Linear Diophantine HyperFuzzy Set and SuperHyperFuzzy Set

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Abstract

Uncertainty modeling underpins decision-making across diverse domains, and numerous frameworks—such as Fuzzy Sets [1, 2], Rough Sets [3, 4], Hesitant Fuzzy Sets [5, 6], and Plithogenic Sets [7, 8]—have been developed to capture different facets of imprecision. Hyperfuzzy Sets and their recursive generalization, SuperHyperfuzzy Sets, assign set-valued membership degrees at multiple hierarchical levels to represent uncertainty more richly [9]. The Linear Diophantine Fuzzy Set further refines this approach by imposing weighted linear Diophantine constraints on membership and non-membership grades [10–13]. In this paper, we define two new constructs—the Linear Diophantine Hyperfuzzy Set and the Linear Diophantine SuperHyperfuzzy Set—by integrating Diophantine constraints with hyperfuzzy and superhyperfuzzy frameworks, and we present a concise application example. These extensions offer a more structured, hierarchical means of applying Linear Diophantine Fuzzy Set methodology in practical uncertain environments.

Keywords: Fuzzy set, HyperFuzzy Set, SuperHyperFuzzy Set, Linear Diophantine Fuzzy Set

Structure of this paper

The format of this paper is described below.

1 Preliminaries

In this section, we summarize the key definitions and notation used throughout this paper. Unless otherwise specified, all sets are assumed finite.

1.1 Fuzzy, Hyperfuzzy, and Superhyperfuzzy Sets

A *fuzzy set* assigns to each element of the universe U a membership degree in the interval $[0, 1]$, allowing for gradual inclusion rather than a strict binary classification [1, 14–18]. Extensions of the fuzzy set concept include the Pythagorean fuzzy set [19,20], the bipolar fuzzy set [21,22], and the intuitionistic fuzzy set [23–25]. A *hyperfuzzy set* refines this idea by mapping each element to a nonempty subset of $[0, 1]$, thereby representing uncertainty in the membership grade itself [26–29]. In the most general form, an (m, n)-*superhyperfuzzy set* associates each nonempty m-level subset of U with a family of nonempty n-level membership sets, capturing hierarchical layers of uncertainty [30, 31].

Definition 1.1 (Fuzzy Set)**.** [1, 17] A *fuzzy set* F on a universe U is specified by a membership function

$$\mu_F : U \longrightarrow [0, 1],$$

so that each element $x \in U$ is assigned a degree of membership $\mu_F(x)$.

Definition 1.2 (Fuzzy Relation)**.** [32, 33] Let F be a fuzzy set on U. A *fuzzy relation* R on U is a map

$$R : U \times U \longrightarrow [0, 1],$$

satisfying

$$R(x, y) \;\le\; \min\{\mu_F(x),\; \mu_F(y)\} \quad \text{for all } x, y \in U.$$

Definition 1.3 (Power Set)**.** [34, 35] The *power set* of U is

$$\mathcal{P}(U) \;=\; \{\, A \mid A \subseteq U \}.$$

Definition 1.4 (Hyperfuzzy Set)**.** [26, 36–39] A *hyperfuzzy set* $\widetilde{F}$ on U is given by a function

$$\widetilde{\mu}\colon U \longrightarrow \mathcal{P}([0,1]) \setminus \{\emptyset\},$$

where for each $x \in U$, the nonempty subset $\widetilde{\mu}(x) \subseteq [0,1]$ represents all possible membership grades of x.

Example 1.5 (Smartphone Battery Life Satisfaction)**.** We assess "satisfactory battery life" for three smartphone models via independent consumer surveys. Let

$$U = \{\text{Model X},\ \text{Model Y},\ \text{Model Z}\}.$$

For each model $x \in U$, three regional surveys report normalized satisfaction scores in $[0, 1]$. Due to sampling variability and differing usage patterns, we aggregate these into a *hyper-membership set* $\widetilde{\mu}(x) \subseteq [0, 1]$, so that $\widetilde{\mu}\colon U \to \mathcal{P}([0,1]) \setminus \{\emptyset\}$.

