

Equivalence Theorem for Simple Coordination Games

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Abstract

In this note, we consider simple coordination games, with each player having the same number of pure strategies to choose from. We model the problem as a “pie-division” problem. Let ‘ n ’ denote the number of strategies available to each of the two players. One player called the “row player” chooses one of the rows of a square matrix of size ‘ n ’. The other player called the “column player” chooses one of the columns of a square matrix of size ‘ n ’. There is a permutation (one-to-one function from a non-empty finite domain to itself) on the set of first ‘ n ’ positive integers, such that if the row player chooses a row and the column player chooses the column assigned by the permutation to itself, then each get a positive share of the pie. Otherwise, they get nothing. We call such two-person games, “simple coordination games”. We show, that for each simple coordination game, there is a class of “mixed integer linear programming problems”, such that the set of pure-strategy equilibria of the game is a “projection” of subset of the set of solutions of any mixed integer linear programming problem in the class and another very closely related class of mixed integer linear programming problem whose solutions for each problem in the class yield the set of pure-strategy equilibria of the simple coordination game. These two types of mixed integer linear programming problems depend only on the location of the positive pay-offs, and not on their magnitude. The mixed integer linear programs are far from obvious and the second one in particular- incorporates in it “multiplicative” or “interdependent” *non-linear features*, that would not be possible unless we required some additional variables to be either ‘0’ or ‘1’.

Keywords: simple coordination games, mixed integer linear programming

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1. Introduction: For interactive decision-making problems, with a non-empty finite strategy set for each decision maker (player), two-person zero-sum (TPZS) games are useful when the decision-makers are in direct conflict with one another, so that a gain for a decision-maker is an equal amount of loss for its rival. A sufficiently comprehensive discussion about TPZS games can be found in chapter 20 of Mote and Madhavan (2016). However, what TPZS games fail to capture is the potential for mutual gains arising from cooperation among the decision makers. This, deficiency has been addressed and remedied in a somewhat more general model, called two-person additively-separable sum (TPASS) games, in Lahiri (2025). The merit of the generalization is that, as in the case of TPZS games, the generalization preserves the possibility of characterizing the set of equilibria as the solution to an extremely plausible linear programming problem and its dual. Is this the most we can do if we want to preserve the property of characterizing the set of equilibria as the sets of solution to a linear

programming problem and its dual? What are the types of bi-matrix games whose set of equilibria (pure-strategy equilibria?) coincide with the set of solutions of a linear programming problem? These are the kind of questions motivating the current investigation. Our interpretation of equilibrium in the context of interactive decision-making problems, is that of a “steady state” as discussed in some detail in Lahiri (2017).

The seminal paper of Mangasarian and Stone (1964), provides us with an “Equivalence Theorem”, which characterizes the set of all equilibria of a bi-matrix game as the set of all solutions to a quadratic programming problem. The proof appeals to the existence result concerning equilibrium of a class of games that includes the set of bi-matrix games as a proper subset, due to Nash (1951). Hence, establishing the equivalence between equilibria of bi-matrix games and solutions of an optimization problem was achieved as early as 1964. However, to the best of our knowledge, there does not seem to be available such an equivalence between bi-matrix games and linear programming problems, beyond the set of TPASS games. There is a class of bi-matrix games called “coordination games”, which does not seem to allow being modelled as TPASS games. In these bi-matrix games, there are multiple pure-strategy equilibria, with each equilibrium rewarding each player differently. Such games, introduce the possibility of coordination and hence negotiation among the players, to decide on one among several pure-strategy equilibria.

