overline{\tilde{A}}(X), & 1 \leq \delta(X) \leq 2, \\ \overline{\tilde{A}}(X), & 2 \leq \delta(X) \leq 3, \\ (1 - \lambda_2) \overline{\tilde{A}}(X) \cup \lambda_2 \underline{\tilde{A}}(X), & 3 \leq \delta(X) \leq 4, \\ \overline{\tilde{A}}(X), & 4 \leq \delta(X) \leq 5, \\ (1 - \lambda_3) \overline{\tilde{A}}(X) \cup \lambda_3 \underline{\tilde{A}}(X), & 5 \leq \delta(X) \leq 6, \\ \{\{0\}\}, & \delta(X) \geq 6. \end{cases}$$

This (2, 2)-hexagonal superhyperfuzzy number captures how adding more sensor clusters (from single-sensor to all-sensor sets) increases both the pessimistic and optimistic reliability intervals in a hexagonal pattern, blending conservatism and optimism at alternating regime boundaries.

Theorem 6.9 (Reduction Properties). *The (m, n) -hexagonal superhyperfuzzy number $\tilde{A}^{(m,n)}$ specializes as follows:*

- (i) **Hexagonal fuzzy reduction:** *If $m = n = 0$, then each $X = \{x\}$ and $\underline{\tilde{A}}(X) = \{\mu_\ell(x)\}$, $\overline{\tilde{A}}(X) = \{\mu_u(x)\}$ induce the classical hexagonal fuzzy membership via μ_ℓ, μ_u . Moreover $\mu_\ell = \mu_u$ recovers the usual hexagonal fuzzy number.*
- (ii) **Hexagonal hyperfuzzy reduction:** *If $m = 0$ but $n \geq 1$, then for each x , $\tilde{\mu}_{0,n}(x) \subseteq [0, 1]$ is the interval $[\underline{\mu}(x), \overline{\mu}(x)]$ of a hexagonal hyperfuzzy number.*
- (iii) **Superhyperfuzzy reduction:** *If $\underline{\tilde{A}}(X) = \overline{\tilde{A}}(X)$ for all X , then each convex combination collapses and $\tilde{\mu}_{m,n}(X) = \underline{\tilde{A}}(X)$, yielding the underlying (m, n) -superhyperfuzzy number.*

Proof. (i) When $m = n = 0$, $\mathcal{P}_0^*(\mathbb{R}) = \mathbb{R}$. Writing $X = \{x\}$, define $\mu_\ell(x)$ and $\mu_u(x)$ as the unique elements of $\underline{\tilde{A}}(X)$ and $\overline{\tilde{A}}(X)$. The piecewise cases then exactly reproduce the hexagonal fuzzy membership function. If $\mu_\ell = \mu_u$, the value is single and matches the classical hexagonal fuzzy number.

(ii) If $m = 0$, each $X = \{x\}$. The map $\tilde{\mu}_{0,n}(x)$ is the union of selected $\underline{\tilde{A}}(X)$ and $\overline{\tilde{A}}(X)$ according to the hexagonal pattern, yielding the interval $[\underline{\mu}(x), \overline{\mu}(x)]$ as in the hexagonal hyperfuzzy definition.

(iii) If $\underline{\tilde{A}}(X) = \overline{\tilde{A}}(X)$ for every X , then each step $(1 - \lambda_i)\underline{\tilde{A}}(X) \cup \lambda_i\overline{\tilde{A}}(X)$ reduces to that one set. Hence $\tilde{\mu}_{m,n}(X) = \underline{\tilde{A}}(X)$ and $\tilde{A}^{(m,n)}$ is the given (m, n) -superhyperfuzzy number. \square

7 Octagonal Fuzzy Numbers

7.1 Octagonal Fuzzy numbers

An *octagonal fuzzy number* employs eight parameters to define three nested trapezoidal membership functions, capturing three plateau levels of membership [14, 88, 89].

Definition 7.1 (Octagonal Fuzzy Number). (cf. [13, 90, 91]) An *octagonal fuzzy number* \tilde{O} on \mathbb{R} is specified by eight real parameters

$$a \leq b \leq c \leq d \leq e \leq f \leq g \leq h,$$

and three piecewise-linear membership functions $\mu_A, \mu_B, \mu_C : \mathbb{R} \rightarrow [0, 1]$ defined by

$$\mu_A(x) = \begin{cases} 0, & x \leq a, \\ \frac{x-a}{d-a}, & a \leq x \leq d, \\ 1, & d \leq x \leq e, \\ \frac{h-x}{h-e}, & e \leq x \leq h, \\ 0, & x \geq h, \end{cases} \quad \mu_B(x) = \begin{cases} 0, & x \leq a, \\ \frac{x-a}{c-a}, & a \leq x \leq c, \\ 0.8, & c \leq x \leq f, \\ \frac{h-x}{h-f}, & f \leq x \leq h, \\ 0, & x \geq h, \end{cases}$$

$$\mu_C(x) = \begin{cases} 0, & x \leq a, \\ \frac{x-a}{b-a}, & a \leq x \leq b, \\ 0.6, & b \leq x \leq g, \\ \frac{h-x}{h-g}, & g \leq x \leq h, \\ 0, & x \geq h. \end{cases}$$

The triple $\tilde{O} = (A, B, C)$ with these three fuzzy membership profiles is called an octagonal fuzzy number. Its support is $[a, h]$, and it exhibits three nested plateaux at heights 0.6, 0.8, and 1.

Example 7.2 (Office Thermal Comfort as an Octagonal Fuzzy Number). A building engineer models occupants' comfort with respect to indoor temperature x (in °C). Based on ergonomic studies, the following ranges are identified:

- *Ideal comfort* between 21 °C and 23 °C.
- *Tolerable comfort* between 20 °C and 24 °C.
- *Acceptable range* between 18 °C and 26 °C.
- *Unacceptable* below 16 °C or above 28 °C.