Concretely, suppose the regional survey averages are:

Model	$\widetilde{\mu}(x)$
Model X	$\{\,0.70,\ 0.75,\ 0.80\}$
Model Y	$\{\,0.60,\ 0.65,\ 0.68\}$
Model Z	$\{\,0.85,\ 0.88,\ 0.90\}$

Table 1: Hyper-membership sets for "satisfactory battery life"

Here:

- For Model X, three regions yielded average satisfaction scores of 0.70, 0.75, and 0.80.
- For Model Y, scores were 0.60, 0.65, and 0.68.
- For Model Z, scores were 0.85, 0.88, and 0.90.

Thus the hyperfuzzy set $\widetilde{F}$ is

$$\widetilde{F} = \bigl\{(\,\text{Model X},\ \{0.70, 0.75, 0.80\}),\ (\,\text{Model Y},\ \{0.60, 0.65, 0.68\}),\ (\,\text{Model Z},\ \{0.85, 0.88, 0.90\})\bigr\}.$$

This example shows how a hyperfuzzy set captures multiple plausible membership degrees, reflecting real-world survey variability and providing a richer uncertainty model than a single crisp grade.

Definition 1.6 (Iterated Powerset)**.** [40–43] For each integer $n \geq 1$, the n-fold iterated powerset of U is defined by

$$\mathcal{P}^1(U) = \mathcal{P}(U), \quad \mathcal{P}^{n+1}(U) = \mathcal{P}\bigl(\mathcal{P}^n(U)\bigr).$$

If one wishes to exclude the empty set at each iteration, replace $\mathcal{P}$ with

$$\mathcal{P}^*(X) \;=\; \mathcal{P}(X) \setminus \{\emptyset\}.$$

Definition 1.7 ((m, n)-SuperHyperfuzzy Set)**.** [9, 44–46] Fix integers $m, n \geq 0$. Define

$$\mathcal{P}^*_m(U) = (\underbrace{\mathcal{P}^* \circ \cdots \circ \mathcal{P}^*}_{m \text{ times}})(U), \quad \mathcal{P}^*_n([0,1]) = (\underbrace{\mathcal{P}^* \circ \cdots \circ \mathcal{P}^*}_{n \text{ times}})([0,1]),$$

where $\mathcal{P}^*(X) = \mathcal{P}(X) \setminus \{\emptyset\}$. An (m, n)-*superhyperfuzzy set* on U is a mapping

$$\widetilde{\mu}_{m,n} \colon \mathcal{P}^*_m(U) \longrightarrow \mathcal{P}\bigl(\mathcal{P}_n([0,1])\bigr) \setminus \{\emptyset\},$$

which assigns each nonempty m-level subset of U a nonempty family of n-level membership-value sets, thereby capturing hierarchical uncertainty.

Example 1.8 (Multi-Expert Product Reliability Assessment)**.** We demonstrate an (m, n)-SuperHyperfuzzy Set with $m = 1$, $n = 2$ in the context of assessing product reliability by two independent expert panels. Let

$$U = \{\text{Smartphone, Laptop, Headphones}\}, \quad m = 1, \ n = 2.$$

Then

$$\mathcal{P}_1^*(U) = \big\{\{x\} \mid x \in U\big\} \quad \text{(all nonempty singletons of products)},$$

and

$$\mathcal{P}_2^*([0,1]) = \big\{\, S \subseteq \mathcal{P}([0,1]) \setminus \{\emptyset\} \mid S \neq \emptyset \big\} \quad \text{(families of nonempty membership-value sets)}.$$

We define the mapping $\widetilde{\mu}_{1,2} \colon \mathcal{P}_1^*(U) \to \mathcal{P}\big(\mathcal{P}_2([0,1])\big) \setminus \{\emptyset\}$ so that for each product x, two expert panels each provide a "cluster" of possible reliability grades in $[0,1]$. Concretely:

$$\begin{aligned}
\widetilde{\mu}_{1,2}\big(\{\text{Smartphone}\}\big) &= \big\{\{0.80,\ 0.85,\ 0.90\}, \{0.75,\ 0.82\}\big\}, \\
\widetilde{\mu}_{1,2}\big(\{\text{Laptop}\}\big) &= \big\{\{0.65,\ 0.70,\ 0.75\}, \{0.60,\ 0.68\}\big\}, \\
\widetilde{\mu}_{1,2}\big(\{\text{Headphones}\}\big) &= \big\{\{0.50,\ 0.55,\ 0.60\}, \{0.45,\ 0.52\}\big\}.
\end{aligned}$$

Here:

- Panel 1's grades are the first sets (e.g. $\{0.80, 0.85, 0.90\}$ for the Smartphone).
- Panel 2's grades are the second sets (e.g. $\{0.75, 0.82\}$ for the Smartphone).