In this note, we consider a type of coordination games, with each player having the same number of pure strategies to choose from. We model the problem as a “pie-division” problem. Let ‘ n ’ denote the number of strategies available to each of the two players. One player called the “row player” chooses one of the rows of a square matrix of size ‘ n ’. The other player called the “column player” chooses one of the columns of a square matrix of size ‘ n ’. There is a permutation (one-to-one function from a non-empty finite domain to itself) on the set of first ‘ n ’ positive integers, such that if the row player chooses a row and the column player chooses the column assigned by the permutation to itself, then each get a positive share of the pie. Otherwise, they get nothing. We call such two-person games, “simple coordination games”. The set of pure-strategy equilibria of such games do not depend on the magnitude of the positive pay-offs, so long as they-and only they- are strictly positive.

We show, that for each simple coordination game, there is a class of “mixed integer linear programming problems”, such that the set of pure-strategy equilibria of the game is a “projection” of the subset of the set of solutions of each mixed integer linear programming problem in this class, and another very closely related class of mixed integer linear programming problems whose solutions yield the set of pure-strategy equilibria of the simple coordination game. These two classes of mixed integer linear programming problems depend only on the location of the positive pay-offs, and not on their magnitude. The integer linear programs are far from obvious and the second one in particular- incorporates in it “multiplicative” or “interdependent” *non-linear features*, that would not be possible unless we required some additional variables to be either ‘0’ or ‘1’.

Although the solutions each problem in the second class of mixed integer linear programming problem, generate the set of pure-strategy equilibria of the coordination game, the contribution of this note is not algorithmic, since the definition of the integer linear programs depend on the set of pure-strategy equilibria. The main contribution of our results is

modelling the known set of equilibria as solutions to a “not so obvious” mixed integer linear programming problem.

2. Notations: Let \mathbb{R} denote the set of real numbers and let \mathbb{R}_+ denote the set of all non-negative real numbers. For a positive integer ℓ , and any non-empty set X , let X^ℓ be the set of all ordered ℓ -tuples with coordinates belonging to X .

For a positive integer ℓ and $j \in \{1, \dots, \ell\}$, let $E^{(\ell,j)}$ denote the point in \mathbb{R}^ℓ whose j^{th} coordinate is 1, and all other coordinates are 0. $E^{(\ell,j)}$ is said to be the j^{th} **unit coordinate vector** in \mathbb{R}^ℓ .

For any non-negative integer ℓ , let $\Delta^{\ell-1} = \{x \in \mathbb{R}_+^\ell \mid \sum_{k=1}^{\ell} x_k = 1\}$. Clearly, $\Delta^{\ell-1}$ is the convex combination of points in the set $\{E^{(\ell,i)} \mid i \in \{1, \dots, \ell\}\}$.

Unless otherwise mentioned, for any positive integer ℓ , we will interpret a point x in \mathbb{R}^ℓ to be a column vector, and its transpose represented by x^T to be a row vector.

Let m and n be positive integers. For any $m \times n$ real-valued matrix (i.e., a real-valued matrix with m rows and n columns) A and any $(i, j) \in \{1, \dots, m\} \times \{1, \dots, n\}$, let a_{ij} denote the $(i, j)^{\text{th}}$ entry of A (i.e., entry at the intersection of the i^{th} row and j^{th} column of A). For $(i, j) \in \{1, \dots, m\} \times \{1, \dots, n\}$, let A_i denote the i^{th} row A and let A^j denote the j^{th} column of A . Further, let A^T be the $n \times m$ matrix denoting the transpose of A .

3. Bi-matrix Games: For positive integers m and n , consider a two-player game between a “row player” and a “column player”, with their pay-off matrices being $m \times n$ matrices A and B respectively. The interpretation of the pair (A, B) , referred to as (an) a $(m \times n)$ **bi-matrix game**, is that if the row player chooses row i and the column player chooses column j , then the pay-off to the row player is a_{ij} and the pay-off to the column player is b_{ij} .

If $B = -A$, then the corresponding bi-matrix decision making problem is called a **matrix** (or **two-person zero-sum (TPZS) game**), which is discussed in chapter 20 of Mote and Madhavan (2019).

A **(randomized or mixed) strategy for the row player** is a point in Δ^{m-1} and a **(randomized or mixed) strategy for the column player** is a point in Δ^{n-1} .

