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# A Unified Framework for $U$-Structures and Functorial Structure: Managing Super, Hyper, SuperHyper, Tree, and Forest Uncertain Over/Under/Off Models

Takaaki Fujita[1]∗, Florentin Smarandache[2]

[1]∗ Independent Researcher, Shinjuku, Shinjuku-ku, Tokyo, Japan. Takaaki.fujita060@gmail.com

[2] University of New Mexico, Gallup Campus, NM 87301, USA. fsmarandache@gmail.com

**Abstract**. A *Classical Structure* is a mathematical framework on a nonempty set $H$ equipped with one or more operations satisfying specified axioms. Here, "Structure" refers broadly to any formal system—from set theory and logic to probability, statistics, algebra, geometry, and beyond. In the domain of fuzzy sets and their extensions, a multitude of models has emerged—fuzzy sets, vague sets, intuitionistic fuzzy sets, neutrosophic sets, picture fuzzy sets, hesitant fuzzy sets, plithogenic sets, and more. While this diversity reflects an active research community, it also introduces conceptual overlap. We therefore group all such variants under the term *Uncertain Sets*.

We propose a unified framework for their systematic management by defining the notions of the *Uncertain Model*, the *U-Structure*, and the *U-Off Structure*. We then revisit and extend the HyperUncertain, SuperUncertain, SuperHyperUncertain, TreeUncertain, and ForestUncertain structures, along with their Off counterparts. To encompass systems beyond Uncertain Sets, we further introduce *Functorial Set* and *Functorial Structure*. This comprehensive approach establishes a robust foundation for organizing and applying uncertainty models across diverse mathematical and applied domains.

**Keywords:** Uncertain set, Powerset, $n$-th powerset, HyperUncertain Set, SuperHyperUncertain Set, Fuzzy Set, Neutrosophic Set, Plithogenic Set, Uncertain Structure

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## 1. Introduction

### 1.1. *Various Uncertain Concepts*

Real-world data often involve uncertainty. To model this, various set-theoretic frameworks have been introduced, including fuzzy sets [1], intuitionistic fuzzy sets [2], picture fuzzy sets [3], bipolar fuzzy sets [4], neutrosophic sets [5], hesitant fuzzy sets [6], quadripartitioned neutrosophic sets [7], and plithogenic sets [8]. These concepts were developed to facilitate the modeling of complex uncertain phenomena, and have attracted a large volume of research in recent years. Representative concepts are listed in Table 1.

Moreover, these concepts have found applications far beyond set theory—in graph theory [18], neural networks [19], decision-making [20], logic [21], topology [22], group theory [23], probability theory [24, 25], automata theory [26, 27], programming [28], vector spaces [29, 30], matroid theory [31], image processing [32], biology [33, 34], social sciences [35], chemistry [36], physics [37], and many other domains. However, as new uncertainty models continue to proliferate, it is not sufficient to arbitrarily increase the number of membership functions or

| Model | Domain |
|---|---|
| Fuzzy [1] | $[0,1]$ |
| Intuitionistic Fuzzy [9] | $\{(\mu,\nu) \in [0,1]^2 \mid \mu+\nu \leq 1\}$ |
| Inconsistent Intuitionistic Fuzzy (Picture Fuzzy) [3] | $\{(\mu,\nu,\pi) \in [0,1]^3 \mid \mu+\nu+\pi \leq 2\}$ |
| Pythagorean Fuzzy [10] | $\{(\mu,\nu) \in [0,1]^2 \mid \mu^2+\nu^2 \leq 1\}$ |
| $q$-Rung Orthopair Fuzzy ($q \geq 1$) [11] | $\{(\mu,\nu) \in [0,1]^2 \mid \mu^q+\nu^q \leq 1\}$ |
| Neutrosophic [5] | $\{(T,I,F) \in [0,1]^3 \mid 0 \leq T+I+F \leq 3\}$ |
| Refined Neutrosophic [12] | $\{\,(T_1,\ldots,T_k;\, I_1,\ldots,I_k;\, F_1,\ldots,F_k) \in [0,1]^{3k} \mid \sum_{j=1}^{k}(T_j+I_j+F_j) \leq 3k\}$ |
| Plithogenic [8, 13] | $\{\,(x,v,\mathrm{pdf}(x,v),\mathrm{pCF}(v_1,v_2)) \mid x \in P,\ v \in P_v,\ \mathrm{pdf}(x,v) \in [0,1]^s,\ \mathrm{pCF}(v_1,v_2) \in [0,1]^t\}$ |
| Hesitant Fuzzy [6] | $\{\,H \subseteq [0,1] \mid H$ is a nonempty finite set$\,\}$ |
| Vague [14] | $\{(\tau,\varphi) \in [0,1]^2 \mid \tau+\varphi \leq 1\}$ |
| Bipolar Fuzzy [15] | $\{\,(\mu^+,\mu^-) \in [0,1]^2\,\}$ |
| $m$-polar Fuzzy ($m \geq 2$) [16] | $\{(\mu_1,\ldots,\mu_m) \in [0,1]^m\}$ |
| Spherical Fuzzy [17] | $\{(P,I,N) \in [0,1]^3 \mid P^2+I^2+N^2 \leq 1\}$ |
| Single-Valued Quadripartitioned Neutrosophic [7] | $\{(T,C,U,F) \in [0,1]^4 \mid T+C+U+F \leq 4\}$ |

Table 1. Representative Uncertainty Models and Their Domains

the model's complexity. From the standpoint of applied mathematics and scientific practice, there is a growing need for a unified framework to systematically organize and manage these diverse uncertainty models.

Table 2 presents an overview of various mathematical structures. The term *structure* here broadly refers to fundamental formal systems used to model diverse phenomena. By incorporating uncertainty conditions—such as those based on fuzzy logic or neutrosophic logic—into these structures, we can make them more adaptable and effective for real-world applications.

| Structure | Description |
|---|---|
| Set | A carrier with distinguished elements or relations but no additional operations. |
| Logic | $(L,\wedge,\vee,\neg)$ with binary and unary connectives satisfying logical axioms. |
| Probability space [38] | $(\Omega,\mathcal{F},P)$ where $P$ is a probability measure on the sigma-algebra $\mathcal{F}$. |
| Statistical model [39] | $(X,\mathcal{A},\theta)$ where $\theta$ maps experimental data $X$ to statistical parameters. |
| Group [40] | $(G,*)$ with an associative operation, identity element, and inverses for every element. |
| Ring [41] | $(R,+,\times)$ with addition and multiplication satisfying the ring axioms. |
| Vector space [42] | $(V,+,\cdot)$ over a field $\mathbb{F}$ with vector addition and scalar multiplication. |
| Metric space [43] | $(X,\mathrm{dist})$ with a distance function obeying non-negativity, symmetry, and triangle inequality. |
| Graph [44] | $(V,E)$ where edges $E$ connect pairs of vertices in either undirected or directed form. |
| Automaton [45] | $(Q,\Sigma,\delta,q_0,F)$ defining a finite-state machine with transitions and accepting states. |
| Game [46] | $(N,\{A_i\},\{u_i\})$ with players $N$, action sets $A_i$, and utility functions $u_i$. |

Table 2. Overview of Structure Types

### 1.2. *HyperUncertain, SuperUncertain, SuperHyperUncertain, TreeUncertain, and ForestUncertain Sets*

Smarandache extended the classical uncertain set framework by employing the powerset and its iterative application to define *HyperUncertain*, *SuperUncertain*, and *SuperHyperUncertain* sets [47, 48]. Representative examples of HyperUncertain sets include *HyperFuzzy* [49, 50], *HyperNeutrosophic* [51, 52], and *HyperPlithogenic* sets [53]. Similarly, *SuperFuzzy*, *SuperNeutrosophic*, and *SuperPlithogenic* sets are typical instances of SuperUncertain sets [54]. For SuperHyperUncertain sets, examples such as *SuperHyperFuzzy*, *SuperHyperNeutrosophic*, and *SuperHyperPlithogenic* have been proposed [47, 55]. More recently, the concepts of *TreeUncertain Sets* and *ForestUncertain Sets* were introduced in the literature [48]. These layered extensions enable the explicit representation of hierarchical and intricately structured uncertainties. As research into HyperUncertainty and SuperHyperUncertainty continues to progress, the development of a unified framework becomes essential for systematically organizing and governing these advanced models. An overview of these Uncertain Set types is provided in Table 3.

| Type | Definition |
|---|---|
| HyperUncertain Set | $\tau : A \to \mathcal{P}^n\big([0,1]^r\big)$, assigning each element a set of $r$-tuples. |
| SuperUncertain Set | $\tau : \mathcal{P}^m(A) \to [0,1]^r$, assigning each $m$-th-level subset a single $r$-tuple. |
| SuperHyperUncertain Set | $\tau : \mathcal{P}^m(A) \to \mathcal{P}^n\big([0,1]^r\big)$, assigning each subset a set of $r$-tuples. |
| TreeUncertain Set | $T : \mathcal{P}(\mathcal{T}) \to \big([0,1]^r\big)^U$, mapping each tree-node subset to a membership function $U \to [0,1]^r$. |
| ForestUncertain Set | $F : \mathcal{P}(\mathcal{F}) \to \big([0,1]^r\big)^U$, aggregating TreeUncertain functions by coordinate-wise maximum. |

TABLE 3. Overview of Uncertain Set Types

### 1.3. *Uncertain Overset, Underset, and Offset*

In classical fuzzy-set theory, all membership degrees are confined to the interval $[0, 1]$. However, in practice, uncertainty can extend beyond these bounds in both directions. To accommodate such cases, Smarandache introduced three generalizations (see Table 4) [56, 57]. These extensions offer a more flexible framework for modeling uncertainty in applied and scientific contexts.

| **Concept** | **Definition** |
|---|---|
| *Uncertain Overset* (U-Over) | Membership values lie in $[1, \psi]$, exceeding the classical maximum of 1, to model overconfidence or excessive affirmation. |
| *Uncertain Underset* (U-Under) | Membership values lie in $[\varphi, 0]$, falling below 0, to capture negation or underconfidence. |
| *Uncertain Offset* (U-Off) | Membership values may lie outside the standard $[0, 1]$ bounds— either below 0 or above 1— thereby representing both under- and over-degrees simultaneously. |

TABLE 4. Definitions of Uncertain Overset, Underset, and Offset

### 1.4. *Motivation and Our Contribution*

In the realm of fuzzy sets and their extensions, new models appear almost daily. While this reflects a vibrant research community, it also risks inundating the field with an overwhelming variety of approaches—often developed to address the very goal of fuzzy theory, namely the modeling of real-world phenomena. Some of these concepts may ultimately offer limited practical benefit or theoretical insight. We therefore argue for a unified, systematic framework to organize and manage both classical fuzzy models and their many extensions.

To this end, we introduce the notions of the *Uncertain Model*, the *$U$-Structure*, and the *$U$-Off Structure*. We then revisit and extend the definitions of HyperUncertain, SuperUncertain, SuperHyperUncertain, TreeUncertain, and ForestUncertain structures, together with their corresponding Off variants, thereby establishing a comprehensive foundation for handling uncertainty across diverse modeling paradigms. Moreover, to encompass structures that fall outside the scope of Uncertain Sets, we define the concepts of *Functorial Set* and *Functorial Structure*. Due to space constraints, we cannot detail all mathematical properties and operations here; we hope that domain experts will advance these avenues in future work.