Since $\alpha = \beta = 0$ in this pure membership-only example, all entries lie in $[0,1]$ and no further Diophantine condition is needed. Thus

$$\widetilde{D} = \big\{(\{x\},\ \widetilde{\mu}_{1,2}(\{x\})) \ \big| \ x \in U\big\}$$

is an $(1,2)$-SuperHyperfuzzy Set capturing two levels of hierarchical uncertainty: one for choice of product ($m = 1$) and one for expert-panel disagreement ($n = 2$).

Example 1.9 ((2,3)-SuperHyperfuzzy Set: Product Bundle Success)**.** Consider a retailer offering three products:

$$U = \{\text{Laptop, Phone, Tablet}\}, \quad m = 2, \ n = 3.$$

We model two "bundle-group" options in $\mathcal{P}_2^*(U)$:

$$A_1 = \big\{\{\text{Laptop, Phone}\},\ \{\text{Phone, Tablet}\}\big\}, \quad A_2 = \big\{\{\text{Laptop, Tablet}\},\ \{\text{Laptop, Phone}\}\big\}.$$

For each A_i, success probabilities are assessed in three nested levels:

1. *Level 1:* Each sub-panel gives a fuzzy set of likely success rates (values in $[0,1]$).
2. *Level 2:* Aggregate these fuzzy sets into a set of Level 1 assessments (an element of $\mathcal{P}_2([0,1])$).
3. *Level 3:* Collect Level 2 aggregates under different market scenarios (an element of $\mathcal{P}_3([0,1])$).

Case A_1:

$$\begin{aligned}
W_{1,\text{up}} &= \{\{0.85, 0.90\},\ \{0.80, 0.85\}\}, \\
W_{1,\text{down}} &= \{\{0.75, 0.80\},\ \{0.70, 0.75\}\}, \\
S_{1,\text{holiday}} &= \{\{0.90, 0.95\},\ \{0.88, 0.92\}\}, \\
S_{1,\text{weekend}} &= \{\{0.80, 0.85\},\ \{0.78, 0.83\}\}.
\end{aligned}$$

Each of these is in $\mathcal{P}_2([0,1])$. Form two Level 3 scenario-sets in $\mathcal{P}_3([0,1])$:

$$\Sigma_{1,\text{scenario}} = \{W_{1,\text{up}},\ W_{1,\text{down}}\}, \quad \Sigma_{1,\text{season}} = \{S_{1,\text{holiday}},\ S_{1,\text{weekend}}\}.$$

Then

$$\widetilde{\mu}_{2,3}(A_1) = \{\Sigma_{1,\text{scenario}},\ \Sigma_{1,\text{season}}\}.$$

Case A_2:

$$\begin{aligned} W_{2,\text{up}} &= \{\{0.80, 0.85\},\ \{0.78, 0.82\}\}, \\ W_{2,\text{down}} &= \{\{0.70, 0.75\},\ \{0.68, 0.72\}\}, \\ S_{2,\text{holiday}} &= \{\{0.88, 0.93\},\ \{0.85, 0.90\}\}, \\ S_{2,\text{weekend}} &= \{\{0.75, 0.80\},\ \{0.73, 0.78\}\}. \end{aligned}$$

Form

$$\Sigma_{2,\text{scenario}} = \{W_{2,\text{up}},\ W_{2,\text{down}}\}, \quad \Sigma_{2,\text{season}} = \{S_{2,\text{holiday}},\ S_{2,\text{weekend}}\},$$

so that

$$\widetilde{\mu}_{2,3}(A_2) = \{\Sigma_{2,\text{scenario}},\ \Sigma_{2,\text{season}}\}.$$

Together, these assignments $\widetilde{\mu}_{2,3}(A_1)$ and $\widetilde{\mu}_{2,3}(A_2)$ define a concrete $(2,3)$-SuperHyperfuzzy Set capturing hierarchical uncertainty—first in bundle-group selection, then in panel aggregation, and finally across market scenarios.