We encode this as an octagonal fuzzy number \tilde{C} with parameters

$$a = 16, b = 18, c = 20, d = 21, e = 23, f = 24, g = 26, h = 28.$$

Define three piecewise-linear membership profiles:

$$\mu_A(x) = \begin{cases} 0, & x \leq 16, \\ \frac{x-16}{21-16}, & 16 < x \leq 21, \\ 1, & 21 < x \leq 23, \\ \frac{28-x}{28-23}, & 23 < x \leq 28, \\ 0, & x > 28, \end{cases}$$

$$\mu_B(x) = \begin{cases} 0, & x \leq 16, \\ 0.8 \frac{x-16}{20-16}, & 16 < x \leq 20, \\ 0.8, & 20 < x \leq 24, \\ 0.8 \frac{28-x}{28-24}, & 24 < x \leq 28, \\ 0, & x > 28, \end{cases}$$

$$\mu_C(x) = \begin{cases} 0, & x \leq 16, \\ 0.6 \frac{x-16}{18-16}, & 16 < x \leq 18, \\ 0.6, & 18 < x \leq 26, \\ 0.6 \frac{28-x}{28-26}, & 26 < x \leq 28, \\ 0, & x > 28. \end{cases}$$

Thus $\tilde{C} = (\mu_A, \mu_B, \mu_C)$ has support $[16, 28]$ °C and exhibits three nested comfort plateaux at heights 1.0 (ideal), 0.8 (tolerable), and 0.6 (acceptable).

Sample evaluations:

- At $x = 22$ °C: $\mu_A = 1.0$, $\mu_B = 0.8$, $\mu_C = 0.6$. (Fully ideal and within all comfort bands.)
- At $x = 19$ °C: $\mu_A = (19 - 16)/(21 - 16) = 0.6$, $\mu_B = 0.8 \times (19 - 16)/(20 - 16) = 0.6$, $\mu_C = 0.6$. (Approaching tolerable comfort.)

- At $x = 17$ °C: $\mu_A = (17 - 16)/5 = 0.2$, $\mu_B = 0.8 \times (17 - 16)/4 = 0.2$, $\mu_C = 0.6 \times (17 - 16)/2 = 0.3$. (Low comfort, nearing unacceptable.)
- At $x = 27$ °C: $\mu_A = (28 - 27)/5 = 0.2$, $\mu_B = 0.8 \times (28 - 27)/4 = 0.2$, $\mu_C = 0.6 \times (28 - 27)/2 = 0.3$. (Symmetrically low comfort at the high end.)

This octagonal fuzzy model captures the dual-range comfort zones and graded decline outside them, providing a nuanced tool for HVAC control and occupant satisfaction analysis.

7.2 Octagonal Hyperfuzzy numbers

Below we present the definition of *Octagonal Hyperfuzzy Numbers* and a detailed illustrative example.

Definition 7.3 (Octagonal Hyperfuzzy Number). Let

$$a \leq b \leq c \leq d \leq e \leq f \leq g \leq h,$$

be the parameters of an octagonal fuzzy number with membership functions μ_A, μ_B, μ_C as in the definition. Then the *octagonal hyperfuzzy number* \tilde{O} is defined by the lower and upper membership functions

$$\underline{\mu}(x) = \mu_C(x), \quad \bar{\mu}(x) = \mu_A(x),$$

and the hyperfuzzy membership

$$\tilde{\mu}(x) = [\underline{\mu}(x), \bar{\mu}(x)] \subseteq [0, 1], \quad x \in \mathbb{R}.$$

Example 7.4 (Office Thermal Comfort as an Octagonal Hyperfuzzy Number). Building on the octagonal fuzzy comfort model ($a = 16, b = 18, c = 20, d = 21, e = 23, f = 24, g = 26, h = 28$), we account for measurement and modeling uncertainty by defining a hexagonal hyperfuzzy membership interval $\tilde{\mu}(x) = [\underline{\mu}(x), \bar{\mu}(x)]$ with

$$\underline{\mu}(x) = \mu_C(x) = \begin{cases} 0, & x \leq 16, \\ \frac{x-16}{18-16} \cdot 0.6, & 16 < x \leq 18, \\ 0.6, & 18 < x \leq 26, \\ 0.6 \frac{28-x}{28-26}, & 26 < x \leq 28, \\ 0, & x \geq 28, \end{cases}$$

$$\bar{\mu}(x) = \mu_A(x) = \begin{cases} 0, & x \leq 16, \\ \frac{x-16}{21-16}, & 16 < x \leq 21, \\ 1, & 21 < x \leq 23, \\ \frac{28-x}{28-23}, & 23 < x \leq 28, \\ 0, & x \geq 28. \end{cases}$$

Hence for any temperature x , the comfort grade lies in the interval

$$\tilde{\mu}(x) = [\underline{\mu}(x), \bar{\mu}(x)] \subseteq [0, 1].$$

Sample evaluations:

$$\begin{aligned} x = 22 \text{ °C} : \quad & \underline{\mu} = 0.6, \quad \bar{\mu} = 1.0, \quad \tilde{\mu}(22) = [0.6, 1.0], \\ x = 19 \text{ °C} : \quad & \underline{\mu} = 0.6, \quad \bar{\mu} = \frac{19-16}{5} = 0.6, \quad \tilde{\mu}(19) = [0.6, 0.6], \\ x = 27 \text{ °C} : \quad & \underline{\mu} = 0.6 \cdot \frac{28-27}{2} = 0.3, \quad \bar{\mu} = \frac{28-27}{5} = 0.2, \quad \tilde{\mu}(27) = [0.2, 0.3]. \end{aligned}$$

This octagonal hyperfuzzy model captures two nested comfort plateaux and quantifies both conservative (μ_C) and optimistic (μ_A) assessments under uncertainty.

