Pure-Strategy Equilibrium for Bi-matrix Games and Coordination Games

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Abstract

We consider pure-strategy equilibrium for bi-matrix games and prove a version of the seminal "Equivalence Theorem" of Mangasarian and Stone, when it is restricted to pure-strategy profiles. We then consider bi-matrix games where both players have the same pay-off matrix. The existence of a pure strategy equilibrium for such games, that we refer to as "two-person identical pay-off matrix" (TPIPM) games, is established easily. We subsequently define coordination games as the class of all symmetric bi-matrix games, each such game having at least two pure-strategy equilibria and the set of pure-strategy equilibria for each of which is identical to the set of pure-strategy equilibria of the corresponding TPIPM game where the pay-off matrix is the average of the two pay-off matrices. We discuss two bi-matrix games well-known for problems with regard to coordination in choice of strategy, to show that they agree with our definition of coordination games.

Keywords: bi-matrix games, pure-strategy equilibrium, identical pay-off matrix games, coordination games

1. Notations: Let $\mathbb{R}$ denote the set of real numbers and let $\mathbb{R}_+$ denote the set of all non-negative real numbers. For a positive integer ℓ, and any non-empty set X, let X^ℓ be the set of all ordered ℓ-tuples with coordinates belonging to X.

For a positive integer ℓ and $j \in \{1, \ldots, \ell\}$, let $E^{(\ell,j)}$ denote the point in $\mathbb{R}^\ell$ whose jth coordinate is 1, and all other coordinates are 0. $E^{(\ell,j)}$ is said to be the jth **unit coordinate vector** in $\mathbb{R}^\ell$.

For any non-negative integer ℓ, let $\Delta^{\ell-1} = \{x \in \mathbb{R}^\ell_+ \mid \sum_{k=1}^{\ell} x_k = 1\}$. Clearly, $\Delta^{\ell-1}$ is the convex combination of points in the set $\{E^{(\ell,j)} \mid j \in \{1, \ldots, \ell\}\}$.

Unless otherwise mentioned, for any positive integer ℓ, we will interpret a point x in $\mathbb{R}^\ell$ to be a column vector, and its transpose represented by x^T to be a row vector.

Let m and n be positive integers. For any m×n real-valued matrix (i.e., a real-valued matrix with m rows and n columns) A and any $(i, j) \in \{1, \ldots, m\} \times \{1, \ldots, n\}$, let a_{ij} denote the $(i, j)^{th}$ entry of A (i.e., entry at the intersection of the ith row and jth column of A). For $(i, j) \in \{1, \ldots, m\} \times \{1, \ldots, n\}$, let A_i denote the ith row A and let A^j denote the jth column of A. Further, let A^T be the n×m matrix denoting the transpose of A.

2. Bi-matrix Games: For positive integers m and n, consider a two-player game between a "row player" and a "column player", with their pay-off matrices being m×n matrices A and B respectively. The interpretation of the pair (A, B), referred to as (an) a (m×n) **bi-matrix game**, is that if the row player chooses row i and the column player chooses column j, then the pay-off to the row player is a_{ij} and the pay-off to the column player is b_{ij}.

If B = -A, then the corresponding bi-matrix decision making problem is called a **matrix** (or **two-person zero-sum** (**TPZS**)) **game**, which is discussed in chapter 20 of Mote and Madhavan (2019).

A (**randomized or mixed**) **strategy for the row player** is a point in Δ^{m-1} and a (**randomized or mixed**) **strategy for the column player** is a point in Δ^{n-1}.

A strategy for the row player is said to be a **pure-strategy for the row player** if the strategy belongs to the set $\{E^{(m,i)} | i \in \{1, \ldots, m\}$.

A strategy for the column player is said to be a **pure-strategy for the column player** if the strategy belongs to $\{E^{(n,j)} | j \in \{1, \ldots, n\}$.

A pair $(p, q) \in \Delta^{m-1} \times \Delta^{n-1}$ is a (**randomized or mixed) strategy profile**.

If $(p, q) = (E^{(m,i)}, E^{(n,j)})$ for some $(i, j) \in \{1, \ldots, m\} \times \{1, \ldots, n\}$, then we refer to the strategy pair as a **pure-strategy profile** and when there is no scope for mis-understanding we may denote the same simply by (i, j).