1.2 Linear Diophantine Fuzzy Set

A Linear Diophantine Fuzzy Set assigns to each element a membership and a nonmembership grade whose weighted sum satisfies a specified linear Diophantine equation [10–13, 47–49]. Related concepts include Spherical Linear Diophantine Fuzzy Sets [11, 50], Linear Diophantine Neutrosophic Sets [51], and q-rung orthopair fuzzy sets [52–56].

Definition 1.10 (Linear Diophantine Fuzzy Set)**.** [10, 11] Let Q be a nonempty universe. Fix reference parameters $\alpha, \beta \in [0,1]$ with

$$0 \ \le\ \alpha + \beta \ \le\ 1.$$

A *linear Diophantine fuzzy set* (LDFS) $\widetilde{D}$ on Q is a collection of triples

$$\widetilde{D} = \left\{ \big(x, \langle A_D(x),\ S_D(x)\rangle, \langle \alpha, \beta\rangle\big) \ \big|\ x \in Q \right\},$$

where

- $A_D, S_D \colon Q \to [0,1]$ assign to each $x \in Q$ its membership grade $A_D(x)$ and non-membership grade $S_D(x)$, and
- these must satisfy

$$0 \ \le\ \alpha\, A_D(x) \ +\ \beta\, S_D(x) \ \le\ 1, \quad \forall x \in Q.$$

The residual (hesitation) degree is then

$$\pi_D(x) = 1 - \big(\alpha\, A_D(x) + \beta\, S_D(x)\big).$$

We often abbreviate each triple $\big(x, (A_D(x), S_D(x)), (\alpha, \beta)\big)$ simply as the *linear Diophantine fuzzy number*

$$\big\langle A_D(x),\ S_D(x)\big\rangle_{(\alpha,\beta)}.$$

Example 1.11 (E-Commerce Product Satisfaction)**.** Consider three products sold on an online marketplace over the past month:

$$Q = \{\text{Wireless Headphones},\ \text{Bluetooth Speaker},\ \text{Smart Watch}\}, \quad (\alpha, \beta) = (0.7,\ 0.2).$$

For each product $x \in Q$:

- Let $N(x)$ be the total number of customer reviews.

- Let $P(x)$ be the number of positive reviews (4–5 stars).
- Let $Ng(x)$ be the number of negative reviews (1–2 stars).
- Define
$$A_D(x) = \frac{P(x)}{N(x)}, \qquad S_D(x) = \frac{Ng(x)}{N(x)}, \qquad \pi_D(x) = 1 - \big(\alpha\, A_D(x) + \beta\, S_D(x)\big).$$
- By construction $0 \le \alpha\, A_D(x) + \beta\, S_D(x) \le 1$.

Suppose the raw review counts are as follows:

$$\begin{aligned}&\text{Wireless Headphones: } N = 500,\ P = 380,\ Ng = 45,\\ &\text{Bluetooth Speaker: } N = 300,\ P = 210,\ Ng = 60,\\ &\text{Smart Watch: } N = 400,\ P = 320,\ Ng = 20.\end{aligned}$$

Then we compute:

Product	$A_D(x)$	$S_D(x)$	$\alpha A_D + \beta S_D$	$\pi_D(x)$
Wireless Headphones	$\frac{380}{500} = 0.76$	$\frac{45}{500} = 0.09$	$0.7 \cdot 0.76 + 0.2 \cdot 0.09 = 0.55$	$1 - 0.55 = 0.45$
Bluetooth Speaker	$\frac{210}{300} = 0.70$	$\frac{60}{300} = 0.20$	$0.7 \cdot 0.70 + 0.2 \cdot 0.20 = 0.53$	$1 - 0.53 = 0.47$
Smart Watch	$\frac{320}{400} = 0.80$	$\frac{20}{400} = 0.05$	$0.7 \cdot 0.80 + 0.2 \cdot 0.05 = 0.57$	$1 - 0.57 = 0.43$