A strategy for the row player is said to be a **pure-strategy for the row player** if the strategy belongs to the set $\{E^{(m,i)} \mid i \in \{1, \dots, m\}\}$.

A strategy for the column player is said to be a **pure-strategy for the column player** if the strategy belongs to $\{E^{(n,j)} \mid j \in \{1, \dots, n\}\}$.

A pair $(p, q) \in \Delta^{m-1} \times \Delta^{n-1}$ is a **(randomized or mixed) strategy profile**.

If $(p, q) = (E^{(m,i)}, E^{(n,j)})$ for some $(i, j) \in \{1, \dots, m\} \times \{1, \dots, n\}$, then we refer to the strategy pair as a **pure-strategy profile** and when there is no scope for mis-understanding we may denote the same simply by (i, j) .

Given an $m \times n$ bi-matrix a strategy profile (x^*, y^*) is said to be an **equilibrium** if for all $(i, j) \in \{1, \dots, m\} \times \{1, \dots, n\}$, $A_i y^* \leq x^{*T} A y^*$ and $x^{*T} B_j \leq x^{*T} B y^*$.

For a concise discussion about equilibria of bi-matrix games one may refer to Lahiri (2025).

Given an $m \times n$ bi-matrix a pure-strategy profile (i, j) is said to be a **pure-strategy equilibrium** if for all $(h, k) \in \{1, \dots, m\} \times \{1, \dots, n\}$, $a_{hj} \leq a_{ij}$ and $b_{ik} \leq b_{ij}$.

It is easy to see that if (i, j) is a pure-strategy equilibrium, then for all $x \in \Delta^{m-1}$, $x^T A^j \leq a_{ij}$ and for all $y \in \Delta^{n-1}$, $B_i y \leq b_{ij}$.

4. Simple Coordination Games: A $n \times n$ bi-matrix game (A, B) where n is a positive integer is said to be a **square bi-matrix game of size n** .

A square bi-matrix-game of size ' n ' (A, B) is said to be a **simple coordination game of size n** , if there exists a permutation π on $\{1, \dots, n\}$, (i.e., π is a one-to-one function from $\{1, \dots, n\}$ to itself, and hence an onto function on $\{1, \dots, n\}$) such that for all $i, j \in \{1, \dots, n\}$, $a_{ij} = b_{ij} = 0$ if $j \neq \pi(i)$, and for all $i \in \{1, \dots, n\}$, $a_{i\pi(i)}, b_{i\pi(i)} > 0$, with $a_{i\pi(i)} + b_{i\pi(i)} = 1$.

Assumption 1: For what follows in this section and the next assume that the positive integer ' n ' denoting the size of the square matrices, is given and fixed.

Hence, instead of writing a "simple coordination game of size n ", we will write "simple coordination game".

Note 1: It easy to see that the **set of pure-strategy equilibria for a simple coordination game** is $\{(i, \pi(i)) | i \in \{1, \dots, n\}\}$. Further, for all $x, y \in \Delta^{n-1}$, $x^T A y = \sum_{i=1}^n a_{i\pi(i)} x_i y_{\pi(i)}$ and $x^T B y = \sum_{i=1}^n b_{i\pi(i)} x_i y_{\pi(i)}$.

A simple coordination game (A, B) is said to be a **diagonal coordination game** if for all $i \in \{1, \dots, n\}$, $\pi(i) = i$.

A simple coordination game (A, B) is said to be an **cross-diagonal coordination game** if for all $i \in \{1, \dots, n\}$, $\pi(i) = n-i+1$

5. Equivalence Theorem for Simple Coordination Games: In this section we present the main result of this note. However, before doing so, we note the following.