Given an m×n bi-matrix a strategy profile (x^*, y^*) is said to be an **equilibrium** if for all (i, j) $\in \{1, \ldots, m\} \times \{1, \ldots, n\}$, $A_i y^* \leq x^{*T} A y^*$ and $x^{*T} B^j \leq x^{*T} B y^*$.

For a concise discussion about equilibria of bi-matrix games one may refer to Lahiri (2025).

Given an m×n bi-matrix a pure-strategy profile (i, j) is said to be a **pure-strategy equilibrium** if for all (h, k) $\in \{1, \ldots, m\} \times \{1, \ldots, n\}$, $a_{hj} \leq a_{ij}$ and $b_{ik} \leq bij$.

It is easy to see that if (i, j) is a pure-strategy equilibrium, then for all $x \in \Delta^{m-1}$, $x^T A^j \leq a_{ij}$ and for all $y \in \Delta^{n-1}$, $B_i y \leq b_{ij}$.

It is quite possible that a bi-matrix game does not have any pure-strategy equilibrium as the following 2×2 TPZS game, often referred to as "matching pennies", reveals.

Example 1: Let A = $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, B = -A. It is easy to see that (A, B) has no pure-strategy equilibrium.

A m×n bi-matrix game (A, B) where A = B, is said to be a **two-person identical pay-off matrix (TPIPM) game**.

A bi-matrix game (A, B) is said to be a **symmetric bi-matrix game** if $B = A^T$.

Clearly if (A, B) is a symmetric bi-matrix game, then A must be a square matrix.

3. Characterization Theorem for Pure-Strategy Equilibria: In this section we first provide a characterization result for the set of pure-strategy equilibria of a bi-matrix game that has at least one pure-strategy equilibrium. The result is analogous to the "Equivalence Theorem" of Mangasarian and Stone (1964).

Theorem 1: Suppose that (A, B) is an m×n bi-matrix which has at least one pure-strategy equilibrium. Then, $(i, j) \in \{1, \ldots, m\} \times \{1, \ldots, n\}$ is a pure-strategy equilibrium for (A, B) if and only there exists real number u^*, v^* such that i, j, u^*, v^* solves the following maximization problem:

Maximize $(a_{hk} + b_{hk}) - u - v$, subject to $a_{rk} - u \leq 0$ for all $r \in \{1, \ldots, m\}$, $b_{hs} - v \leq 0$ for all $s \in \{1, \ldots, n\}$, $u, v \in \mathbb{R}$.

Proof: First note that for any $(h, k) \in \{1, \ldots, m\} \times \{1, \ldots, n\}$ and $u, v \in \mathbb{R}$ if $a_{rk} - u \leq 0$ for all $r \in \{1, \ldots, m\}$, $b_{hs} - v \leq 0$ for all $s \in \{1, \ldots, n\}$, then $(a_{hk} + b_{hk}) - u - v \leq 0$.

Suppose (i, j) is a pure strategy equilibrium. Then, with $u^* = a_{ij}$ and $v^* = b_{ij}$ we have $a_{rj} - u^* \leq 0$ for all $r \in \{1, \ldots, m\}$, $b_{is} - v^* \leq 0$ for all $s \in \{1, \ldots, n\}$ and $(a_{ij} + b_{ij}) - u^* - v^* = 0$.

Thus, i, j, u^*, v^* solves the maximization problem.

Now suppose, i, j, u^*, v^* solves the maximization problem. By hypothesis, we know that a pure-strategy equilibrium (h, k) exists and from the previous step we know that there exists real numbers u, v such that $a_{rk} - u \leq 0$ for all $r \in \{1, \ldots, m\}$, $b_{hs} - v \leq 0$ for all $s \in \{1, \ldots, n\}$, $(a_{hk} + b_{hk}) - u - v = 0$.

Thus, it must be the case that $a_{rj} - u^* \leq 0$ for all $r \in \{1, \ldots, m\}$, $b_{is} - v^* \leq 0$ for all $s \in \{1, \ldots, n\}$ and $(a_{ij} + b_{ij}) - u^* - v^* = 0$.

Since $a_{ij} \leq u^*$, $b_{ij} \leq v^*$ and $(a_{ij} + b_{ij}) - u^* - v^* = 0$, it must be that $a_{ij} = u^*$, $b_{ij} = v^*$.