Table 2: LDFS parameters for three products

Thus the Linear Diophantine Fuzzy Set $\widetilde{D}$ is

$$\widetilde{D} = \big\{(\text{ Wireless Headphones}, \langle 0.76, 0.09\rangle_{(0.7,0.2)}, 0.45),$$
$$(\text{ Bluetooth Speaker}, \langle 0.70, 0.20\rangle_{(0.7,0.2)}, 0.47),$$
$$(\text{ Smart Watch}, \langle 0.80, 0.05\rangle_{(0.7,0.2)}, 0.43)\big\},$$

where each triple $(x, \langle A_D(x), S_D(x)\rangle_{(\alpha,\beta)}, \pi_D(x))$ shows the product's positive-review grade, negative-review grade, and residual hesitation. This concrete example illustrates how LDFS can model both positive and negative sentiment with weighted parameters and quantify remaining uncertainty in a straightforward decision-making scenario.

2 Main Results

2.1 Linear Diophantine HyperFuzzy Set

A Linear Diophantine HyperFuzzy Set assigns each element set-valued membership and nonmembership grades, constrained by a linear Diophantine relation.

Definition 2.1 (Linear Diophantine HyperFuzzy Set)**.** Let U be a nonempty universe. Fix real weights $\alpha, \beta \in [0, 1]$ with
$$\alpha + \beta \ \le\ 1.$$
A *Linear Diophantine HyperFuzzy Set* (LDHFS) $\widetilde{D}$ on U with parameters (α, β) is given by two set-valued mappings
$$\widetilde{\mu},\ \widetilde{\nu}\colon U \ \longrightarrow\ \mathcal{P}([0,1]) \setminus \{\emptyset\},$$
called the *hyper-membership* and *hyper-nonmembership* functions, respectively, such that for every $x \in U$ and for all $u \in \widetilde{\mu}(x),\ v \in \widetilde{\nu}(x)$ we have the *Linear Diophantine condition*

$$0 \ \le\ \alpha\, u \ +\ \beta\, v \ \le\ 1.$$

The associated *hyper-hesitation set* $\widetilde{\pi}$ is defined by

$$\widetilde{\pi}(x) = \big\{\, 1 - (\alpha\, u + \beta\, v) \,\big|\, u \in \widetilde{\mu}(x),\ v \in \widetilde{\nu}(x)\big\}, \quad x \in U.$$

Example 2.2 (Credit Risk Assessment)**.** Consider three loan applicants evaluated by three models and two expert committees. Let

$$U = \{\text{Alice, Bob, Carol}\}, \quad (\alpha, \beta) = (0.6,\ 0.3).$$

For each $x \in U$:

- $\widetilde{\mu}(x) \subseteq [0, 1]$ is the set of possible *creditworthiness* grades from three sources.
- $\widetilde{\nu}(x) \subseteq [0, 1]$ is the set of possible *default-risk* grades from two experts.
- All $u \in \widetilde{\mu}(x)$, $v \in \widetilde{\nu}(x)$ must satisfy $0 \le 0.6\,u + 0.3\,v \le 1$.

Suppose the assessments are:

$$\begin{aligned} \widetilde{\mu}(\text{Alice}) &= \{0.80,\ 0.85,\ 0.90\}, & \widetilde{\nu}(\text{Alice}) &= \{0.10,\ 0.15\},\\ \widetilde{\mu}(\text{Bob}) &= \{0.60,\ 0.65,\ 0.70\}, & \widetilde{\nu}(\text{Bob}) &= \{0.25,\ 0.30\},\\ \widetilde{\mu}(\text{Carol}) &= \{0.40,\ 0.50\}, & \widetilde{\nu}(\text{Carol}) &= \{0.40,\ 0.45,\ 0.50\}. \end{aligned}$$