### 1.5. *Structure of This Research Paper*

The remainder of this paper is organized as follows:

- In Section 2, we introduce the fundamental constructions used throughout this work: the powerset, the $n$th powerset, trees over an attribute set, and the notion of a classical structure.
- Section 3 formalizes $U$-Structures as well as $U$-Over and $U$-Under Structures from the perspective of fuzzy and fuzzy-extension uncertainty.
- In Section 4, we present the definitions and illustrative examples of SuperUncertain Sets, HyperUncertain Sets, SuperHyperUncertain Sets, and their non-classical counterparts.
- Section 5 is devoted to TreeUncertain and ForestUncertain Sets, together with concrete examples.
- Section 6 examines SuperUncertain Structures, HyperUncertain Structures, and SuperHyperUncertain Structures.
- Section 7 explores Hyper-$U$-Off, SuperHyper-$U$-Off, Tree-$U$-Off, and Forest-$U$-Off Structures in detail.
- In Section 8, we introduce the new notions of Functorial Sets and Functorial Structures.
- Finally, Section 9 offers our concluding remarks and directions for future research.

## 2. **Preliminaries**

In this section, we introduce the foundational concepts and notations used throughout this paper. All sets are assumed to be finite, unless otherwise stated.

### 2.1. *Basic Set-Theoretic Constructions*

The powerset of $S$ is the set of all possible subsets of $S$, including the empty set and $S$ itself. The $n$-th powerset of $S$ iteratively applies the powerset operation $n$ times, constructing sets of sets repeatedly from $S$ itself [58–60].

**Definition 2.1** (Base Set)**.** A *base set* $S$ is any nonempty set that serves as the underlying universe for more complex constructions. Formally,

$$S = \{ x \mid x \text{ is an element in the specified domain} \}.$$

All subsequent constructions such as powersets, hyperstructures, and uncertainty-based structures draw their elements ultimately from $S$.

**Definition 2.2** (Powerset). [61] Let $S$ be a base set. The *powerset* of $S$, denoted $\mathcal{P}(S)$, is the collection of all subsets of $S$. That is,

$$\mathcal{P}(S) = \{ A \mid A \subseteq S \}.$$

**Definition 2.3** ($n$-th Powerset). [58, 62, 63] Let $S$ be a base set and $n \geq 1$ an integer. The *$n$-th powerset* of $S$, denoted $\mathcal{P}^n(S)$, is defined inductively by

$$\begin{cases} \mathcal{P}^1(S) = \mathcal{P}(S), \\ \mathcal{P}^{k+1}(S) = \mathcal{P}\big(\mathcal{P}^k(S)\big), \quad \text{for } k \geq 1. \end{cases}$$

Similarly, the *$n$-th nonempty powerset* $\mathcal{P}_*^n(S)$ is defined by

$$\begin{cases} \mathcal{P}_*^1(S) = \mathcal{P}_*(S) = \mathcal{P}(S) \setminus \{\varnothing\}, \\ \mathcal{P}_*^{k+1}(S) = \mathcal{P}_*\big(\mathcal{P}_*^k(S)\big) = \mathcal{P}\big(\mathcal{P}_*^k(S)\big) \setminus \{\varnothing\}, \quad \text{for } k \geq 1. \end{cases}$$

**Example 2.4** (Meal Planning via $n$-th Powerset). Let the set of available dishes be

$$S = \{\text{Soup, Salad, Entrée, Dessert}\}.$$

- *First powerset* $\mathcal{P}^1(S)$ lists all possible daily menus:

$$\mathcal{P}^1(S) = \big\{\emptyset, \{\text{Soup}\}, \{\text{Salad}\}, \{\text{Entrée}\}, \{\text{Dessert}\}, \{\text{Soup,Salad}\}, \ldots, \{\text{Soup,Salad,Entrée,Dessert}\}\big\}.$$

  Each element represents the choice of dishes served on one day.

- *Second powerset* $\mathcal{P}^2(S)$ collects all possible weekly menus (as 7-element subsets of daily menus):

$$\mathcal{P}^2(S) = \mathcal{P}\big(\mathcal{P}^1(S)\big).$$

  For example,

$$\big\{\{\text{Soup,Salad}\}, \{\text{Entrée,Dessert}\}, \{\text{Soup,Entrée,Dessert}\}, \ldots\big\} \in \mathcal{P}^2(S)$$

  encodes one possible seven-day meal plan.

- *Third powerset* $\mathcal{P}^3(S)$ yields all possible monthly plans, each being a collection of weekly plans:

$$\mathcal{P}^3(S) = \mathcal{P}\big(\mathcal{P}^2(S)\big).$$

  For instance,

$$\big\{\{\{\text{Soup,Salad}\}, \{\text{Entrée}\}, \ldots\}, \{\{\text{Salad,Dessert}\}, \{\text{Soup,Entrée}\}, \ldots\}\big\} \in \mathcal{P}^3(S)$$

  represents two distinct weekly menus within a monthly plan.

Thus, the $n$-th powerset $\mathcal{P}^n(S)$ models hierarchical meal planning:

$$\mathcal{P}^1(S) : \text{daily menus}, \quad \mathcal{P}^2(S) : \text{weekly plans}, \quad \mathcal{P}^3(S) : \text{monthly schedules}, \ \ldots$$

### 2.2. *Tree and Forest over an Attribute Set*

A *tree over an attribute set* is a finite rooted, directed acyclic graph in which each leaf is uniquely labeled by a single attribute from the set. A *forest over an attribute set* is a disjoint union of such rooted trees whose leaves collectively partition the attribute set, with each attribute appearing exactly once. Related concepts in the literature include the *Tree-soft Set* framework [64, 65] and the *Forest-soft Set* framework [66, 67].

**Definition 2.5** (Tree over an Attribute Set)**.** Let $A$ be a finite set of *attributes*. A *rooted tree over* $A$, denoted $\mathrm{Tree}(A)$, is a finite rooted, acyclic directed graph $T = (V, E)$ satisfying:

- There is a distinguished *root* $r \in V$.
- Each node $v \in V$ corresponds to either a singleton $\{a\}$ for some $a \in A$ (called a *leaf*) or to a nonempty subset of $A$ (an *internal node* of compatible attributes).
- For every $v \in V$, there is exactly one directed path from $r$ to $v$.
- The set of leaves is in bijection with $A$. In particular, each $a \in A$ appears exactly once as a leaf label.

We write $\mathrm{Tree}(A)$ both for the graph $T$ and, with slight abuse of notation, for its underlying set of vertices $V(T)$ when unambiguous from context.

**Example 2.6** (Tree over an Attribute Set: Product Feature Hierarchy)**.** Let the attribute set be

$$A = \{\text{CPU, RAM, Storage, Brand, Price}\}.$$

A rooted tree $T = (V, E)$ over $A$ can be organized as follows:

- **Root:** $v_0 = \{\text{CPU}, \text{RAM}, \text{Storage}, \text{Brand}, \text{Price}\}$.
- **Internal nodes:**
  - $v_1 = \{\text{CPU}, \text{RAM}, \text{Storage}\}$ (‘‘Technical specs’’).
  - $v_2 = \{\text{Brand}\}$.
  - $v_3 = \{\text{Price}\}$.
- **Leaves:** $\{\{\text{CPU}\}, \{\text{RAM}\}, \{\text{Storage}\}, \{\text{Brand}\}, \{\text{Price}\}\}$, each singleton corresponding to exactly one attribute.
- **Edges:**

$$(v_0 \to v_1), \quad (v_0 \to v_2), \quad (v_0 \to v_3),$$

$$(v_1 \to \{\text{CPU}\}),\ (v_1 \to \{\text{RAM}\}),\ (v_1 \to \{\text{Storage}\}).$$

Every attribute in $A$ appears exactly once as a leaf, and there is a unique directed path from the root $v_0$ to each leaf.

**Definition 2.7** (Forest over an Attribute Set)**.** Let $A$ be a finite set of attributes. A *forest over* $A$, denoted $\mathrm{Forest}(A)$, is a finite acyclic directed graph $F = (V, E)$ whose connected components are each rooted trees over subsets of $A$, such that:

- Each component is a rooted tree as in Definition 2.5, whose leaves partition $A$.

- No two components share a vertex; equivalently, $F$ is a disjoint union of rooted trees $\{T_i\}_{i=1}^{m}$, where each $T_i$ is a rooted tree over some subset $A_i \subseteq A$.
- Every element $a \in A$ appears exactly once as a leaf of exactly one component $T_i$.

Again, we may denote by $\mathrm{Forest}(A)$ both the graph and its vertex set when clear from context.

**Example 2.8** (Forest over an Attribute Set: Separate Feature Modules)**.** Let the attribute set be

$$A = \{\text{CPU, RAM, Storage, Brand, Price, ReleaseDate}\}.$$

We form a forest $F$ consisting of two disjoint rooted trees $T_1$ and $T_2$:

- **Tree** $T_1$ **over** $A_1 = \{\text{CPU}, \text{RAM}, \text{Storage}, \text{Brand}, \text{Price}\}$:
  - Root $u_0 = A_1$.
  - Internal node $u_1 = \{\text{CPU}, \text{RAM}, \text{Storage}\}$.
  - Leaves $\{\text{CPU}\}, \{\text{RAM}\}, \{\text{Storage}\}, \{\text{Brand}\}, \{\text{Price}\}$.
  - Edges as in the previous example.
- **Tree** $T_2$ **over** $A_2 = \{\text{ReleaseDate}\}$:
  - Root and leaf coincide: $w_0 = \{\text{ReleaseDate}\}$.
  - No further internal nodes or edges.

These two trees are vertex-disjoint, their leaves partition $A$, and each attribute appears exactly once as a leaf in exactly one component. Thus $F = T_1 \cup T_2$ is a forest over $A$.

### 2.3. *Classical Structure*

In this paper, the term *Structure* refers to an arbitrary mathematical structure, including but not limited to those from the domains of Set Theory, Logic, Probability, Statistics, Algebra, and Geometry.

**Definition 2.9** (Classical Structure)**.** A *Classical Structure* $C$ is a mathematical structure drawn from one of various domains—such as Set theory, Logic, Probability, Statistics, Algebra, Geometry, Graph theory, Automata theory, Game theory, etc.—and can be formalized as a pair

$$C = \big(H,\ \{\#^{(m)}\}_{m\in\mathcal{I}}\big),$$

where:

- $H$ is a nonempty set (the *carrier* or *universe*).
- For each $m \in \mathcal{I} \subseteq \mathbb{Z}_{>0}$, there is an $m$-ary operation

  $$\#^{(m)} : H^m \longrightarrow H,$$

  subject to specified *axioms* (e.g., associativity, commutativity, identity laws) depending on the particular type of structure.

We say that $C$ is of *type* $\{\#^{(m)} : m \in \mathcal{I}\}$. Examples include:

- A *Set* $(S, \varnothing)$ viewed as a carrier with distinguished elements or relations, but no additional operations.

