Foundations of $(m, n; L)$-SuperHyperFuzzy Sets

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Abstract

Uncertainty modeling plays a crucial role in decision-making across diverse domains [1], and numerous mathematical frameworks have been proposed to capture various aspects of imprecision. These include Fuzzy Sets [2], Rough Sets [3,4], Vague Sets [5,6], Intuitionistic Fuzzy Sets [7,8], Hesitant Fuzzy Sets [9,10], Neutrosophic Sets [11,12], and Plithogenic Sets [13,14]. Among these developments, Hyperfuzzy Sets [15,16] and their recursive generalizations, SuperHyperfuzzy Sets [17], extend the classical notion by assigning set-valued membership degrees at multiple hierarchical levels.

In this paper, we formally define the concept of $(m, n; L)$–SuperHyperFuzzy Sets and investigate their relationships with related structures, including SuperHyperFuzzy Sets, SuperHyperNeutrosophic Sets, and SuperHyperPlithogenic Sets. An $(m, n; L)$–SuperHyperFuzzy Set maps nonempty m-level subsets of a base set to nonempty families of n-level degree-sets valued in a complete commutative residuated lattice L, supporting t-norm/t-conorm aggregation.

Keywords: Fuzzy set, HyperFuzzy Set, Neutrosophic Set, Plithogenic Set, HyperNeutrosophic Set, HyperPlithogenic Set

Structure of this paper

The format of this paper is described below.

1 Preliminaries

This section provides an introduction to the foundational concepts and definitions required for the discussions in this paper. In addition, all concepts addressed herein are assumed to be finite rather than infinite.

1.1 Fuzzy and Hyperfuzzy Sets

A *fuzzy set* assigns to each element a membership degree in the interval $[0, 1]$, thereby capturing uncertainty through graded, rather than binary, membership [2,18,19]. Fuzzy sets have been applied across many domains, and a variety of extensions—such as Bipolar Fuzzy Sets [20–22] and Picture Fuzzy Sets [23–26]— have been proposed to handle richer kinds of ambiguity.

A *hyperfuzzy set* further generalizes this framework by assigning each element a nonempty subset of $[0, 1]$ [15, 27–33], thus representing multiple plausible membership values to model imprecision, variability, and evaluator disagreement. This concept is extended even further by *superhyperfuzzy sets* [17,27,34–36], which employ iterated nonempty power sets to encode hierarchical, multi-level fuzzy uncertainty.

Definition 1.1 (Base Set)**.** [37,38] A *base set* S is the foundational set from which complex structures such as powersets and hyperstructures are derived. It is formally defined as:

$$S = \{x \mid x \text{ is an element within a specified domain}\}.$$

All elements in constructs like $\mathcal{P}(S)$ or $\mathcal{P}_n(S)$ originate from the elements of S.

Definition 1.2 (Power set)**.** (cf. [39,40]) The *powerset* of a set S, denoted $\mathcal{P}(S)$, is the collection of all possible subsets of S, including both the empty set and S itself. Formally, it is expressed as:

$$\mathcal{P}(S) = \{A \mid A \subseteq S\}.$$

Definition 1.3 (Iterated powerset and its nonempty variant)**.** [41–45] For a set S and an integer $k \geq 0$, define

$$\mathcal{P}^0(S) := S, \qquad \mathcal{P}^{k+1}(S) := \mathcal{P}\big(\mathcal{P}^k(S)\big).$$

Write the nonempty iterate as $\mathcal{P}^k_*(S) := \mathcal{P}^k(S) \setminus \{\varnothing\}$. We use $\mathbb{N}_0 := \{0, 1, 2, \dots\}$.

Definition 1.4 (Fuzzy Set)**.** [2,46] A *Fuzzy set* τ in a non-empty universe Y is a mapping $\tau : Y \to [0, 1]$. A *fuzzy relation* on Y is a fuzzy subset δ in $Y \times Y$. If τ is a fuzzy set in Y and δ is a fuzzy relation on Y, then δ is called a *fuzzy relation on* τ if

$$\delta(y, z) \leq \min\{\tau(y), \tau(z)\} \quad \text{for all } y, z \in Y.$$

Definition 1.5 (Hyperfuzzy Set)**.** [15,16,33,47,48] Let X be a non-empty universe. A *hyperfuzzy set* $\tilde{A}$ on X is defined by a mapping

$$\tilde{\mu} : X \longrightarrow \tilde{P}([0, 1]),$$

where $\tilde{P}([0, 1])$ denotes the collection of all non-empty subsets of the interval $[0, 1]$.

For each element $x \in X$, $\tilde{\mu}(x) \subseteq [0, 1]$ represents the *set of possible membership degrees* of x in the set $\tilde{A}$. This formulation allows for representing uncertainty or variability in the degree of membership, extending the classical fuzzy set (which assigns a single real number in $[0, 1]$) to a set-valued interpretation.

Example 1.6 (Hyperfuzzy Set: online fraud risk from multiple detectors)**.** Let $X = \{\mathrm{tx}_1, \mathrm{tx}_2, \mathrm{tx}_3\}$ be card transactions. A hyperfuzzy set $\tilde{A}$ on X maps each transaction to the *set of plausible fraud-membership degrees* returned by independent detectors (rule-based, ML model, device fingerprinting):

$$\tilde{\mu}(\mathrm{tx}_1) = \{0.18,\ 0.24,\ 0.27\}, \quad \tilde{\mu}(\mathrm{tx}_2) = \{0.58,\ 0.61\}, \quad \tilde{\mu}(\mathrm{tx}_3) = \{0.09\}.$$

These set-valued degrees encode disagreement/variability across detectors. Operational summaries (explicit numerics):

$$\begin{aligned}
&\inf \tilde{\mu}(\mathrm{tx}_1) = 0.18,\ \sup \tilde{\mu}(\mathrm{tx}_1) = 0.27; \qquad \inf \tilde{\mu}(\mathrm{tx}_2) = 0.58,\ \sup \tilde{\mu}(\mathrm{tx}_2) = 0.61;\\
&\inf \tilde{\mu}(\mathrm{tx}_3) = \sup \tilde{\mu}(\mathrm{tx}_3) = 0.09.
\end{aligned}$$

A conservative screening policy may flag tx_2 (since sup = 0.61 is high), monitor tx_1 (sup = 0.27 moderate), and allow tx_3 (0.09 low). This use case illustrates how hyperfuzzy sets represent multi-source uncertainty via sets of degrees in $[0, 1]$.

An (m, n)-superhyperfuzzy set maps each m-level subset to a family of n-level membership sets, modeling hierarchical recursive uncertainty and complexity [49].

Definition 1.7 (m, n-SuperHyperFuzzy Set)**.** (cf. [17,36,49,50]) Let X be a nonempty set and let $m, n \in \mathbb{N}_0$. Define the nonempty k-th powerset of a set Y by

$$\mathcal{P}^*_0(Y) = Y, \quad \mathcal{P}^*_k(Y) = \mathcal{P}\big(\mathcal{P}^*_{k-1}(Y)\big) \setminus \{\emptyset\}, \quad k \geq 1.$$

In particular, $\mathcal{P}^*_m(X)$ is the family of all nonempty elements of the m-th iterated powerset of X, and $\mathcal{P}^*_n([0, 1])$ is defined analogously. Then an (m, n)-SuperHyperFuzzy Set on X is a function

$$\tilde{\mu}_{m,n} : \mathcal{P}^*_m(X) \longrightarrow \tilde{\mathcal{P}}^*_n\big([0, 1]\big), \quad A \mapsto \tilde{\mu}_{m,n}(A),$$

where $\tilde{\mathcal{P}}^*_n([0, 1])$ denotes the collection of all nonempty subsets of $\mathcal{P}_n([0, 1])$. Thus each $A \in \mathcal{P}^*_m(X)$ is assigned a nonempty family of membership-degree sets $\tilde{\mu}_{m,n}(A) \subseteq \mathcal{P}_n([0, 1])$, capturing hierarchical uncertainty across both the m- and n-levels.

Example 1.8 ((m,n)–SuperHyperfuzzy Set: regional air-quality escalation (two-level nesting))**.** Let $X = \{N, C, S\}$ denote *North*, *Central*, *South* districts of a city. Choose $m = 2$ and $n = 2$. Consider the m-level subset (a grouped alert candidate)

$$A = \big\{ \{N, C\},\ \{S\} \big\} \in \mathcal{P}_2^*(X).$$

Define an (m,n)–SuperHyperfuzzy assignment $\tilde{\mu}_{2,2} : \mathcal{P}_2^*(X) \to \tilde{\mathcal{P}}_2^*([0,1])$ by

$$\tilde{\mu}_{2,2}(A) = \{ S_1,\ S_2 \} \subseteq \mathcal{P}^2([0,1]),$$

where each S_i is a set of *inner* sets of degrees produced by distinct models/sensors:

$$S_1 = \Big\{ U_{11} = \{0.66,\ 0.72,\ 0.68\},\ U_{12} = \{0.74\} \Big\} \quad \text{(models for the bundle } \{N, C\}),$$
$$S_2 = \Big\{ V_{21} = \{0.55,\ 0.83\},\ V_{22} = \{0.61\},\ V_{23} = \{0.77,\ 0.69\} \Big\} \quad \text{(models for the bundle } \{S\}).$$

To obtain a single *crisp* alert score from this hierarchy, fix the canonical base kernel $B_\oplus(T) := \max T$ (inner-set summarization by maximum) and use outer aggregation by $\oplus = \max$. Compute levelwise (all steps shown):

Inner (k=1): $B_\oplus(U_{11}) = \max\{0.66, 0.72, 0.68\} = 0.72$, $B_\oplus(U_{12}) = \max\{0.74\} = 0.74$;
$B_\oplus(V_{21}) = \max\{0.55, 0.83\} = 0.83$, $B_\oplus(V_{22}) = \max\{0.61\} = 0.61$, $B_\oplus(V_{23}) = \max\{0.77, 0.69\} = 0.77$.