Theorem 7.5 (Generalization of Octagonal Fuzzy and Hyperfuzzy Numbers). *Every octagonal hyperfuzzy number \tilde{O} is a hyperfuzzy number. Moreover:*

- (a) *If $\mu_C(x) = \mu_A(x)$ for all x , then $\tilde{\mu}(x) = \{\mu_A(x)\}$ and \tilde{O} reduces to the original octagonal fuzzy number.*
- (b) *Conversely, any hyperfuzzy number whose membership at each x is a singleton interval may be viewed as an octagonal hyperfuzzy number with coincident lower and upper profiles.*

Hence the class of octagonal hyperfuzzy numbers sits between octagonal fuzzy numbers and the full class of hyperfuzzy numbers.

Proof. 1. Hyperfuzzy axioms. Let $\tilde{\mu}(x) = [\underline{\mu}(x), \bar{\mu}(x)]$.

- *Normality:* Since $\mu_A(x) = 1$ on $[d, e]$, we have $\sup \bigcup_x \tilde{\mu}(x) = 1$.
- *Hyper-convexity:* For any x, y and $\lambda \in [0, 1]$, pick $u \in [\underline{\mu}(x), \bar{\mu}(x)]$, $v \in [\underline{\mu}(y), \bar{\mu}(y)]$. By convexity of μ_C and μ_A ,

$$\underline{\mu}(\lambda x + (1 - \lambda)y) \geq \min\{\underline{\mu}(x), \underline{\mu}(y)\}, \quad \bar{\mu}(\lambda x + (1 - \lambda)y) \geq \min\{\bar{\mu}(x), \bar{\mu}(y)\}.$$

Thus any $w \in \tilde{\mu}(\lambda x + (1 - \lambda)y)$ satisfies $w \geq \min\{u, v\}$.

- *Upper semi-continuity:* Each α -cut $\{x \mid \exists u \geq \alpha, u \in \tilde{\mu}(x)\}$ is the union of the closed α -cuts of μ_C and μ_A , hence closed.
- *Compact support:* $\text{supp}(\tilde{\mu}) = [a, h]$ is compact.

2. Reduction to octagonal fuzzy. If $\mu_C(x) = \mu_A(x)$ for every x , then $\underline{\mu} = \bar{\mu} = \mu_A$ and $\tilde{\mu}(x) = \{\mu_A(x)\}$, recovering exactly the octagonal fuzzy number.

3. Embedding into hyperfuzzy. By definition \tilde{O} is a hyperfuzzy set on \mathbb{R} , so every octagonal hyperfuzzy number belongs to $\mathcal{HF}(\mathbb{R})$.

Therefore the class of octagonal hyperfuzzy numbers strictly contains the octagonal fuzzy numbers and is contained in the full class of hyperfuzzy numbers, as claimed. \square

7.3 Octagonal SuperHyperFuzzy numbers

Below we present the definition of (m, n) -Octagonal SuperHyperfuzzy Numbers and a detailed illustrative example.

Definition 7.6 ((m, n) -Octagonal SuperHyperfuzzy Number). Let $m, n \geq 0$. Choose four nonempty m -level sets

$$X_a \subseteq X_d \subseteq X_e \subseteq X_h \in \mathcal{P}_m^*(\mathbb{R}),$$

and two (m, n) -superhyperfuzzy membership maps

$$\tilde{A}, \bar{A} : \mathcal{P}_m^*(\mathbb{R}) \longrightarrow \mathcal{P}(\mathcal{P}_n([0, 1])) \setminus \{\emptyset\},$$

satisfying $\tilde{A}(X) \subseteq \bar{A}(X)$ for all X . Let $\delta : \mathcal{P}_m^*(\mathbb{R}) \rightarrow \mathbb{R}_{\geq 0}$ be any strictly increasing “size” measure. Define

$$\lambda_1(X) = \frac{\delta(X) - \delta(X_a)}{\delta(X_d) - \delta(X_a)}, \quad \lambda_2(X) = \frac{\delta(X) - \delta(X_e)}{\delta(X_h) - \delta(X_e)}.$$

Then the (m, n) -octagonal superhyperfuzzy number $\tilde{A}^{(m,n)}$ is the map $\tilde{\mu}_{m,n} : \mathcal{P}_m^*(\mathbb{R}) \rightarrow \mathcal{P}(\mathcal{P}_n([0, 1]))$ defined by

$$\tilde{\mu}_{m,n}(X) = \begin{cases} \{\{0\}\}, & \delta(X) \leq \delta(X_a), \\ (1 - \lambda_1(X)) \underline{\tilde{A}}(X) \cup \lambda_1(X) \overline{\tilde{A}}(X), & \delta(X_a) \leq \delta(X) \leq \delta(X_d), \\ \overline{\tilde{A}}(X), & \delta(X_d) \leq \delta(X) \leq \delta(X_e), \\ (1 - \lambda_2(X)) \overline{\tilde{A}}(X) \cup \lambda_2(X) \underline{\tilde{A}}(X), & \delta(X_e) \leq \delta(X) \leq \delta(X_h), \\ \{\{0\}\}, & \delta(X) \geq \delta(X_h). \end{cases}$$

Example 7.7 (Intrusion Detection Reliability as a $(1, 2)$ -Octagonal SuperHyperfuzzy Number). An IoT security system uses three sensor types: Camera (C), Radar (R), and Infrared (I). We model the reliability of detecting an intrusion within a cluster X of sensors under two layers of uncertainty (sensor noise and expert judgment) via a $(1, 2)$ -octagonal superhyperfuzzy number.