- A *Logic* $(L, \wedge, \vee, \neg)$, where $\wedge$ and $\vee$ are binary connectives and $\neg$ is a unary connective satisfying logical axioms.
- A *Probability* structure $(\Omega, \mathcal{F}, P)$, where $P : \mathcal{F} \to [0, 1]$ is a measure on a sigma-algebra $\mathcal{F} \subseteq \mathcal{P}(\Omega)$.
- A *Statistics* model $(X, \mathcal{A}, \theta)$, where $\theta$ maps data $X$ to statistical parameters.
- A *Algebraic* structure:
  - A *Group* $(G, *)$ with $* : G \times G \to G$ satisfying associativity, identity, and inverses.
  - A *Ring* $(R, +, \times)$ with two binary operations satisfying ring axioms.
  - A *Vector Space* $(V, +, \cdot)$ over a field $\mathbb{F}$, where $\cdot : \mathbb{F} \times V \to V$.
- A *Geometric* structure $(X, \text{dist})$, where $\text{dist} : X \times X \to \mathbb{R}$ satisfies the metric axioms.
- A *Graph* $(V, E)$, where $E \subseteq \{\{u, v\} \mid u, v \in V\}$ in the undirected case (or $E \subseteq V \times V$ in the directed case), with additional notions of adjacency and incidence.
- An *Automaton* $(Q, \Sigma, \delta, q_0, F)$, where $Q$ is a finite set of states, $\Sigma$ is an input alphabet, $\delta : Q \times \Sigma \to Q$ is the transition function, $q_0 \in Q$ is the start state, and $F \subseteq Q$ is the set of accepting states.
- A *Game* $(N, \{A_i\}, \{u_i\})$, where $N$ is a set of players, $A_i$ is the action set for player $i$, and $u_i : \prod_{j \in N} A_j \to \mathbb{R}$ is the utility (payoff) function for player $i$.

## 3. Fuzzy and Fuzzy-Extension Types of Uncertainty

In this section, we present the unified concepts of the *U-Structure* and the *U-Off Structure*, which generalize and subsume fuzzy and fuzzy-extension based models of uncertainty.

### 3.1. *U-Structure*

A variety of uncertainty models have been proposed in the literature, including the Fuzzy Set [1], Vague Set [14], Intuitionistic Fuzzy Set [68], Neutrosophic Set [69], Plithogenic Set [8], and Hesitant Fuzzy Set [6]. These models have been extended and applied to a wide range of underlying *Structures*—that is, arbitrary mathematical structures including, but not limited to, those from Set Theory, Logic, Probability, Statistics, Algebra, and Geometry. In this paper, we collectively refer to these models as *Uncertain Sets*. We investigate a unified framework for systematically managing these models by introducing the concepts of the *Uncertain Model*, the *U-Structure*, and the *U-Off Structure*.

**Definition 3.1** (Uncertain Model)**.** Let $\mathbb{U}$ denote the collection of all *Uncertain* models, where each $U \in \mathbb{U}$ is characterized by a mapping

$$\mu_U : X_U \longrightarrow \text{Dom}(U),$$

from some universe of discourse $X_U$ to a set of admissible *membership-degree tuples* $\text{Dom}(U) \subseteq [0, 1]^r$ (for an appropriate integer $r \geq 1$), subject to specific algebraic or geometric constraints depending on $U$. In particular, we include at least the following types:

(1) **Fuzzy.**

$$\text{Dom}(\text{Fuzzy}) = [0, 1].$$

(2) **Intuitionistic Fuzzy.**

$$\mathrm{Dom}\bigl(\text{Intuitionistic Fuzzy}\bigr) \;=\; \bigl\{(\mu,\nu)\in[0,1]^2 \;\big|\; \mu+\nu\le 1\bigr\}.$$

(3) **Inconsistent Intuitionistic Fuzzy (Picture Fuzzy, Ternary Fuzzy).**

$$\mathrm{Dom}\bigl(\text{Inconsistent Intuitionistic Fuzzy}\bigr) \;=$$

$$\bigl\{(\mu,\nu,\pi)\in[0,1]^3 \;\big|\; \mu+\nu+\pi\le 2\bigr\},$$

where $\mu$, $\nu$, and $\pi$ represent membership, non-membership, and refusal degrees respectively.

(4) **Pythagorean Fuzzy.**

$$\mathrm{Dom}\bigl(\text{Pythagorean Fuzzy}\bigr) \;=$$

$$\bigl\{(\mu,\nu)\in[0,1]^2 \;\big|\; \mu^2+\nu^2\le 1\bigr\}.$$

(5) **q-Rung Orthopair Fuzzy.**

$$\mathrm{Dom}\bigl(\text{q-Rung Orthopair Fuzzy}\bigr) \;=$$

$$\bigl\{(\mu,\nu)\in[0,1]^2 \;\big|\; \mu^q+\nu^q\le 1\bigr\},\quad q\ge 1.$$

(6) **Neutrosophic.**

$$\mathrm{Dom}\bigl(\text{Neutrosophic}\bigr) \;=$$

$$\bigl\{(T,I,F)\in[0,1]^3 \;\big|\; 0\le T+I+F\le 3\bigr\}.$$

(7) **Refined Neutrosophic.**

$$\mathrm{Dom}\bigl(\text{Refined Neutrosophic}\bigr) \;=$$

$$\bigl\{\bigl(T_1,T_2,\dots,T_k;\; I_1,I_2,\dots,I_k;\; F_1,F_2,\dots,F_k\bigr)\in[0,1]^{3k} \;\big|\; \sum_{j=1}^{k}(T_j+I_j+F_j)\le 3k\bigr\}.$$

(8) **Plithogenic.** Let $P$ be a set of primary attributes and $P_v$ the set of their possible values. Then

$$\mathrm{Dom}\bigl(\text{Plithogenic}\bigr) \;=\; \Bigl\{\bigl(x,\ v,\ \mathrm{pdf}(x,v),\ \mathrm{pCF}(v_1,v_2)\bigr) \;\Big|\; \begin{array}{l} x\in P,\ v\in P_v,\\ \mathrm{pdf}(x,v)\in[0,1]^s,\ \mathrm{pCF}(v_1,v_2)\in[0,1]^t \end{array}\Bigr\}.$$

(9) . . . Additional uncertainty models may be included, each with its own domain $\mathrm{Dom}(U)\subseteq[0,1]^r$ defined by suitable constraints.

We refer to any element of $\mathrm{Dom}(U)$ as a *$U$-membership degree*.

Let $U\in\mathbb{U}$ be a fixed uncertainty model with degree-domain $\mathrm{Dom}(U)$. Recall that in this paper, the term *Structure* refers to an arbitrary mathematical structure, including but not limited to those from the domains of Set Theory, Logic, Probability, Statistics, Algebra, and Geometry.

**Definition 3.2** ($U$-Structure)**.** Let $C=(H,\{\#^{(m)}\})$ be a Classical Structure as in Definition 2.9, with carrier $H$. A *$U$-Structure $C^U$* is the pair

$$C^U \;=\; \Bigl(H,\ \mu\colon H\longrightarrow \mathrm{Dom}(U),\ \{\#^{(m)}\}\Bigr),$$

where

$$\mu(x)\in\mathrm{Dom}(U)\quad\text{is the }U\text{-membership degree of }x\in H.$$

The underlying classical operations $\{\#^{(m)}\}$ remain those of $C$, but each element of $H$ carries a $U$-membership label $\mu(x)$. In effect, $C^U$ is a *fuzzy- or neutrosophic*-enrichment of the classical structure $C$.

**Example 3.3** (Fuzzy Social Network)**.** Let $G = (V, E)$ be a classical undirected graph representing a social network, where

$$V = \{v_1, v_2, \ldots, v_n\} \quad \text{is the set of users, and} \quad E \subseteq \big\{\{v_i, v_j\} \mid v_i, v_j \in V,\ i \neq j\big\}$$

is the set of friendship relations. We choose $U = \text{Fuzzy}$, so that

$$\text{Dom}(U) = [0, 1].$$

Define a mapping

$$\mu : V \longrightarrow [0, 1]$$

by assigning to each user $v_i \in V$ a *fuzzy membership degree* $\mu(v_i) \in [0, 1]$ indicating the *"activity level"* of $v_i$ in the network (e.g. frequency of posts, comments, or logins). Concretely:

$$\mu(v_i) = \frac{\text{number of posts by } v_i \text{ in the last month}}{\text{maximum posts by any user in the last month}}.$$

Then

$$G^U = \big(V,\ \mu : V \to [0, 1],\ E\big)$$

is a *Fuzzy-Social-Network* (a Fuzzy $U$-Structure). Here:

- The underlying classical structure is $G = (V, E)$.
- The map $\mu(v_i)$ assigns each user $v_i$ a value in $[0, 1]$, quantifying how active or "influential" they are.
- The edge set $E$ remains unchanged, encoding the binary "friendship" relation.

Such a Fuzzy Social Network can be used to model and analyze how user influence (as a fuzzy degree) interacts with the network topology.

**Example 3.4** (Neutrosophic Decision-Support Database)**.** Consider a decision-support system for medical diagnoses. Let

$$R = \{r_1, r_2, \ldots, r_m\}$$

be a finite set of patient records in a database. Each record $r_j$ contains raw laboratory values, symptoms, and preliminary observations. We choose

$$U = \text{Neutrosophic}, \quad \text{Dom}(U) = \big\{(T, I, F) \in [0, 1]^3 \;\big|\; 0 \leq T + I + F \leq 3\big\}.$$

Here:

- $T$ denotes the *truth-membership degree* (confidence that the patient has a certain disease).
- $I$ denotes the *indeterminacy-degree* (level of uncertainty due to inconclusive tests).
- $F$ denotes the *falsity-degree* (confidence that the patient does *not* have the disease).

Define a mapping

$$\mu \,:\, R \longrightarrow \mathrm{Dom}(U) \quad \text{by} \quad \mu(r_j) = \big(T_j,\, I_j,\, F_j\big),$$

where $T_j$, $I_j$, and $F_j$ are computed as follows for patient record $r_j$:

$$T_j = \frac{\text{(number of strongly positive test results for } r_j)}{\text{(total number of relevant tests)}},$$
$$I_j = \frac{\text{(number of borderline or conflicting results for } r_j)}{\text{(total number of relevant tests)}},$$
$$F_j = \frac{\text{(number of strongly negative test results for } r_j)}{\text{(total number of relevant tests)}}.$$

By construction, $T_j + I_j + F_j \le 3$ (since each ratio lies in $[0, 1]$). Then

$$R^U \;=\; \big(R,\, \mu : R \to \mathrm{Dom}(U)\big)$$

is a *Neutrosophic Database-Structure* (a Neutrosophic $U$-Structure). In this example:

- The carrier set $R$ is the collection of patient records.
- The mapping $\mu(r_j) = (T_j, I_j, F_j)$ captures, for each patient, the degrees of confirmation, indeterminacy, and negation for a given disease.
- There is no additional classical operation here—this structure is simply a set equipped with a neutrosophic label on each element.

Such a Neutrosophic Database permits algorithms to retrieve and rank patient records by their combined truth, indeterminacy, and falsity degrees when suggesting diagnoses or recommending further tests.

### 3.2. *U-Over, U-Under, and U-Off Structures*

A $U$-Over Structure allows membership degrees exceeding 1 to model overconfidence. A $U$-Under Structure allows degrees below 0 to capture underconfidence. A $U$-Off Structure permits components outside the $[0, 1]$ range.

