Equivalence Theorem for Simple Coordination Games

By

Somdeb Lahiri (Email: somdeb.lahiri@gmail.com)

ORCID: https://orcid.org/0000-0002-5247-3497

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Abstract

In this note, we consider simple coordination games, with each player having the same number of pure strategies to choose from. We model the problem as a “pie-division” problem. Let ‘n’ denote the number of strategies available to each of the two players. One player called the “row player” chooses one of the rows of a square matrix of size ‘n’. The other player called the “column player” chooses of the columns of a square matrix of size ‘n’. There is a permutation (one-to-one function from a non-empty finite domain to itself) on the set of first ‘n’ positive integers, such that if the row player chooses a row and the column player chooses the column assigned by the permutation to itself, then each get a positive pay-off. Otherwise, they get nothing. We call such two-person games, “simple coordination games”. We show, that for each simple coordination game, there are two “linear programming problems”, such that the set of pure-strategy equilibria of the game is in “one-to-one correspondence” with the set of solutions of each of the two linear programming problems. We provide a second characterization of pure-strategy equilibrium in terms of solutions to ‘n’ pairs of linear programming problems. We subsequently address the problem of coordination between the two players and show that a way to solve this problem is the “leader-follower” method. where one of the players is pre-committed to its best pure strategy and the other chooses its best response to the pre-committed strategy. Such a solution arises by solving one of two quadratic programming problems.

Keywords: simple coordination games, pure-strategy, equilibrium, linear programming, quadratic programming

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1. Introduction: For interactive decision-making problems, with a non-empty finite strategy set for each decision maker (player), two-person zero-sum (TPZS) games are useful when the decision-makers are in direct conflict with one another, so that a gain for a decision-maker is an equal amount of loss for its rival. A sufficiently comprehensive discussion of TPZS games can be found in chapter 20 of Mote and Madhavan (2016). However, what TPZS games fail to capture is the potential for mutual gains arising from cooperation among the decision makers. This, deficiency has been addressed and remedied in a somewhat more general model, called two-person additively-separable sum (TPASS) games, in Lahiri (2025). The merit of the generalization is that, as in the case of TPZS games, the generalization preserves the possibility of characterizing the set of equilibria as the solution to an extremely plausible linear programming problem and its dual. Is this the most we can do if we want to preserve the property of characterizing the set of equilibria as the sets of solution to a linear programming problem and its dual? What are the types of bi-matrix games whose set of equilibria (pure-strategy equilibria?) coincide with the set of solutions of a linear programming problem? These are the kind of questions motivating the current investigation. Our interpretation of equilibrium in the context of bi-matrix interactive decision-making problems, is that of a "steady state" as discussed in some detail in Lahiri (2017).

The seminal paper of Mangasarian and Stone (1964), provides us with an "Equivalence Theorem", which characterizes the set of all equilibria of a bi-matrix game as the set of all solutions to a quadratic programming problem. The proof appeals to the existence result concerning equilibrium of a class of games that includes the set of bi-matrix games as a proper subset, due to Nash (1951). Hence, establishing the equivalence between equilibria of bi-matrix games and solutions of an optimization problem was achieved as early as 1964. However, to the best of our knowledge, there does not seem to be available such an equivalence between bi-matrix games and linear programming problems, beyond the set of TPASS games. There is a class of bi-matrix games called "coordination games", which does not seem to allow being modelled as TPASS games. In these bi-matrix games, there are multiple pure-strategy equilibria, with each equilibrium rewarding each player differently. Such games, introduce the possibility of coordination and hence negotiation among the players, to decide on one among several pure-strategy equilibria.

In this note, we consider a type of coordination games, with each player having the same number of pure strategies to choose from. Let 'n' denote the number of strategies available to each of the two players. One player called the "row player" chooses one of the rows of a square matrix of size 'n'. The other player called the "column player" chooses one of the columns of a square matrix of size 'n'. There is a permutation (one-to-one function from a non-empty finite domain to itself) on the set of first 'n' positive integers, such that if the row player chooses a row and the column player chooses the column assigned by the permutation to itself, then each get a positive pay-off. Otherwise, they get nothing. We call such two-person games, "simple coordination games". The set of pure-strategy equilibria of such games do not depend on the magnitude of the positive pay-offs, so long as they-and only they- are strictly positive.

We show, that for each simple coordination game, there are two "linear programming problems", such that the set of pure-strategy equilibria of the game is in "one-to-one correspondence" with the set of solutions of each of the two linear programming problems. We provide a second characterization of pure-strategy equilibrium in terms of solutions to 'n' pairs of linear programming problems.

