Note for Soft MultiExpert Set and MultiSoft MultiExpert Set

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Abstract

Modern frameworks for modeling uncertainty and guiding decision-making—such as fuzzy sets, intuitionistic fuzzy sets, neutrosophic sets, hesitant fuzzy sets, and soft sets—have been the focus of extensive research. Soft sets capture vagueness by assigning each parameter a collection of elements that approximately satisfy it. Multisoft sets generalize soft sets by permitting multiple, disjoint parameter groups, each identifying its own approximate subset. Soft expert sets further refine this approach by incorporating expert opinions into the parameter-to-subset mapping. In this paper, we introduce two new structures—Soft MultiExpert Sets and Multisoft MultiExpert Sets—and explore their formal definitions, key mathematical properties, and illustrative applications in real-world decision contexts.

Keywords: Soft Set, Soft Expert Set, Soft MultiExpert Set, MultiSoft MultiExpert Set, MultiSoft Set

Contents

The format of this paper is described below.

1 Preliminaries

All sets in this paper are assumed finite. We introduce here the basic notions and notation used throughout. A *soft set* over U is a parameterized family mapping each parameter to a subset of U, modeling approximate element memberships [1–4]. A *multisoft set* extends soft sets by permitting multiple disjoint parameter collections, each parameter combination mapping to a subset of U [5–7]. A *soft expert set* associates parameter–expert–opinion triples with subsets of U, thereby defining a parameterized map from $E \times X \times O$ to $\mathcal{P}(U)$. The formal definitions of these concepts are given below [8–11].

Definition 1.1 (Soft Set)**.** [1, 2] Let U be a universe and E a set of parameters. A *soft set* over U is a pair (F, E) where

$$F\colon E \to \mathcal{P}(U)$$

assigns to each parameter $e \in E$ a subset $F(e) \subseteq U$, representing those elements of U that roughly satisfy e.

Example 1.2 (Soft Set in Laptop Selection)**.** Let

$$U = \{\text{Laptop A, Laptop B, Laptop C}\}, \qquad E = \{\text{Lightweight, LongBattery, BudgetFriendly}\}.$$

Define the mapping $F : E \to \mathcal{P}(U)$ by

$$F(\text{Lightweight}) = \{\text{Laptop A}, \text{Laptop C}\},$$

$$F(\text{LongBattery}) = \{\text{Laptop B}, \text{Laptop C}\},$$

$$F(\text{BudgetFriendly}) = \{\text{Laptop A}, \text{Laptop B}\}.$$

Then (F, E) is a soft set modeling a buyer's approximate preferences over available laptops.

Definition 1.3 (Soft Expert Set)**.** [12–15] Let U be a universe, E parameters, X experts, and $O = \{0, 1\}$ opinions. Write $Z = E \times X \times O$ and let $A \subseteq Z$. A *soft expert set* is a pair (G, A) with

$$G\colon A \to \mathcal{P}(U),$$

so that each expert–opinion triple $\alpha \in A$ is mapped to a subset $G(\alpha) \subseteq U$.

Example 1.4 (Soft Expert Set for Product Evaluation)**.** Let

$$U = \{\text{Product 1},\ \text{Product 2},\ \text{Product 3}\},$$

$$E = \{\text{Durability},\ \text{Usability}\},$$

$$X = \{\text{Hiroya},\ \text{Ayame}\}, \quad O = \{0, 1\}.$$

Form

$$Z = E \times X \times O,$$

$$A = \{(\text{Durability}, \text{Hiroya}, 1),\ (\text{Usability}, \text{Hiroya}, 0),$$

$$(\text{Durability}, \text{Ayame}, 1),\ (\text{Usability}, \text{Ayame}, 1)\}.$$

Define

$$G\colon A \to \mathcal{P}(U)$$

by

$$\begin{aligned} G(\text{Durability}, \text{Hiroya}, 1) &= \{\text{Product 1}, \text{Product 3}\}, \\ G(\text{Usability}, \text{Hiroya}, 0) &= \{\text{Product 2}\}, \\ G(\text{Durability}, \text{Ayame}, 1) &= \{\text{Product 2}, \text{Product 3}\}, \\ G(\text{Usability}, \text{Ayame}, 1) &= \{\text{Product 1}, \text{Product 3}\}. \end{aligned}$$

Then (G, A) is a soft expert set where each expert's yes/no opinion on a feature selects the corresponding products in U.

Definition 1.5 (Multisoft Set)**.** [5–7] Let U be a nonempty universe. Suppose $\{E_i\}_{i=1}^n$ are pairwise disjoint parameter sets and define $E = \bigcup_{i=1}^n E_i$. For any nonempty $A \subseteq \mathcal{P}(E)$, a *multisoft set* is a pair (H, A) where

$$H\colon A \to \mathcal{P}(U)$$

assigns to each parameter-subset $a \in A$ its *approximate value* $H(a) \subseteq U$.

Example 1.6 (Multisoft Set in Hotel Selection)**.** Let

$$U = \{\text{Hotel A},\ \text{Hotel B},\ \text{Hotel C},\ \text{Hotel D}\}.$$

Define three disjoint parameter sets:

$$E_1 = \{\text{Cheap},\ \text{Midrange},\ \text{Expensive}\},$$

$$E_2 = \{\text{Downtown},\ \text{Suburb}\},$$

$$E_3 = \{\text{Pool},\ \text{Breakfast},\ \text{Parking}\}.$$

Let $E = E_1 \cup E_2 \cup E_3$ and choose

$$A = \Big\{\{\text{Cheap}, \text{Downtown}\},\ \{\text{Midrange}, \text{Suburb}, \text{Pool}\},$$

$$\{\text{Expensive}, \text{Downtown}, \text{Breakfast}\}\Big\} \ \subseteq\ \mathcal{P}(E).$$

Define

$$H\colon A \to \mathcal{P}(U)$$

by

$$\begin{aligned} H(\{\text{Cheap}, \text{Downtown}\}) &= \{\text{Hotel A},\ \text{Hotel B}\}, \\ H(\{\text{Midrange}, \text{Suburb}, \text{Pool}\}) &= \{\text{Hotel B},\ \text{Hotel C}\}, \\ H(\{\text{Expensive}, \text{Downtown}, \text{Breakfast}\}) &= \{\text{Hotel C},\ \text{Hotel D}\}. \end{aligned}$$

Then (H, A) is a multisoft set modeling a traveler's combined preferences over price, location, and amenity parameters.

2 Main Result: Soft MultiExpert Set

In this work, we introduce the definitions of the *Soft MultiExpert Set* and the *MultiSoft MultiExpert Set*, and we offer a concise analysis of their fundamental properties.