If (A, B) be a simple coordination game such that for a given permutation π on $\{1, \dots, n\}$, $i, j \in \{1, \dots, n\}$, $a_{ij} = b_{ij} = 0$ if $j \neq \pi(i)$, and for all $i \in \{1, \dots, n\}$, $a_{i\pi(i)}, b_{i\pi(i)} > 0$, with $a_{i\pi(i)} + b_{i\pi(i)} = 1$, then for any $\alpha \in (0, 1)$, $(\alpha I^{(\pi)}, (1-\alpha)I^{(\pi)})$ is also a simple coordination where $i, j \in \{1, \dots, n\}$, the $(i, j)^{\text{th}}$ entry of $I^{(\pi)}$ denoted by c_{ij} is 1 if $j = \pi(i)$ and c_{ij} is 0 if $j \neq \pi(i)$. $I^{(\pi)}$ is said to an n -dimensional "permutation matrix".

Further (x^*, y^*) is a pure-strategy equilibrium for (A, B) if and only if (x^*, y^*) is a pure-strategy equilibrium for $(\alpha I^{(\pi)}, (1-\alpha)I^{(\pi)})$.

For $i \in \{1, \dots, n\}$, let $I_i^{(\pi)}$ denote the i^{th} row of $I^{(\pi)}$, i^{th} column of $I^{(\pi)}$. Thus, $i \in \{1, \dots, n\}$, $I_i^{(\pi)}$ is an n -dimensional row vector and $I^{(\pi),i}$ is an n -dimensional column vector.

Theorem 1: Let (A, B) be a simple coordination game such that for a given permutation π on $\{1, \dots, n\}$, $i, j \in \{1, \dots, n\}$, $a_{ij} = b_{ij} = 0$ if $j \neq \pi(i)$, and for all $i \in \{1, \dots, n\}$, $a_{i\pi(i)}, b_{i\pi(i)} > 0$, with $a_{i\pi(i)} + b_{i\pi(i)} = 1$.

(i) If (x^*, y^*) is a pure-strategy equilibrium for (A, B) , $i, j \in \{1, \dots, n\}$, then there exists two real numbers u^*, v^* and $z^* \in \mathbb{R}^n$, such that for any $\alpha \in (0, 1)$, x^*, y^*, z^*, u^*, v^* solve the following mixed integer linear programming problem denoted MILP1(α):

Maximize $\sum_{i=1}^n z_i - \frac{1}{2}u - \frac{1}{2}v$, subject to $z_i = (1-\alpha)x_i + \alpha y_{\pi(i)}$, $I_i^{(\pi)} y \leq u$, $x^T I^{(\pi),i} \leq v$, $x_i \in \{0, 1\}$, $y_j \in \{0, 1\}$, $z_i \in \mathbb{R}$ for all $i \in \{1, \dots, n\}$, $x, y \in \Delta^{n-1}$, $u, v \in \mathbb{R}$.

Further, $z_i^* \in \{0, 1\}$ for all $i \in \{1, \dots, n\}$.

(ii) If for some $\alpha \in (0, 1)$, $x^*, y^*, z^* \in \mathbb{R}^n$, $u^*, v^* \in \mathbb{R}$ solves the following mixed integer linear programming problem denoted MILP2(α):

Maximize $\sum_{i=1}^n z_i - \alpha u - (1-\alpha)v$, subject to $z_i = (1-\alpha)x_i + \alpha y_{\pi(i)}$, $I_i^{(\pi)} y \leq u$, $x^T I^{(\pi),i} \leq v$, $x_i \in \{0, 1\}$, $y_j \in \{0, 1\}$, $z_i \in \{0, 1\}$ for all $i \in \{1, \dots, n\}$, $x, y \in \Delta^{n-1}$, $u, v \in \mathbb{R}$.

then (x^*, y^*) is a pure-strategy equilibrium for (A, B) .

Proof: First note that for some $\alpha \in (0, 1)$, if x, y, z, u, v satisfies the constraints of either MILP1(α) or MILP2(α), then there exists $j, k \in \{1, \dots, n\}$ so that $x_j = 1$, $x_i = 0$ if $i \neq j$, $y_{\pi(k)} = 1$, $y_i = 0$ if $i \neq \pi(k)$.