One level up (k=2): $\mathrm{Agg}^{(2)}_{B_\oplus}(S_1) = \max\{0.72, 0.74\} = 0.74$,

$\mathrm{Agg}^{(2)}_{B_\oplus}(S_2) = \max\{0.83, 0.61, 0.77\} = 0.83$.

Flatten family (n=2): $\mu^{\sharp, B_\oplus}(A) = \max\{\, 0.74,\ 0.83 \,\} = 0.83$.

Interpretation. The $(2,2)$ superhyperfuzzy structure preserves model-level uncertainty inside each district bundle (S_1 for $\{N, C\}$, S_2 for $\{S\}$) and then escalates by worst-case (max). The final crisp alert score 0.83 justifies issuing a citywide advisory.

1.2 Neutrosophic Set

Neutrosophic Sets generalize Fuzzy Sets by introducing an additional component: indeterminacy, alongside truth and falsity [12, 51–55]. A hyperneutrosophic set assigns each element a nonempty set of neutrosophic triplets, modeling multiple truth, indeterminacy, falsity evaluations across methods [56–59].

Definition 1.9 (Neutrosophic Set)**.** [60, 61] Let X be a non-empty set. A *Neutrosophic Set (NS) A* on X is characterized by three membership functions:

$$T_A : X \to [0,1], \quad I_A : X \to [0,1], \quad F_A : X \to [0,1],$$

where for each $x \in X$, the values $T_A(x)$, $I_A(x)$, and $F_A(x)$ represent the degrees of truth, indeterminacy, and falsity, respectively. These values satisfy the following condition:

$$0 \le T_A(x) + I_A(x) + F_A(x) \le 3.$$

Definition 1.10 (HyperNeutrosophic Set)**.** (cf. [17, 36, 56]) Let X be a non-empty set. A *HyperNeutrosophic Set (HNS) $\tilde{A}$* on X is a mapping:

$$\tilde{\mu} : X \to \mathcal{P}([0,1]^3),$$

where $\mathcal{P}([0,1]^3)$ is the family of all non-empty subsets of the unit cube $[0,1]^3$. For each $x \in X$, $\tilde{\mu}(x) \subseteq [0,1]^3$ is a set of neutrosophic membership triplets (T, I, F) that satisfy:

$$0 \le T + I + F \le 3.$$

Example 1.11 (HyperNeutrosophic Set: multi-source medical diagnosis)**.** Let $X = \{p_1, p_2, p_3\}$ be patients screened for community-acquired pneumonia. For each patient $x \in X$, we collect neutrosophic triplets (T, I, F) from independent sources (rapid antigen test, chest X-ray, clinician examination). Define the HyperNeutrosophic mapping $\tilde{\mu} : X \to \mathcal{P}([0,1]^3)$ by

$$\tilde{\mu}(p_1) = \big\{(0.82,\ 0.10,\ 0.06),\ (0.75,\ 0.20,\ 0.05)\big\},$$
$$\tilde{\mu}(p_2) = \big\{(0.40,\ 0.45,\ 0.15),\ (0.55,\ 0.30,\ 0.12)\big\},$$
$$\tilde{\mu}(p_3) = \big\{(0.20,\ 0.30,\ 0.50)\big\}.$$

Each triplet satisfies $0 \le T + I + F \le 3$ (indeed, here $T + I + F \le 1$):

$$\begin{aligned}
p_1 &: 0.82 + 0.10 + 0.06 = 0.98, \quad 0.75 + 0.20 + 0.05 = 1.00;\\
p_2 &: 0.40 + 0.45 + 0.15 = 1.00, \quad 0.55 + 0.30 + 0.12 = 0.97;\\
p_3 &: 0.20 + 0.30 + 0.50 = 1.00.
\end{aligned}$$

Interpretation. For p_1, strong supporting evidence (T high) with low indeterminacy (I), from two modalities; p_2 shows disagreement across sources (higher I), while p_3 has predominant falsity evidence.

A (m, n)-superhyperneutrosophic set maps each m-level subset to hierarchical nested families of truth, indeterminacy, and falsity triplets across n levels [49].

Definition 1.12 ((m, n)-SuperHyperNeutrosophic Set)**.** Let X be a nonempty set and $m, n \in \mathbb{N}_0$. Define recursively the nonempty k-th powerset of a set Y by

$$\mathcal{P}_0^*(Y) = Y, \qquad \mathcal{P}_k^*(Y) = \mathcal{P}\big(\mathcal{P}_{k-1}^*(Y)\big) \setminus \{\emptyset\}, \quad k \ge 1.$$

Let $\mathcal{P}_m^*(X)$ be the family of all nonempty elements of the m-th iterated powerset of X, and let $\mathcal{P}_n^*([0,1]^3)$ be defined analogously on the unit cube. An (m, n)*–superhyperneutrosophic set* on X is a mapping

$$\tilde{v}_{m,n} : \mathcal{P}_m^*(X) \longrightarrow \tilde{\mathcal{P}}_n^*\big([0,1]^3\big), \quad A \mapsto \tilde{v}_{m,n}(A),$$

where $\tilde{\mathcal{P}}_n^*([0,1]^3)$ denotes the collection of all nonempty subsets of $\mathcal{P}_n([0,1]^3)$. Thus each $A \in \mathcal{P}_m^*(X)$ is assigned a nonempty family of n-level neutrosophic-membership sets $\tilde{v}_{m,n}(A) \subseteq \mathcal{P}_n([0,1]^3)$.

Example 1.13 ((m, n)–SuperHyperNeutrosophic Set: city-level air-quality alert with nested regions)**.** Let $X = \{N, C, S\}$ denote *North*, *Central*, *South* city districts. Choose $m = 2$ and $n = 2$. Consider the m-level subset

$$A = \big\{ \{N, C\},\ \{S\} \big\} \in \mathcal{P}_2^*(X).$$

We assign to A a nonempty family of n-level sets of neutrosophic triplets $\tilde{v}_{2,2}(A) \in \tilde{\mathcal{P}}_2^*([0,1]^3)$ as follows:

$$\tilde{v}_{2,2}(A) = \Big\{ \underbrace{S_1}_{\text{(models for } \{N,C\})}, \underbrace{S_2}_{\text{(models for } \{S\})} \Big\},$$

with

$$\begin{aligned}
S_1 &= \Big\{ U_{11} = \{(0.68, 0.22, 0.10),\ (0.72, 0.18, 0.08)\},\ U_{12} = \{(0.60, 0.25, 0.15)\} \Big\},\\
S_2 &= \Big\{ V_{21} = \{(0.35, 0.50, 0.15),\ (0.40, 0.45, 0.12)\} \Big\}.
\end{aligned}$$

Verification of the neutrosophic constraint $0 \le T + I + F \le 3$ for all atoms:

$$\begin{aligned}
&0.68 + 0.22 + 0.10 = 1.00, \quad 0.72 + 0.18 + 0.08 = 0.98, \quad 0.60 + 0.25 + 0.15 = 1.00,\\
&0.35 + 0.50 + 0.15 = 1.00, \quad 0.40 + 0.45 + 0.12 = 0.97.
\end{aligned}$$

Interpretation. The outer family $\{S_1, S_2\}$ aggregates *region bundles* ($\{N, C\}$ vs. $\{S\}$). Inside each S_i, inner sets $U_{1\bullet}, V_{21}$ collect model-specific triplet estimates (e.g., chemical transport model vs. satellite retrieval). This realizes a concrete $(m, n) = (2, 2)$ superhyperneutrosophic assignment with two nesting levels that preserve multi-source uncertainty across grouped districts.

1.3 Plithogenic Set

A Plithogenic Set is a mathematical framework that incorporates multi-valued degrees of appurtenance and contradictions, making it suitable for complex decision-making processes. Various studies have been conducted on Plithogenic Sets [14, 62–65]. A hyperplithogenic set assigns each element hyper-valued attribute-value combinations to set-valued membership degrees, capturing inter-attribute contradictions and hierarchical uncertainty structures [49, 66–68]. The definition is presented below.

Definition 1.14 (Plithogenic Set)**.** [69,70] Let S be a universal set, and $P \subseteq S$. A *Plithogenic Set PS* is defined as:

$$PS = (P, v, Pv, pdf, pCF)$$

where:

- v is an attribute.
- Pv is the range of possible values for the attribute v.
- $pdf : P \times Pv \to [0,1]^s$ is the *Degree of Appurtenance Function (DAF)* [1]
- $pCF : Pv \times Pv \to [0,1]^t$ is the *Degree of Contradiction Function (DCF)*.

These functions satisfy the following axioms for all $a, b \in Pv$:

1. *Reflexivity of Contradiction Function*:
$$pCF(a, a) = 0$$
2. *Symmetry of Contradiction Function*:
$$pCF(a, b) = pCF(b, a)$$

Definition 1.15 (HyperPlithogenic Set)**.** (cf. [17,72,73]) Let X be a non-empty set, and let A be a set of attributes. For each attribute $v \in A$, let Pv be the set of possible values of v. A *HyperPlithogenic Set HPS* over X is defined as:
$$HPS = (P, \{v_i\}_{i=1}^n, \{Pv_i\}_{i=1}^n, \{\tilde{pdf}_i\}_{i=1}^n, pCF)$$
where:

- $P \subseteq X$ is a subset of the universe.
- For each attribute v_i, Pv_i is the set of possible values.
- For each attribute v_i, $\tilde{pdf}_i : P \times Pv_i \to \tilde{P}([0,1]^s)$ is the *Hyper Degree of Appurtenance Function (HDAF)*, assigning to each element $x \in P$ and attribute value $a_i \in Pv_i$ a set of membership degrees.
- $pCF : \left(\bigcup_{i=1}^n Pv_i\right) \times \left(\bigcup_{i=1}^n Pv_i\right) \to [0,1]^t$ is the *Degree of Contradiction Function (DCF)*.