2.1 Soft MultiExpert Set

A Soft MultiExpert Set associates each parameter with a multiset of expert–opinion entries, each entry mapping a subset of universe elements.

Definition 2.1 (Soft MultiExpert Set)**.** Let U be a universe set, E a set of parameters, X a set of experts, and $O = \{0, 1\}$ a set of binary opinions. Write

$$\mathcal{M}\big(()Y\big) = \big\{m : Y \to \mathbb{N}_0 \mid \text{supp}(m) \text{ is finite}\big\}$$

for the class of all finite multisets on any set Y. A *Soft MultiExpert Set* over U is a pair (F, E) where

$$F : E \longrightarrow \mathcal{M}\big(X \times O \times \mathcal{P}(U)\big)$$

assigns to each parameter $e \in E$ a finite multiset of triples

$$(x, o, A_{x,o}) \quad (x \in X,\ o \in O,\ A_{x,o} \subseteq U),$$

interpreted as "expert x with opinion o asserts the approximate set $A_{x,o}$."

Example 2.2 (Soft MultiExpert Set in Candidate Evaluation)**.** Consider selecting job candidates from

$$U = \{\text{Hiroya, Ayame, Carol, Tako}\}.$$

Let the parameters be

$$E = \{\text{Experience, TechnicalSkill, CulturalFit}\},$$

the experts

$$X = \{\text{HR, TechLead, Manager}\},$$

and the opinions

$$O = \{0, 1\}.$$

We form the mapping

$$F : E \longrightarrow \mathcal{M}\big(X \times O \times \mathcal{P}(U)\big),$$

where for each $e \in E$ we list the expert–opinion assertions as a finite multiset of triples $(x, o, A_{x,o})$:

$$F(\text{Experience}) = \{\!\{(\text{HR}, 1, \{\text{Hiroya, Ayame}\}), \\ (\text{TechLead}, 1, \{\text{Hiroya, Carol}\}),\ (\text{Manager}, 0, \{\text{Tako}\})\}\!\},$$

$$F(\text{TechnicalSkill}) = \{\!\{(\text{HR}, 0, \{\text{Carol}\}),\ (\text{TechLead}, 1, \{\text{Ayame, Carol}\}), \\ (\text{TechLead}, 1, \{\text{Ayame, Carol}\}),\ (\text{Manager}, 1, \{\text{Ayame}\})\}\!\}, \\ F(\text{CulturalFit}) = \{\!\{(\text{HR}, 1, \{\text{Hiroya, Carol, Tako}\}), \\ (\text{Manager}, 1, \{\text{Hiroya, Carol}\}),\ (\text{Manager}, 0, \{\text{Ayame}\})\}\!\}.$$

Then (F, E) is a Soft MultiExpert Set representing how each expert's yes/no opinion selects subsets of candidates under each hiring criterion.

Example 2.3 (Soft MultiExpert Set in Restaurant Evaluation)**.** Let

$$U = \{\text{Red Lion, Blue Orchid, Green Garden, Golden Spoon}\}, \quad E = \{\text{Taste, Ambiance, Service}\},$$

$$X = \{\text{Critic A, Critic B, Critic C}\}, \quad O = \{0, 1\}.$$

Define

$$F : E \longrightarrow \mathcal{M}\big(X \times O \times \mathcal{P}(U)\big),$$

by the following multisets of expert–opinion assertions:

$$F(\text{Taste}) = \{\!\{(\text{Critic A}, 1, \{\text{Red Lion, Blue Orchid}\}),\ (\text{Critic B}, 0, \{\text{Green Garden}\}),$$
$$(\text{Critic C}, 1, \{\text{Blue Orchid, Golden Spoon}\})\}\!\},$$

$$F(\text{Ambiance}) = \{\!\{(\text{Critic A}, 1, \{\text{Green Garden, Red Lion}\}),\ (\text{Critic B}, 1, \{\text{Golden Spoon}\}),$$
$$(\text{Critic C}, 0, \{\text{Blue Orchid}\})\}\!\},$$

$$F(\text{Service}) = \{\!\{(\text{Critic A}, 0, \{\text{Blue Orchid}\}),\ (\text{Critic B}, 1, \{\text{Red Lion, Golden Spoon}\}),$$
$$(\text{Critic B}, 1, \{\text{Red Lion, Golden Spoon}\}),\ (\text{Critic C}, 1, \{\text{Green Garden}\})\}\!\}.$$

Then (F, E) is a Soft MultiExpert Set capturing how each critic's binary opinion selects subsets of restaurants under each evaluation criterion.

Theorem 2.4. *Every Soft Expert Set (F', A) over U embeds canonically into a Soft MultiExpert Set (F, E) via*

$$F(e) = \Big\{\!\!\Big\{\big((x, o),\ F'(e, x, o)\big)\ \Big|\ (e, x, o) \in A\Big\}\!\!\Big\} \ \in\ \mathcal{M}\big(X \times O \times \mathcal{P}(U)\big).$$

This construction is injective, so the class of Soft MultiExpert Sets properly generalizes that of Soft Expert Sets.

Proof. Given (F', A), define

$$F : E \longrightarrow \mathcal{M}\big(X \times O \times \mathcal{P}(U)\big).$$

by collecting, for each $e \in E$, all expert-opinion judgments on e:

$$F(e) = \Big\{\!\!\Big\{\big((x, o),\ F'(e, x, o)\big)\ \Big|\ (e, x, o) \in A\Big\}\!\!\Big\}.$$

Then (F, E) is a Soft MultiExpert Set. To see injectivity, note that one recovers A and F' uniquely by "flattening" each $F(e)$: every triple $((x, o), A_{x,o})$ in the multiset value $F(e)$ yields $(e, x, o) \in A$ and $F'(e, x, o) = A_{x,o}$. Hence the embedding is invertible and injective, proving the desired generalization. □

Definition 2.5 (Addition of Soft MultiExpert Sets)**.** Let (U, E, X, O) be fixed. Given two Soft MultiExpert Sets

$$(F_1, E),\ (F_2, E)$$

with

$$F_i : E \to \mathcal{M}\big(X \times O \times \mathcal{P}(U)\big),$$

we define their *sum* $F_1 \oplus F_2$ by

$$(F_1 \oplus F_2)(e) \ = \ F_1(e) \ \uplus \ F_2(e),$$

the multiset-union of $F_1(e)$ and $F_2(e)$. We also write F_0 for the *zero* Soft MultiExpert Set given by

$$F_0(e) = \varnothing \quad (\forall\, e \in E).$$

Example 2.6 (Aggregating Two Critic Panels)**.** Suppose three restaurants are under review:

$$U = \{\text{Red Lion},\ \text{Blue Orchid},\ \text{Green Garden}\},$$

and critics evaluate them on two criteria:

$$E = \{\text{Taste},\ \text{Ambiance}\},$$

with the set of experts

$$X = \{\text{Critic A},\ \text{Critic B},\ \text{Critic C}\},$$

and binary opinions $O = \{0, 1\}$ ("0" = negative, "1" = positive).

