Equivalence Theorem for Simple Coordination Games

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Abstract

In this note, we consider simple coordination games, with each player having the same number of pure strategies to choose from. We model the problem as a "pie-division" problem. Let 'n' denote the number of strategies available to each of the two players. One player called the "row player" chooses one of the rows of a square matrix of size 'n'. The other player called the "column player" chooses of the columns of a square matrix of size 'n'. There is a permutation (one-to-one function from a non-empty finite domain to itself) on the set of first 'n' positive integers, such that if the row player chooses a row and the column player chooses the column assigned by the permutation to itself, then each get a positive share of the pie. Otherwise, they get nothing. We call such two-person games, "simple coordination games". We show, that for each simple coordination game, there is an "integer linear programming problem", such that the set of pure-strategy equilibria of the game is a subset of the set of solutions of the integer linear programming problem and another very closely related integer programming problem whose solutions yield the set of pure strategy equilibria of the simple coordination game. These two integer linear programming problems depend only on the location of the positive pay-offs, and not on their magnitude. If one examines the objective function of the two integer linear programming problems, then one will find implicit in them both "altruism" as well as "antagonism". The integer linear programs are far from obvious and the second one-in particular- surreptitiously incorporates in it "multiplicative" or "interdependent" *non-linear features*, that would not be possible unless we required some additional variables to be either '0' or '1'.

Keywords: simple coordination games, integer linear programming

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1. Introduction: For interactive decision-making problems, with a non-empty finite strategy set for each decision maker (player), two-person zero-sum (TPZS) games are useful when the decision-makers are in direct conflict with one another, so that a gain for a decision-maker is an equal amount of loss for its rival. A sufficiently comprehensive discussion about TPZS games can be found in chapter 20 of Mote and Madhavan (2016). However, what TPZS games fail to capture is the potential for mutual gains arising from cooperation among the decision makers. This, deficiency has been addressed and remedied in a somewhat more general model, called two-person additively-separable sum (TPASS) games, in Lahiri (2025). The merit of the generalization is that, as in the case of TPZS games, the generalization preserves the possibility of characterizing the set of equilibria as the solution to an extremely plausible linear programming problem and its dual. Is this the most we can do if we want to preserve the property of characterizing the set of equilibria as the sets of solution to a linear programming problem and its dual? What are the types of bi-matrix games whose set of equilibria (pure-strategy equilibria?) coincide with the set of solutions of a linear programming problem? These are the kind of questions motivating the current investigation. Our interpretation of equilibrium in the context of interactive decision-making problems, is that of a "steady state" as discussed in some detail in Lahiri (2017).

The seminal paper of Mangasarian and Stone (1964), provides us with an "Equivalence Theorem", which characterizes the set of all equilibria of a bi-matrix game as the set of all solutions to a quadratic programming problem. The proof appeals to the existence result concerning equilibrium of a class of games that includes the set of bi-matrix games as a proper subset, due to Nash (1951). Hence, establishing the equivalence between equilibria of bi-matrix games and solutions of an optimization problem was achieved as early as 1964. However, to the best of our knowledge, there does not seem to be available such an equivalence between bi-matrix games and linear programming problems, beyond the set of TPASS games. There is a class of bi-matrix games called "coordination games", which does not seem to allow being modelled as TPASS games. In these bi-matrix games, there are multiple pure-strategy equilibria, with each equilibrium rewarding each player differently. Such games, introduce the possibility of coordination and hence negotiation among the players, to decide on one among several pure strategy equilibria.

In this note, we consider a type of coordination games, with each player having the same number of pure strategies to choose from. We model the problem as a "pie-division" problem. Let 'n' denote the number of strategies available to each of the two players. One player called the "row player" chooses one of the rows of a square matrix of size 'n'. The other player called the "column player" chooses of the columns of a square matrix of size 'n'. There is a permutation (one-to-one function from a non-empty finite domain to itself) on the set of first 'n' positive integers, such that if the row player chooses a row and the column player chooses the column assigned by the permutation to itself, then each get a positive share of the pie. Otherwise, they get nothing. We call such two-person games, "simple coordination games". The set of pure strategy equilibria of such games do not depend on the magnitude of the positive pay-offs, so long as they-and only they- are strictly positive.