Thus, $I_k^{(\pi)} y = 1 > 0$, $I_i^{(\pi)} y = 0$ if $i \neq k$, so that $u \geq 1 > 0$, $x^T I^{(\pi),\pi(j)} = 1$, $x^T I^{(\pi),i} = 0$ if $i \neq \pi(j)$ so that $v \geq 1 > 0$.

Further, $z_j = (1-\alpha)x_j + \alpha y_{\pi(j)} = (1-\alpha) + \alpha y_{\pi(j)}$, $z_k = (1-\alpha)x_k + \alpha y_{\pi(k)} = (1-\alpha)x_k + \alpha$, and $z_i = (1-\alpha)x_i + \alpha y_{\pi(i)} = 0$, if $i \notin \{j, k\}$.

Thus if $j \neq k$, then $\sum_{i=1}^n z_i - \alpha u - (1-\alpha)v = z_j + z_k - \alpha u - (1-\alpha)v = (1-\alpha) + \alpha y_{\pi(j)} + (1-\alpha)x_k + \alpha - \alpha u - (1-\alpha)v = 1 - \alpha u - (1-\alpha)v = 1 - \alpha u - (1-\alpha)v$, since $x_k = 0 = y_{\pi(j)}$.

If $j = k$, then $\sum_{i=1}^n z_i - \alpha u - (1-\alpha)v = z_j - \alpha u - (1-\alpha)v = \alpha x_j + (1-\alpha)y_{\pi(j)} - \alpha u - (1-\alpha)v = 1 - \alpha u - (1-\alpha)v$, since $x_j = 1 = y_{\pi(j)}$ and $x_i = 0 = y_{\pi(i)}$ if $i \neq j$.

Since $u \geq 1$ and $v \geq 1$, $1 - \alpha u - (1-\alpha)v \leq 0$.

Thus, $\sum_{i=1}^n z_i - \alpha u - (1-\alpha)v \leq 0$

(i) Given a pure-strategy equilibrium (x^*, y^*) for (A, B) , let $u^* = v^* = x^{*T} I^{(\pi)} y^* = \sum_{i=1}^n c_{i\pi(i)} x_i^* y_{\pi(i)}^* = \sum_{i=1}^n x_i^* y_{\pi(i)}^*$

If (x^*, y^*) is a pure-strategy equilibrium for (A, B) , then let $u^* = v^* = x^{*T} I^{(\pi)} y^* = \sum_{i=1}^n c_{i\pi(i)} x_i^* y_{\pi(i)}^* = \sum_{i=1}^n x_i^* y_{\pi(i)}^*$

From Note 1 in section 4, we know that since (A, B) is a simple coordination game (x^*, y^*) is a pure-strategy equilibrium for (A, B) , that there exists $k \in \{1, \dots, n\}$ such that $x_k^* = y_{\pi(k)}^* = 1$ and $x_i^* = y_{\pi(i)}^* = 0$, if $i \neq k$.

Thus, $\sum_{i=1}^n x_i^* y_{\pi(i)}^* = x_k^* y_{\pi(k)}^* = 1$.

Thus, $u^* = v^* = 1$.

Further, since for all $\alpha \in (0, 1)$, if $z_i^* = (1 - \alpha)x_i^* + \alpha y_{\pi(i)}^*$ for all $i \in \{1, \dots, n\}$, it follows that $z_k^* = 1$ and $z_i^* = 0$, for all $\alpha \in (0, 1)$.

We have, $I_k^{(\pi)} y^* = 1, I_i^{(\pi)} y^* = 0$ for $i \neq k, x^{*\Gamma I^{(\pi), \pi(k)}} = 1$ and $x^{*\Gamma I^{(\pi), \pi(i)}} = 0$ for $i \neq k$.