**Definition 3.5** ($U$-Over, U-Under, and U-Off Structures)**.** Let $U$ be an uncertainty model with classical domain $\mathrm{Dom}(U) \subseteq [0, 1]^r$. For real scalars $\varphi < 0 < 1 < \psi$, define the *nonclassical degree-range*

$$\mathrm{Dom}_{\varphi,\psi}(U) \;=\; \Big\{\, d \in \mathbb{R}^r \;\Big|\; \exists\, e \in \mathrm{Dom}(U) \text{ with } d_i \in [\,\varphi,\, \psi\,] \text{ for } 1 \le i \le r \Big\}.$$

We then define variants of $U$-Structures as follows.

(1) A *classical U-Structure* is one for which $\mu(x) \in \mathrm{Dom}(U) \subseteq [0, 1]^r$ for every $x \in H$.

(2) A *U-Over Structure* is one for which $\mu(x) \in [1, \psi]^r \subset \mathrm{Dom}_{\varphi,\psi}(U)$. That is, each membership component may exceed 1, but never drop below 1.

(3) A *U-Under Structure* is one for which $\mu(x) \in [\varphi, 0]^r \subset \mathrm{Dom}_{\varphi,\psi}(U)$. In particular, membership values may be negative (below 0), but never exceed 0.

(4) A *U-Off Structure* is one for which $\mu(x) \in \mathrm{Dom}_{\varphi,\psi}(U) \setminus \mathrm{Dom}(U)$, i.e. each membership tuple may lie outside the classical cube $[0, 1]^r$, possibly partially negative and partially greater than 1.

When $\varphi < 0 < 1 < \psi$ are understood from context, we simply speak of *U-Over*, *U-Under*, and *U-Off Structures*.

**Example 3.6** ($U$-Over Graph Structure)**.** Let $G = (V, E)$ be a classical undirected graph. Choose the fuzzy uncertainty model $U$ = Fuzzy with classical domain $\mathrm{Dom}(U) = [0, 1]$ and set $\psi = 1.5$. Define

$$\mu : V \longrightarrow [1, \psi], \qquad \mu(v) = 1 + 0.5 \frac{\deg(v)}{\max_{u \in V} \deg(u)},$$

so that each $\mu(v) \in [1, 1.5]$. Then

$$G^{U\text{-Over}} = \bigl(V,\ \mu,\ E\bigr)$$

is a *Fuzzy-Over Graph Structure*: every vertex carries a membership degree exceeding the classical maximum of 1, modeling overconfidence in the node's importance.

**Example 3.7** ($U$-Under Group Structure)**.** Let $(\mathbb{Z}_n, +)$ be the cyclic group of order $n$. Again take $U$ = Fuzzy with $\mathrm{Dom}(U) = [0, 1]$ and choose $\varphi = -0.2$. Define

$$\mu : \mathbb{Z}_n \longrightarrow [\varphi, 0], \qquad \mu(x) = -0.2 \frac{\mathrm{ord}(x)}{n},$$

where $\mathrm{ord}(x)$ is the order of $x$ in the group. Thus each $\mu(x) \in [-0.2, 0]$. Then

$$\bigl(\mathbb{Z}_n,\ \mu,\ +\bigr)$$

is a *Fuzzy-Under Group Structure*: every element's membership lies below 0, capturing underconfidence in its group-theoretic significance.

**Example 3.8** ($U$-Off Neutrosophic Database)**.** Consider a set of patient records $R = \{r_1, \ldots, r_m\}$. Let $U$ = Neutrosophic with classical domain $\mathrm{Dom}(U) = \{(T, I, F) \in [0, 1]^3 \mid 0 \le T + I + F \le 3\}$ and choose $\varphi = -0.1$, $\psi = 1.2$. Define the nonclassical range $\mathrm{Dom}_{\varphi,\psi}(U) = [\varphi, \psi]^3$, and set

$$\mu : R \longrightarrow \mathrm{Dom}_{\varphi,\psi}(U), \qquad \mu(r_j) = \bigl(T_j, I_j, F_j\bigr),$$

where

$$T_j = \min\Bigl(1.2,\ \tfrac{\#\text{positive}}{\#\text{tests}} \times 1.1\Bigr), \quad I_j = \max\Bigl(-0.1,\ \tfrac{\#\text{borderline}}{\#\text{tests}} - 0.05\Bigr), \quad F_j = \tfrac{\#\text{negative}}{\#\text{tests}}.$$

Some $T_j$ may exceed 1 and some $I_j$ may fall below 0, so $\mu(r_j) \notin \mathrm{Dom}(U)$. Then

$$R^{U\text{-Off}} = \bigl(R,\ \mu\bigr)$$

is a *Neutrosophic Off Structure*: membership tuples lie partially outside $[0, 1]^3$, simultaneously capturing both over- and under-confidence.

## 4. New Concepts: SuperHyperUncertain Set

A SuperHyperUncertain Set assigns to each m-subset of a base set a family of distinct possible r-tuples of membership degrees.

4.1. *Normal SuperHyperUncertain Set*

We begin with the foundational concepts of the *HyperUncertain Set* and the *SuperUncertain Set*. And the formal definition of a *Normal SuperHyperUncertain Set* is provided below.

**Definition 4.1** ($n$-HyperUncertain Set)**.** Let $A$ be a nonempty base set, let $r \geq 1$ be a fixed integer, and let $n \geq 1$ be a fixed integer. A *HyperUncertain Set* (classical version) over $A$ is a mapping

$$\tau : A \longrightarrow \mathcal{P}^n\big([0,1]^r\big),$$

such that for each $x \in A$,

$$\tau(x) \subseteq [0,1]^r.$$

In other words, each element $x \in A$ is assigned a set of possible $r$-tuples of membership-degrees, where $\tau(x) \in \mathcal{P}^n([0,1]^r)$. Equivalently, one may regard a HyperUncertain Set as $\tau : A \to \{ B \mid B \subseteq [0,1]^r \}$, with the understanding that the codomain is the $n$-th powerset of $[0,1]^r$.

**Example 4.2** (HyperUncertain Set)**.**

- If $r = 1$, a *HyperFuzzy Set* is $\tau : A \to \mathcal{P}^1\big([0,1]\big) = \mathcal{P}([0,1])$, e.g. $\tau(a) = \{0.2, 0.3\} \subseteq [0,1]$.
- If $r = 2$ a *HyperIntuitionistic Fuzzy Set* is

$$\tau : A \to \mathcal{P}^1\big([0,1]^2\big) = \mathcal{P}([0,1]^2)$$

, e.g.

$$\tau(a) = \{ (0.2, 0.5),\ (0.3, 0.4)\} \subseteq [0,1]^2$$

.

- If $r = 3$, a *HyperNeutrosophic Set* is $\tau : A \to \mathcal{P}([0,1]^3)$.

**Definition 4.3** ($m$-SuperUncertain Set)**.** Let $A$ be a nonempty base set, let $r \geq 1$ be a fixed integer, and let $m \geq 1$ be a fixed integer. A *SuperUncertain Set* (classical version) over $A$ is a mapping

$$\tau : \mathcal{P}^m(A) \longrightarrow [0,1]^r,$$

such that for each $X \in \mathcal{P}^m(A)$,

$$\tau(X) = \big(d_1(X),\ d_2(X),\ \ldots,\ d_r(X)\big) \in [0,1]^r.$$

In other words, each $m$-th-powerset-level subset $X \subseteq A$ is assigned a single $r$-tuple of membership-degrees. Equivalently, $\tau : \mathcal{P}^m(A) \to [0,1]^r$.

**Example 4.4** (SuperUncertain Set)**.**

- If $r = 1$, a *SuperFuzzy Set* is $\tau : \mathcal{P}^1(A) = \mathcal{P}(A) \to [0,1]$, e.g. $\tau(\{a,b\}) = 0.9$.
- If $r = 2$, a *SuperIntuitionistic Fuzzy Set* is $\tau : \mathcal{P}(A) \to [0,1]^2$, e.g. $\tau(\{a,b\}) = (0.5,\ 0.7)$.
- If $r = 3$, a *SuperNeutrosophic Set* is $\tau : \mathcal{P}(A) \to [0,1]^3$.

The definition of an $(m,n)$-SuperHyperUncertain Set is presented below. Note that in some papers, the case $m = n$ is assumed by default. Here, $m$ and $n$ are non-negative integers.

**Definition 4.5** ($(m, n)$-SuperHyperUncertain Set)**.** Let $A$ be a nonempty base set, let $r \geq 1$ be a fixed integer, let $m \geq 1$ be a fixed integer for the "super" level, and let $n \geq 1$ be a fixed integer for the "hyper" level. A *SuperHyperUncertain Set* (classical version) over $A$ is a mapping

$$\tau : \mathcal{P}^m(A) \longrightarrow \mathcal{P}^n\big([0,1]^r\big),$$

such that for each $X \in \mathcal{P}^m(A)$,

$$\tau(X) \subseteq [0,1]^r.$$

Equivalently, each $X \in \mathcal{P}^m(A)$ is assigned a set of possible $r$-tuples of membership-degrees drawn from $[0, 1]^r$, and this "candidate-set of degrees" itself lies in the $n$-th powerset of $[0, 1]^r$.

**Example 4.6** (General $(m, n)$-SuperHyperUncertain Sets)**.** Let $A$ be a nonempty set and fix integers $r, m, n \geq 1$. A $(m, n)$-SuperHyperUncertain Set over $A$ is a mapping

$$\tau : \mathcal{P}^m(A) \longrightarrow \mathcal{P}^n\big([0,1]^r\big),$$

which assigns to each $m$-fold subset of $A$ a collection of $r$-tuples in $[0, 1]^r$, itself regarded as an element of the $n$th powerset of $[0, 1]^r$. In particular:

- If $r = 1$, then $\tau : \mathcal{P}^m(A) \to \mathcal{P}^n([0, 1])$ is called a *SuperHyperFuzzy Set*.
- If $r = 2$, then $\tau : \mathcal{P}^m(A) \to \mathcal{P}^n\big([0, 1]^2\big)$ is called a *SuperHyperIntuitionistic Fuzzy Set*.
- If $r = 3$, then $\tau : \mathcal{P}^m(A) \to \mathcal{P}^n\big([0, 1]^3\big)$ is called a *SuperHyperNeutrosophic Set*.

**Example 4.7** (Environmental Sensor Reliability as a $(1, 2)$-SuperHyperUncertain Set)**.** Let $A = \{Temp, Humid, AQ\}$ be three environmental sensors measuring temperature, humidity, and air quality. We take $r = 1$, $m = 1$, and $n = 2$, so that

$$\tau : \mathcal{P}(A) \longrightarrow \mathcal{P}^2\big([0,1]\big).$$

Hence each subset $X \subseteq A$ is assigned two candidate fuzzy–reliability sets (one for the dry season, one for the wet season). For instance,

$$\tau(\{Temp\}) = \big\{\{0.85, 0.90\},\ \{0.70, 0.75\}\big\}, \quad \tau(\{Humid\}) = \big\{\{0.80, 0.85\},\ \{0.65, 0.70\}\big\},$$

$$\tau(\{AQ\}) = \big\{\{0.90, 0.95\},\ \{0.60, 0.65\}\big\}, \quad \tau(\{Temp, Humid\}) = \big\{\{0.82, 0.88\},\ \{0.62, 0.68\}\big\},$$

and similarly for all nonempty $X \subseteq A$. Each inner set $\{d_1, d_2\} \subseteq [0, 1]$ gives the possible membership degrees under one seasonal condition. Thus $\tau$ is a $(1, 2)$-SuperHyperUncertain Set, which in the special cases of superhyperfuzzy or superhyperneutrosophic would assign, respectively, fuzzy or neutrosophic degree–sets at the subset level.