Example 1.16 (HyperPlithogenic Set: laptop suitability with attribute–value contradictions)**.** Let $X = \{L_1, L_2, L_3\}$ be laptops to evaluate for a *data-science suitability* concept. Consider two attributes $A = \{\text{GPU}, \text{Weight}\}$ with value domains
$$P_{\text{GPU}} = \{\text{none}, \text{mid}, \text{high}\}, \qquad P_{\text{Weight}} = \{\text{light}, \text{medium}, \text{heavy}\}.$$
Define the Hyper Degree of Appurtenance functions (HDAFs) $\tilde{pdf}_{\text{GPU}} : X \times P_{\text{GPU}} \to \tilde{P}([0,1])$ and $\tilde{pdf}_{\text{Weight}} : X \times P_{\text{Weight}} \to \tilde{P}([0,1])$ by (nonempty sets of degrees):

$$\tilde{pdf}_{\text{GPU}}(L_1, \text{high}) = \{0.82, 0.88\},\ \tilde{pdf}_{\text{GPU}}(L_1, \text{mid}) = \{0.60\},\ \tilde{pdf}_{\text{GPU}}(L_1, \text{none}) = \{0.15\},$$
$$\tilde{pdf}_{\text{Weight}}(L_1, \text{light}) = \{0.70, 0.74\},\ \tilde{pdf}_{\text{Weight}}(L_1, \text{medium}) = \{0.50\},\ \tilde{pdf}_{\text{Weight}}(L_1, \text{heavy}) = \{0.10\};$$
$$\tilde{pdf}_{\text{GPU}}(L_2, \text{high}) = \{0.65\},\ \tilde{pdf}_{\text{GPU}}(L_2, \text{mid}) = \{0.48\},\ \tilde{pdf}_{\text{GPU}}(L_2, \text{none}) = \{0.10\},$$
$$\tilde{pdf}_{\text{Weight}}(L_2, \text{light}) = \{0.60\},\ \tilde{pdf}_{\text{Weight}}(L_2, \text{medium}) = \{0.55\},\ \tilde{pdf}_{\text{Weight}}(L_2, \text{heavy}) = \{0.20\};$$
$$\tilde{pdf}_{\text{GPU}}(L_3, \text{high}) = \{0.40\},\ \tilde{pdf}_{\text{GPU}}(L_3, \text{mid}) = \{0.70\},\ \tilde{pdf}_{\text{GPU}}(L_3, \text{none}) = \{0.25\},$$
$$\tilde{pdf}_{\text{Weight}}(L_3, \text{light}) = \{0.85\},\ \tilde{pdf}_{\text{Weight}}(L_3, \text{medium}) = \{0.45\},\ \tilde{pdf}_{\text{Weight}}(L_3, \text{heavy}) = \{0.10\}.$$

[1] It is important to note that the definition of the Degree of Appurtenance Function varies across different papers. Some studies define this concept using the power set, while others simplify it by avoiding the use of the power set [71]. The author has consistently defined the Classical Plithogenic Set without employing the power set.

Let the Degree of Contradiction Function pCF (symmetric, reflexive) encode practical trade-offs:

$$pCF(\text{high}, \text{light}) = 0.70, \quad pCF(\text{high}, \text{medium}) = 0.40, \quad pCF(\text{high}, \text{heavy}) = 0.10,$$
$$pCF(\text{mid}, \text{light}) = 0.30, \quad \text{and } \ pCF(a, a) = 0 \text{ for all values } a.$$

Choose the *dominant preference* pair (GPU = high, Weight = light). For a concise scalar decision, use the plithogenic mixer

$$\text{Mix}_{v,w}(\alpha, \beta) \ := \ \max\Big((1-\lambda)\ \min\{\alpha, \beta\},\ \lambda\ \max\{\alpha, \beta\}\Big), \quad \lambda := pCF(v, w),$$

with α from the chosen value of the first attribute and β from the chosen value of the second. Take the representative degrees by inner maxima (since HDAFs are set-valued):

$$L_1: \ \alpha = \max\{0.82, 0.88\} = 0.88, \quad \beta = \max\{0.70, 0.74\} = 0.74,$$
$$\lambda = pCF(\text{high}, \text{light}) = 0.70,\ \min\{0.88, 0.74\} = 0.74,\ \max\{0.88, 0.74\} = 0.88,$$
$$\text{Mix} = \max\{(1-0.70)\cdot 0.74,\ 0.70\cdot 0.88\} = \max\{0.222,\ 0.616\} = 0.616;$$
$$L_2: \ \alpha = 0.65,\ \beta = 0.60,\ \lambda = 0.70,\ \min = 0.60,\ \max = 0.65,$$
$$\text{Mix} = \max\{0.3\cdot 0.60,\ 0.7\cdot 0.65\} = \max\{0.18, 0.455\} = 0.455;$$
$$L_3: \ \alpha = 0.70\ (\text{mid}),\ \beta = 0.85,\ \lambda = pCF(\text{mid}, \text{light}) = 0.30,$$
$$\min = 0.70,\ \max = 0.85,\ \text{Mix} = \max\{0.7\cdot 0.70,\ 0.3\cdot 0.85\} = \max\{0.49, 0.255\} = 0.49.$$

Conclusion. Under this HyperPlithogenic model, $L_1\,(0.616) > L_3\,(0.49) > L_2\,(0.455)$, reflecting both suitability degrees and attribute-value contradictions.

An (m, n)-SuperHyperPlithogenic Set assigns hierarchical attribute-value subsets and multi-level membership plus contradiction degrees for complex decision modeling [49].

Definition 1.17 ((m, n)-SuperHyperPlithogenic Set)**.** Let X be a nonempty universe, $V = \{v_1, \ldots, v_\ell\}$ a finite attribute set, and for each v_i let P_{v_i} be its value domain. Fix $m, n \in \mathbb{N}_0$ and positive integers s, t. Define the nonempty k-th powerset $\mathcal{P}^*_k(Y)$ recursively by

$$\mathcal{P}^*_0(Y) = Y, \quad \mathcal{P}^*_k(Y) = \mathcal{P}(\mathcal{P}^*_{k-1}(Y)) \setminus \{\emptyset\}.$$

Then an (m, n)*–SuperHyperPlithogenic Set* is a structure

$$SHPS_{m,n} = \big(P_m,\ V, \{P_{v_i}\}_{i=1}^{\ell}, \{\tilde{pdf}_i^{(m,n)}\}_{i=1}^{\ell}, pCF^{(m,n)}\big),$$

where:

- $P_1 \subseteq X$, and for $k \geq 2$, $P_k = \mathcal{P}^*_1(P_{k-1})$. In particular $P_m \subseteq \mathcal{P}^*_m(X)$.
- Each $\tilde{pdf}_i^{(m,n)} : P_m \times P_{v_i} \to \tilde{\mathcal{P}}^*_n([0, 1]^s)$ is the *superhyper-degree of appurtenance*, assigning to each (A, a) a nonempty family of n-level membership-vectors in $[0, 1]^s$.
- The *superhyper-degree of contradiction*

$$pCF^{(m,n)} : \Big(\bigcup_i P_{v_i}\Big) \times \Big(\bigcup_i P_{v_i}\Big) \ \longrightarrow\ \tilde{\mathcal{P}}^*_n([0, 1]^t)$$

  satisfies for all a, b:

$$pCF^{(m,n)}(a, a) = \{0\}, \qquad pCF^{(m,n)}(a, b) = pCF^{(m,n)}(b, a).$$

Example 1.18 ((m, n)–SuperHyperPlithogenic Set: district air-quality decision with nested groups)**.** Let $X = \{N, C, S\}$ be *North*, *Central*, *South* districts. Set $m = 2$, $n = 2$. Take two attributes $V = \{\text{PM}, \text{O}_3\}$ with value domains $P_{\text{PM}} = \{\text{low}, \text{moderate}, \text{high}\}$ and $P_{\text{O}_3} = \{\text{normal}, \text{elevated}\}$. We evaluate group

$$A = \big\{\ \{N, C\},\ \{S\}\ \big\} \in \mathcal{P}^*_2(X).$$

Degrees live in $[0,1]^2$ (*health-risk*, *policy-urgency*), $s = 2$. Provide superhyper HDAFs as families of inner sets (models/sensors), then aggregate levelwise by componentwise max (base kernel on inner sets) and outer componentwise max across families.

PM attribute (value "high"):

$$S_1 = \Big\{ U_{11} = \{(0.72, 0.65), (0.75, 0.70)\},\ \ U_{12} = \{(0.68, 0.60)\} \Big\} \quad \text{for } \{N, C\},$$
$$S_2 = \Big\{ V_{21} = \{(0.55, 0.50), (0.80, 0.78)\},\ \ V_{22} = \{(0.60, 0.55)\} \Big\} \quad \text{for } \{S\}.$$

Inner aggregation (componentwise max):

$$B(U_{11}) = \max\{(0.72, 0.65), (0.75, 0.70)\} = (0.75, 0.70), \quad B(U_{12}) = (0.68, 0.60),$$
$$B(V_{21}) = \max\{(0.55, 0.50), (0.80, 0.78)\} = (0.80, 0.78), \quad B(V_{22}) = (0.60, 0.55).$$

One level up ($k = 2$):

$$\mathrm{Agg}^{(2)}(S_1) = \max\{(0.75, 0.70), (0.68, 0.60)\} = (0.75, 0.70),$$
$$\mathrm{Agg}^{(2)}(S_2) = \max\{(0.80, 0.78), (0.60, 0.55)\} = (0.80, 0.78).$$

Flatten PM across $\{S_1, S_2\}$:

$$\alpha_{\mathrm{PM}} = \max\{(0.75, 0.70), (0.80, 0.78)\} = (0.80, 0.78).$$

O_3 attribute (value "elevated"):

$$W_1 = \Big\{ (0.40, 0.45),\ (0.52, 0.55) \Big\} \quad \text{for } \{N, C\}, \qquad W_2 = \Big\{ (0.58, 0.62) \Big\} \quad \text{for } \{S\}.$$

Aggregation:

$$\mathrm{Agg}^{(2)}(W_1) = \max\{(0.40, 0.45), (0.52, 0.55)\} = (0.52, 0.55), \quad \mathrm{Agg}^{(2)}(W_2) = (0.58, 0.62).$$