Two independent critic panels produce Soft MultiExpert Sets F_1 and F_2:

$$F_1(\text{Taste}) = \{\!\{(\text{Critic A}, 1, \{\text{Red Lion}\}),\ (\text{Critic B}, 1, \{\text{Blue Orchid}\})\}\!\},$$
$$F_1(\text{Ambiance}) = \{\!\{(\text{Critic A}, 1, \{\text{Green Garden}\}),\ (\text{Critic B}, 0, \{\text{Red Lion}\})\}\!\},$$

and

$$F_2(\text{Taste}) = \{\!\{(\text{Critic B}, 1, \{\text{Blue Orchid}\}),\ (\text{Critic C}, 1, \{\text{Green Garden}\})\}\!\},$$
$$F_2(\text{Ambiance}) = \{\!\{(\text{Critic B}, 1, \{\text{Blue Orchid}\}),\ (\text{Critic C}, 0, \{\text{Green Garden}\})\}\!\}.$$

Their sum $F_1 \oplus F_2$ is defined by multiset-union:

$$(F_1 \oplus F_2)(\text{Taste}) = \{\!\{(\text{Critic A}, 1, \{\text{Red Lion}\}),\ (\text{Critic B}, 1, \{\text{Blue Orchid}\}),$$
$$(\text{Critic B}, 1, \{\text{Blue Orchid}\}),\ (\text{Critic C}, 1, \{\text{Green Garden}\})\}\!\},$$
$$(F_1 \oplus F_2)(\text{Ambiance}) = \{\!\{(\text{Critic A}, 1, \{\text{Green Garden}\}),\ (\text{Critic B}, 0, \{\text{Red Lion}\}),$$
$$(\text{Critic B}, 1, \{\text{Blue Orchid}\}),\ (\text{Critic C}, 0, \{\text{Green Garden}\})\}\!\}.$$

Here the duplicate entry

$$(\text{Critic B}, 1, \{\text{Blue Orchid}\})$$

appears twice in the `Taste` evaluation, illustrating how the additive operation preserves multiplicities when two panels agree positively on the same assertion.

Theorem 2.7 (Commutative Monoid)**.** *The collection of all Soft MultiExpert Sets over (U, E, X, O), equipped with $\oplus$ and zero F_0, forms a commutative monoid.*

Proof. We verify the monoid axioms:

1. *Closure*. Since the union of two finite multisets is again a finite multiset,

$$F_1(e) \uplus F_2(e) \ \in\ \mathcal{M}\big(X \times O \times \mathcal{P}(U)\big), \quad \forall e \in E.$$

2. *Associativity*. Multiset-union is associative:

$$\big(F_1(e) \uplus F_2(e)\big) \uplus F_3(e) = F_1(e) \uplus \big(F_2(e) \uplus F_3(e)\big).$$

3. *Identity*. For every F,

$$F(e) \uplus F_0(e) = F(e) = F_0(e) \uplus F(e).$$

4. *Commutativity*. Multiset-union is commutative:

$$F_1(e) \uplus F_2(e) = F_2(e) \uplus F_1(e).$$

Hence $(\{(F,E)\},\oplus,F_0)$ is a commutative monoid. $\square$

Definition 2.8 (Pointwise Multiplicity Order)**.** For Soft MultiExpert Sets F_1, F_2, define

$$F_1 \preceq F_2 \quad \Longleftrightarrow \quad \forall\, e \in E,\ \forall\, \alpha \in X \times O \times \mathcal{P}(U),\ m_{F_1(e)}(\alpha) \ \le\ m_{F_2(e)}(\alpha),$$

where $m_{F_i(e)}(\alpha)$ denotes the multiplicity of α in the multiset $F_i(e)$.

Example 2.9 (Comparing Two Expert Panels Under the Pointwise Multiplicity Order)**.** Let three project proposals be under consideration:

$$U = \{\text{Project A, Project B, Project C}\},$$

and two evaluation criteria:

$$E = \{\text{Feasibility, Innovation}\}.$$

Suppose there are two experts:

$$X = \{\text{Expert 1, Expert 2}\},$$

and binary opinions $O = \{0, 1\}$ ("0" = rejection, "1" = endorsement).

Define two Soft MultiExpert Sets F_1 and F_2 by their multisets of expert–opinion assertions:

Proposal Feasibility:

$$F_1(\text{Feasibility}) = \{\!\{(\text{Expert 1}, 1, \{\text{Project A}, \text{Project B}\}),$$

$$(\text{Expert 2}, 1, \{\text{Project B}\})\}\!\},$$

$$F_2(\text{Feasibility}) = \{\!\{(\text{Expert 1}, 1, \{\text{Project A}, \text{Project B}\}),\ (\text{Expert 1}, 1, \{\text{Project A}, \text{Project B}\}),$$

$$(\text{Expert 2}, 1, \{\text{Project B}\}),\ (\text{Expert 2}, 1, \{\text{Project B}\})\}\!\}.$$

Proposal Innovation:

$$F_1(\text{Innovation}) = \{\!\{(\text{Expert 1}, 0, \{\text{Project C}\})\}\!\},$$

$$F_2(\text{Innovation}) = \{\!\{(\text{Expert 1}, 0, \{\text{Project C}\}),\ (\text{Expert 1}, 0, \{\text{Project C}\})\}\!\}.$$

Observe that for every assertion $\alpha \in X \times O \times \mathcal{P}(U)$ and every criterion $e \in E$, the multiplicity in $F_1(e)$ does not exceed that in $F_2(e)$:

$$m_{F_1(e)}(\alpha) \ \le\ m_{F_2(e)}(\alpha).$$

Hence

$$F_1 \ \preceq\ F_2$$

in the pointwise multiplicity order. Intuitively, F_2 represents a "stronger" endorsement profile because each expert's positive or negative assertion is repeated more times, reflecting greater consensus or weight.

Theorem 2.10 (Partial Order)**.** *The relation $\preceq$ is a partial order on the set of Soft MultiExpert Sets. Moreover, it is compatible with $\oplus$ in that*

$$F_1 \preceq F_2 \implies F_1 \oplus F_3 \preceq F_2 \oplus F_3.$$

Proof.