Thus, $a_{rj} \leq a_{ij}$ for all $r \in \{1, \ldots, m\}$, $b_{is} \leq b_{ij}$ for all $s \in \{1, \ldots, n\}$.

Thus, (i, j) is a pure-strategy equilibrium. Q.E.D.

Note 1: The assumption about the existence of a pure strategy equilibrium is crucial for the validity of theorem 1, since for (A, B) as in example any $(i, j) \in \{1, 2\} \times \{1, 2\}$, with $u^* = v^* = 1$, solves the maximization problem, even though (A, B) has no pure-strategy equilibrium.

In view of note 1, the following result is of some interest.

Proposition 1: Given any m×n matrix A, consider the TPIPM game (A, A). Then, this game has a pure-strategy equilibrium.

Proof: Let $a_{ij} = \max \{a_{hk} | (h, k) \in \{1, \ldots, m\} \times \{1, \ldots, n\}\}$. Then, $a_{ij} \geq a_{rj}$ for all $r \in \{1, \ldots, m\}$ and $a_{ij} \geq a_{is}$ for all $s \in \{1, \ldots, n\}$.

Thus, (i, j) is a pure-strategy equilibrium. Q.E.D.

4. Coordination Games: We are now in a position to define what the purpose of this note is.

A symmetric bi-matrix game (A, A^T) is said to be a **coordination game** if it has at least two pure-strategy equilibria and the set of pure strategy equilibria of (A, A^T) is identical to the set of equilibria of the TPIPM game $(\frac{A+A^T}{2}, \frac{A+A^T}{2})$

The following two examples illustrate the concept of coordination games. The essential ideas behind the two examples are discussed lucidly in Nguyen and Wait (2016). For what follows let A be a 2×2 real-valued matrix of the form $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ for some positive real numbers a, b, c, d. Thus, $A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ and $\frac{A+A^T}{2} = \begin{bmatrix} a & \frac{b+c}{2} \\ \frac{b+c}{2} & d \end{bmatrix}$.

Example 2: Suppose min {a, d} > max {b, c}. Hence, a > $\frac{b+c}{2}$ and d > $\frac{b+c}{2}$.

Thus, a > c, which implies (1, 1) is a pure-strategy equilibrium for (A, A^T) and d > b, which implies (2, 2) is a pure strategy equilibrium for (A, A^T).

If we consider the TPIPM game ($\frac{A+A^T}{2}$, $\frac{A+A^T}{2}$), then since a > $\frac{b+c}{2}$ and d > $\frac{b+c}{2}$, both (1, 1) and (2, 2) are pure-strategy equilibria for ($\frac{A+A^T}{2}$, $\frac{A+A^T}{2}$).

Thus, (A, A^T) is a coordination game.

Example 3: Suppose min {b, c} > max {a, d}. Hence, $\frac{b+c}{2}$ > d and $\frac{b+c}{2}$ > a.

b > d and c > a implies (1, 2) is an equilibrium for (A, A^T). c > a and b > d implies (2, 1) is an equilibrium for (A, A^T).

If we consider the TPIPM game ($\frac{A+A^T}{2}$, $\frac{A+A^T}{2}$), then $\frac{b+c}{2}$ > d and $\frac{b+c}{2}$ > a imply (1, 2) and (2, 1) are equilibria for ($\frac{A+A^T}{2}$, $\frac{A+A^T}{2}$).

Thus, (A, A^T) is a coordination game.

Note 2: The significance of example 3, is that neither (1, 2) nor (2, 1) is a "symmetric equilibrium", the latter being a concept that is discussed in some detail in Lahiri (2025). As shown towards the end of section 4 in Lahiri (2025), the proof of the existence of a "symmetric equilibrium" for a 2×2 symmetric game is quite straightforward, and hence the symmetric game in example 3 has a symmetric equilibrium in randomized strategies. What is problematic for the players in example 3, is coordinating their actions to arrive at one of the two non-symmetric strategies. However, that is a characteristic feature of every "interactive decision-making problem" which has multiple equilibria where each equilibrium rewards the different players differently. Hence, the coordination problem is as true for example 2 as it is for example 3.

References

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