Level-1 sets:

$$X_a = \{\{C\}\}, \quad X_d = \{\{C\}, \{R\}\}, \quad X_e = \{\{C\}, \{R\}, \{I\}\}, \quad X_h = \{\{C\}, \{R\}, \{I\}, \{C, R\}\}.$$

Here $X_a \subseteq X_d \subseteq X_e \subseteq X_h$. Let $\delta(X) = |X|$ be the number of first-level clusters, so $\delta(X_a) = 1$, $\delta(X_d) = 2$, $\delta(X_e) = 3$, $\delta(X_h) = 4$.

Pessimistic membership families $\underline{\tilde{A}}$:

$$\underline{\tilde{A}}(X) = \begin{cases} \{\{0.60, 0.70\}, \{0.55, 0.65\}\}, & X = X_a, \\ \{\{0.65, 0.75\}, \{0.60, 0.70\}\}, & X = X_d, \\ \{\{0.70, 0.80\}, \{0.65, 0.75\}\}, & X = X_e, \\ \{\{0.75, 0.85\}, \{0.70, 0.80\}\}, & X = X_h. \end{cases}$$

Optimistic membership families $\overline{\tilde{A}}$:

$$\overline{\tilde{A}}(X) = \begin{cases} \{\{0.80, 0.90\}, \{0.75, 0.85\}\}, & X = X_a, \\ \{\{0.85, 0.95\}, \{0.80, 0.90\}\}, & X = X_d, \\ \{\{0.90, 0.98\}, \{0.88, 0.95\}\}, & X = X_e, \\ \{\{0.92, 0.99\}, \{0.90, 0.97\}\}, & X = X_h. \end{cases}$$

Weight functions:

$$\lambda_1(X) = \frac{\delta(X) - 1}{2 - 1}, \quad \lambda_2(X) = \frac{\delta(X) - 3}{4 - 3}.$$

Octagonal superhyperfuzzy membership $\tilde{\mu}_{1,2}(X)$:

$$\tilde{\mu}_{1,2}(X) = \begin{cases} \underline{\tilde{A}}(X), & \delta(X) \leq 1, \\ (1 - \lambda_1) \underline{\tilde{A}}(X) \cup \lambda_1 \overline{\tilde{A}}(X), & 1 \leq \delta(X) \leq 2, \\ \overline{\tilde{A}}(X), & 2 \leq \delta(X) \leq 3, \\ (1 - \lambda_2) \overline{\tilde{A}}(X) \cup \lambda_2 \underline{\tilde{A}}(X), & 3 \leq \delta(X) \leq 4, \\ \{\{0\}\}, & \delta(X) \geq 4. \end{cases}$$

Interpretation:

- For $X_a = \{C\}$, reliability is conservative intervals $\{0.60, 0.70\}$, $\{0.55, 0.65\}$.
- For $X_d = \{C, R\}$, we blend pessimistic $\{0.65, 0.75\}$ with optimistic $\{0.85, 0.95\}$ at equal weight $\lambda_1 = 1$.
- For $X_e = \{C, R, I\}$, reliability is optimistic intervals $\{0.90, 0.98\}$, $\{0.88, 0.95\}$.
- For X_h , we blend back to pessimistic intervals $\{0.75, 0.85\}$ and $\{0.70, 0.80\}$ with weight $\lambda_2 = 1$.

This (1,2)-octagonal superhyperfuzzy number captures how adding sensors initially increases reliability, plateaus at three nested levels, and then blends back as cluster complexity grows, under two uncertainty layers.

Theorem 7.8 (Reduction to Octagonal Fuzzy, Hyperfuzzy, and SuperHyperfuzzy). *The (m, n) -octagonal superhyperfuzzy number $\tilde{A}^{(m,n)}$ specializes as follows:*

- (i) **Octagonal fuzzy reduction:** *If $m = n = 0$, then each $X = \{x\}$ and $\underline{\tilde{A}}(X) = \{\mu_\ell(x)\}$, $\overline{\tilde{A}}(X) = \{\mu_u(x)\}$ recover the classical octagonal fuzzy membership functions μ_ℓ, μ_u . Moreover if $\mu_\ell = \mu_u$, $\tilde{A}^{(0,0)}$ is exactly the original octagonal fuzzy number.*
- (ii) **Octagonal hyperfuzzy reduction:** *If $m = 0$ but $n \geq 1$, then for each x , $\tilde{\mu}_{0,n}(x) = [\underline{\mu}(x), \overline{\mu}(x)]$ is the interval-valued membership of an octagonal hyperfuzzy number.*
- (iii) **Superhyperfuzzy reduction:** *If $\underline{\tilde{A}}(X) = \overline{\tilde{A}}(X)$ for every X , then each convex combination collapses and $\tilde{\mu}_{m,n}(X) = \underline{\tilde{A}}(X)$, so $\tilde{A}^{(m,n)}$ coincides with the underlying (m, n) -superhyperfuzzy number.*

Proof. (i) With $m = n = 0$, $\mathcal{P}_0^*(\mathbb{R}) = \mathbb{R}$. Writing $X = \{x\}$, let $\mu_\ell(x)$ and $\mu_u(x)$ be the unique elements of $\underline{\tilde{A}}(X)$ and $\overline{\tilde{A}}(X)$. Then the piecewise definition of $\tilde{\mu}_{0,0}(x)$ exactly matches the octagonal fuzzy membership profile. If $\mu_\ell = \mu_u$, this reduces to the crisp single-valued membership $\mu(x)$.

(ii) If $m = 0$, each $X = \{x\}$. The map $\tilde{\mu}_{0,n}(x)$ is the union of $\underline{\tilde{A}}(X)$ and $\overline{\tilde{A}}(X)$ according to the octagonal pattern, yielding the interval $[\underline{\mu}(x), \overline{\mu}(x)]$ of an octagonal hyperfuzzy number.