While the two results discussed in the previous paragraph provide complete characterizations of the set of pure-strategy equilibria for simple coordination games, the problem of coordination is not reflected in any of them. This is the problem we deal with subsequently in this paper. It seems, that a way to solve the coordination problem is the "leader-follower" method, where one of the players is pre-committed to its best pure strategy and the other chooses its best response to the pre-committed strategy. Such a solution arises by solving one of two quadratic programming problems.

2. Notations: Let $\mathbb{R}$ denote the set of real numbers and let $\mathbb{R}_+$ denote the set of all non-negative real numbers. For a positive integer ℓ, and any non-empty set X, let X^ℓ be the set of all ordered ℓ-tuples with coordinates belonging to X.

For a positive integer ℓ and $j \in \{1, \ldots, \ell\}$, let $E^{(\ell,j)}$ denote the point in $\mathbb{R}^{\ell}$ whose j^{th} coordinate is 1, and all other coordinates are 0. $E^{(\ell,j)}$ is said to be the j^{th} **unit coordinate vector** in $\mathbb{R}^{\ell}$.

For any non-negative integer ℓ, let $\Delta^{\ell-1} = \{x \in \mathbb{R}^{\ell}_{+} \mid \sum_{k=1}^{\ell} x_k = 1\}$. Clearly, $\Delta^{\ell-1}$ is the convex combination of points in the set $\{E^{(\ell,j)} \mid j \in \{1, \ldots, \ell\}\}$.

Unless otherwise mentioned, for any positive integer ℓ, we will interpret a point x in $\mathbb{R}^{\ell}$ to be a column vector, and its transpose represented by x^T to be a row vector.

Let m and n be positive integers. For any m×n real-valued matrix (i.e., a real-valued matrix with m rows and n columns) A and any $(i, j) \in \{1, \ldots, m\} \times \{1, \ldots, n\}$, let a_{ij} denote the $(i, j)^{th}$ entry of A (i.e., entry at the intersection of the i^{th} row and j^{th} column of A). For $(i, j) \in \{1, \ldots, m\} \times \{1, \ldots, n\}$, let A_i denote the i^{th} row A and let A^j denote the j^{th} column of A. Further, let A^T be the n×m matrix denoting the transpose of A.

3. Bi-matrix Games: For positive integers m and n, consider a two-player game between a "row player" and a "column player", with their pay-off matrices being m×n matrices A and B respectively. The interpretation of the pair (A, B), referred to as (an) a (m×n) **bi-matrix game**, is that if the row player chooses row i and the column player chooses column j, then the pay-off to the row player is a_{ij} and the pay-off to the column player is b_{ij}.

A (**randomized or mixed**) **strategy for the row player** is a point in Δ^{m-1} and a (**randomized or mixed**) **strategy for the column player** is a point in Δ^{n-1}.

A strategy for the row player is said to be a **pure-strategy for the row player** if the strategy belongs to the set $\{E^{(m,i)} \mid i \in \{1, \ldots, m\}\}$.

A strategy for the column player is said to be a **pure-strategy for the column player** if the strategy belongs to $\{E^{(n,j)} \mid j \in \{1, \ldots, n\}\}$.

A pair $(x, y) \in \Delta^{m-1} \times \Delta^{n-1}$ is a (**randomized or mixed) strategy profile**.

If $(x, y) = (E^{(m,i)}, E^{(n,j)})$ for some $(i, j) \in \{1, \ldots, m\} \times \{1, \ldots, n\}$, then we refer to the strategy pair as a **pure-strategy profile** and when there is no scope for mis-understanding we may denote the same simply by (i, j).

Given an m×n bi-matrix a strategy profile (x^*, y^*) is said to be an **equilibrium** if for all (i, j) $\in \{1, \ldots, m\} \times \{1, \ldots, n\}$, $A_i y^* \leq x^{*T} A y^*$ and $x^{*T} B^j \leq x^{*T} B y^*$.

For a concise discussion about equilibria of bi-matrix games one may refer to Lahiri (2025).

Given an m×n bi-matrix a pure-strategy profile (i, j) is said to be a **pure-strategy equilibrium** if for all (h, k) $\in \{1, \ldots, m\} \times \{1, \ldots, n\}$, $a_{hj} \leq a_{ij}$ and $b_{ik} \leq bij$.

It is easy to see that if (i, j) is a pure-strategy equilibrium, then for all $x \in \Delta^{m-1}$, $x^T A^j \leq a_{ij}$ and for all $y \in \Delta^{n-1}$, $B_i y \leq b_{ij}$.