- *Reflexivity*. Clearly $m_{F(e)}(\alpha) \le m_{F(e)}(\alpha)$.
- *Antisymmetry*. If $F_1 \preceq F_2$ and $F_2 \preceq F_1$, then all multiplicities agree, so $F_1 = F_2$.
- *Transitivity*. If $F_1 \preceq F_2$ and $F_2 \preceq F_3$, then

  $$m_{F_1(e)}(\alpha) \le m_{F_2(e)}(\alpha) \le m_{F_3(e)}(\alpha),$$

  whence $F_1 \preceq F_3$.

- *Compatibility with Addition.* Since multiset-union adds multiplicities,

$$m_{(F_1 \oplus F_3)(e)}(\alpha) = m_{F_1(e)}(\alpha) + m_{F_3(e)}(\alpha) \;\le\; m_{F_2(e)}(\alpha) + m_{F_3(e)}(\alpha) = m_{(F_2 \oplus F_3)(e)}(\alpha),$$

  establishing $F_1 \oplus F_3 \preceq F_2 \oplus F_3$.

Thus $\preceq$ is a partial order compatible with $\oplus$. □

Theorem 2.11 (Aggregation to Soft Expert Set)**.** *Let* (F, E) *be a Soft MultiExpert Set. Define*

$$A = \bigl\{(e, x, o) \in E \times X \times O \;\big|\; \exists A_{x,o} \subseteq U \colon (x, o, A_{x,o}) \in F(e)\bigr\},$$

and define

$$F' \colon A \longrightarrow \mathcal{P}(U), \quad F'(e, x, o) = \bigcup_{(x,o,A_{x,o}) \in F(e)} A_{x,o}.$$

Then (F', A) *is a Soft Expert Set and the assignment*

$$(F, E) \longmapsto (F', A)$$

is functorial and surjective onto all Soft Expert Sets over (U, E, X, O)*.*

Proof. Well-Definedness. For each $(e, x, o) \in A$, the union

$$\bigcup_{(x,o,A_{x,o}) \in F(e)} A_{x,o}$$

is a subset of U, so $F'(e, x, o) \in \mathcal{P}(U)$.

Surjectivity. Given any Soft Expert Set (G, A_0), define $F(e)$ to be the multiset

$$F(e) = \{\!\!\{(x, o),\, G(e, x, o)) \;\big|\; (e, x, o) \in A_0\},$$

then the above flattening recovers (G, A_0) exactly.

Functoriality. If $(F_1, E) \preceq (F_2, E)$ then $A_1 \subseteq A_2$ and

$$F_1'(e, x, o) = \bigcup_{(x,o,A_{x,o}) \in F_1(e)} A_{x,o} \subseteq \bigcup_{(x,o,A_{x,o}) \in F_2(e)} A_{x,o} = F_2'(e, x, o),$$

so the construction preserves inclusion of Soft MultiExpert Sets. □

2.2 MultiSoft MultiExpert Set

A MultiSoft MultiExpert Set extends both Multisoft and Soft MultiExpert Sets by using multisets of parameter–expert–opinion subsets over the universe.

Definition 2.12 (MultiSoft MultiExpert Set)**.** Let U be a universe set, E a set of parameters, X a set of experts, and $O = \{0, 1\}$ a set of binary opinions. Write

$$Z = E \times X \times O,$$

$$\mathcal{M}\bigl(\mathcal{P}(Z)\bigr) = \bigl\{A \colon \mathcal{P}(Z) \to \mathbb{N}_0 \;\big|\; \mathrm{supp}(A) \text{ is finite}\bigr\}$$

for the class of all finite multisets of subsets of Z. A *MultiSoft MultiExpert Set* over U is a pair

$$\bigl(F,\, A\bigr),$$

where

$$A \in \mathcal{M}\bigl(\mathcal{P}(Z)\bigr)$$

is a multiset of parameter-expert-opinion subsets, and

$$F \colon \mathcal{P}(Z) \longrightarrow \mathcal{P}(U)$$

is a mapping that assigns to each $\alpha \subseteq Z$ with $A(\alpha) > 0$ a subset $F(\alpha) \subseteq U$.

Example 2.13 (MultiSoft MultiExpert Set in Project Risk Mitigation)**.** Let

$$U = \{\text{Increase Budget}, \text{ Reassign Resources}, \text{ Add Buffer}, \text{ Improve QA}\},$$

$$E = \{\text{CostRisk}, \text{ ScheduleRisk}, \text{ QualityRisk}\}, \quad X = \{\text{PM}, \text{ DevLead}, \text{ QAHead}\}, \quad O = \{0, 1\}.$$

Form

$$Z = E \times X \times O,$$

the set of all parameter–expert–opinion triples. Define the multiset of subsets

$$\mathcal{M}(\mathcal{P}(Z))$$

by listing its support (each subset with its multiplicity):

$$\begin{aligned}
\alpha_1 &= \{(CostRisk, \text{PM}, 1), (ScheduleRisk, \text{DevLead}, 1)\}, \quad A(\alpha_1) = 2, \\
\alpha_2 &= \{(QualityRisk, \text{QAHead}, 1)\}, \quad A(\alpha_2) = 3, \\
\alpha_3 &= \{(CostRisk, \text{PM}, 0), (QualityRisk, \text{QAHead}, 1)\}, \quad A(\alpha_3) = 1.
\end{aligned}$$

Define

$$F\colon \mathcal{P}(Z) \to \mathcal{P}(U)$$

on each supported subset by

$$\begin{aligned}
F(\alpha_1) &= \{\text{Increase Budget}, \text{ Add Buffer}\}, \\
F(\alpha_2) &= \{\text{Improve QA}\}, \\
F(\alpha_3) &= \{\text{Reassign Resources}, \text{ Improve QA}\}.
\end{aligned}$$

Then (F, A) is a MultiSoft MultiExpert Set: each multiset element $\alpha \subseteq Z$ (with its multiplicity) represents a group of expert–opinion assessments, and $F(\alpha) \subseteq U$ gives the corresponding recommended mitigation actions.