(iii) If $\underline{\tilde{A}}(X) = \overline{\tilde{A}}(X)$ for all X , then each convex-combination step $(1 - \lambda_i)\underline{\tilde{A}}(X) \cup \lambda_i\overline{\tilde{A}}(X)$ collapses to the single set $\underline{\tilde{A}}(X)$. Hence $\tilde{\mu}_{m,n}(X) = \underline{\tilde{A}}(X)$ for every X , showing that $\tilde{A}^{(m,n)}$ is exactly the underlying (m, n) -superhyperfuzzy number. \square

8 Conclusion

In this paper, we have introduced hyperfuzzy and superhyperfuzzy generalizations of several well-known fuzzy-number constructs—namely, triangular, trapezoidal, pentagonal, hexagonal, and octagonal fuzzy numbers—and of ranking functions on fuzzy numbers. By assigning set-valued membership degrees at multiple hierarchical levels, our framework enriches the representation of uncertainty and provides a unified approach to modeling complex, graded information in decision support and related fields.

Looking ahead, we anticipate several fruitful directions for future research:

- *Computational studies:* implementing these new number types in numerical experiments and case studies to evaluate their practical utility.
- *Operational extensions:* defining additional arithmetic and aggregation operators, and investigating their algebraic and order-theoretic properties.
- *Mathematical structure:* exploring deeper structural aspects, such as lattice and topological features, of hyperfuzzy and superhyperfuzzy numbers.
- *Further generalizations:* integrating our constructs with other uncertainty frameworks—such as neutrosophic sets [8, 92–94], quadripartitioned neutrosophic sets [95–97], and plithogenic sets [9, 98–102]—to capture even richer forms of indeterminacy and conflict.

We hope that these developments will pave the way for robust, hierarchically structured models of uncertainty across a wide range of applications.

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Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

References

- [1] Lotfi A Zadeh. Fuzzy sets. *Information and control*, 8(3):338–353, 1965.
- [2] Lotfi A Zadeh. A note on z-numbers. *Information sciences*, 181(14):2923–2932, 2011.
- [3] Zdzisław Pawlak. Rough sets. *International journal of computer & information sciences*, 11:341–356, 1982.
- [4] Said Broumi, Florentin Smarandache, and Mamoni Dhar. Rough neutrosophic sets. *Infinite Study*, 32:493–502, 2014.
- [5] Vicenç Torra and Yasuo Narukawa. On hesitant fuzzy sets and decision. In *2009 IEEE international conference on fuzzy systems*, pages 1378–1382. IEEE, 2009.
- [6] Vicenç Torra. Hesitant fuzzy sets. *International journal of intelligent systems*, 25(6):529–539, 2010.
- [7] Madeleine Al Tahan, Saba Al-Kaseasbeh, and Bijan Davvaz. Neutrosophic quadruple hv-modules and their fundamental module. *Neutrosophic Sets and Systems*, 72:304–325, 2024.
- [8] Said Broumi, Mohamed Talea, Assia Bakali, and Florentin Smarandache. Single valued neutrosophic graphs. *Journal of New theory*, (10):86–101, 2016.
- [9] Fazeelat Sultana, Muhammad Gulistan, Mumtaz Ali, Naveed Yaqoob, Muhammad Khan, Tabasam Rashid, and Tauseef Ahmed. A study of plithogenic graphs: applications in spreading coronavirus disease (covid-19) globally. *Journal of ambient intelligence and humanized computing*, 14(10):13139–13159, 2023.
- [10] Nivetha Martin. Plithogenic swara-topsis decision making on food processing methods with different normalization techniques. *Advances in Decision Making*, 69, 2022.
- [11] Takaaki Fujita. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*. Biblio Publishing, 2025.