4. Simple Coordination Games: A n×n bi-matrix game (A, B) where n is a positive integer is said to be a **square bi-matrix game of size n.**

A square bi-matrix-game of size 'n' (A, B) is said to be a **simple coordination game of size n**, if there exists a permutation π on $\{1, \ldots, n\}$, (i.e., π is a one-to-one function from $\{1, \ldots,$

n} to itself, and hence an onto function on {1, …, n}) such that for all i, j∈{1, …, n}, $a_{ij} = b_{ij} = 0$ if $j \neq \pi(i)$, and for all i ∈ {1, …, n}, $a_{i\pi(i)}, b_{i\pi(i)} > 0$.

Assumption 1: For what follows in this section and the next assume that the positive integer 'n' denoting the size of the square matrices, is given and fixed.

Hence, instead of writing a "simple coordination game of size n", we will write "simple coordination game".

Note 1: It easy to see that the **set of pure-strategy equilibria for a simple coordination game** is {(i, π(i))|i∈{1, …, n}}. Further, for all x, y∈Δ^{n-1}, $x^TAy = \sum_{i=1}^{n} a_{i\pi(i)}x_iy_{\pi(i)}$ and $x^TBy = \sum_{i=1}^{n} b_{i\pi(i)}x_iy_{\pi(i)}$.
A simple coordination game (A, B) is said to be a **diagonal coordination game** if for all i ∈{1, …, n}, π(i) = i.

A simple coordination game (A, B) is said to be an **cross-diagonal coordination game** if for all i∈{1, …, n}, π(i) = n-i+1

5. Equivalence Theorem for Simple Coordination Games: In this section we present two equivalence theorems, that completely characterize the set of pure-strategy equilibria for simple coordination games. Both theorems characterize the set of equilibria as solutions to linear programming problems.

Theorem 1: Let (A, B) be a simple coordination game such that for a given permutation π on {1, …, n}, i, j∈{1, …, n}, $a_{ij} = b_{ij} = 0$ if $j \neq \pi(i)$, and for all i ∈ {1, …, n}, $a_{i\pi(i)}, b_{i\pi(i)} > 0$.
Then,

(i) The array <$y^{*(i)}$|i∈{1, …, n}> solves the linear programming problem [Maximize $\sum_{i=1}^{n} B_iy^{(i)}$, subject to $y^{(i)} \in \Delta^{n-1}$ for all i∈{1, …, n}] <u>if and only if</u> [for all i∈{1, …, n}, $y^{*(i)} \in \Delta^{n-1}$ and {j|$y_j^{*(i)}$> 0} = {π(i)}].

(ii) The array <$x^{*(i)}$|i∈{1, …, n}> solves the linear programming problem [Maximize $\sum_{i=1}^{n} x^{(i)T}A^i$, subject to $x^{(i)} \in \Delta^{n-1}$ for all i∈{1, …, n}] <u>if and only if</u> [for all i∈{1, …, n}, $x^{*(i)} \in \Delta^{n-1}$ and {j|$x_j^{*(i)}$> 0} = {i}].

Proof: The proof of the theorem follows from the following observations:

(i) For all i∈{1, …, n} and $y^{(i)} \in \Delta^{n-1}$: $B_iy^{(i)} \leq b_{i\pi(i)}$ & [$B_iy^{(i)} = b_{i\pi(i)}$ if and only if $y_{\pi(i)}^{(i)} = 1$].

(ii) For all i∈{1, …, n} and $x^{(i)} \in \Delta^{n-1}$: $x^{(i)T}A^i \leq a_{i\pi(i)}$ & [$x^{(i)T}A^i = a_{i\pi(i)}$ if and only if $x_i^{(i)} = 1$].
Q.E.D.

A sharper version of theorem 1 whose proof is similar to the proof of theorem 1, is the following.

Theorem 2: Let (A, B) be a simple coordination game such that for a given permutation π on {1, …, n}, i, j∈{1, …, n}, $a_{ij} = b_{ij} = 0$ if $j \neq \pi(i)$, and for all i ∈ {1, …, n}, $a_{i\pi(i)}, b_{i\pi(i)} > 0$.

(i) (x^*, y^*) is a pure-strategy equilibrium for (A, B), <u>if and only if</u> for some i∈{1, …, n}, x^* solves [Maximize x^TA^i, subject to x∈Δ^{n-1}] and y^* solves [Maximize B_iy, subject to y∈Δ^{n-1}].