For example, for Alice with $u = 0.85$ and $v = 0.15$:

$$0.6 \times 0.85 + 0.3 \times 0.15 = 0.51 \quad (\text{lies in } [0, 1]).$$

The *hyper-hesitation set* is

$$\widetilde{\pi}(x) = \{\, 1 - (0.6\,u + 0.3\,v) \mid u \in \widetilde{\mu}(x),\ v \in \widetilde{\nu}(x)\}.$$

Hence the LDHFS $\widetilde{D}$ is

$$\big\{\, (\,x,\ \widetilde{\mu}(x),\ \widetilde{\nu}(x),\ \widetilde{\pi}(x)) \,\big|\, x \in \{\text{Alice,Bob,Carol}\}\big\}.$$

This example captures multiple credit-scoring opinions (hyper-membership), expert default-risk judgments (hyper-nonmembership), and the residual uncertainty (hyper-hesitation) in a coherent LDHFS framework.

Theorem 2.3 (Generalization of LDFS and HyperFuzzy Set)**.** *Every Linear Diophantine Fuzzy Set and every HyperFuzzy Set can be obtained as a special case of an LDHFS:*

(i) (LDFS case) *If for each $x \in U$ the sets $\widetilde{\mu}(x)$ and $\widetilde{\nu}(x)$ are singletons, $\widetilde{\mu}(x) = \{A_D(x)\}$, $\widetilde{\nu}(x) = \{S_D(x)\}$, then $\widetilde{D}$ reduces to the usual Linear Diophantine Fuzzy Set $\big\{(x, A_D(x), S_D(x))\big\}$.*

(ii) (HyperFuzzy case) *If we choose $\alpha = 1$, $\beta = 0$, and set $\widetilde{\nu}(x) = \{0\}$ for all x, then the only nontrivial data is the mapping $\widetilde{\mu}\colon U \to \mathcal{P}([0, 1]) \setminus \{\emptyset\}$, recovering exactly a HyperFuzzy Set.*

Proof. (i) Under the singleton assumption $\widetilde{\mu}(x) = \{A_D(x)\}$, $\widetilde{\nu}(x) = \{S_D(x)\}$, the Linear Diophantine condition

$$0 \ \le\ \alpha\,u + \beta\,v = \alpha\,A_D(x) + \beta\,S_D(x) \ \le\ 1 \quad \big(u = A_D(x),\ v = S_D(x)\big)$$

is exactly the feasibility requirement for a Linear Diophantine Fuzzy Set. The hyper-hesitation set $\widetilde{\pi}(x) = \{1 - (\alpha A_D(x) + \beta S_D(x))\}$ collapses to the single hesitation degree of the LDFS. Hence $\widetilde{D}$ coincides with the standard LDFS.

(ii) If $\alpha = 1$ and $\beta = 0$, then for any $\widetilde{\mu}(x) \subseteq [0, 1]$ nonempty and $\widetilde{\nu}(x) = \{0\}$, the condition

$$0 \ \le\ 1 \cdot u + 0 \cdot v = u \ \le\ 1 \quad \forall\, u \in \widetilde{\mu}(x)$$

holds automatically. The mapping $\widetilde{\mu}$ alone carries all uncertainty information, exactly as in a HyperFuzzy Set. The auxiliary sets $\widetilde{\nu}(x)$ and $\widetilde{\pi}(x)$ play no substantive role. Thus $\widetilde{D}$ restricts to a HyperFuzzy Set.

□

2.2 $(m,n)-$Linear Diophantine SuperHyperFuzzy Set

A $(m,n)-$Linear Diophantine SuperHyperFuzzy Set assigns each element set-valued membership and non-membership grades, constrained by a linear Diophantine relation.