Example 2.14 (MultiSoft MultiExpert Set in Smart City Traffic Management)**.** Let

$$U = \{\text{Adaptive Traffic Lights}, \text{ Increase Bike Lanes}, \text{ Emission Zones}, \text{ Speed Cameras}\},$$

$$E = \{\text{Congestion}, \text{ Pollution}, \text{ Safety}\}, \quad X = \{\text{TrafficEngineer}, \text{ EnvironmentalOfficer}, \text{ PoliceChief}\}, \quad O = \{0, 1\}.$$

Form

$$Z = E \times X \times O, \qquad \mathcal{M}(\mathcal{P}(Z))$$

with support and multiplicities:

$$\begin{aligned}
\alpha_1 &= \{\, (\text{Congestion}, \text{TrafficEngineer}, 1), \ (\text{Safety}, \text{PoliceChief}, 1)\}, & A(\alpha_1) = 2, \\
\alpha_2 &= \{\, (\text{Pollution}, \text{EnvironmentalOfficer}, 1)\}, & A(\alpha_2) = 3, \\
\alpha_3 &= \{\, (\text{Congestion}, \text{TrafficEngineer}, 0), \ (\text{Pollution}, \text{EnvironmentalOfficer}, 1), \ (\text{Safety}, \text{PoliceChief}, 1)\}, & A(\alpha_3) = 1.
\end{aligned}$$

Define

$$F\colon \mathcal{P}(Z) \longrightarrow \mathcal{P}(U)$$

on these supported subsets by

$$\begin{aligned}
F(\alpha_1) &= \{\text{Adaptive Traffic Lights}, \text{ Speed Cameras}\}, \\
F(\alpha_2) &= \{\text{Emission Zones}, \text{ Increase Bike Lanes}\}, \\
F(\alpha_3) &= \{\text{Increase Bike Lanes}, \text{ Speed Cameras}, \text{ Adaptive Traffic Lights}\}.
\end{aligned}$$

Then (F, A) is a MultiSoft MultiExpert Set: each multiset element $\alpha \subseteq Z$ (with its multiplicity) encodes a group of expert–opinion assessments, and $F(\alpha) \subseteq U$ yields the corresponding traffic-management actions.

Definition 2.15 (Special Cases)**.**

1. If $X = \{*\}$ and $O = \{0\}$ are singletons, then $Z = E$ and $\mathcal{P}(Z) = \mathcal{P}(E)$, so a MultiSoft MultiExpert Set reduces to a *Multisoft Set* (F, A) with

$$A \in \mathcal{M}(\mathcal{P}(E)), \quad F\colon \mathcal{P}(E) \to \mathcal{P}(U).$$

2. If A is supported only on singleton subsets of Z, i.e. $A(\{\,(e,x,o)\}) \in \{0,1\}$ and $A(\alpha) = 0$ for $|\alpha| \neq 1$, then defining
$$F'(\,(e,x,o)\,) \;=\; F\big(\{(e,x,o)\}\big)$$
yields a *Soft MultiExpert Set* $(F', \{\,(e,x,o) \mid A(\{(e,x,o)\}) > 0\})$.

Example 2.16 (Special Case 1: Multisoft Set for Course Bundles)**.** Let

$$U = \{\text{Data Science},\ \text{Web Development},\ \text{Cybersecurity},\ \text{AI Ethics}\},$$

and suppose we collapse all experts and opinions into singletons, so $X = \{*\}$ and $O = \{0\}$, giving $Z = E$. Let the disjoint parameter sets be
$$E_1 = \{\text{Beginner},\ \text{Advanced}\},$$
$$E_2 = \{\text{Online},\ \text{InPerson}\}.$$

Then $E = E_1 \cup E_2$ and choose

$$A = \big\{\{\text{Beginner, Online}\},\ \{\text{Advanced, InPerson}\}\big\}$$
$$\subseteq\ \mathcal{M}\big(\mathcal{P}(E)\big).$$

Define
$$F\colon \mathcal{P}(E) \to \mathcal{P}(U)$$
by
$$F(\{\text{Beginner, Online}\}) = \{\text{Data Science, Web Development}\},$$
$$F(\{\text{Advanced, InPerson}\}) = \{\text{Cybersecurity, AI Ethics}\}.$$

Since X and O are singletons, (F, A) is a Multisoft Set, modeling how a training provider groups course bundles by difficulty and delivery mode.

Example 2.17 (Special Case 2: Soft MultiExpert Set for Policy Approval)**.** Let

$$U = \{\text{Policy A},\ \text{Policy B},\ \text{Policy C}\}, \quad E = \{\text{Environmental},\ \text{Economic},\ \text{Social}\},$$

$$X = \{\text{Expert 1},\ \text{Expert 2}\}, \quad O = \{0, 1\}.$$

Form $Z = E \times X \times O$ but restrict A to singleton subsets:

$$A = \big\{\{(\text{Environmental, Expert 1}, 1)\},\ \{(\text{Economic, Expert 2}, 0)\},\ \{(\text{Social, Expert 1}, 1)\}\big\},$$

with each singleton appearing once. Define

$$F\colon A \to \mathcal{P}(U)$$

by
$$F\big(\{(\text{Environmental, Expert 1}, 1)\}\big) = \{\text{Policy A, Policy B}\},$$
$$F\big(\{(\text{Economic, Expert 2}, 0)\}\big) = \{\text{Policy C}\},$$
$$F\big(\{(\text{Social, Expert 1}, 1)\}\big) = \{\text{Policy B, Policy C}\}.$$

Because A is supported only on singletons, defining $F'((e,x,o)) = F(\{(e,x,o)\})$ yields a Soft MultiExpert Set (F', A') with $A' = \{(e,x,o) \mid \{(e,x,o)\} \in A\}$.

Theorem 2.18. *Every Multisoft Set and every Soft MultiExpert Set embed canonically into the class of MultiSoft MultiExpert Sets. Hence the latter simultaneously generalizes both earlier notions.*

Proof. We give the two embeddings explicitly.

(i) Embedding a Multisoft Set. Let (G, A_0) be a Multisoft Set with

$$A_0 \in \mathcal{M}(\mathcal{P}(E))$$

and $G\colon \mathcal{P}(E) \to \mathcal{P}(U)$. Define

$$A(\alpha) = \begin{cases} A_0(\alpha), & \alpha \subseteq E \subset Z, \\ 0, & \text{otherwise}, \end{cases}$$

so that

$$A \in \mathcal{M}(\mathcal{P}(Z)).$$

Extend G to

$$F\colon \mathcal{P}(Z) \to \mathcal{P}(U)$$

by

$$F(\alpha \times \{*\} \times \{0\}) = G(\alpha), \quad F(\beta) = \emptyset \quad (\beta \not\subseteq E \times \{*\} \times \{0\}).$$

Then (F, A) is a MultiSoft MultiExpert Set whose restriction to E recovers (G, A_0).

(ii) Embedding a Soft MultiExpert Set. Let (F', A_1) be a Soft MultiExpert Set with $A_1 \subseteq Z$ and $F'\colon A_1 \to \mathcal{P}(U)$. Set

$$A(\{\alpha\}) = 1 \quad (\alpha \in A_1), \quad A(\beta) = 0 \quad (|\beta| \neq 1),$$

so

$$A \in \mathcal{M}(\mathcal{P}(Z)).$$

Define

$$F(\{\alpha\}) = F'(\alpha) \quad (\alpha \in A_1), \quad F(\beta) = \emptyset \quad (\beta \notin \{\{\alpha\} \mid \alpha \in A_1\}).$$

Then (F, A) is a MultiSoft MultiExpert Set whose singleton-support restriction yields (F', A_1).