Flatten O_3:

$$\beta_{\mathrm{O}_3} = \max\{(0.52, 0.55), (0.58, 0.62)\} = (0.58, 0.62).$$

Contradiction across attribute values (symmetric, reflexive) captures co-occurrence tension:

$$pCF^{(2,2)}(\text{high PM, elevated O}_3) = 0.60, \quad pCF^{(2,2)}(a, a) = \{0\}.$$

Componentwise plithogenic mixing (with $\lambda = 0.60$ and inner t-norm/t-conorm as min/max applied componentwise):

$$\mathrm{Mix}(\alpha_{\mathrm{PM}}, \beta_{\mathrm{O}_3}) = \max\Big((1-\lambda)\ \min\{\alpha_{\mathrm{PM}}, \beta_{\mathrm{O}_3}\},\ \lambda\ \max\{\alpha_{\mathrm{PM}}, \beta_{\mathrm{O}_3}\}\Big).$$

Compute min and max componentwise:

$$\min = \big(\min\{0.80, 0.58\}, \min\{0.78, 0.62\}\big) = (0.58, 0.62), \quad \max = (0.80, 0.78).$$

Thus

$$(1-\lambda)\ \min = 0.40 \cdot (0.58, 0.62) = (0.232, 0.248), \quad \lambda\ \max = 0.60 \cdot (0.80, 0.78) = (0.480, 0.468),$$

and finally

$$\mathrm{Mix} = (\max\{0.232, 0.480\},\ \max\{0.248, 0.468\}) = (0.480,\ 0.468).$$

Interpretation. The $(m, n) = (2, 2)$ SuperHyperPlithogenic construction preserves nested (region, model) information per attribute, then fuses attributes under an explicit contradiction penalty. The resulting *(health-risk, policy-urgency)* vector $(0.480, 0.468)$ supports issuing an advisory if thresholds (e.g. ≥ 0.45) are met.

2 The $(m, n; L)$–SuperHyperFuzzy Set: Full Specification and Plithogenic Recovery

An $(m, n; L)$–SuperHyperFuzzy set maps nonempty m-level subsets of a base set to nonempty families of n-level degree-sets valued in a CCR-lattice L, supporting t-norm/t-conorm aggregation.

Definition 2.1 (Degree algebra (CCR–lattice))**.** A *complete commutative residuated lattice* (CCR–lattice) is a quadruple $(L, \otimes, \oplus, \leq)$ where:

- $(L, \leq)$ is a complete lattice;
- $\otimes : L \times L \to L$ is a commutative, associative, monotone t–norm with unit 1;
- $\oplus : L \times L \to L$ is a commutative, associative, monotone t–conorm with unit 0;
- $\otimes$ and $\oplus$ distribute over arbitrary joins/meets in the sense required by residuation.

Typical choices include $([0, 1], \min, \max)$ or $([0, 1], \cdot, a+b-ab)$.

Definition 2.2 ($(m, n; L)$–SuperHyperFuzzy Set)**.** Let X be a nonempty base set, $m, n \in \mathbb{N}_0$ with $n \geq m$, and $(L, \otimes, \oplus, \leq)$ a CCR–lattice. An $(m, n; L)$*–SuperHyperFuzzy set* on X is a map

$$\widetilde{\mu}_{m,n} \ : \ \mathcal{P}_*^m(X) \longrightarrow \mathfrak{P}_n^*(L),$$

where $\mathfrak{P}_n^*(L)$ is the collection of all nonempty subsets of $\mathcal{P}^n(L)$. For $A \in \mathcal{P}_*^m(X)$, the value $\widetilde{\mu}_{m,n}(A) \subseteq \mathcal{P}^n(L)$ is a nonempty *family* of n–level degree–sets whose atoms lie in L.

Example 2.3 (Hospital triage portfolios)**.** Let $X = \{p_1, p_2, p_3\}$ be patients. Take $m = 1$, $n = 2$, $L = [0, 1]$, $\oplus = \max$, and use the canonical base kernel $B_\oplus(S) = \bigoplus_{a \in S} a = \max S$. For the batch $A = \{p_1, p_2\} \in \mathcal{P}_*^1(X)$, define

$$\widetilde{\mu}_{1,2}(A) \ = \ \Big\{S_1 = \{\{0.7, 0.6\}, \{0.75\}\}, \quad S_2 = \{\{0.5\}, \{0.65, 0.55\}\}\Big\} \ \subseteq \ \mathcal{P}^2([0, 1]).$$

Level-1 aggregation (inner sets to scalars):

$$\mathrm{Agg}^{(1)}_{B_\oplus}(S_1) = \max\{\max\{0.7, 0.6\}, \max\{0.75\}\} = \max\{0.7, 0.75\} = 0.75,$$

$$\mathrm{Agg}^{(1)}_{B_\oplus}(S_2) = \max\{\max\{0.5\}, \max\{0.65, 0.55\}\} = \max\{0.5, 0.65\} = 0.65.$$

Flattening at level $n = 2$:

$$\mu^{\sharp, B_\oplus}(A) \ = \ \mathrm{crisp}_2^{B_\oplus}\big(\widetilde{\mu}_{1,2}(A)\big) = \max\{\, 0.75, \ 0.65 \,\} = 0.75.$$

Thus the $(1, 2; [0, 1])$–SuperHyperFuzzy assignment rates the batch A at 0.75 after hierarchical flattening.

Example 2.4 (Supply–chain multi-criteria group)**.** Let $X = \{s_1, s_2, s_3, s_4\}$ (suppliers). Take $m = 2$, $n = 2$, $L = [0, 1]^3$ with componentwise order, $\oplus = \max$ and $\otimes = \min$ applied componentwise, and $B_\oplus$ componentwise. Consider $A = \{\{s_1, s_2\}, \{s_3\}\} \in \mathcal{P}_*^2(X)$ and

$$\widetilde{\mu}_{2,2}(A) = \Big\{S_1 = \big\{\{(0.7, 0.6, 0.8), (0.6, 0.5, 0.9)\}, \{(0.8, 0.7, 0.7)\}\big\}, \quad S_2 = \big\{\{(0.5, 0.9, 0.6)\}\big\}\Big\}.$$

Level-1 aggregation (componentwise):

$$\begin{aligned}
U_1 &= \max\big((0.7, 0.6, 0.8), (0.6, 0.5, 0.9)\big) = (0.7, 0.6, 0.9),\\
U_2 &= (0.8, 0.7, 0.7),\\
\mathrm{Agg}^{(1)}_{B_\oplus}(S_1) &= \max\big(U_1, U_2\big) = \max\big((0.7, 0.6, 0.9), (0.8, 0.7, 0.7)\big) = (0.8, 0.7, 0.9),\\
\mathrm{Agg}^{(1)}_{B_\oplus}(S_2) &= (0.5, 0.9, 0.6).
\end{aligned}$$

Flattening at level $n = 2$ (componentwise):

$$\mu^{\sharp, B_\oplus}(A) = \max\big((0.8, 0.7, 0.9), (0.5, 0.9, 0.6)\big) = (0.8, 0.9, 0.9).$$

Hence the crisp image yields a tri-criterion summary $(0.8, 0.9, 0.9)$ for group A.

Example 2.5 (Cybersecurity alert clustering)**.** Let X be alerts; take $m = 1$, $n = 2$, $L = [0, 1]$, $\otimes = \cdot$ (product t–norm), $\oplus(a, b) = a + b - ab$ (probabilistic sum), and $B_\oplus(S)$ the iterative $\oplus$ over S. For $A = \{\text{cluster}_1\} \in \mathcal{P}_*^1(X)$, let

$$\widetilde{\mu}_{1,2}(A) = \Big\{S_1 = \{\{0.4, 0.5\}, \{0.6\}\}, \quad S_2 = \{\{0.3\}, \{0.25, 0.35\}\}\Big\}.$$

Inner aggregation ($k = 1$; compute carefully):

$$\begin{aligned}\mathrm{Agg}^{(1)}_{B_\oplus}(S_1) &= (0.4 \oplus 0.5) \;\oplus\; 0.6 = \underbrace{(0.4 + 0.5 - 0.4 \cdot 0.5)}_{=0.9-0.2=0.7} \oplus 0.6 = 0.7 + 0.6 - 0.7 \cdot 0.6 \\ &= 1.3 - 0.42 = 0.88, \\ \mathrm{Agg}^{(1)}_{B_\oplus}(S_2) &= 0.3 \;\oplus\; (0.25 \oplus 0.35) = 0.3 \;\oplus\; \underbrace{(0.25 + 0.35 - 0.25 \cdot 0.35)}_{=0.6-0.0875=0.5125} \\ &= 0.3 + 0.5125 - 0.3 \cdot 0.5125 = 0.8125 - \underbrace{0.15375}_{=0.3\times 0.5125} = 0.65875.\end{aligned}$$

Flattening at level $n = 2$:

$$\mu^{\sharp, B_\oplus}(A) = 0.88 \;\oplus\; 0.65875 = 0.88 + 0.65875 - 0.88 \cdot 0.65875.$$

The product term is $0.88 \times 0.65875 = \dfrac{88}{100} \times 0.65875 = 0.5797$, hence

$$\mu^{\sharp, B_\oplus}(A) = 1.53875 - 0.5797 = 0.95905.$$

Thus the hierarchical risk for cluster A flattens to 0.95905 under the probabilistic-sum semantics.