(ii) $\{(x^{*(i)}, y^{*(i)})|\ i\in\{1, \ldots, n\}\}$ is the set of pure-strategy equilibria of (A, B) <u>if and only if</u> for all $i\in\{1, \ldots, n\}$, $x^{*(i)}$ solves [Maximize x^TA^i, subject to $x\in\Delta^{n-1}$] and $y^{*(i)}$ solves [Maximize B_iy, subject to $y\in\Delta^{n-1}$].

6. Coordinated outcomes for Simple Coordination Games: While so-far we have been concerned with characterizing the set of all equilibria for simple co-ordination, let us now allow for the possibility of coordination among the two players, so that the outcome is more specific than the entire set of pure-strategy equilibria.

The row player would naturally seek a pure strategy in $\underset{i\in\{1,\ldots,n\}}{argmax}\, a_{i\pi(i)}$ and it is reasonable for the column player to seek a pure strategy in $\underset{i\in\{1,\ldots,n\}}{\text{argmax}}\, b_{i\pi(i)}$.

Thus, if $\{(x^{*(i)}, y^{*(i)})|\ i\in\{1, \ldots, n\}\}$ is the set of pure-strategy equilibria of (A, B), then the row player would seek a pure strategy in $\underset{i\in\{1,\ldots,n\}}{\text{argmax}}\, x^{*(i)T}A^i$ and the column player would seek a pure strategy in $\underset{i\in\{1,\ldots,n\}}{\text{argmax}}\, B_i\, y^{*(i)}$.

There is no guarantee that $\underset{i\in\{1,\ldots,n\}}{\text{argmax}}\, x^{*(i)T}A^i \cap \underset{i\in\{1,\ldots,n\}}{\text{argmax}}\, B_i\, y^{*(i)} \neq \phi$ and this precisely what creates a "coordination problem".

The mathematical resolution of such problems is reflected in the quadratic programming problems as in the following theorem.

Theorem 3: Let (A, B) be a simple coordination game such that for a given permutation π on $\{1, \ldots, n\}$, $i, j\in\{1, \ldots, n\}$, $a_{ij} = b_{ij} = 0$ if $j \neq \pi(i)$, and for all $i \in \{1, \ldots, n\}$, $a_{i\pi(i)}, b_{i\pi(i)} > 0$.

(i) $j\in\underset{i\in\{1,\ldots,n\}}{argmax}\, a_{i\pi(i)}$ <u>if and only if</u> $\theta^*\in\Delta^{n-1}$ satisfying $\theta_j^*= 1$ along with the array $<x^{*(i)}|i\in\{1, \ldots, n\}>$ in Δ^{n-1} satisfying $x_i^{*(i)} = 1$ for all $i\in\{1, \ldots, n\}$ solve [Maximize $\sum_{i=1}^{n}\theta_i x^{(i)T}A^i$, subject to $x^{(i)}\in \Delta^{n-1}$ for all $i\in\{1, \ldots, n\}$, $\theta\in\Delta^{n-1}$].

(ii) $j\in\underset{i\in\{1,\ldots,n\}}{\text{argmax}}\, b_{i\pi(i)}$ <u>if and only if</u> $\theta^*\in\Delta^{n-1}$ satisfying $\theta_j^*= 1$ along with the array $<y^{*(i)}|i\in\{1, \ldots, n\}>$ in Δ^{n-1} satisfying $y_{\pi(i)}^{*(i)} = 1$ for all $i\in\{1, \ldots, n\}$ solve [Maximize $\sum_{i=1}^{n}\theta_i B_i y^{(i)}$, subject to $y^{(i)}\in \Delta^{n-1}$ for all $i\in\{1, \ldots, n\}$, $\theta\in\Delta^{n-1}$].

References

1. Lahiri, S. (2017): Competitive Equilibrium in Generalized Games: A New Interpretation. Studies in Microeconomics, Volume 5, Issue 1, Pages 35- 52. Available at: https://doi.org/10.1177/2321022217696119

2. Lahiri, S. (2025): Two-Person Additively-Separable Sum Games. https://doi.org/10.31224/4775

3. Mangasarian, O. L. and Stone, H. (1964): Two-Person Nonzero-Sum Games and Quadratic Programming. *Journal of Mathematical Analysis And Applications*, Vol. 9, Pages 348-355.

4. Mote, V. L. and Madhavan, T. (2016): Operations Research. *Wiley India Private Ltd.*

5. Nash, J. (1951): Non-cooperative games. *Annals of Mathematics*. 2nd Ser. 54 (2), Pages 286–295.