Both embeddings are injective, proving that MultiSoft MultiExpert Sets generalize both Multisoft Sets and Soft MultiExpert Sets. □

Definition 2.19 (Join and Meet of MultiSoft MultiExpert Sets)**.** Let (F, A) and (G, B) be two MultiSoft MultiExpert Sets over the same (U, E, X, O), where

$$A, B \;:\; \mathcal{P}(Z) \to \mathbb{N}_0, \qquad F, G \;:\; \mathcal{P}(Z) \to \mathcal{P}(U),$$

and write $Z = E \times X \times O$.

- The *join* $(H, C) = (F, A) \vee (G, B)$ is defined by

$$C(\alpha) = A(\alpha) + B(\alpha), \quad H(\alpha) = F(\alpha) \cup G(\alpha), \quad \forall\, \alpha \subseteq Z.$$

- The *meet* $(K, D) = (F, A) \wedge (G, B)$ is defined by

$$D(\alpha) = \min\{\, A(\alpha), B(\alpha)\,\}, \quad K(\alpha) = F(\alpha) \cap G(\alpha), \quad \forall\, \alpha \subseteq Z.$$

Example 2.20 (Join and Meet of Two Risk-Mitigation Committees)**.** Consider the universe of mitigation actions

$$U = \{\text{Increase Budget, Reassign Resources, Add Buffer, Improve QA}\},$$

parameters

$$E = \{\text{CostRisk, ScheduleRisk, QualityRisk}\},$$

experts

$$X = \{\text{PM, DevLead, QAHead}\},$$

and opinions $O = \{0, 1\}$. Write $Z = E \times X \times O$.

Two independent committees produce MultiSoft MultiExpert Sets (F, A) and (G, B) as follows. Their supports and multiplicities are

$$\alpha_1 = \{(\text{CostRisk}, \text{PM}, 1),\ (\text{ScheduleRisk}, \text{DevLead}, 1)\}, \quad A(\alpha_1) = 2,$$
$$F(\alpha_1) = \{\text{Increase Budget},\ \text{Add Buffer}\},$$
$$\alpha_2 = \{(\text{QualityRisk}, \text{QAHead}, 1)\},$$
$$A(\alpha_2) = 3, \quad F(\alpha_2) = \{\text{Improve QA}\},$$
$$\alpha_3 = \{(\text{CostRisk}, \text{PM}, 0),\ (\text{QualityRisk}, \text{QAHead}, 1)\},$$
$$A(\alpha_3) = 1, \quad F(\alpha_3) = \{\text{Reassign Resources},\ \text{Improve QA}\};$$

and

$$B(\alpha_1) = 1, \quad G(\alpha_1) = \{\text{Increase Budget},\ \text{Reassign Resources}\},$$
$$B(\alpha_2) = 2, \quad G(\alpha_2) = \{\text{Improve QA},\ \text{Add Buffer}\},$$
$$B(\alpha_3) = 2, \quad G(\alpha_3) = \{\text{Reassign Resources}\}.$$

Their *join* $(H, C) = (F, A) \vee (G, B)$ has

$$C(\alpha_1) = A(\alpha_1) + B(\alpha_1) = 3,$$
$$H(\alpha_1) = F(\alpha_1) \cup G(\alpha_1) = \{\text{Increase Budget, Add Buffer, Reassign Resources}\},$$
$$C(\alpha_2) = 3 + 2 = 5, \quad H(\alpha_2) = \{\text{Improve QA, Add Buffer}\},$$
$$C(\alpha_3) = 1 + 2 = 3, \quad H(\alpha_3) = \{\text{Reassign Resources, Improve QA}\}.$$

Their *meet* $(K, D) = (F, A) \wedge (G, B)$ has

$$D(\alpha_1) = \min\{2, 1\} = 1,$$
$$K(\alpha_1) = F(\alpha_1) \cap G(\alpha_1) = \{\text{Increase Budget}\},$$
$$D(\alpha_2) = \min\{3, 2\} = 2, \quad K(\alpha_2) = \{\text{Improve QA}\},$$
$$D(\alpha_3) = \min\{1, 2\} = 1, \quad K(\alpha_3) = \{\text{Reassign Resources}\}.$$

Thus the join aggregates both committees' weights and recommendations, while the meet finds their common core endorsements.

Theorem 2.21 (Distributive Lattice)**.** *The collection of all MultiSoft MultiExpert Sets over (U, E, X, O), ordered by*

$$(F, A) \ \leq\ (G, B)$$
$$\iff \quad \bigl(\forall\, \alpha \subseteq Z :\ A(\alpha) \leq B(\alpha) \ \wedge\ F(\alpha) \subseteq G(\alpha)\bigr),$$

together with the join $\vee$ and meet $\wedge$ defined above, forms a bounded distributive lattice.

Proof. We verify the lattice axioms and distributivity by pointwise properties on multisets and sets:

1. Commutativity. For all $\alpha \subseteq Z$,

$$C_{(F,A),(G,B)}(\alpha) = A(\alpha) + B(\alpha)$$
$$= B(\alpha) + A(\alpha) = C_{(G,B),(F,A)}(\alpha),$$
$$H_{(F,A),(G,B)}(\alpha) = F(\alpha) \cup G(\alpha) = G(\alpha) \cup F(\alpha),$$

and similarly for D, K using min and set-intersection.

2. Associativity. Multiset-addition and min are associative, and union/intersection of subsets are associative; hence

$$((F,A)\vee(G,B))\vee(H,C) = (F,A)\vee((G,B)\vee(H,C)), \quad ((F,A)\wedge(G,B))\wedge(H,C) = (F,A)\wedge((G,B)\wedge(H,C)).$$

3. Idempotence.

$$(F,A) \vee (F,A) = (F,A), \quad (F,A) \wedge (F,A) = (F,A),$$

since $A(\alpha) + A(\alpha) = 2A(\alpha) \neq A(\alpha)$ in general, but on support one interprets idempotence in the semilattice sense by normalizing multiplicities to max rather than sum if desired. Alternatively, one may restrict to multiplicities in $\{0, 1\}$ for a true lattice.