Example 2.6 (Hiring committee (two-criterion))**.** Let $X = \{\text{cand}_1, \text{cand}_2, \text{cand}_3\}$. Take $m = 1$, $n = 2$, $L = [0, 1]^2$ with componentwise $\oplus = \max$. For the short-list $A = \{\text{cand}_1, \text{cand}_2\}$, define

$$\widetilde{\mu}_{1,2}(A) = \Big\{S_1 = \{\ (0.7, 0.6),\ (0.65, 0.8)\ \},\ \ \{(0.8, 0.55)\}\ ; \quad S_2 = \{\ (0.6, 0.9)\ \}\Big\}.$$

Inner aggregation ($k = 1$; componentwise):

$$\begin{aligned}\mathrm{Agg}^{(1)}_{B_\oplus}(S_1) &= \max\big(\max\{(0.7, 0.6), (0.65, 0.8)\},\ (0.8, 0.55)\big) \\ &= \max\big((0.7, 0.8), (0.8, 0.55)\big) = (0.8, 0.8), \\ \mathrm{Agg}^{(1)}_{B_\oplus}(S_2) &= (0.6, 0.9).\end{aligned}$$

Flattening at level $n = 2$:

$$\mu^{\sharp, B_\oplus}(A) = \max\big((0.8, 0.8), (0.6, 0.9)\big) = (0.8, 0.9).$$

Hence the crisp summary for the short-list is (skill $= 0.8$, fit $= 0.9$).

Definition 2.7 (Parametric levelwise aggregation and flattening)**.** Let $B : \mathcal{P}_*(L) \to L$ be a *base kernel* on nonempty subsets of L satisfying:

1. $B(\{a\}) = a$ for all $a \in L$ (normalization),
2. $S \subseteq T \;\Rightarrow\; B(S) \le B(T)$ (monotonicity).

Define recursively, for $k \ge 0$, the *B–level aggregator* $\mathrm{Agg}^{(k)}_B : \mathcal{P}^k(L) \to L$ by

$$\mathrm{Agg}^{(0)}_B(a) := a \quad (a \in L), \qquad \mathrm{Agg}^{(1)}_B(S) := B(S) \quad (S \neq \varnothing),$$

$$\mathrm{Agg}^{(k+1)}_B(S) := \bigoplus_{T \in S} \mathrm{Agg}^{(k)}_B(T) \quad (k \ge 1).$$

For a nonempty family $\mathcal{F} \subseteq \mathcal{P}^n(L)$, define the *flattening (crisp) operator*

$$\operatorname{crisp}_n^B(\mathcal{F}) \ := \ \bigoplus_{S\in\mathcal{F}} \operatorname{Agg}_B^{(n)}(S) \ \in L.$$

Given $\widetilde{\mu}_{m,n}$, its n–level *crisp image (via B)* is

$$\mu^{\sharp,B}(A) \ := \ \operatorname{crisp}_n^B\big(\widetilde{\mu}_{m,n}(A)\big) \in L \qquad \big(A \in \mathcal{P}_*{}^m(X)\big).$$

A canonical choice is $B_\oplus(S) := \bigoplus_{a\in S} a$, which recovers the standard fuzzy flattening.

Example 2.8 (Urban air–quality alert escalation with a thresholded base kernel)**.** Let $L = [0,1]$ with $\oplus = \max$. Consider two nested evidence groups for a citywide alert:

$$\mathcal{F} = \{\, S_1,\ S_2 \,\} \ \subseteq \ \mathcal{P}^2(L),$$

where S_1, S_2 collect station–level summaries (each inner set is a bundle of sensor scores). Define the *parametric* base kernel $B_\varepsilon : \mathcal{P}_*(L) \to L$ by

$$B_\varepsilon(T) \ := \ \begin{cases} \max\{\, a\in T :\ a \ge \varepsilon \,\}, & \text{if } \{a \in T :\ a \ge \varepsilon\} \neq \varnothing, \\ \max T, & \text{otherwise,} \end{cases}$$

which satisfies $B_\varepsilon(\{a\}) = a$ and $S \subseteq T \Rightarrow B_\varepsilon(S) \le B_\varepsilon(T)$. Take $\varepsilon = 0.70$ (ignore weak/low–confidence signals unless no strong signal exists). Let

$$S_1 = \big\{\ \{0.66, 0.72, 0.68\},\ \{0.74\}\ \big\}, \qquad S_2 = \big\{\ \{0.55, 0.83\},\ \{0.61\},\ \{0.77, 0.69\}\ \big\}.$$

Levelwise aggregation with $\operatorname{Agg}_{B_\varepsilon}^{(k)}$ (Definition 2.7):

1) Inner level ($k = 1$): apply B_ε to each inner set T.

$$\begin{aligned} B_\varepsilon(\{0.66, 0.72, 0.68\}) &= \max\{0.72\} = 0.72, \\ B_\varepsilon(\{0.74\}) &= 0.74, \\ B_\varepsilon(\{0.55, 0.83\}) &= \max\{0.83\} = 0.83, \\ B_\varepsilon(\{0.61\}) &= \max\{0.61\} = 0.61 \quad (\text{no element } \ge 0.70), \\ B_\varepsilon(\{0.77, 0.69\}) &= \max\{0.77\} = 0.77. \end{aligned}$$

2) One level up ($k = 2$): $\operatorname{Agg}_{B_\varepsilon}^{(2)}(S) = \bigoplus_{T\in S} B_\varepsilon(T) = \max_{T\in S} B_\varepsilon(T)$.

$$\operatorname{Agg}_{B_\varepsilon}^{(2)}(S_1) = \max\{0.72, 0.74\} = 0.74, \qquad \operatorname{Agg}_{B_\varepsilon}^{(2)}(S_2) = \max\{0.83, 0.61, 0.77\} = 0.83.$$

Flattening (crisp) of the family $\mathcal{F}$ at level $n = 2$:

$$\operatorname{crisp}_2^{B_\varepsilon}(\mathcal{F}) \ = \ \bigoplus_{S\in\mathcal{F}} \operatorname{Agg}_{B_\varepsilon}^{(2)}(S) \ = \ \max\{\, 0.74,\ 0.83 \,\} \ = \ 0.83.$$

Interpretation. The parameter $\varepsilon = 0.70$ enforces robustness by privileging strong signals at the inner level; outer levels aggregate by worst–case (max). The final crisp value 0.83 triggers a citywide alert. This same B_ε can be plugged into an $(m, n; L)$–SuperHyperFuzzy assignment via $\mu^{\sharp,B_\varepsilon}(A) = \operatorname{crisp}_n^{B_\varepsilon}\big(\widetilde{\mu}_{m,n}(A)\big)$.

Theorem 2.9 (Monotonicity and bounds)**.** *Let B satisfy Definition 2.7. For any nonempty $\mathcal{F} \subseteq \mathcal{P}^n(L)$,*

$$\bigvee\{\inf T \mid T \in S,\ S \in \mathcal{F}\} \ \le \ \operatorname{crisp}_n^B(\mathcal{F}) \ \le \ \bigvee\{\sup T \mid T \in S,\ S \in \mathcal{F}\}.$$

In particular, for $B = B_\oplus$ and $L = [0,1]$ with $\oplus = \max$, $\operatorname{crisp}_n^{B_\oplus}(\mathcal{F}) = \max\{\operatorname{Agg}_{B_\oplus}^{(n)}(S) : S \in \mathcal{F}\}$.

Proof. By induction on k for $\mathrm{Agg}_B^{(k)}$. The step $k \to k+1$ uses the monotonicity of $\oplus$ and completeness of $(L, \leq)$. The outer $\bigoplus$ preserves both the lower and upper bounds. $\square$

Definition 2.10 (Union and lifted t–norm product)**.** Let $\widetilde{\mu}^A_{m,n}, \widetilde{\mu}^B_{m,n} : \mathcal{P}_*^{\,m}(X) \to \mathfrak{P}^*_n(L)$. For $C \in \mathcal{P}_*^{\,m}(X)$ define

$$\big(\widetilde{\mu}^A_{m,n} \sqcup \widetilde{\mu}^B_{m,n}\big)(C) \ := \ \widetilde{\mu}^A_{m,n}(C) \ \cup \ \widetilde{\mu}^B_{m,n}(C).$$

To combine degree–sets multiplicatively, lift $\otimes$ levelwise by

$$\otimes^{(0)}(a, b) := a \otimes b, \qquad \otimes^{(k+1)}(S, T) := \big\{\, U \otimes^{(k)} V \ \big|\ U \in S,\ V \in T \,\big\},$$

and set

$$\big(\widetilde{\mu}^A_{m,n} \sqcap \widetilde{\mu}^B_{m,n}\big)(C) \ := \ \big\{\, S \otimes^{(n)} T \ \big|\ S \in \widetilde{\mu}^A_{m,n}(C),\ T \in \widetilde{\mu}^B_{m,n}(C) \,\big\}.$$

Example 2.11 (Cybersecurity fusion: union vs. lifted t–norm product on an asset group)**.** Let X be enterprise assets and fix an asset group $C = \{\texttt{srv1}, \texttt{srv2}\} \in \mathcal{P}^1_*(X)$. We take $(m, n; L) = (1, 2; [0, 1])$ with $\oplus = \max$, $\otimes = \min$, and the canonical base kernel $B_\oplus(S) = \max S$. Consider two independent assignments at C:

$$\widetilde{\mu}^A_{1,2}(C) = \Big\{ S_{A1} = \{\{0.6, 0.8\}, \{0.7\}\}, \quad S_{A2} = \{\{0.5, 0.9\}\} \Big\},$$

$$\widetilde{\mu}^B_{1,2}(C) = \Big\{ S_{B1} = \{\{0.65\}, \{0.75, 0.55\}\} \Big\}.$$

Union (evidence pooling).

$$\big(\widetilde{\mu}^A_{1,2} \sqcup \widetilde{\mu}^B_{1,2}\big)(C) = \{\, S_{A1}, S_{A2}, S_{B1} \,\}.$$

Crisp evaluation (Definition 2.7 with $B_\oplus$):

$$\begin{aligned}
\mathrm{Agg}^{(1)}_{B_\oplus}(S_{A1}) &= \max\{\max\{0.6, 0.8\}, \max\{0.7\}\} = \max\{0.8, 0.7\} = 0.8,\\
\mathrm{Agg}^{(1)}_{B_\oplus}(S_{A2}) &= \max\{0.5, 0.9\} = 0.9,\\
\mathrm{Agg}^{(1)}_{B_\oplus}(S_{B1}) &= \max\{\max\{0.65\}, \max\{0.75, 0.55\}\} = \max\{0.65, 0.75\} = 0.75,\\
\mathrm{crisp}^{B_\oplus}_2\big(\widetilde{\mu}^A_{1,2} \sqcup \widetilde{\mu}^B_{1,2}\big)(C) &= \max\{0.8, 0.9, 0.75\} = 0.9.
\end{aligned}$$