**Example 4.8** (Smartphone Feature Satisfaction as a $(2, 2)$-SuperHyperUncertain Set)**.** Let

$$A = \{Battery, Camera, Durability\}$$

be three smartphone features. We set $r = 1$, $m = 2$, and $n = 2$, so that

$$\tau : \mathcal{P}^2(A) \longrightarrow \mathcal{P}^2\big([0,1]\big).$$

Here $\mathcal{P}^2(A)$ consists of all unordered pairs of features, and for each $X \in \mathcal{P}^2(A)$, $\tau(X)$ is a pair of fuzzy–satisfaction sets corresponding to two user segments (casual vs. power users). Concretely,

$$\tau(\{Battery, Camera\}) = \big\{\{0.75, 0.80\},\ \{0.90, 0.95\}\big\},$$

$$\tau(\{Battery, Durability\}) = \big\{\{0.70, 0.78\},\ \{0.88, 0.92\}\big\},$$

$$\tau(\{Camera, Durability\}) = \big\{\{0.68, 0.74\},\ \{0.85, 0.89\}\big\}.$$

Each inner set $\{d_1, d_2\}$ gives the typical satisfaction degrees for casual and power users, respectively. Hence $\tau$ realizes a $(2, 2)$-SuperHyperUncertain Set. In contexts of superhyperfuzzy or superhyperneutrosophic modeling, the analogous constructions would assign, instead of fuzzy numbers, sets of neutrosophic triplets to each feature–pair.

**Example 4.9** (Culinary Preference as a SuperHyperIntuitionistic Fuzzy Set)**.** Let $A = \{\text{Pasta, Sushi, Curry}\}$ be three dishes. We choose

$$r = 2, \quad m = 1, \quad n = 1, \qquad \tau\colon \mathcal{P}(A) \longrightarrow \mathcal{P}\big([0,1]^2\big).$$

Each singleton $\{d\} \subseteq A$ is assigned two candidate *(membership, non-membership)* pairs, for instance:

$$\tau(\{\text{Pasta}\}) = \big\{(0.90, 0.10),\ (0.70, 0.20)\big\},$$

$$\tau(\{\text{Sushi}\}) = \big\{(0.85, 0.05),\ (0.75, 0.15)\big\},$$

$$\tau(\{\text{Curry}\}) = \big\{(0.80, 0.10),\ (0.65, 0.25)\big\}.$$

Here the first pair models evaluations by food critics, the second by casual diners. This set is a SuperHyperUncertain Set with $r = 2$ (intuitionistic case), generalizing the $r = 1$ SuperHyperFuzzy scenario.

**Example 4.10** (Clinical Symptom Assessment as a SuperHyperNeutrosophic Set)**.** Let $A = \{\text{Fever, Cough, Fatigue}\}$ be three symptoms. We set

$$r = 3, \quad m = 1, \quad n = 1, \qquad \tau\colon \mathcal{P}(A) \longrightarrow \mathcal{P}\big([0,1]^3\big).$$

Each symptom is assigned two candidate *(truth, indeterminacy, falsity)* triplets, for example:

$$\tau(\{\text{Fever}\}) = \big\{(0.90, 0.05, 0.05),\ (0.85, 0.10, 0.05)\big\}, \quad \tau(\{\text{Cough}\}) = \big\{(0.80, 0.10, 0.10),\ (0.75, 0.15, 0.10)\big\},$$

$$\tau(\{\text{Fatigue}\}) = \big\{(0.70, 0.20, 0.10),\ (0.65, 0.25, 0.10)\big\}.$$

The first triplet represents a physician's assessment, the second a nurse's. This is a SuperHyperUncertain Set with $r = 3$ (neutrosophic case), extending the $r = 1$ SuperHyperFuzzy and $r = 2$ SuperHyperIntuitionistic cases to full neutrosophic evaluations.

4.2. *Non-Classical HyperUncertain, SuperUncertain, SuperHyperUncertain Set*

Non-classical HyperUncertain sets extend hyper-level uncertainty by allowing membership degrees in $[\varphi, \psi]$. Non-classical SuperUncertain assign off-range degrees to $m$-fold subsets. Non-classical SuperHyperUncertain combine both, mapping $m$-subsets to $n$-fold off-degree sets.

**Definition 4.11** (Non-Classical HyperUncertain, SuperUncertain, SuperHyperUncertain)**.** Let $\varphi < 0 < 1 < \psi$ be real bounds, and let $r \geq 1$, $m \geq 1$, $n \geq 1$. We replace $[0,1]$ by $[\varphi, \psi]$ and $[0,1]^r$ by $[\varphi, \psi]^r$. Then:

(1) A *Non-Classical HyperUncertain Set* is $\tau : A \to \mathcal{P}^n\big([\varphi, \psi]^r\big)$, so that $\tau(x) \subseteq [\varphi, \psi]^r$ for each $x \in A$.

(2) A *Non-Classical SuperUncertain Set* is $\tau : \mathcal{P}^m(A) \to [\varphi, \psi]^r$, so that $\tau(X) \in [\varphi, \psi]^r$ for each $X \in \mathcal{P}^m(A)$.

(3) A *Non-Classical SuperHyperUncertain Set* is $\tau : \mathcal{P}^m(A) \to \mathcal{P}^n\big([\varphi, \psi]^r\big)$, so that $\tau(X) \subseteq [\varphi, \psi]^r$.

**Example 4.12** (Non-Classical HyperUncertain Set)**.** Let $A = \{\text{S1, S2, S3}\}$ be three sensors. Fix $\varphi = -0.1$, $\psi = 1.1$, $r = 1$, $n = 1$. Then

$$\tau\colon A \longrightarrow \mathcal{P}\big([\varphi, \psi]\big),$$

with, for instance,

$$\tau(\text{S1}) = \{\, 1.05,\ -0.05\}, \quad \tau(\text{S2}) = \{\, 1.10,\ 0.90\}, \quad \tau(\text{S3}) = \{\, 0.95,\ -0.10\}.$$

Each $\tau(\text{Si}) \subseteq [\varphi, \psi]$ allows reliability degrees above 1 or below 0.

**Example 4.13** (Non-Classical SuperUncertain Set)**.** Let $A = \{\text{P1, P2, P3}\}$ be three products. Fix $\varphi = -0.2$, $\psi = 1.2$, $r = 2$, $m = 2$. Then

$$\tau\colon \mathcal{P}^2(A) \longrightarrow [\varphi, \psi]^2,$$

with, for example,

$$\tau(\{\text{P1}, \text{P2}\}) = (1.10,\ -0.10), \quad \tau(\{\text{P1}, \text{P3}\}) = (0.95,\ 1.15), \quad \tau(\{\text{P2}, \text{P3}\}) = (1.20,\ 0.00).$$

Here each pair of products receives a two-component "off" membership degree.

**Example 4.14** (Non-Classical SuperHyperUncertain Set)**.** Let $A = \{\text{R1, R2}\}$ be two regions. Fix $\varphi = -0.1$, $\psi = 1.2$, $r = 3$, $m = 1$, $n = 2$. Then

$$\tau\colon \mathcal{P}(A) \longrightarrow \mathcal{P}^2\big([\varphi, \psi]^3\big),$$

with, for instance,

$$\tau(\{\text{R1}\}) = \big\{\, (1.10, 0.05, 0.05),\ (-0.05, 1.00, 0.10)\big\}, \quad \tau(\{\text{R2}\}) = \big\{\, (1.20, 0.00, 0.00),\ (0.00, 1.10, -0.05)\big\}.$$

Each region is assigned two candidate neutrosophic triplets, some components lying outside $[0, 1]$.

**Theorem 4.15** (Generalization of the Classical $(m, n)$-SuperHyperUncertain Set)**.** *Let* $\tau_{\text{NC}}\colon \mathcal{P}^m(A) \to \mathcal{P}^n\big([\varphi, \psi]^r\big)$ *be a Non-Classical* $(m, n)$*-SuperHyperUncertain Set as in Definition. If* $\varphi = 0$ *and* $\psi = 1$*, then*

$$[\varphi, \psi]^r = [0, 1]^r \quad \Longrightarrow \quad \mathcal{P}^n\big([\varphi, \psi]^r\big) = \mathcal{P}^n\big([0, 1]^r\big),$$

*and hence* $\tau_{\mathrm{NC}}$ *is exactly a classical* $(m, n)$*-SuperHyperUncertain Set* $\tau\colon \mathcal{P}^m(A) \to \mathcal{P}^n([0,1]^r)$*. Conversely, any classical* $(m, n)$*-SuperHyperUncertain Set arises by taking* $\varphi = 0$*,* $\psi = 1$*. Therefore the non-classical notion strictly generalizes the classical one.*

*Proof.* By definition, $\tau_{\mathrm{NC}}(X) \subseteq [\varphi, \psi]^r$ for each $X$. When $\varphi = 0$ and $\psi = 1$, we have $[\varphi, \psi]^r = [0,1]^r$, so

$$\tau_{\mathrm{NC}} : \mathcal{P}^m(A) \longrightarrow \mathcal{P}^n([0,1]^r),$$

which is precisely the classical Definition of an $(m, n)$-SuperHyperUncertain Set. No further modification is needed.

Conversely, given any classical $\tau\colon \mathcal{P}^m(A) \to \mathcal{P}^n([0,1]^r)$, one obtains a Non-Classical $(m, n)$-SuperHyperUncertain Set by declaring $\varphi = 0$, $\psi = 1$ and setting $\tau_{\mathrm{NC}} = \tau$. Since $[0,1]^r \subseteq [\varphi, \psi]^r$, all axioms of the non-classical definition are satisfied.

Thus the non-classical construction contains the classical one as the special case $\varphi = 0, \psi = 1$, proving the theorem. □

## 5. TreeUncertain Set and ForestUncertain Set

We examine the TreeUncertain Set and the ForestUncertain Set.

### 5.1. *Classical TreeUncertain Set*

A Classical TreeUncertain Set is a mapping from subsets of a rooted tree's nodes to functions assigning each element an $r$-vector in $[0,1]^r$. The definition of the *Classical TreeUncertain Set* is presented below.