In view of the observation preceding the statement of theorem 1, if (x^*, y^*) is a pure-strategy equilibrium for (A, B) , if and only if for all $\alpha \in (0, 1)$, (x^*, y^*) is a pure-strategy equilibrium for $(\alpha I^{(\pi)}, (1-\alpha)I^{(\pi)})$.

Let $\alpha \in (0, 1)$. Thus, (x^*, y^*) is a pure-strategy equilibrium for $(\alpha I^{(\pi)}, (1-\alpha)I^{(\pi)})$ implies $\alpha x^{*\Gamma I^{(\pi)}} y^* \geq \alpha I_i^{(\pi)} y^*$ for all $i \in \{1, \dots, n\}$, so that $1 = u^* = x^{*\Gamma I^{(\pi)}} y^* \geq I_i^{(\pi)} y^*$ for all $i \in \{1, \dots, n\}$.

Similarly, $1 = v^* = x^{*\Gamma I^{(\pi)}} y^* \geq x^{*\Gamma I^{(\pi), \pi(i)}}$ for all $i \in \{1, \dots, n\}$.

Thus, x^*, y^*, z^*, u^*, v^* satisfies all the constraints of the mixed integer linear programming problem.

Further, $\sum_{i=1}^n z_i^* - \alpha u^* - (1-\alpha)v^* = z_k^* - \alpha u^* - (1-\alpha)v^* = 1 - \alpha - (1-\alpha) = 0$.

Thus, x^*, y^*, z^*, u^*, v^* solves MILP1(α).

Note that $z_k^* = 1$ and $z_i^* = 0$ if $i \neq k$.

This proves (i).

(ii) Now suppose, for some $\alpha \in (0, 1)$, x^*, y^*, z^*, u^*, v^* solves MILP2(α).

By note 1, we know that for all $i \in \{1, \dots, n\}$, $(i, \pi(i))$ is a pure-strategy equilibrium for the simple coordination game $(\alpha I^{(\pi)}, (1-\alpha)I^{(\pi)})$.

From the argument above we know that each such equilibrium solves MILP1(α), and the maximum value of the objective function is 0, which is attained at every pure-strategy equilibrium. Further, for all $i \in \{1, 2, \dots, n\}$ the corresponding value of $z_i \in \{0, 1\}$.

Thus, it must be that $\sum_{i=1}^n z_i^* - \alpha u^* - (1-\alpha)v^* = 0$, where for all $i \in \{1, \dots, n\}$, $z_i^* = (1 - \alpha)x_i^* + \alpha y_{\pi(i)}^*$ and $z_i^* \in \{0, 1\}$.

Further, $I_i^{(\pi)} y^* \leq u^*$ for all $i \in \{1, \dots, n\}$, $x^{*\Gamma I^{(\pi), \pi(i)}} \leq v^*$, $x_i^*, y_i^* \in \{0, 1\}$ for all $i \in \{1, \dots, n\}$ and $x^*, y^* \in \Delta^{n-1}$, implies $c_{i\pi(i)} y_{\pi(i)}^* = I_i^{(\pi)} y^* \leq u^*$ and $c_{i\pi(i)} x_i^* = x^{*\Gamma I^{(\pi), \pi(i)}} \leq v^*$ for all $i \in \{1, \dots, n\}$.

Since, $x^*, y^* \in \Delta^{n-1}$ and $x_i^*, y_i^* \in \{0, 1\}$ for all $i \in \{1, \dots, n\}$, there exists $j, k \in \{1, \dots, n\}$ such that $x_j^* = 1, x_i^* = 0$ for $i \neq j, y_{\pi(k)}^* = 1, y_i^* = 0$ for $i \neq \pi(k)$.

Towards a contradiction suppose $j \neq k$.

Thus, $z_j^* = (1-\alpha) \in (0, 1), z_k^* = \alpha \in (0, 1)$ and $z_i^* = 0$ for $i \neq j, i \neq k$.

This contradicts, $z_i^* \in \{0, 1\}$ for all $i \in \{1, \dots, n\}$.