We show, that for each simple coordination game, there is an "integer linear programming problem", such that the set of pure-strategy equilibria of the game is a subset of the set of solutions of the integer linear programming problem and another very closely related integer programming problem whose solutions yield the set of pure strategy equilibria of the simple coordination game. These two integer linear programming problems depend only on the location of the positive pay-offs, and not on their magnitude. If one examines the objective function of the two integer linear programming problems, then one will find implicit in them both "altruism" as well as "antagonism". The integer linear programs are far from obvious and the second one-in particular- surreptitiously incorporates in it "multiplicative" or "interdependent" *non-linear features*, that would not be possible unless we required some additional variables to be either '0' or '1'.

Although the solutions to the second integer linear programming problem, generates the set of pure strategy equilibria of the coordination game, the contribution of this note is not algorithmic, since the definition of the integer linear programs depend on the set of pure strategy equilibria. The main contribution of our results is modelling the known set of equilibria as solutions to a "not so obvious" integer linear programming problem. It is precisely this modelling, that reveals the interplay of "altruism" and "antagonism" between the decision makers, which is probably the core and inherent aspect of coordination games.

2. Notations: Let $\mathbb{R}$ denote the set of real numbers and let $\mathbb{R}_+$ denote the set of all non-negative real numbers. For a positive integer ℓ, and any non-empty set X, let X^ℓ be the set of all ordered ℓ-tuples with coordinates belonging to X.

For a positive integer ℓ and $j \in \{1, \ldots, \ell\}$, let $E^{(\ell,j)}$ denote the point in $\mathbb{R}^\ell$ whose j^{th} coordinate is 1, and all other coordinates are 0. $E^{(\ell,j)}$ is said to be the j^{th} **unit coordinate vector** in $\mathbb{R}^\ell$.

For any non-negative integer ℓ, let $\Delta^{\ell-1} = \{x \in \mathbb{R}_+^\ell \mid \sum_{k=1}^{\ell} x_k = 1\}$. Clearly, $\Delta^{\ell-1}$ is the convex combination of points in the set $\{E^{(\ell,j)} \mid j \in \{1, \ldots, \ell\}\}$.

Unless otherwise mentioned, for any positive integer ℓ, we will interpret a point x in $\mathbb{R}^\ell$ to be a column vector, and its transpose represented by x^T to be a row vector.

Let m and n be positive integers. For any m×n real-valued matrix (i.e., a real-valued matrix with m rows and n columns) A and any $(i, j) \in \{1, \ldots, m\} \times \{1, \ldots, n\}$, let a_{ij} denote the $(i, j)^{th}$ entry of A (i.e., entry at the intersection of the i^{th} row and j^{th} column of A). For $(i, j) \in \{1, \ldots, m\} \times \{1, \ldots, n\}$, let A_i denote the i^{th} row A and let A^j denote the j^{th} column of A. Further, let A^T be the n×m matrix denoting the transpose of A.

3. Bi-matrix Games: For positive integers m and n, consider a two-player game between a "row player" and a "column player", with their pay-off matrices being m×n matrices A and B respectively. The interpretation of the pair (A, B), referred to as (an) a (m×n) **bi-matrix game**, is that if the row player chooses row i and the column player chooses column j, then the pay-off to the row player is a_{ij} and the pay-off to the column player is b_{ij}.

If B = -A, then the corresponding bi-matrix decision making problem is called a **matrix** (or **two-person zero-sum** (**TPZS**)) **game**, which is discussed in chapter 20 of Mote and Madhavan (2019).

A (**randomized or mixed**) **strategy for the row player** is a point in Δ^{m-1} and a (**randomized or mixed**) **strategy for the column player** is a point in Δ^{n-1}.

A strategy for the row player is said to be a **pure-strategy for the row player** if the strategy belongs to the set $\{E^{(m,i)} \mid i \in \{1, \ldots, m\}\}$.

A strategy for the column player is said to be a **pure-strategy for the column player** if the strategy belongs to $\{E^{(n,j)} \mid j \in \{1, \ldots, n\}\}$.

A pair $(p, q) \in \Delta^{m-1} \times \Delta^{n-1}$ is a (**randomized or mixed) strategy profile**.