**Definition 5.1** (TreeUncertain Set)**.** Let $U$ be a nonempty set and let $\mathcal{T} = (A, E)$ be a finite rooted tree with node–set $A$. Fix $r \geq 1$. A *TreeUncertain Set* on $U$ indexed by $\mathcal{T}$ is a map

$$T : \mathcal{P}(A) \longrightarrow \{\, \mu : U \to [0,1]^r \},\quad S \longmapsto T(S) = \mu_S,$$

where each $\mu_S(x) \in [0,1]^r$ for $x \in U$. Equivalently, for each $S \subseteq A$ and $x \in U$,

$$\mu_S(x) = \big(d_1^{(S)}(x), \dots, d_r^{(S)}(x)\big),\quad d_i^{(S)}(x) \in [0,1].$$

**Remark 5.2.**
- If $r = 1$, $T(S)\colon U \to [0,1]$ is a *TreeFuzzy Set*.
- If $r = 2$, $\mu_S(x) = (\mu(x), \nu(x))$ recovers a *TreeIntuitionistic Fuzzy Set*.
- If $r = 3$, $\mu_S(x) = (T, I, F)$ gives a *TreeNeutrosophic Set*.

**Example 5.3** (E-commerce Preferences)**.** Let $U = \{\text{Alice}, \text{Bob}, \text{Carol}\}$ and $\mathcal{T}$ be the tree with root "Electronics" and children "Mobile", "Laptop". For $r = 1$, define

$$T(\{\text{Mobile}\})(x) = \begin{cases} 0.90 & x = \text{Alice},\\ 0.45 & x = \text{Bob},\\ 0.75 & x = \text{Carol}, \end{cases}\qquad T(\{\text{Laptop}\})(x) = \begin{cases} 0.30 & x = \text{Alice},\\ 0.80 & x = \text{Bob},\\ 0.50 & x = \text{Carol}. \end{cases}$$

Setting $T(\varnothing)(x) = 0.50$ completes the TreeUncertain Set.

5.2. *Classical ForestUncertain Set*

A Classical ForestUncertain Set us a mapping from subsets of a forest's nodes to functions assigning each element the maximum of tree-specific $r$-vectors in $[0,1]^r$. The definition of the *Classical ForestUncertain Set* is presented below.

**Definition 5.4** (ForestUncertain Set)**.** Let $\{\mathcal{T}_t = (V_t, E_t) \mid t \in T\}$ be a finite family of rooted trees, and set $V = \bigcup_{t\in T} V_t$. Fix $r \geq 1$ and a nonempty universe $U$. For each $t \in T$, let

$$T_t : \mathcal{P}(V_t) \longrightarrow \{\mu : U \to [0,1]^r\}$$

be a TreeUncertain Set. Then the *ForestUncertain Set* is the map

$$F : \mathcal{P}(V) \longrightarrow \{\nu : U \to [0,1]^r\}, \quad F(X)(u) = \big(\max_{t: X\cap V_t \neq \emptyset} T_t(X \cap V_t)(u)_i\big)_{i=1}^r, \quad u \in U.$$

By convention, if $X \cap V_t = \emptyset$ for all $t$, one defines $F(X)(u) = (0,\ldots,0) \in [0,1]^r$.

**Remark 5.5.**

- If $r = 1$, $F(X)\colon U \to [0,1]$ is a *ForestFuzzy Set*.
- If $r = 2$, $F(X)(u) = (\mu(u), \nu(u))$ recovers a *ForestIntuitionistic Fuzzy Set*.
- If $r = 3$, $F(X)(u) = (T, I, F)$ gives a *ForestNeutrosophic Set*.

**Example 5.6** (E-commerce Consumer Preferences as a ForestUncertain Set)**.** Let

$$U = \{\text{Alice}, \text{Bob}\},$$

and consider two disjoint attribute-trees:

$$\mathcal{T}_1 : \text{Electronics} \to \{\text{Mobile}, \text{Laptop}\}, \quad \mathcal{T}_2 : \text{Clothing} \to \{\text{Shirts}, \text{Pants}\}.$$

Fix $r = 1$. Define the TreeUncertain Sets

$$T_1 : \mathcal{P}(\{\text{Mobile}, \text{Laptop}\}) \to \{\mu : U \to [0,1]\}, \quad T_2 : \mathcal{P}(\{\text{Shirts}, \text{Pants}\}) \to \{\mu : U \to [0,1]\},$$

by

$$T_1(\{\text{Mobile}\})(\text{Alice}) = 0.80,\ T_1(\{\text{Mobile}\})(\text{Bob}) = 0.50,$$

$$T_1(\{\text{Laptop}\})(\text{Alice}) = 0.60,\ T_1(\{\text{Laptop}\})(\text{Bob}) = 0.70,$$

$$T_2(\{\text{Shirts}\})(\text{Alice}) = 0.90,\ T_2(\{\text{Shirts}\})(\text{Bob}) = 0.40,$$

$$T_2(\{\text{Pants}\})(\text{Alice}) = 0.30,\ T_2(\{\text{Pants}\})(\text{Bob}) = 0.80.$$

Form the forest $\mathcal{F} = \mathcal{T}_1 \sqcup \mathcal{T}_2$. The ForestUncertain Set

$$F : \mathcal{P}\big(\{\text{Mobile}, \text{Laptop}, \text{Shirts}, \text{Pants}\}\big)$$

$$\longrightarrow \{\nu : U \to [0,1]\}$$

is defined by

$$F(X)(u) = \max\big\{T_1(X \cap \{\text{Mobile}, \text{Laptop}\})(u),\ T_2(X \cap \{\text{Shirts}, \text{Pants}\})(u)\big\}.$$

For example, for $X = \{\text{Mobile}, \text{Pants}\}$:

$$F(X)(\text{Alice}) = \max(0.80, 0.30) = 0.80, \quad F(X)(\text{Bob}) = \max(0.50, 0.80) = 0.80.$$

**Theorem 5.7** (ForestUncertain Set Generalizes TreeUncertain Set)**.** *Let $\mathcal{T} = (V, E)$ be a finite rooted tree and let*

$$T : \mathcal{P}(V) \longrightarrow \{\mu : U \to [0,1]^r\}$$

*be a TreeUncertain Set on $U$. Form the singleton forest $\{\mathcal{T}\}$. Then the corresponding ForestUncertain Set*

$$F : \mathcal{P}(V) \longrightarrow \{\nu : U \to [0,1]^r\}, \quad F(X)(u) = \max\big\{T(X \cap V)(u)_i\big\}_{i=1}^{r} = T(X)(u), \quad u \in U,$$

*satisfies $F(X) = T(X)$ for every $X \subseteq V$. Hence every TreeUncertain Set arises as a special case of a ForestUncertain Set.*

*Proof.* Since our forest consists of exactly one tree $\mathcal{T} = (V, E)$, we have $V_1 = V$ and no other components. By definition of the ForestUncertain Set,

$$F(X)(u) = \max_{t: X \cap V_t \neq \emptyset} T_t(X \cap V_t)(u) = T_1(X \cap V_1)(u) = T(X)(u),$$

for all $X \subseteq V$ and $u \in U$. Here:

(1) There is only one index $t = 1$, so the maximum over $t$ reduces to the single value $T_1(X \cap V)(u)$.

(2) Since $V_1 = V$, we have $X \cap V_1 = X$.

(3) Therefore $F(X)(u) = T_1(X)(u)$, and by identifying $T_1$ with $T$, we get $F(X) = T(X)$.

Thus $F$ and $T$ coincide on all subsets of $V$, proving that the TreeUncertain Set $T$ is exactly the ForestUncertain Set $F$ for this singleton forest. This establishes that the ForestUncertain Set construction indeed generalizes TreeUncertain Sets. □

### 5.3. *Non-Classical TreeUncertain Set*

A Non-Classical TreeUncertain Set assigns each subset of tree nodes an uncertainty mapping from universe elements to $r$-tuples drawn from $[\varphi, \psi]^r$, modeling over- and under-confidence. The definition of the *Non-Classical TreeUncertain Set* is presented below.

**Definition 5.8** (Non-Classical TreeUncertain Set)**.** Let $U$ be a nonempty universe, let $\mathcal{T} = \text{Tree}(A)$ be a finite rooted tree of attributes $A = \{A_1, \ldots, A_m\}$, and fix real bounds $\varphi < 0 < 1 < \psi$ and an integer $r \geq 1$. A *Non-Classical TreeUncertain Set* over $U$ with respect to $\mathcal{T}$ is a mapping

$$T_{\text{NC}} : \mathcal{P}(\mathcal{T}) \longrightarrow \big([\varphi, \psi]^r\big)^U,$$

such that for each $S \subseteq \mathcal{T}$,

$$T_{\text{NC}}(S) = \mu_S : U \longrightarrow [\varphi, \psi]^r,$$

where

$$\mu_S(x) = \big(d_1^{(S)}(x),\, d_2^{(S)}(x),\, \ldots,\, d_r^{(S)}(x)\big) \in [\varphi, \psi]^r, \quad \forall\, x \in U,$$

and each coordinate $d_i^{(S)}(x)$ satisfies $\varphi \leq d_i^{(S)}(x) \leq \psi$. In other words, each element $x \in U$ and each attribute-subset $S \subseteq \mathcal{T}$ is assigned an $r$-tuple of "non-classical" uncertainty-degrees drawn from $[\varphi, \psi]$.

**Remark 5.9.**

- If $r = 1$, $T_{\mathrm{NC}}(S) : U \to [\varphi, \psi]$ is a *Non-Classical TreeFuzzy Set*.
- If $r = 2$, $T_{\mathrm{NC}}(S)(x) = (\mu(x), \nu(x)) \in [\varphi, \psi]^2$ is a *Non-Classical TreeIntuitionistic Fuzzy Set*.
- If $r = 3$, $T_{\mathrm{NC}}(S)(x) = (T(x), I(x), F(x)) \in [\varphi, \psi]^3$ is a *Non-Classical TreeNeutrosophic Set*.

**Example 5.10** (Non-Classical TreeIntuitionistic Fuzzy Preferences)**.** Let $U = \{\text{Alice}, \text{Bob}\}$ and consider the attribute-tree

$$\mathcal{T} : \quad \text{Food} \to \{\text{Fruit}, \text{Vegetable}\}.$$

Fix $\varphi = -0.1$, $\psi = 1.1$, and $r = 2$. Define

$$T_{\mathrm{NC}} : \mathcal{P}(\{\text{Fruit}, \text{Vegetable}\}) \longrightarrow \{\mu : U \to [\varphi, \psi]^2\},$$

by assigning to each subset $S \subseteq \{\text{Fruit}, \text{Vegetable}\}$ two "off-intuitionistic" degrees $(\mu, \nu)$. For example,

$$T_{\mathrm{NC}}(\{\text{Fruit}\})(\text{Alice}) = (1.05, 0.00), \quad T_{\mathrm{NC}}(\{\text{Fruit}\})(\text{Bob}) = (0.90, -0.05),$$

$$T_{\mathrm{NC}}(\{\text{Vegetable}\})(\text{Alice}) = (0.80, 0.10), \quad T_{\mathrm{NC}}(\{\text{Vegetable}\})(\text{Bob}) = (1.10, -0.10),$$

$$T_{\mathrm{NC}}(\{\text{Fruit}, \text{Vegetable}\})(\text{Alice}) = (0.95, 0.05), \quad T_{\mathrm{NC}}(\{\text{Fruit}, \text{Vegetable}\})(\text{Bob}) = (0.85, 0.00).$$

Each pair $(\mu, \nu) \in [-0.1, 1.1]^2$ may exceed 1 or drop below 0, modeling over- and under-confidence in the attribute-combination.