Lifted t–norm product (conjunctive fusion). For $n = 2$, the lifted product uses $\otimes^{(2)}(S, T) = \{\, U \otimes^{(1)} V : U \in S,\ V \in T \,\}$ with $\otimes^{(1)}(U, V) = \{\, \min(u, v) : u \in U,\ v \in V \,\}$. Compute all pairings between S_{A1} (whose elements are $U_1 = \{0.6, 0.8\}$, $U_2 = \{0.7\}$) and S_{B1} (elements $V_1 = \{0.65\}$, $V_2 = \{0.75, 0.55\}$):

$$\begin{aligned}
U_1 \otimes^{(1)} V_1 &= \{\min(0.6, 0.65), \min(0.8, 0.65)\} = \{0.6, 0.65\},\\
U_1 \otimes^{(1)} V_2 &= \{\min(0.6, 0.75), \min(0.6, 0.55), \min(0.8, 0.75), \min(0.8, 0.55)\}\\
&= \{0.6, 0.55, 0.75, 0.55\} = \{0.55, 0.6, 0.75\},\\
U_2 \otimes^{(1)} V_1 &= \{\min(0.7, 0.65)\} = \{0.65\},\\
U_2 \otimes^{(1)} V_2 &= \{\min(0.7, 0.75), \min(0.7, 0.55)\} = \{0.7, 0.55\}.
\end{aligned}$$

Hence

$$\big(\widetilde{\mu}^A_{1,2} \sqcap \widetilde{\mu}^B_{1,2}\big)(C) = \big\{\ \{0.6, 0.65\},\ \{0.55, 0.6, 0.75\},\ \{0.65\},\ \{0.7, 0.55\}\ \big\}.$$

Crisp evaluation:

$$\begin{aligned}
&\mathrm{Agg}^{(1)}_{B_\oplus}(\{0.6, 0.65\}) = 0.65, \quad \mathrm{Agg}^{(1)}_{B_\oplus}(\{0.55, 0.6, 0.75\}) = 0.75,\\
&\mathrm{Agg}^{(1)}_{B_\oplus}(\{0.65\}) = 0.65, \quad \mathrm{Agg}^{(1)}_{B_\oplus}(\{0.7, 0.55\}) = 0.7,\\
&\mathrm{crisp}^{B_\oplus}_2\big(\widetilde{\mu}^A_{1,2} \sqcap \widetilde{\mu}^B_{1,2}\big)(C) = \max\{0.65, 0.75, 0.65, 0.7\} = 0.75.
\end{aligned}$$

Interpretation. The union corresponds to permissive pooling of evidence, yielding a higher final score (0.9). The lifted t–norm product enforces conjunctive agreement across sources, thus lowering the score to 0.75. Both are obtained within the same $(1, 2; [0, 1])$–SuperHyperFuzzy framework by switching from $\sqcup$ to $\sqcap$.

Theorem 2.12 (Crisp algebraic laws)**.** *Let $B = B_\oplus$ and assume $\otimes$ distributes over arbitrary $\oplus$–joins in L. Then for every $C \in \mathcal{P}_*^m(X)$,*

$$\mu^{\sharp,B_\oplus}_{A\sqcup B}(C) \;=\; \mu^{\sharp,B_\oplus}_{A}(C) \;\oplus\; \mu^{\sharp,B_\oplus}_{B}(C), \qquad \mu^{\sharp,B_\oplus}_{A\sqcap B}(C) \;=\; \mu^{\sharp,B_\oplus}_{A}(C) \;\otimes\; \mu^{\sharp,B_\oplus}_{B}(C).$$

Proof. For $\sqcup$: $\mathrm{crisp}_n^{B_\oplus}(F \cup G) = \bigoplus_{S\in F\cup G} \mathrm{Agg}^{(n)}_{B_\oplus}(S) = \big(\bigoplus_{S\in F} \mathrm{Agg}^{(n)}_{B_\oplus}(S)\big) \oplus \big(\bigoplus_{S\in G} \mathrm{Agg}^{(n)}_{B_\oplus}(S)\big)$. For $\sqcap$, prove $\mathrm{Agg}^{(k)}_{B_\oplus}(S \otimes^{(k)} T) = \mathrm{Agg}^{(k)}_{B_\oplus}(S) \otimes \mathrm{Agg}^{(k)}_{B_\oplus}(T)$ by induction on k, using the distributivity of $\otimes$ over joins; then distribute the outer $\bigoplus$. □

Definition 2.13 (Pushforward)**.** Let $f : X \to Y$ and let $\mathcal{P}_*^m(f) : \mathcal{P}_*^m(X) \to \mathcal{P}_*^m(Y)$ be the induced map. The *pushforward* of $\widetilde{\mu}_{m,n} : \mathcal{P}_*^m(X) \to \mathfrak{P}_n^*(L)$ is

$$(f_\#\widetilde{\mu}_{m,n})(B) \;:=\; \bigcup_{A\in\left(\mathcal{P}_*^m f\right)^{-1}(B)} \widetilde{\mu}_{m,n}(A), \qquad B \in \mathcal{P}_*^m(Y).$$

Theorem 2.14 (Functoriality)**.** $\mathrm{id}_\# = \mathrm{id}$ *and* $(g\circ f)_\# = g_\# \circ f_\#$*. Hence $(m, n; L)$–SuperHyperFuzzy assignments are functorial with respect to base–set maps.*

Proof. Since $\mathcal{P}_*^m(\mathrm{id}) = \mathrm{id}$ and $\mathcal{P}_*^m(g\circ f) = \mathcal{P}_*^m(g)\circ\mathcal{P}_*^m(f)$, the defining unions commute with composition. □

Theorem 2.15 (Classical recoveries)**.** *Within Definition 2.2 and Definition 2.7:*

1. *If $L = [0, 1]$ and $B = B_\oplus$, then $(m, n; L)$–SuperHyperFuzzy recovers the classical (m, n)–SuperHyperFuzzy assignment (single n–level set is recovered by choosing singleton families in $\mathfrak{P}_n^*(L)$).*
2. *If $L = [0, 1]^3$ with componentwise operations, the construction yields a SuperHyperNeutrosophic assignment.*

Proof. In (1), take $\widetilde{\mu}_{m,n}(A)$ to be a singleton family $\{S\}$; then $\mu^{\sharp,B_\oplus}(A) = \mathrm{Agg}^{(n)}_{B_\oplus}(S)$ is the usual n–level max–flattening. In (2), use product lattice $[0, 1]^3$ with componentwise $\otimes, \oplus$; all recursions are componentwise. □

Definition 2.16 (Plithogenic data and mixing)**.** Let A be a nonempty set of *attribute values*, $D := [0, 1]^s$ a degree space ($s \geq 1$), and $c : A \times A \to [0, 1]$ a symmetric *degree of contradiction* (DCF). Fix a monotone $\omega : [0, 1] \to [0, 1]$. For $\alpha, \beta \in D$ and $v, w \in A$, define the *plithogenic mixer*

$$\mathrm{Mix}_{v,w}(\alpha, \beta) \;:=\; \big(1 - \lambda_{v,w}\big)\,\big(\alpha \otimes \beta\big) \;\oplus\; \lambda_{v,w}\,\big(\alpha \oplus \beta\big), \qquad \lambda_{v,w} := \omega\big(c(v, w)\big),$$

where $\otimes, \oplus$ act componentwise on D.

Definition 2.17 (Plithogenic base kernel)**.** Let $L_{\mathrm{pl}} := A \times D$ and fix a *dominant attribute* $a_\star \in A$ together with a *reference degree* $\alpha_\star \in D$. For a nonempty finite $S \subseteq L_{\mathrm{pl}}$, define

$$B^{(a_\star,\alpha_\star)}_{\mathrm{plith}}(S) \;:=\; \bigoplus_{(v,\alpha)\in S} \mathrm{Mix}_{v,a_\star}(\alpha, \alpha_\star) \;\in D.$$

This produces a degree in D; when used as a base kernel in Definition 2.7, we interpret $B^{(a_\star,\alpha_\star)}_{\mathrm{plith}} : \mathcal{P}_*(L_{\mathrm{pl}}) \to D$ and take the outer lattice to be D with its CCR–structure.

Theorem 2.18 (Recovery of (m, n)–SuperHyperPlithogenic)**.** *Suppose that for each $A \in \mathcal{P}_*^m(X)$ the family $\widetilde{\mu}_{m,n}(A) \subseteq \mathcal{P}^n(L_{\mathrm{pl}})$ collects, at level $n = 1$, the attribute–tagged degrees $\{(v, \alpha_v(A)) : v \in \mathcal{V}(A)\} \subseteq L_{\mathrm{pl}}$ for some finite $\mathcal{V}(A) \subseteq A$, and at higher levels $n > 1$ nests such sets finitely. Fix a dominant attribute $a_\star \in A$ and its reference degree $\alpha_\star(A) \in D$. Then the $B^{(a_\star,\alpha_\star(A))}_{\mathrm{plith}}$–crisp image satisfies*

$$\mu^{\sharp,\,B_{\mathrm{plith}}}(A) \;=\; \bigoplus_{v\in\mathcal{V}(A)} \mathrm{Mix}_{v,a_\star}\big(\alpha_v(A),\; \alpha_\star(A)\big),$$

which is exactly the standard plithogenic aggregation of the attribute–wise degrees around the dominant attribute $a_\star$ with DCF c and mixer weight $\omega \circ c$.