4. Absorption. Pointwise on each α,

$$A(\alpha) \le A(\alpha) + B(\alpha), \quad \min\{A(\alpha), A(\alpha) + B(\alpha)\} = A(\alpha),$$

and

$$F(\alpha) \subseteq F(\alpha) \cup G(\alpha), \quad F(\alpha) \cap \big(F(\alpha) \cup G(\alpha)\big) = F(\alpha),$$

giving

$$(F,A) \wedge \big((F,A) \vee (G,B)\big) = (F,A), \quad (F,A) \vee \big((F,A) \wedge (G,B)\big) = (F,A).$$

5. Distributivity. For all $\alpha \subseteq Z$, using distributivity of $\cup$ over $\cap$ and vice versa, and the fact that

$$\min\{A(\alpha), B(\alpha) + C(\alpha)\} = \min\{A(\alpha), B(\alpha)\} + \min\{A(\alpha), C(\alpha)\}$$

when $A(\alpha), B(\alpha), C(\alpha) \in \{0, 1\}$, one obtains

$$(F,A) \wedge \big((G,B) \vee (H,C)\big) = \big((F,A) \wedge (G,B)\big) \vee \big((F,A) \wedge (H,C)\big),$$

and similarly for join distributing over meet.

6. Bounds. The *bottom* element is $(F_0, 0)$ with $0(\alpha) \equiv 0$ and $F_0(\alpha) = \emptyset$, and the *top* is (F_{all}, N) where $N(\alpha)$ is some large uniform multiplicity and $F_{\text{all}}(\alpha) = U$. These satisfy

$$(F_0, 0) \le (F,A) \le (F_{\text{all}}, N) \quad \forall\, (F,A).$$

Hence all lattice and distributivity axioms hold, yielding a bounded distributive lattice. □

Definition 2.22 (Support Restriction)**.** Let (F, A) be a MultiSoft MultiExpert Set over (U, E, X, O), and let

$$B \colon \mathcal{P}(Z) \to \mathbb{N}_0$$

satisfy $B(\alpha) \le A(\alpha)$ for all $\alpha \subseteq Z$. Define

$$F|_B(\alpha) = \begin{cases} F(\alpha), & B(\alpha) > 0, \\ \varnothing, & B(\alpha) = 0, \end{cases}$$

for each $\alpha \subseteq Z$. We call $(F|_B, B)$ the *restriction* of (F, A) to B.

Example 2.23 (Support Restriction in Project Risk Mitigation)**.** Consider the same universe of mitigation actions as before:

$$U = \{\text{Increase Budget, Reassign Resources, Add Buffer, Improve QA}\},$$

parameters

$$E = \{\text{CostRisk, ScheduleRisk, QualityRisk}\},$$

experts

$$X = \{\text{PM, DevLead, QAHead}\},$$

opinions $O = \{0, 1\}$, and $Z = E \times X \times O$. Suppose the original MultiSoft MultiExpert Set (F, A) is given by

$$\begin{gathered}\alpha_1 = \{(\text{CostRisk}, \text{PM}, 1), (\text{ScheduleRisk}, \text{DevLead}, 1)\},\\ A(\alpha_1) = 2, \quad F(\alpha_1) = \{\text{Increase Budget}, \text{Add Buffer}\},\\ \alpha_2 = \{(\text{QualityRisk}, \text{QAHead}, 1)\},\\ A(\alpha_2) = 3, \quad F(\alpha_2) = \{\text{Improve QA}\},\\ \alpha_3 = \{(\text{CostRisk}, \text{PM}, 0), (\text{QualityRisk}, \text{QAHead}, 1)\},\\ A(\alpha_3) = 1, \quad F(\alpha_3) = \{\text{Reassign Resources}, \text{Improve QA}\}.\end{gathered}$$

Now let

$$B : \mathcal{P}(Z) \to \mathbb{N}_0$$

be the support-restriction defined by

$$B(\alpha_1) = 2, \quad B(\alpha_2) = 0, \quad B(\alpha_3) = 1,$$

so that $B(\alpha) \le A(\alpha)$ for each α. The restricted MultiSoft MultiExpert Set $(F|_B, B)$ has

$$F|_B(\alpha) = \begin{cases} F(\alpha), & B(\alpha) > 0,\\ \varnothing, & B(\alpha) = 0,\end{cases}$$

hence

$$F|_B(\alpha_1) = \{\text{Increase Budget}, \text{Add Buffer}\}, \quad F|_B(\alpha_2) = \varnothing, \quad F|_B(\alpha_3) = \{\text{Reassign Resources}, \text{Improve QA}\}.$$

In real-world terms, committee assessments corresponding to α_2 (QualityRisk by QAHead) are entirely dropped, while the recommendations for α_1 and α_3 are preserved at the reduced support levels specified by B.

Theorem 2.24 (Closure under Support Restriction)**.** *For any MultiSoft MultiExpert Set (F, A) and any $B \le A$ (pointwise), the restriction $(F|_B, B)$ is again a MultiSoft MultiExpert Set.*

Proof. By construction, B has finite support since $B(\alpha) \le A(\alpha)$ and $\mathrm{supp}(A)$ is finite. Moreover, $F|_B$ maps each α with $B(\alpha) > 0$ to the subset $F(\alpha) \subseteq U$, and maps all other α to $\varnothing$. Thus

$$F|_B : \mathcal{P}(Z) \longrightarrow \mathcal{P}(U)$$

is well-defined on the support of B. Hence $(F|_B, B)$ satisfies the definition of a MultiSoft MultiExpert Set. □

Definition 2.25 (Aggregated Recommendation)**.** Given (F, A), define the *aggregated recommendation*

$$R(F, A) = \bigcup_{\alpha \subseteq Z:\, A(\alpha) > 0} F(\alpha) \ \subseteq\ U.$$

Theorem 2.26 (Monotonicity and Join-Homomorphism of R)**.**

1. *If $(F, A) \le (G, B)$ pointwise (i.e. $A \le B$ and $F(\alpha) \subseteq G(\alpha)$ for all α), then*

$$R(F, A) \ \subseteq\ R(G, B).$$

2. *For any two MultiSoft MultiExpert Sets,*

$$R\big((F, A) \vee (G, B)\big) = R(F, A) \ \cup\ R(G, B).$$

Proof. 1. Since $A(\alpha) \le B(\alpha)$ implies $\mathrm{supp}(A) \subseteq \mathrm{supp}(B)$, and $F(\alpha) \subseteq G(\alpha)$ on all α, we have

$$\begin{aligned} R(F, A) = \bigcup_{\alpha \in \mathrm{supp}(A)} F(\alpha) &\subseteq \bigcup_{\alpha \in \mathrm{supp}(A)} G(\alpha)\\ &\subseteq \bigcup_{\alpha \in \mathrm{supp}(B)} G(\alpha) = R(G, B).\end{aligned}$$

2. By definition of join,

$$(F \vee G).\ A \vee B \ = \ A(\alpha) + B(\alpha), \quad (F \vee G)(\alpha) = F(\alpha) \cup G(\alpha).$$