**Theorem 5.11** (Non-Classical Generalizes Classical TreeUncertain Set)**.** *Let $\mathcal{T} = (A, E)$ be a finite rooted tree on nodes $A$ and let $U$ be any nonempty universe. Fix real bounds $\varphi < 0 < 1 < \psi$ and an integer $r \geq 1$. Suppose*

$$T_{\mathrm{NC}} : \mathcal{P}(A) \longrightarrow \left([\varphi, \psi]^r\right)^U$$

*is a Non-Classical TreeUncertain Set. If $\varphi = 0$ and $\psi = 1$, then $T_{\mathrm{NC}}$ is exactly a classical TreeUncertain Set*

$$T : \mathcal{P}(A) \longrightarrow \{\, \mu : U \to [0, 1]^r\}, \quad T(S) = T_{\mathrm{NC}}(S).$$

*Conversely, any classical $T$ extends to a Non-Classical $T_{\mathrm{NC}}$ by choosing $\varphi = 0$ and $\psi = 1$. Hence the Non-Classical notion strictly generalizes the classical one.*

*Proof.* By definition, a Non-Classical TreeUncertain Set assigns to each $S \subseteq A$ a function $\mu_S : U \to [\varphi, \psi]^r$. When $\varphi = 0$ and $\psi = 1$, the codomain $[\varphi, \psi]^r$ coincides with $[0, 1]^r$. Thus

$$T_{\mathrm{NC}}(S) : \ U \longrightarrow [0, 1]^r$$

satisfies exactly the requirements of a classical TreeUncertain Set. No further modification is needed: one simply sets $T(S) = T_{\mathrm{NC}}(S)$.

Conversely, given any classical

$$T : \mathcal{P}(A) \longrightarrow \{\mu : U \to [0, 1]^r\},$$

one obtains a Non-Classical TreeUncertain Set by declaring $\varphi = 0$, $\psi = 1$ and defining $T_{\mathrm{NC}}(S) = T(S)$. Since $[0,1]^r \subseteq [\varphi,\psi]^r$, all axioms of the non-classical definition hold automatically.

Therefore, the family of Non-Classical TreeUncertain Sets (for arbitrary $\varphi < 0 < 1 < \psi$) contains the classical TreeUncertain Sets as the special case $\varphi = 0$, $\psi = 1$, proving that the Non-Classical concept is a true generalization. □

### 5.4. *Non-Classical ForestUncertain Set*

A Non-Classical ForestUncertain Set maps each attribute-subset in a forest to a function assigning universe elements coordinate-wise maximum $r$-tuples from $[\varphi,\psi]^r$, modeling over- and under-confidence. The definition of the *Non-Classical ForestUncertain Set* is presented below.

**Definition 5.12** (Non-Classical ForestUncertain Set)**.** Let $\{\mathcal{T}_t = \mathrm{Tree}(A^{(t)}) \mid t \in T\}$ be a finite collection of attribute-trees and form the forest $\mathcal{F} = \bigsqcup_{t\in T} \mathcal{T}_t$. Fix real bounds $\varphi < 0 < 1 < \psi$ and integer $r \geq 1$. Suppose each $\mathcal{T}_t$ carries a Non-Classical TreeUncertain Set

$$T_{t,\mathrm{NC}} : \mathcal{P}(\mathcal{T}_t) \longrightarrow \left([\varphi,\psi]^r\right)^U.$$

Then a *Non-Classical ForestUncertain Set* over $U$ with respect to $\mathcal{F}$ is the mapping

$$F_{\mathrm{NC}} : \mathcal{P}(\mathcal{F}) \longrightarrow \left([\varphi,\psi]^r\right)^U$$

given, for each $X \subseteq \mathcal{F}$ and each $x \in U$, by

$$F_{\mathrm{NC}}(X)(x) = \max_{t\in T:\, X\cap\mathcal{T}_t\neq\emptyset} \left\{T_{t,\mathrm{NC}}\big(X\cap\mathcal{T}_t\big)(x)\right\},$$

where the “max” is taken coordinate-wise on the $r$-tuples in $[\varphi,\psi]^r$. If $X \cap \mathcal{T}_t = \emptyset$ for all $t \in T$, one may define $F_{\mathrm{NC}}(\emptyset)(x) = (\varphi,\varphi,\ldots,\varphi)$ or another fixed baseline in $[\varphi,\psi]^r$.

**Remark 5.13.**
- If $r = 1$, $F_{\mathrm{NC}}(X) : U \to [\varphi,\psi]$ is a *Non-Classical ForestFuzzy Set*.
- If $r = 2$, $F_{\mathrm{NC}}(X)(u) = (\mu(u), \nu(u)) \in [\varphi,\psi]^2$ is a *Non-Classical ForestIntuitionistic Fuzzy Set*.
- If $r = 3$, $F_{\mathrm{NC}}(X)(u) = (T(u), I(u), F(u)) \in [\varphi,\psi]^3$ is a *Non-Classical ForestNeutrosophic Set*.

**Example 5.14** (Non-Classical ForestUncertain Set)**.** Let

$$U = \{\mathrm{Alice}, \mathrm{Bob}\},$$

and consider two disjoint attribute-trees:

$$\mathcal{T}_1 : \text{ Electronics} \to \{\text{Mobile}, \text{Laptop}\}, \quad \mathcal{T}_2 : \text{ Clothing} \to \{\text{Shirts}, \text{Pants}\}.$$

Fix $\varphi = -0.1$, $\psi = 1.1$, and $r = 2$. For each $t = 1, 2$, define a Non-Classical TreeUncertain Set

$$T_{t,\mathrm{NC}} : \mathcal{P}(V_t) \to \{\mu : U \to [\varphi,\psi]^2\},$$

where $V_1 = \{\text{Mobile}, \text{Laptop}\}$ and $V_2 = \{\text{Shirts}, \text{Pants}\}$. For instance:

$$T_{1,\mathrm{NC}}(\{\text{Mobile}\})(\text{Alice}) = (1.05, -0.05), \quad T_{1,\mathrm{NC}}(\{\text{Mobile}\})(\text{Bob}) = (0.80, -0.10),$$

$$T_{1,\mathrm{NC}}(\{\text{Laptop}\})(\text{Alice}) = (0.95, 0.00), \quad T_{1,\mathrm{NC}}(\{\text{Laptop}\})(\text{Bob}) = (1.10, -0.10),$$

$$T_{2,\mathrm{NC}}(\{\text{Shirts}\})(\text{Alice}) = (0.90, -0.05), \quad T_{2,\mathrm{NC}}(\{\text{Shirts}\})(\text{Bob}) = (1.00, -0.10),$$

$$T_{2,\mathrm{NC}}(\{\text{Pants}\})(\text{Alice}) = (1.10, -0.10), \quad T_{2,\mathrm{NC}}(\{\text{Pants}\})(\text{Bob}) = (0.85, 0.00).$$

Let $\mathcal{F} = \mathcal{T}_1 \sqcup \mathcal{T}_2$. The Non-Classical ForestUncertain Set

$$F_{\mathrm{NC}} : \mathcal{P}(V_1 \cup V_2) \to \{\nu : U \to [\varphi, \psi]^2\}$$

is defined by

$$F_{\mathrm{NC}}(X)(u) = \Big(\max\{\, T_{1,\mathrm{NC}}(X \cap V_1)(u)_i,\ T_{2,\mathrm{NC}}(X \cap V_2)(u)_i\}\Big)_{i=1,2}.$$

For example, for $X = \{\text{Mobile}, \text{Pants}\}$,

$$F_{\mathrm{NC}}(X)(\text{Alice}) = (\max\{1.05, 1.10\},\ \max\{-0.05, -0.10\}) = (1.10, -0.05),$$

$$F_{\mathrm{NC}}(X)(\text{Bob}) = (\max\{0.80, 0.85\},\ \max\{-0.10, 0.00\}) = (0.85, 0.00).$$

**Theorem 5.15** (Generalization by Non-Classical ForestUncertain Set)**.** *Let $\{\mathcal{T}_t = (V_t, E_t) \mid t \in T\}$ be a finite family of rooted trees on disjoint node-sets $V_t$, and form the forest $\mathcal{F} = \bigsqcup_{t\in T} \mathcal{T}_t$. Fix $\varphi < 0 < 1 < \psi$ and $r \geq 1$, and suppose each $\mathcal{T}_t$ carries a Non-Classical TreeUncertain Set*

$$T_{t,\mathrm{NC}} : \mathcal{P}(V_t) \longrightarrow \{\mu : U \to [\varphi, \psi]^r\}.$$

*Let*

$$F_{\mathrm{NC}} : \mathcal{P}\big(\cup_t V_t\big) \longrightarrow \{\nu : U \to [\varphi, \psi]^r\}$$

*be the induced Non-Classical ForestUncertain Set (Definition 5.12). Then:*

*(1) If $T = \{t_0\}$ is a singleton, so that $\mathcal{F} = \mathcal{T}_{t_0}$, then*

$$F_{\mathrm{NC}}(X) = T_{t_0,\mathrm{NC}}(X) \quad \forall\, X \subseteq V_{t_0}.$$

*Hence $F_{\mathrm{NC}}$ reduces to the Non-Classical TreeUncertain Set on $\mathcal{T}_{t_0}$.*

*(2) If $\varphi = 0$ and $\psi = 1$, then $[\varphi, \psi]^r = [0,1]^r$ and each $T_{t,\mathrm{NC}}$ becomes an ordinary TreeUncertain Set $T_t : \mathcal{P}(V_t) \to \{\mu : U \to [0,1]^r\}$. In that case $F_{\mathrm{NC}}$ coincides with the classical ForestUncertain Set $F$ built from the $T_t$.*

*Thus the Non-Classical ForestUncertain Set simultaneously generalizes both the Non-Classical TreeUncertain Set and the classical ForestUncertain Set.*

*Proof.* **(1) Singleton forest.** If $T = \{t_0\}$, then $\mathcal{F} = V_{t_0}$ and for any $X \subseteq V_{t_0}$,

$$F_{\mathrm{NC}}(X)(u) = \max_{t : X \cap V_t \neq \emptyset} T_{t,\mathrm{NC}}(X \cap V_t)(u) = T_{t_0,\mathrm{NC}}(X)(u), \quad u \in U,$$

since the only nonempty intersection is $X \cap V_{t_0} = X$. This shows $F_{\mathrm{NC}}$ agrees with $T_{t_0,\mathrm{NC}}$.

**(2) Restriction to** $[0,1]$**.** Setting $\varphi = 0$ and $\psi = 1$ makes each

$$T_{t,\mathrm{NC}} : \mathcal{P}(V_t) \to \{\mu : U \to [0,1]^r\},$$

i.e. a classical TreeUncertain Set $T_t$. Then by definition of the Non-Classical ForestUncertain Set,

$$F_{\mathrm{NC}}(X)(u) = \max_{t:X\cap V_t\neq\emptyset} T_t(X\cap V_t)(u),$$

which is exactly the formula for the classical ForestUncertain Set $F$. Hence $F_{\mathrm{NC}} = F$ in this case.