If $(p, q) = (E^{(m,i)}, E^{(n,j)})$ for some $(i, j) \in \{1, \ldots, m\} \times \{1, \ldots, n\}$, then we refer to the strategy pair as a **pure-strategy profile** and when there is no scope for mis-understanding we may denote the same simply by (i, j).

Given an m×n bi-matrix a strategy profile (x^*, y^*) is said to be an **equilibrium** if for all (i, j) $\in \{1, \ldots, m\} \times \{1, \ldots, n\}$, $A_i y^* \leq x^{*T} A y^*$ and $x^{*T} B^j \leq x^{*T} B y^*$.

For a concise discussion about equilibria of bi-matrix games one may refer to Lahiri (2025).

Given an m×n bi-matrix a pure-strategy profile (i, j) is said to be a **pure-strategy equilibrium** if for all (h, k) $\in\{1, \ldots, m\}\times\{1, \ldots, n\}$, $a_{hj} \le a_{ij}$ and $b_{ik} \le b_{ij}$.

It is easy to see that if (i, j) is a pure-strategy equilibrium, then for all $x\in\Delta^{m-1}$, $x^TA^j \le a_{ij}$ and for all $y\in\Delta^{n-1}$, $B_iy \le b_{ij}$.

4. Simple Coordination Games: A n×n bi-matrix game (A, B) where n is a positive integer is said to be a **square bi-matrix game of size n.**

A square bi-matrix-game of size 'n' (A, B) is said to be a **simple coordination game of size n**, if there exists a permutation π on {1, …, n}, (i.e., π is a one-to-one function from {1, …, n} to itself, and hence an onto function on {1, …, n}) such that for all $i, j\in\{1, \ldots, n\}$, $a_{ij} = b_{ij} = 0$ if $j \neq \pi(i)$, and for all $i \in\{1, \ldots, n\}$, $a_{i\pi(i)}, b_{i\pi(i)} > 0$, with $a_{i\pi(i)} + b_{i\pi(i)} = 1$.

Assumption 1: For what follows in this section and the next assume that the positive integer 'n' denoting the size of the square matrices, is given and fixed.

Hence, instead of writing a "simple coordination game of size n", we will write "simple coordination game".

Note 1: It easy to see that the **set of pure-strategy equilibria for a simple coordination game** is $\{(i, \pi(i))|i\in\{1, \ldots, n\}\}$. Further, for all $x, y\in\Delta^{n-1}$, $x^TAy = \sum_{i=1}^{n} a_{i\pi(i)}x_iy_{\pi(i)}$ and $x^TBy = \sum_{i=1}^{n} b_{i\pi(i)}x_iy_{\pi(i)}$.

A simple coordination game (A, B) is said to be a **diagonal coordination game** if for all $i \in\{1, \ldots, n\}$, $\pi(i) = i$.

A simple coordination game (A, B) is said to be an **cross-diagonal coordination game** if for all $i\in\{1, \ldots, n\}$, $\pi(i) = n-i+1$

5. Equivalence Theorem for Simple Coordination Games: In this section we present the main result of this note. However, before doing so, we note the following.

If (A, B) be a simple coordination game such that for a given permutation π on {1, …, n}, $i, j\in\{1, \ldots, n\}$, $a_{ij} = b_{ij} = 0$ if $j \neq \pi(i)$, and for all $i \in\{1, \ldots, n\}$, $a_{i\pi(i)}, b_{i\pi(i)} > 0$, with $a_{i\pi(i)} + b_{i\pi(i)} = 1$, then $(\frac{1}{2}I^{(\pi)}, \frac{1}{2} I^{(\pi)})$ is also a simple coordination where $i, j\in\{1, \ldots, n\}$, the $(i, j)^{th}$ entry of $I^{(\pi)}$ denoted by c_{ij} is 1 if $j = \pi(i)$ and c_{ij} is 0 if $j \neq \pi(i)$. $I^{(\pi)}$ is said to an n-dimensional "permutation matrix".

Further (x^*, y^*) is a pure strategy equilibrium for (A, B) <u>if and only if</u> (x^*, y^*) is a pure strategy equilibrium for $(\frac{1}{2}I^{(\pi)}, \frac{1}{2} I^{(\pi)})$.