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- [12] Takaaki Fujita and Florentin Smarandache. Examples of fuzzy sets, hyperfuzzy sets, and superhyperfuzzy sets in climate change and the proposal of several new concepts. *Climate Change Reports*, 2:1–18, 2025.
- [13] Felbin C Kennedy and SU Malini. Role of octagonal fuzzy numbers in solving some special fuzzy linear programming problems. In *International Forum for Interdisciplinary Mathematics*, pages 152–163. Springer, 2015.
- [14] Samayan Narayanamoorthy and Sengottaiyan Maheswari. The intelligence of octagonal fuzzy number to determine the fuzzy critical path: A new ranking method. *Scientific Programming*, 2016(1):6158208, 2016.
- [15] Lotfi A Zadeh. Biological application of the theory of fuzzy sets and systems. In *The Proceedings of an International Symposium on Biocybernetics of the Central Nervous System*, pages 199–206. Little, Brown and Comp. London, 1969.
- [16] H-J Zimmermann. Fuzzy set theory and mathematical programming. *Fuzzy sets theory and applications*, pages 99–114, 1986.
- [17] Hans-Jürgen Zimmermann. *Fuzzy set theory—and its applications*. Springer Science & Business Media, 2011.
- [18] Muhammad Akram. Bipolar fuzzy graphs. *Information sciences*, 181(24):5548–5564, 2011.
- [19] Wen-Ran Zhang. Bipolar fuzzy sets and relations: a computational framework for cognitive modeling and multiagent decision analysis. *NAFIPS/IFIS/NASA '94. Proceedings of the First International Joint Conference of The North American Fuzzy Information Processing Society Biannual Conference. The Industrial Fuzzy Control and Intelligence*, pages 305–309, 1994.
- [20] Bui Cong Cuong and Vladik Kreinovich. Picture fuzzy sets—a new concept for computational intelligence problems. In *2013 third world congress on information and communication technologies (WICT 2013)*, pages 1–6. IEEE, 2013.
- [21] Young Bae Jun, Kul Hur, and Kyoung Ja Lee. Hyperfuzzy subalgebras of bck/bci-algebras. *Annals of Fuzzy Mathematics and Informatics*, 2017.
- [22] M MAHARIN. An over view on hyper fuzzy subgroups. *Scholar: National School of Leadership*, 9(1.2), 2020.
- [23] Takaaki Fujita. The hyperfuzzy vikor and hyperfuzzy dematel methods for multi-criteria decision-making. *Spectrum of Decision Making and Applications*, 3(1):292–315, 2026.
- [24] Yong Lin Liu, Hee Sik Kim, and J. Neggers. Hyperfuzzy subsets and subgroupoids. *J. Intell. Fuzzy Syst.*, 33:1553–1562, 2017.
- [25] Takaaki Fujita and Florentin Smarandache. A concise introduction to hyperfuzzy, hyperneutrosophic, hyperplithogenic, hypersoft, and hyperrough sets with practical examples. *Neutrosophic Sets and Systems*, 80:609–631, 2025.
- [26] Takaaki Fujita. Note for intuitionistic hyperrough set, one-directional s-hyperrough set, tolerance hyperrough set, and dynamic hyperrough set. *Soft Computing Fusion with Applications*, 2024.
- [27] Takaaki Fujita. Hyperrough topsis method and superhyperrough topsis method. *Uncertainty Discourse and Applications*, 2024.
- [28] Florentin Smarandache. Foundation of superhyperstructure & neutrosophic superhyperstructure. *Neutrosophic Sets and Systems*, 63(1):21, 2024.
- [29] Marzieh Rahmati and Mohammad Hamidi. Extension of g-algebras to superhyper g-algebras. *Neutrosophic Sets and Systems*, 55:557–567, 2023.
- [30] Adel Al-Odhari. A brief comparative study on hyperstructure, super hyperstructure, and n-super superhyperstructure. *Neutrosophic Knowledge*, 6:38–49, 2025.
- [31] Takaaki Fujita. Extensions of multidirected graphs: Fuzzy, neutrosophic, plithogenic, rough, soft, hypergraph, and superhypergraph variants. *International Journal of Topology*, 2(3):11, 2025.
- [32] Takaaki Fujita. Exploration of graph classes and concepts for superhypergraphs and n-th power mathematical structures. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, 3(4):512.
- [33] Prabir Bhattacharya and NP Mukherjee. Fuzzy relations and fuzzy groups. *Information sciences*, 36(3):267–282, 1985.
- [34] JC Dunn. A graph theoretic analysis of pattern classification via tamura’s fuzzy relation. *IEEE Transactions on Systems, Man, and Cybernetics*, (3):310–313, 2010.
- [35] Z Nazari and B Mosapour. The entropy of hyperfuzzy sets. *Journal of Dynamical Systems and Geometric Theories*, 16(2):173–185, 2018.
- [36] Hashem Bordbar, Mohammad Rahim Bordbar, Rajab Ali Borzooei, and Young Bae Jun. N-subalgebras of bck= bci-algebras which are induced from hyperfuzzy structures. *Iranian Journal of Mathematical Sciences and Informatics*, 16(2):179–195, 2021.
- [37] Seok-Zun Song, Seon Jeong Kim, and Young Bae Jun. Hyperfuzzy ideals in bck/bci-algebras. *Mathematics*, 5(4):81, 2017.
- [38] Jayanta Ghosh and Tapas Kumar Samanta. Hyperfuzzy sets and hyperfuzzy group. *Int. J. Adv. Sci. Technol*, 41:27–37, 2012.
- [39] Takaaki Fujita. Hyperfuzzy and superhyperfuzzy promethee methods for multi-criteria decision-making in it service management. *Authorea Preprints*, 2025.
- [40] Tong Shaocheng. Interval number and fuzzy number linear programmings. *Fuzzy sets and systems*, 66(3):301–306, 1994.
- [41] Rakesh Kumar and Gaurav Dhiman. A comparative study of fuzzy optimization through fuzzy number. *International Journal of Modern Research*, 1(1):1–14, 2021.
- [42] Dilnoz Muhamediyeva and Baxtiyor Tagbayev. Method of converting z-number to classic fuzzy number. 2022.
- [43] Didier Dubois and Henri Prade. The mean value of a fuzzy number. *Fuzzy sets and systems*, 24(3):279–300, 1987.
- [44] Didier Dubois and Henri Prade. Fuzzy numbers: an overview. *Readings in Fuzzy Sets for Intelligent Systems*, pages 112–148, 1993.
- [45] Didier Dubois and Henri Prade. Operations on fuzzy numbers. *International Journal of systems science*, 9(6):613–626, 1978.
- [46] Hong-yu Zhang, Hong-gang Peng, Jing Wang, and Jian-qiang Wang. An extended outranking approach for multi-criteria decision-making problems with linguistic intuitionistic fuzzy numbers. *Applied Soft Computing*, 59:462–474, 2017.
- [47] Ali Mert. Defuzzification of non-linear pentagonal intuitionistic fuzzy numbers and application in the minimum spanning tree problem. *Symmetry*, 15(10):1853, 2023.