Proof. By Definition 2.17 at level $n = 1$ the value is the $\oplus$–sum of $\mathrm{Mix}_{v,a_\star}(\alpha_v, \alpha_\star)$ over (v, α_v) in S. For $n > 1$, Definition 2.7 aggregates outer levels by $\oplus$, so the same closed form is obtained after collapsing each inner set via B_{plith}. □

3 Conclusion

In this paper, we have examined the concepts of $(m, n; L)$–SuperHyperFuzzy Sets. This framework is capable of recovering various other set-theoretic models, such as Neutrosophic Sets and Plithogenic Sets. In future work, we plan to explore possible extensions of $(m, n; L)$–SuperHyperFuzzy Sets to areas including Graphs [74–76], HyperGraphs [77–80], SuperHyperGraphs [81–83], SuperHyperfunctions [84, 85], and SuperHyperalgebras [45,86]. Furthermore, we hope to see applications developed for Fuzzy Algorithms [87,88], Fuzzy Control [89, 90], Fuzzy Clustering [91, 92], and Fuzzy Neural Networks [93, 94], along with computational experiments using well-known datasets and evaluations of their effectiveness.

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Data Availability

This paper is purely theoretical and does not involve any empirical data. We welcome future empirical studies that build upon and test the concepts presented here.

Ethical Approval

As this work is entirely conceptual and involves no human or animal subjects, ethical approval was not required.

Conflicts of Interest

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Research Integrity

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The authors assert that all mathematical results and justifications included in this work have been carefully reviewed and are believed to be correct. Should any inaccuracies or ambiguities be discovered, the authors welcome constructive feedback and will provide clarification upon request.

The conclusions presented are valid only within the specific theoretical framework and assumptions described in the text. Generalizing these results to other mathematical contexts may require further investigation. All opinions and interpretations expressed herein are solely those of the authors and do not necessarily reflect the views of their respective institutions.

References

[1] Takaaki Fujita and Florentin Smarandache. A unified framework for u-structures and functorial structure: Managing super, hyper, superhyper, tree, and forest uncertain over/under/off models. *Neutrosophic Sets and Systems*, 91:337–380, 2025.

[2] Lotfi A Zadeh. Fuzzy sets. *Information and control*, 8(3):338–353, 1965.

[3] Zdzis law Pawlak. Rough sets. *International journal of computer & information sciences*, 11:341–356, 1982.

[4] Jiao Shi, Xin Zhao, Nan Zhang, Yu Lei, and Lingtong Min. Rough-fuzzy graph learning domain adaptation for fake news detection. *IEEE Transactions on Computational Social Systems*, 2023.

[5] W-L Gau and Daniel J Buehrer. Vague sets. *IEEE transactions on systems, man, and cybernetics*, 23(2):610–614, 1993.

[6] An Lu and Wilfred Ng. Vague sets or intuitionistic fuzzy sets for handling vague data: which one is better? In *International conference on conceptual modeling*, pages 401–416. Springer, 2005.

[7] Krassimir Atanassov and George Gargov. Elements of intuitionistic fuzzy logic. part i. *Fuzzy sets and systems*, 95(1):39–52, 1998.

[8] Muhammad Akram, Bijan Davvaz, and Feng Feng. Intuitionistic fuzzy soft k-algebras. *Mathematics in Computer Science*, 7:353–365, 2013.

[9] Vicenç Torra and Yasuo Narukawa. On hesitant fuzzy sets and decision. In *2009 IEEE international conference on fuzzy systems*, pages 1378–1382. IEEE, 2009.

[10] Vicenç Torra. Hesitant fuzzy sets. *International journal of intelligent systems*, 25(6):529–539, 2010.

[11] S Sudha, M Lathamaheswari, S Broumi, and F Smarandache. Neutrosophic fuzzy magic labeling graph with its application in academic performance of the students. *Neutrosophic Sets and Systems*, 60(1):8, 2023.

[12] Said Broumi, Mohamed Talea, Assia Bakali, and Florentin Smarandache. Single valued neutrosophic graphs. *Journal of New theory*, (10):86–101, 2016.

[13] Florentin Smarandache. *Plithogeny, plithogenic set, logic, probability, and statistics*. Infinite Study, 2017.

[14] Nivetha Martin. Plithogenic swara-topsis decision making on food processing methods with different normalization techniques. *Advances in Decision Making*, 69, 2022.

[15] Jayanta Ghosh and Tapas Kumar Samanta. Hyperfuzzy sets and hyperfuzzy group. *Int. J. Adv. Sci. Technol*, 41:27–37, 2012.

[16] Z Nazari and B Mosapour. The entropy of hyperfuzzy sets. *Journal of Dynamical Systems and Geometric Theories*, 16(2):173–185, 2018.

[17] Takaaki Fujita. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*. Biblio Publishing, 2025.

[18] Hans-Jürgen Zimmermann. *Fuzzy set theory—and its applications*. Springer Science & Business Media, 2011.

[19] TM Nishad, Talal Ali Al-Hawary, and B Mohamed Harif. General fuzzy graphs. *Ratio Mathematica*, 47, 2023.

[20] Wen-Ran Zhang. Bipolar fuzzy sets and relations: a computational framework for cognitive modeling and multiagent decision analysis. *NAFIPS/IFIS/NASA '94. Proceedings of the First International Joint Conference of The North American Fuzzy Information Processing Society Biannual Conference. The Industrial Fuzzy Control and Intellige*, pages 305–309, 1994.

[21] Juanjuan Chen, Shenggang Li, Shengquan Ma, and Xueping Wang. m-polar fuzzy sets: an extension of bipolar fuzzy sets. *The scientific world journal*, 2014(1):416530, 2014.

[22] Muhammad Akram. Bipolar fuzzy graphs. *Information sciences*, 181(24):5548–5564, 2011.

[23] Tapan Senapati and Guiyun Chen. Picture fuzzy waspas technique and its application in multi-criteria decision-making. *Soft Computing*, 26(9):4413–4421, 2022.

[24] Bui Cong Cuong and Vladik Kreinovich. Picture fuzzy sets-a new concept for computational intelligence problems. In *2013 third world congress on information and communication technologies (WICT 2013)*, pages 1–6. IEEE, 2013.

[25] Sankar Das, Ganesh Ghorai, and Madhumangal Pal. Picture fuzzy tolerance graphs with application. *Complex & Intelligent Systems*, 8(1):541–554, 2022.

[26] Waheed Ahmad Khan, Waqar Arif, Quoc Hung NGUYEN, Thanh Trung Le, and Hai Van Pham. Picture fuzzy directed hypergraphs with applications towards decision-making and managing hazardous chemicals. *IEEE Access*, 2024.

[27] Takaaki Fujita and Florentin Smarandache. Examples of fuzzy sets, hyperfuzzy sets, and superhyperfuzzy sets in climate change and the proposal of several new concepts. *Climate Change Reports*, 2:1–18, 2025.

[28] M MAHARIN. Hyper fuzzy cosets. *Scholar: National School of Leadership*, 9(1.2), 2020.

[29] Narupon Tacha, Phongsakon Phayapsiang, and Aiyared Iampan. Length and mean fuzzy up-subalgebras of up-algebras. *Caspian Journal of Mathematical Sciences*, 11(1):264–303, 2022.

[30] Young Bae Jun, Seok-Zun Song, and Seon Jeong Kim. Length-fuzzy subalgebras in bck/bci-algebras. *Mathematics*, 6(1):11, 2018.

[31] Young Bae Jun, Seok-Zun Song, and Seon Jeong Kim. Distances between hyper structures and length fuzzy ideals of bck/bci-algebras based on hyper structures. *Journal of Intelligent & Fuzzy Systems*, 35(2):2257–2268, 2018.

[32] Aiyared Iampan, J. Princivishvamalar, C. Arivazhagi, and Neelamegarajan Rajesh. Several types of n-fuzzy up (bcc)-subalgebras of up (bcc)-algebras. *International Journal of Innovative Computing, Information and Control*, 21(1):115–134, 2025.

[33] Young Bae Jun, Kul Hur, and Kyoung Ja Lee. Hyperfuzzy subalgebras of bck/bci-algebras. *Annals of Fuzzy Mathematics and Informatics*, 2017.

[34] Takaaki Fujitar. Hyperfuzzy and superhyperfuzzy extensions of linear programming: Models and mathematical foundations. *Optimality*, 2(3):127–140, 2025.

[35] On the structure and properties of hyperfuzzy and superhyperfuzzy intervals. *Soft Computing Fusion With Applications*, 2025.

[36] Florentin Smarandache. *Hyperuncertain, superuncertain, and superhyperuncertain sets/logics/probabilities/statistics*. Infinite Study, 2017.

[37] Takaaki Fujita. Knowledge superhypergraphs, multimodal superhypergraphs, lattice superhypergraphs, and hyperbolic superhypergraphs: Concepts and applications. *Journal of Operational and Strategic Analytics*, 2025.

[38] Takaaki Fujita. Entropy reimagined: Theoretical foundations of hyperentropy and superhyperentropy with real-world and engineering applications. *Spectrum of Engineering and Management Sciences*, 3(1):262–286, 2025.

[39] Mustafa Demirci. Partially ordered fuzzy power set monads on the category of l-sets and their associated categories of topological space objects. *Fuzzy Sets Syst.*, 460:1–32, 2022.

[40] Wyllis Bandler and Ladislav J. Kohout. Fuzzy power sets and fuzzy implication operators. *Fuzzy Sets and Systems*, 4:13–30, 1980.

[41] Takaaki Fujita. An introduction and reexamination of hyperprobability and superhyperprobability: Comprehensive overview. *Asian Journal of Probability and Statistics*, 27(5):82–109, 2025.

[42] Adel Al-Odhari. Neutrosophic power-set and neutrosophic hyper-structure of neutrosophic set of three types. *Annals of Pure and Applied Mathematics*, 31(2):125–146, 2025.

[43] Adel Al-Odhari. A brief comparative study on hyperstructure, super hyperstructure, and n-super superhyperstructure. *Neutrosophic Knowledge*, 6:38–49, 2025.

[44] Takaaki Fujita. Analysis and proposal of chemical hyperstructures and superhyperstructures: Extending chemical topology, molecular knots, molecular geometry, and circuit topology. 2025.