Notation 2.4. *Define the* flattening operator $\mathrm{flat}_n : \mathcal{P}^n([0,1]) \to \mathcal{P}([0,1])$ *recursively by*

$$\mathrm{flat}_0(x) = \{x\}, \qquad \mathrm{flat}_{k+1}(S) = \bigcup_{T \in S} \mathrm{flat}_k(T).$$

Definition 2.5 ((m,n)-Linear Diophantine SuperHyperfuzzy Set)**.** Let U be a nonempty universe. Fix integers $m,n \geq 0$ and weights $\alpha,\beta \in [0,1]$ with $\alpha+\beta \leq 1$. Recall $\mathcal{P}_m^*(U)$ and $\mathcal{P}_n^*([0,1])$ from Definition 1.7. An (m,n)*-linear Diophantine superhyperfuzzy set* on U is a pair of mappings

$$\widetilde{\mu}_{m,n},\ \widetilde{\nu}_{m,n}\ :\ \mathcal{P}_m^*(U)\ \longrightarrow\ \mathcal{P}\big(\mathcal{P}_n([0,1])\big) \setminus \{\emptyset\},$$

called the *hyper-membership* and *hyper-nonmembership* functions, such that for every $A \in \mathcal{P}_m^*(U)$, every $u \in \widetilde{\mu}_{m,n}(A)$, every $v \in \widetilde{\nu}_{m,n}(A)$, and every $s \in \mathrm{flat}_n(u)$, $t \in \mathrm{flat}_n(v)$, the following *linear Diophantine condition* holds:

$$0\ \leq\ \alpha\,s\ +\ \beta\,t\ \leq\ 1.$$

The associated *hyper-hesitation set* $\widetilde{\pi}_{m,n} : \mathcal{P}_m^*(U) \to \mathcal{P}([0,1])$ is defined by

$$\widetilde{\pi}_{m,n}(A) = \big\{1-(\alpha\,s+\beta\,t)\ \big|\ u \in \widetilde{\mu}_{m,n}(A),\ v \in \widetilde{\nu}_{m,n}(A),\ s \in \mathrm{flat}_n(u),\ t \in \mathrm{flat}_n(v)\big\}.$$

Example 2.6 (Hierarchical Credit-Risk Evaluation)**.** Consider two loan applicants:

$$U = \{\text{Alice, Bob}\}, \quad m=1,\ n=2, \quad (\alpha,\beta) = (0.6, 0.3).$$

Then

$$\mathcal{P}_1^*(U) = \big\{\{\text{Alice}\},\{\text{Bob}\}\big\}, \quad \mathcal{P}_2^*([0,1]) = \big\{S \subseteq \mathcal{P}([0,1]) \setminus \{\emptyset\} \mid S \neq \emptyset\big\}.$$

We define two set–valued mappings $\widetilde{\mu}_{1,2}, \widetilde{\nu}_{1,2} : \mathcal{P}_1^*(U) \to \mathcal{P}\big(\mathcal{P}_2([0,1])\big) \setminus \{\emptyset\}$ by, for each $x \in \{\text{Alice}, \text{Bob}\}$,

$$\widetilde{\mu}_{1,2}(\{x\}) = \big\{\{0.80,\ 0.85\},\ \{0.78,\ 0.82\}\big\},$$
$$\widetilde{\nu}_{1,2}(\{x\}) = \big\{\{0.10,\ 0.15\},\ \{0.12,\ 0.18\}\big\}.$$

Here each inner set is a sub-expert's fuzzy grade, and the outer family groups two expert committees.

By Definition 2.5, for any $u \in \widetilde{\mu}_{1,2}(\{x\})$, $v \in \widetilde{\nu}_{1,2}(\{x\})$ and any $s \in \mathrm{flat}_2(u)$, $t \in \mathrm{flat}_2(v)$, the Linear Diophantine constraint

$$0\ \leq\ \alpha\,s+\beta\,t = 0.6\,s+0.3\,t\ \leq\ 1$$

must hold. For instance, taking $u = \{0.85, 0.80\}$, $v = \{0.18, 0.12\}$, and $s = 0.85$, $t = 0.18$, we get

$$0.6 \cdot 0.85 + 0.3 \cdot 0.18 = 0.51 + 0.054 = 0.564 \in [0,1].$$

The associated hyper-hesitation set is

$$\widetilde{\pi}_{1,2}(\{x\}) = \big\{\,1-(0.6\,s+0.3\,t)\ \big|\ s \in \mathrm{flat}_2(u),\ t \in \mathrm{flat}_2(v)\big\}.$$

Thus the pair $\big(\widetilde{\mu}_{1,2},\ \widetilde{\nu}_{1,2}\big)$ defines a $(1,2)$-Linear Diophantine SuperHyperfuzzy Set capturing two hierarchical levels of expert uncertainty under weighted membership vs. nonmembership constraints.