-
- [48] JJ1033879 Buckley. Fuzzy complex numbers. *Fuzzy sets and systems*, 33(3):333–345, 1989.
- [49] Ahmed Hatip. An introduction to weak fuzzy complex numbers. *Galoitica Journal Of Mathematical Structures and Applications*, 3(01):08–13, 2023.
- [50] Angel Garrido. Axiomatic of fuzzy complex numbers. *Axioms*, 1(1):21–32, 2012.
- [51] Mohamed Abdel-Basset, M Saleh, Abdullallah Gamal, and Florentin Smarandache. An approach of topsis technique for developing supplier selection with group decision making under type-2 neutrosophic number. *Applied Soft Computing*, 77:438–452, 2019.
- [52] Irfan Deli and Yusuf Subas. A ranking method of single valued neutrosophic numbers and its applications to multi-attribute decision making problems. *International Journal of Machine Learning and Cybernetics*, 8:1309 – 1322, 2016.
- [53] S Dhouib. Optimization of travelling salesman problem on single valued triangular neutrosophic number using dhouib-matrix-tsp1 heuristic. *International Journal of Engineering*, 34(12):2642–2647, 2021.
- [54] Aliya Fahmi, Saleem Abdullah, Fazli Amin, Asad Ali, and W Ahmad Khan. Some geometric operators with triangular cubic linguistic hesitant fuzzy number and their application in group decision-making. *Journal of Intelligent & Fuzzy Systems*, 35(2):2485–2499, 2018.
- [55] Harish Garg, Abazar Keikha, and Hassan Mishmast Nehi. Multiple-attribute decision-making problem using topsis and choquet integral with hesitant fuzzy number information. *Mathematical problems in engineering*, 2020(1):9874951, 2020.
- [56] Qi Yue, Liezhang Liu, and Yuan Tao. Dynamic person-position matching decision method based on hesitant fuzzy number information. *Scientific Reports*, 14(1):3846, 2024.
- [57] YS Yun. Parametric operations between 3-dimensional triangular fuzzy number and trapezoidal fuzzy set. *Journal of Algebra & Applied Mathematics*, 21(2), 2023.
- [58] San-Chyi Chang. Fuzzy production inventory for fuzzy product quantity with triangular fuzzy number. *Fuzzy sets and systems*, 107(1):37–57, 1999.
- [59] Hung-Chi Chang, Jing-Shing Yao, and Liang-Yuh Ouyang. Fuzzy mixture inventory model with variable lead-time based on probabilistic fuzzy set and triangular fuzzy number. *Mathematical and Computer Modelling*, 39:287–304, 2004.
- [60] Xinjian Guan, Fengjiao Yu, Hongshi Xu, Changwen Li, and Yongle Guan. Flood risk assessment of urban metro system using random forest algorithm and triangular fuzzy number based analytical hierarchy process approach. *Sustainable Cities and Society*, 109:105546, 2024.
- [61] D Bharathi and A Saraswathi. A pivotal operation on triangular fuzzy number for solving fuzzy nonlinear programming problems. *Mathematics and statistics*, 12(2):126–134, 2024.
- [62] Dojin Kim, Hyeonsoo Kim, and Lee-Chae Jang. Some inequalities for generalized choquet integrals of triangular fuzzy number-valued functions and its application. *Iranian Journal of Fuzzy Systems*, 21(6):83–99, 2024.
- [63] Didier Dubois and Henri Prade. Ranking fuzzy numbers in the setting of possibility theory. *Information sciences*, 30(3):183–224, 1983.
- [64] Daud Mohamad, Saidatull Akma Shaharani, and Nor Hanimah Kamis. A z-number-based decision making procedure with ranking fuzzy numbers method. In *AIP conference proceedings*, volume 1635, pages 160–166. American Institute of Physics, 2014.
- [65] Tian-Shy Liou and Mao-Jiun J Wang. Ranking fuzzy numbers with integral value. *Fuzzy sets and systems*, 50(3):247–255, 1992.
- [66] Shan-Huo Chen. Ranking fuzzy numbers with maximizing set and minimizing set. *Fuzzy sets and Systems*, 17(2):113–129, 1985.
- [67] Maryam Sotoudeh-Anvari and Alireza Sotoudeh-Anvari. Setback in ranking fuzzy numbers: a study in fuzzy risk analysis in diabetes prediction. *Artificial Intelligence Review*, 56(5):4591–4639, 2023.
- [68] S Prasad and A Sinha. Ranking fuzzy numbers with unified integral value and comparative reviews. *Journal of Scientific Research*, 14(1):131–151, 2022.
- [69] Jian Lin, Zhangxu Lin, Yan Huang, and Zeshui Xu. Ranking l–r fuzzy numbers based on hybrid two-dimensional deviation degree. *Computational and Applied Mathematics*, 44(5):233, 2025.
- [70] Saeid Abbasbandy and T Hajjari. A new approach for ranking of trapezoidal fuzzy numbers. *Computers & mathematics with applications*, 57(3):413–419, 2009.
- [71] YLP Thorani, P Phani Bushan Rao, and N Ravi Shankar. Ordering generalized Phani trapezoidal fuzzy numbers. *Int. J. Contemp. Math. Sciences*, 7(12):555–573, 2012.
- [72] Amarpreet Kaur and Amit Kumar. A new approach for solving fuzzy transportation problems using generalized trapezoidal fuzzy numbers. *Applied soft computing*, 12(3):1201–1213, 2012.
- [73] Ali Ebrahimnejad. A simplified new approach for solving fuzzy transportation problems with generalized trapezoidal fuzzy numbers. *Applied Soft Computing*, 19:171–176, 2014.
- [74] Wang Jianqiang and Zhang Zhong. Aggregation operators on intuitionistic trapezoidal fuzzy number and its application to multi-criteria decision making problems. *Journal of Systems Engineering and Electronics*, 20(2):321–326, 2009.
- [75] Saeid Abbasbandy and B Asady. The nearest trapezoidal fuzzy number to a fuzzy quantity. *Applied mathematics and computation*, 156(2):381–386, 2004.
- [76] George Biswas, Totan Garai, and Uttaran Santra. A possibility-based multi-criteria decision-making approach for artificial recharge structure selection using pentagonal fuzzy numbers. *Decision Analytics Journal*, 9:100365, 2023.
- [77] Avishek Chakraborty, Sankar Prasad Mondal, Shariful Alam, Ali Ahmadian, Norazak Senu, Debashis De, and Soheil Salahshour. The pentagonal fuzzy number: Its different representations, properties, ranking, defuzzification and application in game problems. *Symmetry*, 11:248, 2019.
- [78] Apurba Panda and Madhumangal Pal. A study on pentagonal fuzzy number and its corresponding matrices. *Pacific Science Review B: Humanities and Social Sciences*, 1(3):131–139, 2015.