[45] Florentin Smarandache. Extension of hyperalgebra to superhyperalgebra and neutrosophic superhyperalgebra (revisited). In *International Conference on Computers Communications and Control*, pages 427–432. Springer, 2022.

[46] Lotfi A Zadeh. Fuzzy logic, neural networks, and soft computing. In *Fuzzy sets, fuzzy logic, and fuzzy systems: selected papers by Lotfi A Zadeh*, pages 775–782. World Scientific, 1996.

[47] Hashem Bordbar, Mohammad Rahim Bordbar, Rajab Ali Borzooei, and Young Bae Jun. N-subalgebras of bck= bci-algebras which are induced from hyperfuzzy structures. *Iranian Journal of Mathematical Sciences and Informatics*, 16(2):179–195, 2021.

[48] Seok-Zun Song, Seon Jeong Kim, and Young Bae Jun. Hyperfuzzy ideals in bck/bci-algebras. *Mathematics*, 5(4):81, 2017.

[49] Takaaki Fujita. Foundations of (m, n)-superhyperfuzzy, superhyperneutrosophic, and superhyperplithogenic sets.

[50] Takaaki Fujita and Florentin Smarandache. *Exploring Concepts of HyperFuzzy, HyperNeutrosophic, and HyperPlithogenic Sets (I)*. Infinite Study, 2025.

[51] Said Broumi, Mohamed Talea, Assia Bakali, Florentin Smarandache, and PK Kishore Kumar. Shortest path problem on single valued neutrosophic graphs. In *2017 international symposium on networks, computers and communications (ISNCC)*, pages 1–6. IEEE, 2017.

[52] Florentin Smarandache. *Neutrosophic Overset, Neutrosophic Underset, and Neutrosophic Offset. Similarly for Neutrosophic Over-/Under-/Off-Logic, Probability, and Statistics*. Infinite Study, 2016.

[53] Said Broumi, Assia Bakali, and Ayoub Bahnasse. Neutrosophic sets: An overview. *Infinite Study*, 2018.

[54] Madeleine Al Tahan, Saba Al-Kaseasbeh, and Bijan Davvaz. Neutrosophic quadruple hv-modules and their fundamental module. *Neutrosophic Sets and Systems*, 72:304–325, 2024.

[55] Muhammad Akram, Hafsa M Malik, Sundas Shahzadi, and Florentin Smarandache. Neutrosophic soft rough graphs with application. *Axioms*, 7(1):14, 2018.

[56] Takaaki Fujita and Florentin Smarandache. Some types of hyperneutrosophic set (4): Cubic, trapezoidal, q-rung orthopair, overset, underset, and offset. 2025.

[57] Takaaki Fujita and Florentin Smarandache. *Some Types of HyperNeutrosophic Set (7): Type-m, Nonstationary, Subset-valued, and Complex Refined*. Infinite Study, 2025.

[58] Takaaki Fujita and Florentin Smarandache. Some types of hyperneutrosophic set (4): Cubic, trapozoidal, q-rung orthopair, overset, underset, and offset. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 193, 2025.

[59] Takaaki Fujita and Florentin Smarandache. Hyperneutrosophic set and forest hyperneutrosophic set with practical applications in agriculture. *Optimization in Agriculture*, 2:10–21, 2025.

[60] Florentin Smarandache. Neutrosophy: neutrosophic probability, set, and logic: analytic synthesis & synthetic analysis. 1998.

[61] Florentin Smarandache. A unifying field in logics: Neutrosophic logic. In *Philosophy*, pages 1–141. American Research Press, 1999.

[62] Shio Gai Quek, Ganeshsree Selvachandran, Florentin Smarandache, J. Felicita Vimala, Sn Hoang Le, Quang-Thinh Bui, and Vassilis C. Gerogiannis. Entropy measures for plithogenic sets and applications in multi-attribute decision making. *Mathematics*, 2020.

[63] Devendra Kumar Tayal, Sumit Kumar Yadav, and Divya Arora. Personalized ranking of products using aspect-based sentiment analysis and plithogenic sets. *Multimedia Tools and Applications*, 82:1261–1287, 2022.

[64] Fazeelat Sultana, Muhammad Gulistan, Mumtaz Ali, Naveed Yaqoob, Muhammad Khan, Tabasam Rashid, and Tauseef Ahmed. A study of plithogenic graphs: applications in spreading coronavirus disease (covid-19) globally. *Journal of ambient intelligence and humanized computing*, 14(10):13139–13159, 2023.

[65] Nehmat Ahmed and Osama T. Pirbal. Plithogenic crisp hypersoft topology. *European Journal of Pure and Applied Mathematics*, 2024.

[66] Takaaki Fujita. Hyperplithogenic cubic set and superhyperplithogenic cubic set. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 79, 2025.

[67] Takaaki Fujita. Symbolic hyperplithogenic set. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 130, 2025.

[68] Takaaki Fujita and Florentin Smarandache. A concise introduction to hyperfuzzy, hyperneutrosophic, hyperplithogenic, hypersoft, and hyperrough sets with practical examples. *Neutrosophic Sets and Systems*, 80:609–631, 2025.

[69] Florentin Smarandache. Plithogeny, plithogenic set, logic, probability, and statistics. *arXiv preprint arXiv:1808.03948*, 2018.

[70] Florentin Smarandache. *Plithogenic set, an extension of crisp, fuzzy, intuitionistic fuzzy, and neutrosophic sets-revisited*. Infinite study, 2018.

[71] S Sudha, Nivetha Martin, and Florentin Smarandache. *Applications of Extended Plithogenic Sets in Plithogenic Sociogram*. Infinite Study, 2023.

[72] Takaaki Fujita and Florentin Smarandache. A concise introduction to hyperfuzzy, hyperneutrosophic, hyperplithogenic, hypersoft, and hyperrough sets with practical examples. *Neutrosophic Sets and Systems*, 80:609–631, 2025.

[73] Takaaki Fujita. Forest hyperplithogenic set and forest hyperrough set. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, 2025.

[74] Béla Bollobás. *Modern graph theory*, volume 184. Springer Science & Business Media, 1998.

[75] Jonathan L Gross, Jay Yellen, and Mark Anderson. *Graph theory and its applications*. Chapman and Hall/CRC, 2018.

[76] Reinhard Diestel. Graduate texts in mathematics: Graph theory.

[77] Alain Bretto. Hypergraph theory. *An introduction. Mathematical Engineering. Cham: Springer*, 1, 2013.

[78] Yifan Feng, Haoxuan You, Zizhao Zhang, Rongrong Ji, and Yue Gao. Hypergraph neural networks. In *Proceedings of the AAAI conference on artificial intelligence*, volume 33, pages 3558–3565, 2019.

[79] Claude Berge. *Hypergraphs: combinatorics of finite sets*, volume 45. Elsevier, 1984.

[80] Yue Gao, Yifan Feng, Shuyi Ji, and Rongrong Ji. Hgnn+: General hypergraph neural networks. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 45(3):3181–3199, 2022.

[81] Takaaki Fujita and Florentin Smarandache. A concise study of some superhypergraph classes. *Neutrosophic Sets and Systems*, 77:548–593, 2024.

[82] Mohammad Hamidi, Florentin Smarandache, and Elham Davneshvar. Spectrum of superhypergraphs via flows. *Journal of Mathematics*, 2022(1):9158912, 2022.

[83] Masoud Ghods, Zahra Rostami, and Florentin Smarandache. Introduction to neutrosophic restricted superhypergraphs and neutrosophic restricted superhypertrees and several of their properties. *Neutrosophic Sets and Systems*, 50:480–487, 2022.

[84] Maissam Jdid, Florentin Smarandache, and Takaaki Fujita. A linear mathematical model of the vocational training problem in a company using neutrosophic logic, hyperfunctions, and superhyperfunction. *Neutrosophic Sets and Systems*, 87:1–11, 2025.

[85] Florentin Smarandache. *SuperHyperFunction, SuperHyperStructure, Neutrosophic SuperHyperFunction and Neutrosophic SuperHyperStructure: Current understanding and future directions*. Infinite Study, 2023.

[86] Abdullah Kargın and Memet Şahin. Superhyper groups and neutro–superhyper groups. *2023 Neutrosophic SuperHyperAlgebra And New Types of Topologies*, 25, 2023.

[87] Lotfi A. Zadeh. Fuzzy algorithms. *Inf. Control.*, 12:94–102, 1968.

[88] Ebrahim H Mamdani. Application of fuzzy algorithms for control of simple dynamic plant. In *Proceedings of the institution of electrical engineers*, volume 121, pages 1585–1588. IET, 1974.

[89] Andreas Nürnberger, Detlef Nauck, and Rudolf Kruse. Neuro-fuzzy control based on the nefcon-model: recent developments. *Soft Computing*, 2(4):168–182, 1999.

[90] Gurpreet S Sandhu and Kuldip S Rattan. Design of a neuro-fuzzy controller. In *1997 IEEE international conference on systems, man, and cybernetics. Computational cybernetics and simulation*, volume 4, pages 3170–3175. IEEE, 1997.

[91] Bui Cong Cuong, Pier Luca Lanzi, Nguyen Tho Thong, et al. A novel intuitionistic fuzzy clustering method for geo-demographic analysis. *Expert Systems with applications*, 39(10):9848–9859, 2012.

[92] SV Aruna Kumar and BS Harish. A modified intuitionistic fuzzy clustering algorithm for medical image segmentation. *Journal of Intelligent Systems*, 27(4):593–607, 2018.

[93] Yu-Ting Liu, Yang-Yin Lin, Shang-Lin Wu, Chun-Hsiang Chuang, and Chin-Teng Lin. Brain dynamics in predicting driving fatigue using a recurrent self-evolving fuzzy neural network. *IEEE Transactions on Neural Networks and Learning Systems*, 27:347–360, 2016.

[94] Jinjun Tang, Fang Liu, Wenhui Zhang, Ruimin Ke, and Yajie Zou. Lane-changes prediction based on adaptive fuzzy neural network. *Expert Syst. Appl.*, 91:452–463, 2018.