Theorem 2.7 (Generalization of LDFS, LDHFS, and (m,n)-SHF Set)**.** *Let $\widetilde{D} = (\widetilde{\mu}_{m,n}, \widetilde{\nu}_{m,n})$ be an (m,n)-Linear Diophantine SuperHyperfuzzy Set on U. Then:*

(i) (LDFS) *If $m=n=0$ and for each $x \in U$, both $\widetilde{\mu}_{0,0}(x)$ and $\widetilde{\nu}_{0,0}(x)$ are singletons $\{A_D(x)\}$, $\{S_D(x)\}$, then $\widetilde{D}$ reduces to the Linear Diophantine Fuzzy Set of Definition 1.1.*

(ii) (LDHFS) *If $m=n=0$, $\alpha=1$, $\beta=0$, and $\widetilde{\nu}_{0,0}(x) = \{0\}$ for all x, then $\widetilde{D}$ is exactly a HyperFuzzy Set as in Definition 1.4.*

(iii) ((m,n)-SHF Set) *If $\beta = 0$ and for each $A \in \mathcal{P}^*_m(U)$, the set $\widetilde{\nu}_{m,n}(A)$ is chosen to be the trivial n-fold nested zero $\{\{\ldots\{0\}\ldots\}\}$, then the only nontrivial mapping is $\widetilde{\mu}_{m,n}$, so $\widetilde{D}$ reduces to the (m,n)-SuperHyperfuzzy Set of Definition 1.7.*

Proof. (i) When $m = n = 0$, $\mathcal{P}^*_0(U) = U$ and $\flat_0(x) = \{x\}$. Singleton images force each condition $0 \le \alpha\, A_D(x) + \beta\, S_D(x) \le 1$, recovering exactly the Linear Diophantine Fuzzy Set.

(ii) With $m = n = 0$, $\alpha = 1$, $\beta = 0$, and $\widetilde{\nu}_{0,0}(x) = \{0\}$, the linear condition $0 \le s \le 1$ holds for every $s \in \widetilde{\mu}_{0,0}(x) \subseteq [0,1]$. Thus only the hyper-membership mapping remains, yielding a HyperFuzzy Set.

(iii) If $\beta = 0$ and $\widetilde{\nu}_{m,n}(A)$ is the nested zero set, then for all $u \in \widetilde{\mu}_{m,n}(A)$ and $s \in \flat_n(u)$,

$$0 \ \le\ \alpha\, s + 0 \ \le\ 1,$$

and the nonmembership part is trivial. Hence $\widetilde{D}$ coincides with the (m,n)-SuperHyperfuzzy Set given by $\widetilde{\mu}_{m,n}$ alone.

□

3 Conclusion

In this paper, we have introduced two novel frameworks—the *Linear Diophantine Hyperfuzzy Set* and the *Linear Diophantine SuperHyperfuzzy Set*—by embedding linear Diophantine constraints into the hyperfuzzy and superhyperfuzzy paradigms. Looking ahead, we plan to extend these models to additional uncertainty frameworks, including Neutrosophic Sets [57–61], Plithogenic Sets [7, 62–64], HyperRough Sets [34, 65, 66], Hesitant Fuzzy Sets [5, 6, 67–69], Z-Numbers [2, 70], and other related frameworks. We will also investigate algorithmic strategies and computational implementations to validate and apply these constructs in real-world decision-making scenarios. Furthermore, we intend to explore graph-based generalizations by integrating the Linear Diophantine Hyperfuzzy Set concept into HyperGraphs [71–74] and SuperHyperGraphs [43, 75–79].

Funding

This study did not receive any financial or external support from organizations or individuals.

Acknowledgments

We extend our sincere gratitude to everyone who provided insights, inspiration, and assistance throughout this research. We particularly thank our readers for their interest and acknowledge the authors of the cited works for laying the foundation that made our study possible. We also appreciate the support from individuals and institutions that provided the resources and infrastructure needed to produce and share this paper. Finally, we are grateful to all those who supported us in various ways during this project.

Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

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