-
- [79] Sankar Prasad Mondal and Manimohan Mandal. Pentagonal fuzzy number, its properties and application in fuzzy equation. *Future Computing and Informatics Journal*, 2(2):110–117, 2017.
- [80] R Srinivasan, T Nakkeeran, and G Saveetha. Evaluation of fuzzy non-preemptive priority queues in intuitionistic pentagonal fuzzy numbers using centroidal approach. *Malaya Journal of Matematik (MJM)*, 1:427–430, 2020.
- [81] R Srinivasan, T Nakkeeran, K Renganathan, and V Vijayan. The performance of pentagonal fuzzy numbers in finite source queue models using pascal’s triangular graded mean. *Materials Today: Proceedings*, 37:947–949, 2021.
- [82] Avishek Chakraborty, Suman Kalyan Maity, Shalini Jain, Sankar Prasad Mondal, and Shariful Alam. Hexagonal fuzzy number and its distinctive representation, ranking, defuzzification technique and application in production inventory management problem. *Granular Computing*, 6:507 – 521, 2020.
- [83] P Rajarajeswari, A Sahaya Sudha, and R Karthika. A new operation on hexagonal fuzzy number. *International journal of fuzzy logic systems*, 3(3):15–26, 2013.
- [84] N Karthikeyan et al. A proposed method to solve transportation problem by generalized pentagonal and hexagonal fuzzy numbers. *Int. J. of Aquatic Science*, 12(2):1499–1509, 2021.
- [85] M Deshmukh, K Ghadle, and O Jadhav. An innovative approach for ranking hexagonal fuzzy numbers to solve linear programming problems. *International Journal on Emerging Technologies*, 11(2):385–388, 2020.
- [86] Ladji Kane, Moctar Diakite, Souleymane Kane, Hawa Bado, and Daouda Diawara. Fully fuzzy transportation problems with pentagonal and hexagonal fuzzy numbers. *Journal of applied research on industrial engineering*, 8(3):251–269, 2021.
- [87] Kamal Hossain Gazi, Sankar Prasad Mondal, Banashree Chatterjee, Neha Ghorui, Arijit Ghosh, and Debashis De. A new synergistic strategy for ranking restaurant locations: A decision-making approach based on the hexagonal fuzzy numbers. *RAIRO-operations research*, 57(2):571–608, 2023.
- [88] SU Malini and Felbin C Kennedy. An approach for solving fuzzy transportation problem using octagonal fuzzy numbers. *Applied mathematical sciences*, 7(54):2661–2673, 2013.
- [89] D Gurukumaresan, C Duraisamy, and R Srinivasan. Optimal solution of fuzzy transportation problem using octagonal fuzzy numbers. *Computer Systems Science & Engineering*, 37(3), 2021.
- [90] Felbin C Kennedy and Malini SU. Fuzzy linear programs with octagonal fuzzy numbers. *Scientiae Mathematicae Japonicae*, 78(4-Special):55–65, 2015.
- [91] Vikas S Jadhava, Sumedh U Buktareb, and Omprakash S Jadhavc. Ranking of octagonal fuzzy numbers for solving fuzzy job sequencing problem using robust ranking technique. *Journal of Statistics, Optimization and Data Science*, 1(2):22–28, 2024.
- [92] Takaaki Fujita and Florentin Smarandache. A concise introduction to hyperfuzzy, hyperneutrosophic, hyperplithogenic, hypersoft, and hyperrough sets with practical examples. *Neutrosophic Sets and Systems*, 80:609–631, 2025.
- [93] Said Broumi, Assia Bakali, Mohamed Talea, and Florentin Smarandache. An isolated bipolar single-valued neutrosophic graphs. In *Information Systems Design and Intelligent Applications: Proceedings of Fourth International Conference INDIA 2017*, pages 816–822. Springer, 2018.
- [94] Muhammad Akram, Hafsa M Malik, Sundas Shahzadi, and Florentin Smarandache. Neutrosophic soft rough graphs with application. *Axioms*, 7(1):14, 2018.
- [95] R Radha, A Stanis Arul Mary, and Florentin Smarandache. Quadripartitioned neutrosophic pythagorean soft set. *International Journal of Neutrosophic Science (IJNS) Volume 14, 2021*, page 11, 2021.
- [96] P Yiarayong. Some weighted aggregation operators of quadripartitioned single-valued trapezoidal neutrosophic sets and their multi-criteria group decision-making method for developing green supplier selection criteria. *OPSEARCH*, pages 1–55, 2024.
- [97] Satham Hussain, Jahir Hussain, Isnaini Rosyida, and Said Broumi. Quadripartitioned neutrosophic soft graphs. In *Handbook of Research on Advances and Applications of Fuzzy Sets and Logic*, pages 771–795. IGI Global, 2022.
- [98] Takaaki Fujita. Short note of extended hyperplithogenic sets and general extended plithogenic graphs. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 57, 2025.
- [99] Nivetha Martin, R Priya, and Florentin Smarandache. Generalized plithogenic sets in multi-attribute decision making. In *Neutrosophic and Plithogenic Inventory Models for Applied Mathematics*, pages 519–546. IGI Global Scientific Publishing, 2025.
- [100] P Sathya, Nivetha Martin, and Florentine Smarandache. Plithogenic forest hypersoft sets in plithogenic contradiction based multi-criteria decision making. *Neutrosophic Sets and Systems*, 73:668–693, 2024.
- [101] Takaaki Fujita and Florentin Smarandache. A review of the hierarchy of plithogenic, neutrosophic, and fuzzy graphs: Survey and applications. In *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond (Second Volume)*. Biblio Publishing, 2024.
- [102] Takaaki Fujita and Florentin Smarandache. Survey of planar and outerplanar graphs in fuzzy and neutrosophic graphs. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 366, 2024.