For $i\in\{1, \ldots, n\}$, let $I_i^{(\pi)}$ denote the i^{th} row of $I^{(\pi),i}$ denote the i^{th} column of $I^{(\pi)}$. Thus, $i\in\{1, \ldots, n\}$, $I_i^{(\pi)}$ is an n-dimensional row vector and $I^{(\pi),i}$ is an n-dimensional column vector.

Theorem 1: Let (A, B) be a simple coordination game such that for a given permutation π on {1, …, n}, $i, j\in\{1, \ldots, n\}$, $a_{ij} = b_{ij} = 0$ if $j \neq \pi(i)$, and for all $i \in\{1, \ldots, n\}$, $a_{i\pi(i)}, b_{i\pi(i)} > 0$, with $a_{i\pi(i)} + b_{i\pi(i)} = 1$.

(i) If (x^*, y^*) is a pure-strategy equilibrium for (A, B), $i, j \in \{1, \ldots, n\}$, then there exists two real numbers u^*, v^* and $z^* \in \mathbb{R}^n$ such that x^*, y^*, z^*, u^*, v^* solves the following integer linear programming problem denoted ILP1:

Maximize $\sum_{i=1}^{n} z_i - \frac{1}{2}u - \frac{1}{2}v$, subject to $z_i = \frac{1}{2}(x_i + y_{\pi(i)})$, $I_i^{(\pi)}y \le u$, $x^T I^{(\pi),i} \le v$, $x_i \in \{0, 1\}$, $y_j \in \{0, 1\}$, $z_i \in \mathbb{R}$ for all $i \in \{1, \ldots, n\}$, $x, y \in \Delta^{n-1}$, $u, v \in \mathbb{R}$.

Further, $z_i^* \in \{0, 1\}$ for all $i \in \{1, \ldots, n\}$.

(ii) If $x^*, y^*, z^* \in \mathbb{R}^n$, $u^*, v^* \in \mathbb{R}$ solves the following integer linear programming problem denoted ILP2:

Maximize $\sum_{i=1}^{n} z_i - \frac{1}{2}u - \frac{1}{2}v$, subject to $z_i = \frac{1}{2}(x_i + y_{\pi(i)})$, $I_i^{(\pi)}y \le u$, $x^T I^{(\pi),i} \le v$, $x_i \in \{0, 1\}$, $y_j \in \{0, 1\}$, $z_i \in \{0, 1\}$ for all $i \in \{1, \ldots, n\}$, $x, y \in \Delta^{n-1}$, $u, v \in \mathbb{R}$.

then (x^*, y^*) is a pure-strategy equilibrium for (A, B).

Proof: In view of the observation preceding the statement of theorem 1, it is enough to prove the result for the simple coordination game $(\frac{1}{2}I^{(\pi)}, \frac{1}{2}I^{(\pi)})$.

First note that if x, y, z, u, v satisfies the constraints of either ILP1 or ILP2, then there exists j, $k \in \{1, \ldots, n\}$ so that $x_j = 1$, $x_i = 0$ if $i \ne j$, $y_{\pi(k)} = 1$, $y_i = 0$ if $i \ne \pi(k)$.

Thus, $I_k^{(\pi)}y = 1 > 0$, $I_i^{(\pi)}y = 0$ if $i \ne k$, so that $u \ge 1 > 0$, $x^T I^{(\pi),\pi(j)} = 1$, $x^T I^{(\pi),i} = 0$ if $i \ne \pi(j)$ so that $v \ge 1 > 0$.

Further, $z_j = \frac{1}{2}(x_j + y_{\pi(j)}) = \frac{1}{2} + \frac{1}{2}y_{\pi(i)})$, $z_k = \frac{1}{2}(x_k + y_{\pi(k)}) = \frac{1}{2}x_k + \frac{1}{2}$, and $z_i = \frac{1}{2}(x_i + y_{\pi(i)}) = 0$, if $i \notin \{j, k\}$.

Thus if $j \ne k$, then $\sum_{i=1}^{n} z_i - \frac{1}{2}u - \frac{1}{2}v = z_j + z_k - \frac{1}{2}u - \frac{1}{2}v = \frac{1}{2} + \frac{1}{2}y_{\pi(i)}) + \frac{1}{2}x_k + \frac{1}{2} - \frac{1}{2}u - \frac{1}{2}v = 1 + \frac{1}{2}(x_k + y_{\pi(i)}) - \frac{1}{2}u - \frac{1}{2}v = 1 - \frac{1}{2}u - \frac{1}{2}v$, since $x_k = 0 = y_{\pi(j)}$.