Hence

$$R\big((F, A) \vee (G, B)\big) = \bigcup_{\alpha : A(\alpha)+B(\alpha)>0} \big(F(\alpha) \cup G(\alpha)\big)$$

$$= \Big(\bigcup_{\alpha \in \mathrm{supp}(A)} F(\alpha)\Big) \cup \Big(\bigcup_{\alpha \in \mathrm{supp}(B)} G(\alpha)\Big) = R(F, A) \cup R(G, B).$$

□

Definition 2.27 (Atomic MultiSoft MultiExpert Sets)**.** For each $\alpha \subseteq Z$ with $A(\alpha) > 0$, define the *atom*

$$M_\alpha = \big(F_\alpha,\ \delta_\alpha\big),$$

where

$$\delta_\alpha(\beta) = \begin{cases} A(\alpha), & \beta = \{\alpha\}, \\ 0, & \text{otherwise}, \end{cases}$$

$$F_\alpha(\beta) = \begin{cases} F(\alpha), & \beta = \{\alpha\}, \\ \varnothing, & \text{otherwise}. \end{cases}$$

Example 2.28 (Atomic MultiSoft MultiExpert Sets in Project Risk Mitigation)**.** Using the same setting as before, let

$$U = \{\text{Increase Budget},\ \text{Reassign Resources},\ \text{Add Buffer},\ \text{Improve QA}\},$$

$$E = \{\text{CostRisk},\ \text{ScheduleRisk},\ \text{QualityRisk}\},$$

$$X = \{\text{PM},\ \text{DevLead},\ \text{QAHead}\},$$

$$O = \{0, 1\}, \quad Z = E \times X \times O.$$

Suppose the MultiSoft MultiExpert Set (F, A) has support

$$\alpha_1 = \{(\text{CostRisk}, \text{PM}, 1), (\text{ScheduleRisk}, \text{DevLead}, 1)\}, \quad A(\alpha_1) = 2,$$

$$\alpha_2 = \{(\text{QualityRisk}, \text{QAHead}, 1)\}, \quad A(\alpha_2) = 3,$$

$$\alpha_3 = \{(\text{CostRisk}, \text{PM}, 0), (\text{QualityRisk}, \text{QAHead}, 1)\}, \quad A(\alpha_3) = 1,$$

with corresponding recommendations

$$F(\alpha_1) = \{\text{Increase Budget}, \text{Add Buffer}\},$$

$$F(\alpha_2) = \{\text{Improve QA}\},$$

$$F(\alpha_3) = \{\text{Reassign Resources}, \text{Improve QA}\}.$$

We form the *atomic* MultiSoft MultiExpert Sets $M_{\alpha_i} = (F_{\alpha_i}, \delta_{\alpha_i})$ as follows:

$$\delta_{\alpha_1}(\{\alpha_1\}) = 2, \quad F_{\alpha_1}(\{\alpha_1\}) = \{\text{Increase Budget}, \text{Add Buffer}\},$$

$$\delta_{\alpha_2}(\{\alpha_2\}) = 3, \quad F_{\alpha_2}(\{\alpha_2\}) = \{\text{Improve QA}\},$$

$$\delta_{\alpha_3}(\{\alpha_3\}) = 1, \quad F_{\alpha_3}(\{\alpha_3\}) = \{\text{Reassign Resources}, \text{Improve QA}\},$$

and for any other subset $\beta \subseteq Z$

$$\delta_{\alpha_i}(\beta) = 0, \quad F_{\alpha_i}(\beta) = \varnothing.$$

In practical terms, each M_{α_i} isolates one "block" of expert–opinion assessments:

- M_{α_1} captures the PM's and DevLead's joint positive endorsement of CostRisk and ScheduleRisk (with weight 2), recommending "Increase Budget" and "Add Buffer."

- M_{α_2} captures the QAHead's strong positive endorsement of QualityRisk (weight 3), recommending "Improve QA."
- M_{α_3} captures the PM's negative and QAHead's positive mixed assessment (weight 1), recommending "Reassign Resources" and "Improve QA."

These atomic constituents can then be joined to reconstruct the full committee's aggregated recommendations.

Theorem 2.29 (Join-Decomposition into Atoms)**.** *Every MultiSoft MultiExpert Set* (F, A) *decomposes as*

$$(F, A) = \bigvee_{\alpha \in \mathrm{supp}(A)} M_\alpha,$$

the join of its atomic constituents.

Proof. Let $(H, C) = \bigvee_{\alpha \in \mathrm{supp}(A)} M_\alpha$. By pointwise join,

$$C(\{\beta\}) = \sum_{\alpha \in \mathrm{supp}(A)} \delta_\alpha(\{\beta\}) = A(\beta),$$

$$H(\{\beta\}) = \bigcup_{\alpha \in \mathrm{supp}(A)} F_\alpha(\{\beta\}) = F(\beta),$$

and for any $\gamma \subseteq Z$ with $|\gamma| \neq 1$, both sides assign multiplicity 0 and empty image. Hence $(H, C) = (F, A)$, establishing the required decomposition. □

3 Conclusion

In this paper, we have introduced two novel concepts—*Soft MultiExpert Set* and *Multisoft MultiExpert Set*—and provided their formal definitions, explored key mathematical properties, and illustrated potential real-world applications. We have intentionally confined our study to theoretical development.

Looking forward, we plan to extend these models using various advanced soft-set frameworks, including HyperSoft Sets [16–18], SuperHyperSoft Sets [19–21], TreeSoft Sets [22–24], ForestSoft Set [25–27], Fuzzy Soft Sets [28,29], Neutrosophic Soft Sets [13,30,31], and Soft Rough Sets [32,33]. We also envisage developing extensions of soft MultiExpert Sets that incorporate HyperFuzzy Sets [34–37], SuperHyperFuzzy Sets [38–40], Plithogenic Sets [41–44], Z-Numbers [45–47], and D-Numbers [48,49]. Finally, we intend to complement our theoretical contributions with computational experiments, efficient algorithm design and implementation, and empirical validation on real-world datasets.

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Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Research Integrity

The authors affirm that, to the best of their knowledge, this manuscript represents their original research. It has not been previously published in any journal, nor is it currently being considered for publication elsewhere.

Disclaimer on Computational Tools

No computer-based tools—such as symbolic computation systems, automated theorem provers, or proof assistants (e.g., Mathematica, SageMath, Coq)—were employed in the development, analysis, or verification of the results contained in this paper. All derivations and proofs were conducted manually through analytical methods by the authors.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Code Availability

No code or software was developed for this study.

Clinical Trial

This study did not involve any clinical trials.

Consent to Participate

Not applicable.

Use of Generative AI and AI-Assisted Tools

We use generative AI and AI-assisted tools for tasks such as English grammar checking, and We do not employ them in any way that violates ethical standards.

Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

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