Thus, it must be the case that $j = k$ and hence $x_j^* = y_{\pi(j)}^* = 1, x_i^* = y_{\pi(i)}^* = 0$ for all $i \neq j$.

Thus, $1 = x_j^* = c_{j\pi(j)} x_i^* = x^{*\Gamma I^{(\pi), \pi(j)}} \leq v^*, 1 = y_{\pi(j)}^* = c_{j\pi(j)} y_{\pi(j)}^* = I_j^{(\pi)} y^* \leq u^*$

Further, $z_j^* = (1 - \alpha)x_j^* + \alpha y_{\pi(j)}^* = 1$ and $z_i^* = 0$ for $i \neq j$.

Hence, $0 = \sum_{i=1}^n z_i^* - \alpha u^* - (1-\alpha)v^* = z_j^* - \alpha u^* - (1-\alpha)v^* = (1 - \alpha)x_j^* + \alpha y_{\pi(j)}^* - \alpha u^* - (1-\alpha)v^*$.

Further, $x^{*T} I^{(\pi)} y^* = x_j^* y_{\pi(j)}^* = 1$.

However, $(1 - \alpha)x_j^* + \alpha y_{\pi(j)}^* = z_j^* = \alpha u^* + (1-\alpha)v^*$.

This, combined with $1 = x_j^* \leq v^*$ and $1 = y_{\pi(j)}^* \leq u^*$ implies $1 = x_j^* = v^*$ and $1 = y_{\pi(j)}^* = u^*$

Thus, $I_i^{(\pi)} y^* \leq 1 = x^{*T} I^{(\pi)} y^*$ and $x^{*T} I^{(\pi), \pi(i)} \leq 1 = x^{*T} I^{(\pi)} y^*$ for all $i \in \{1, \dots, n\}$.

Thus, (x^*, y^*) is a pure-strategy equilibrium for $(\alpha I^{(\pi)}, (1-\alpha)I^{(\pi)})$.

In view of the observation preceding the statement of theorem 1, it must be the case that (x^*, y^*) is a pure-strategy equilibrium for (A, B)

This proves (ii). Q.E.D.

Note 2: The only difference between MILP1(α) and MILP2(α) is that in MILP1(α), the values of z_i can be any real number, though at the optimal solution, they belong to $\{0, 1\}$, whereas in MILP2(α), we impose the constraint $z_i \in \{0, 1\}$ for all $i \in \{1, \dots, n\}$.

The following corollary of theorem 1 is a precise version of what can be called an “Equivalence Theorem” for simple coordination games.

Corollary of theorem 1: Let (A, B) be a simple coordination game such that for a given permutation π on $\{1, \dots, n\}$, $i, j \in \{1, \dots, n\}$, $a_{ij} = b_{ij} = 0$ if $j \neq \pi(i)$, and for all $i \in \{1, \dots, n\}$, $a_{i\pi(i)}, b_{i\pi(i)} > 0$, with $a_{i\pi(i)} + b_{i\pi(i)} = 1$.

(x^*, y^*) is a pure-strategy equilibrium for (A, B) , $i, j \in \{1, \dots, n\}$, if and only if there exists $\alpha \in (0, 1)$, two real numbers u^*, v^* and $z^* \in \mathbb{R}^n$, such that x^*, y^*, z^*, u^*, v^* solve the following mixed integer linear programming problem denoted MILP2(α):

Maximize $\sum_{i=1}^n z_i - \alpha u - (1-\alpha)v$, subject to $z_i = (1-\alpha)x_i + \alpha y_{\pi(i)}$, $I_i^{(\pi)} y \leq u$, $x^T I^{(\pi), i} \leq v$, $x_i \in \{0, 1\}$, $y_j \in \{0, 1\}$, $z_i \in \{0, 1\}$ for all $i \in \{1, \dots, n\}$, $x, y \in \Delta^{n-1}$, $u, v \in \mathbb{R}$.

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