If $j = k$, then $\sum_{i=1}^{n} z_i - \frac{1}{2}u - \frac{1}{2}v = z_j - \frac{1}{2}u - \frac{1}{2}v = \frac{1}{2}(x_j + y_{\pi(i)}) - \frac{1}{2}u - \frac{1}{2}v = 1 - \frac{1}{2}u - \frac{1}{2}v$, since $x_j = 1 = y_{\pi(j)}$.

Since $u \ge 1$ and $v \ge 1$, $1 - \frac{1}{2}u - \frac{1}{2}v \le 0$.

Thus, $\sum_{i=1}^{n} z_i - \frac{1}{2}u - \frac{1}{2}v \le 0$

(i) If (x^*, y^*) is a pure-strategy equilibrium for $(\frac{1}{2}I^{(\pi)}, \frac{1}{2}I^{(\pi)})$, then let $u^* = v^* = x^{*T}I^{(\pi)}y^* = \sum_{i=1}^{n} c_{i\pi(i)} x_i^* y_{\pi(i)}^* = \sum_{i=1}^{n} x_i^* y_{\pi(i)}^*$

Clearly, $x^* \in \Delta^{n-1}$, $y^* \in \Delta^{n-1}$ and there exists $k \in \{1, \ldots, n\}$ such that $x_k^* = y_{\pi(k)}^* = 1$ and $x_i^* = y_{\pi(i)}^* = 0$, if $i \ne k$.

Thus, $\sum_{i=1}^{n} x_i^* y_{\pi(i)}^* = x_k^* y_{\pi(k)}^* = 1$.

Further, $I_k^{(\pi)}y^* = 1$, $I_i^{(\pi)}y^* = 0$ for $i \ne k$, $x^{*T}I^{(\pi),\pi(k)} = 1$ and $x^{*T}I^{(\pi),\pi(i)} = 0$ for $i \ne k$.

(x^*, y^*) is an equilibrium implies $\frac{1}{2}x^{*T}I^{(\pi)}y^* \geq \frac{1}{2}I_i^{(\pi)}y^*$ for all $i \in \{1, \ldots, n\}$, so that $u^* = x^{*T}I^{(\pi)}y^* \geq I_i^{(\pi)}y^*$ for all $i \in \{1, \ldots, n\}$.

Similarly, $v^* = x^{*T}I^{(\pi)}y^* \geq x^{*T}I^{(\pi),\pi(i)}$ for all $i \in \{1, \ldots, n\}$.

However, $u^* = v^* = x^{*T}I^{(\pi)}y^* = x_k^* y_{\pi(k)}^* = 1$.

Let $z^* \in \mathbb{R}^n$ be such that for all $i \in \{1, \ldots, n\}$, $z_i^* = \frac{1}{2}(x_i^* + y_{\pi(i)}^*)$. Thus, $z_k^* = 1$ and $z_i^* = 0$ if $i \neq k$.

Thus, x^*, y^*, z^*, u^*, v^* satisfies all the constraints of the integer linear programming problem.

Further, $\sum_{i=1}^{n} z_i^* - \frac{1}{2}u^* - \frac{1}{2}v^* = z_k^* - \frac{1}{2}u^* - \frac{1}{2}v^* = 1 - \frac{1}{2} - \frac{1}{2} = 0$.

Thus, x^*, y^*, z^*, u^*, v^* solves ILP1.

Note that $z_k^* = 1$ and $z_i^* = 0$ if $i \neq k$.

This proves (i).

(ii) Now suppose, x^*, y^*, z^*, u^*, v^* solves ILP2.

By note 1, we know that $(i, \pi(i))$ is a pure-strategy equilibrium for all $i \in \{1, \ldots, n\}$.

From the argument above we know that each such equilibrium solves ILP1, and the maximum value of the objective function is 0, which is attained at every pure-strategy equilibrium.

From the argument above we also know that each such equilibrium solves ILP 2, since for all $i \in \{1, 2, \ldots, n\}$ the corresponding value of $z_i \in \{0, 1\}$.

Thus, it must be that $\sum_{i=1}^{n} z_i^* - \frac{1}{2}u^* - \frac{1}{2}v^* = 0$, where for all $i \in \{1, \ldots, n\}$, $z_i^* = \frac{1}{2}(x_i^* + y_{\pi(i)}^*)$ and $z_i^* \in \{0, 1\}$.

Further, $I_i^{(\pi)}y^* \leq u^*$ for all $i \in \{1, \ldots, n\}$, $x^{*T}I^{(\pi),\pi(i)} \leq v^*$, $x_i^*, y_i^* \in \{0, 1\}$ for all $i \in \{1, \ldots, n\}$ and $x^*, y^* \in \Delta^{n-1}$, implies $c_{i\pi(i)}\, y_{\pi(i)}^* = A_i y^* \leq u^*$ and $c_{i\pi(i)} x_i^* = x^{*T}I^{(\pi),\pi(i)} \leq v^*$ for all $\in \{1, \ldots, n\}$.

Since, $x^*, y^* \in \Delta^{n-1}$ and $x_i^*, y_i^* \in \{0, 1\}$ for all $i \in \{1, \ldots, n\}$, there exists $j, k \in \{1, \ldots, n\}$ such that $x_j^* = 1$, $x_i^* = 0$ for $i \neq j$, $y_{\pi(k)}^* = 1$, $y_i^* = 0$ for $i \neq \pi(k)$.

Towards a contradiction suppose $j \neq k$.

Thus, $z_j^* = \frac{1}{2} \in (0, 1)$, $z_k^* = \frac{1}{2} \in (0, 1)$ and $z_i^* = 0$ for $i \neq j$, $i \neq k$.

This contradicts, $z_i^* \in \{0, 1\}$ for all $i \in \{1, \ldots, n\}$.

Thus, it must be the case that $j = k$ and hence $x_j^* = y_{\pi(j)}^* = 1$, $x_i^* = y_{\pi(i)}^* = 0$ for all $i \neq j$.

Thus, $z_j^* = \frac{1}{2}(x_j^* + y_{\pi(j)}^*) = 1$ and $z_i^* = 0$ for $i \neq j$.

Hence, $0 = \sum_{i=1}^{n} z_i^* - \frac{1}{2}u^* - \frac{1}{2}v^* = z_j^* - \frac{1}{2}u^* - \frac{1}{2}v^* = \frac{1}{2}(x_j^* + y_{\pi(j)}^*) - \frac{1}{2}u^* - \frac{1}{2}v^*$

Further, $x^{*T}I^{(\pi)}y^* = x_j^* y_{\pi(j)}^* = 1$.

Further, $1 = y_{\pi(j)}^* = c_{j\pi(j)} y_{\pi(j)}^* = I_j^{(\pi)} y^* \leq u^*$ implies $1 \leq u^*$ and $1 = x_j^* = c_{j\pi(j)} x_j^* \leq v^*$ implies $1 \leq v^*$.

However, $\frac{1}{2} x_j^* + \frac{1}{2} y_{\pi(j)}^* = z_j^* = \frac{1}{2} u^* + \frac{1}{2} v^*$.

This, combined with $1 = x_j^* \leq v^*$ and $1 = y_{\pi(j)}^* \leq u^*$ implies $1 = x_j^* = v^*$ and $1 = y_{\pi(j)}^* = u^*$

Thus, $I_i^{(\pi)} y^* \leq 1 = x^{*T}I^{(\pi)}y^*$ and $x^{*T}I^{(\pi),\pi(i)} \leq 1 = x^{*T}I^{(\pi)}y^*$ for all $i \in \{1, \ldots, n\}$.

Thus, (x^*, y^*) is a pure-strategy equilibrium for $(I^{(\pi)}, B^{(\pi)})$.

This proves (ii). Q.E.D.

Note 2: The only difference between ILP1 and ILP2 is that in ILP1, the values of z_i can be any real number, though at the optimal solution, they belong to $\{0, 1\}$, whereas in ILP2, we impose the constraint $z_i \in \{0, 1\}$ for all $i \in \{1, \ldots, n\}$.

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