Combining (1) and (2), we conclude that $F_{\mathrm{NC}}$ indeed generalizes both the Non-Classical TreeUncertain Set (when $\mathcal{F}$ is a single tree) and the classical ForestUncertain Set (when $\varphi = 0, \psi = 1$). □

## 6. Hyper-$U$-structure, SuperHyper-$U$-structure, Tree-$U$-structure, and Forest-$U$-structure

In this section, we examine the Hyper-$U$-structure, SuperHyper-$U$-structure, Tree-$U$-structure, and Forest-$U$-structure. Note that the term Structure here refers broadly to various mathematical concepts such as graphs, topologies, and automata, as well as structures arising in everyday life.

### 6.1. *Hyper-$U$-structure*

A Hyper-$U$-Structure enriches a classical structure by assigning each element a distinct set of possible membership degrees in $\mathrm{Dom}(U)$.

**Definition 6.1** (Hyper-$U$-Structure)**.** Let $H$ be a nonempty carrier set, let $U \in \mathbb{U}$ be a fixed uncertainty model with degree-domain $\mathrm{Dom}(U) \subseteq [0,1]^r$, and fix an integer $n \geq 1$. A *Hyper-$U$-Structure* is a triple

$$\mathcal{H}^U = \Big(H,\ \mu_H : H \longrightarrow \mathcal{P}^n\big(\mathrm{Dom}(U)\big),\ \{\#^{(m)}\}\Big),$$

where

$$\mu_H(x) \subseteq \mathrm{Dom}(U) \quad \text{for each } x \in H,$$

and $\mathcal{P}^n(\mathrm{Dom}(U))$ denotes the $n$-th powerset of $\mathrm{Dom}(U)$. Equivalently, each element $x \in H$ is assigned a set of possible $U$-membership degrees. The classical operations $\{\#^{(m)}\}$ on $H$ remain as in the underlying classical structure.

**Example 6.2** (Hyper-$U$-Structure: Employee Performance Uncertainty)**.** Let

$$H = \{\mathrm{Alice}, \mathrm{Bob}, \mathrm{Carol}\}$$

be a set of employees, and let $U = \mathrm{Fuzzy}$ with $\mathrm{Dom}(U) = [0,1]$. Choose $n = 1$, so that $\mu_H : H \to \mathcal{P}\big([0,1]\big)$. Define

$$\mu_H(\mathrm{Alice}) = \{0.85,\ 0.90\}, \quad \mu_H(\mathrm{Bob}) = \{0.75,\ 0.80\}, \quad \mu_H(\mathrm{Carol}) = \{0.65,\ 0.70\}.$$

Here each $\mu_H(x) \subseteq [0,1]$ is the set of possible "performance" degrees obtained from two different appraisal methods (self-rating and peer-rating). Then

$$\mathcal{H}^U = \big(H,\ \mu_H,\ \text{(no additional operations)}\big)$$

is a Hyper-Fuzzy Structure: each employee carries a set of fuzzy membership degrees rather than a single value.

### 6.2. *Super-U-structure*

A Super-$U$-Structure enriches a classical structure by assigning each $m$-subset of the carrier a single membership degree in $\mathrm{Dom}(U)$.

**Definition 6.3** (Super-$U$-Structure)**.** Let $H$ be a nonempty carrier set, let $U \in \mathbb{U}$ be a fixed uncertainty model with degree-domain $\mathrm{Dom}(U) \subseteq [0,1]^r$, and fix an integer $m \geq 1$. A *Super-$U$-Structure* is a triple

$$\mathcal{S}^U \;=\; \Big(H,\; \mu_S\colon \mathcal{P}^m(H) \;\longrightarrow\; \mathrm{Dom}(U),\; \{\#^{(m)}\}\Big),$$

where

$$\mu_S(X) \;\in\; \mathrm{Dom}(U) \quad \text{for each } X \subseteq H \text{ with } X \in \mathcal{P}^m(H),$$

and $\mathcal{P}^m(H)$ denotes the $m$-th powerset of $H$. In other words, each subset $X \subseteq H$ at "level $m$" is assigned a single $U$-membership degree. The operations $\{\#^{(m)}\}$ remain as in the classical carrier.

**Example 6.4** (Super-$U$-Structure: Team Collaboration Scores)**.** Let

$$H = \{\text{Alice}, \text{Bob}, \text{Carol}\}$$

be the same employee set, and again $U$ = Fuzzy with $\mathrm{Dom}(U) = [0,1]$. Take $m = 1$, so $\mu_S\colon \mathcal{P}(H) \to [0,1]$. Define, for each team $X \subseteq H$,

$$\mu_S(\{\,\text{Alice}, \text{Bob}\}) = 0.95, \quad \mu_S(\{\,\text{Alice}, \text{Carol}\}) = 0.88, \quad \mu_S(\{\,\text{Bob}, \text{Carol}\}) = 0.80,$$

and set $\mu_S(\{x\}) = 0.70$ for each singleton and $\mu_S(\varnothing) = 0$. Then

$$\mathcal{S}^U = \big(H,\; \mu_S,\; \text{(no additional operations)}\big)$$

is a Super-Fuzzy Structure: each subset of employees (each "team") is assigned a single fuzzy collaboration score.

### 6.3. *SuperHyper-U-structure*

A SuperHyper-$U$-Structure assigns to each $m$-level subset of $H$ an $n$-th level subset of $\mathrm{Dom}(U)$, yielding multiple possible $U$-membership degree tuples per subset. We define the SuperHyper-$U$-structure.

**Definition 6.5** (SuperHyper-$U$-Structure)**.** Let $H$ be a nonempty carrier set, let $U \in \mathbb{U}$ be a fixed uncertainty model with degree-domain $\mathrm{Dom}(U) \subseteq [0,1]^r$, and fix integers $m, n \geq 1$. A *SuperHyper-$U$-Structure* is a triple

$$\mathcal{SH}^U \;=\; \Big(H,\; \mu_{SH}\colon \mathcal{P}^m(H) \;\longrightarrow\; \mathcal{P}^n\big(\mathrm{Dom}(U)\big),\; \{\#^{(m)}\}\Big),$$

where

$$\mu_{SH}(X) \;\subseteq\; \mathrm{Dom}(U) \quad \text{for each } X \in \mathcal{P}^m(H).$$

That is, each $m$-th-level subset $X \subseteq H$ is assigned a collection (an $n$-th-level subset) of possible $U$-membership degrees. The classical operations $\{\#^{(m)}\}$ on $H$ remain unchanged.

**Example 6.6** (SuperHyper-$U$-Structure: Employee Team Synergy)**.** Let

$$H = \{\text{Alice}, \text{Bob}, \text{Carol}\}, \quad U = \text{Fuzzy}, \quad \text{Dom}(U) = [0, 1].$$

Choose $m = 2$ and $n = 2$, so that

$$\mu_{SH} : \mathcal{P}^2(H) \longrightarrow \mathcal{P}^2\big([0, 1]\big).$$

We interpret $\mu_{SH}(X)$ as two candidate fuzzy–synergy score sets for the team $X$. For instance:

$$\mu_{SH}(\{\text{Alice}, \text{Bob}\}) = \big\{\{0.75, 0.80\},\ \{0.85, 0.90\}\big\},$$

$$\mu_{SH}(\{\text{Alice}, \text{Carol}\}) = \big\{\{0.70, 0.78\},\ \{0.88, 0.92\}\big\},$$

$$\mu_{SH}(\{\text{Bob}, \text{Carol}\}) = \big\{\{0.65, 0.70\},\ \{0.82, 0.88\}\big\}.$$

Here each inner set $\{d_1, d_2\} \subseteq [0, 1]$ gives possible synergy scores under two evaluation methods (peer review vs. management review). The underlying operations on $H$ remain those of the classical structure.

**Theorem 6.7** (SuperHyper-$U$-Structure generalizes Hyper-$U$ and Super-$U$)**.** *Let*

$$\mathcal{SH}^U = \big(H,\ \mu_{SH} : \mathcal{P}^m(H) \to \mathcal{P}^n(\text{Dom}(U)),\ \{\#^{(m)}\}\big)$$

*be a SuperHyper-$U$-Structure (Definition 6.5). Then:*

*(1)* (Hyper-$U$ specialization) *Defining*

$$\mu_H(x) \;=\; \mu_{SH}\big(\{x\}\big) \;\in\; \mathcal{P}^n(\text{Dom}(U)), \quad \forall\, x \in H,$$

*yields $\mathcal{H}^U = (H, \mu_H, \{\#^{(m)}\})$, a Hyper-$U$-Structure as in Definition 6.1.*

*(2)* (Super-$U$ specialization) *If $n = 1$ and each $\mu_{SH}(X)$ is a singleton $\{d_X\} \subseteq \text{Dom}(U)$, then setting*

$$\mu_S(X) = d_X, \quad \forall\, X \in \mathcal{P}^m(H),$$

*defines $\mathcal{S}^U = (H, \mu_S, \{\#^{(m)}\})$, a Super-$U$-Structure as in Definition 6.3.*

*Conversely, any Hyper-U or Super-$U$-Structure can be obtained by these specializations.*

*Proof.* **(1) Hyper-$U$ specialization.** Since $\{x\} \in \mathcal{P}^m(H)$ for $m \geq 1$, the assignment

$$\mu_H : H \longrightarrow \mathcal{P}^n(\text{Dom}(U)), \quad x \mapsto \mu_{SH}(\{x\})$$

is well-defined. By construction $\mu_H(x) \subseteq \text{Dom}(U)$ and lives in the $n$-th powerset, hence $(H, \mu_H)$ satisfies Definition 6.1.

**(2) Super-$U$ specialization.** If $n = 1$, then $\mathcal{P}^1(\text{Dom}(U)) = \{\{d\} \mid d \in \text{Dom}(U)\}$. Requiring each $\mu_{SH}(X) = \{d_X\}$ forces a unique scalar $d_X \in \text{Dom}(U)$. Defining $\mu_S(X) = d_X$ yields $\mu_S : \mathcal{P}^m(H) \to \text{Dom}(U)$ with $\mu_S(X) \in \text{Dom}(U)$ for all $X$, matching Definition 6.3.

**Converses.** Given any Hyper-$U$-Structure $(H, \mu_H)$, choose $m = 1$, $n \geq 1$, and define $\mu_{SH}(\{x\}) = \mu_H(x)$, extending arbitrarily on larger subsets. Given any Super-$U$-Structure $(H, \mu_S)$, choose $n = 1$ and set $\mu_{SH}(X) = \{\mu_S(X)\}$. Both constructions recover $\mathcal{SH}^U$, completing the proof. □

6.4. *Tree-U-structure*

A Tree-$U$-Structure enriches a universe $H$ by assigning each subset of a rooted tree's nodes a $U$-membership mapping $H \to \mathrm{Dom}(U)$, preserving original operations via U-model semantics. We define the Tree-$U$-structure.

**Definition 6.8** (Tree-$U$-Structure)**.** Let $H$ be a nonempty universe of elements, and let $\mathcal{T} = \mathrm{Tree}(A)$ be a finite rooted tree whose node set is $A = \{A_1, A_2, \ldots, A_k\}$. Fix an uncertainty model $U \in \mathbb{U}$ with degree-domain $\mathrm{Dom}(U) \subseteq [0,1]^r$. A *Tree-U-Structure* is a triple

$$\mathcal{T}^U = \Big(H,\ \mu_T : \mathcal{P}(\mathcal{T}) \longrightarrow \big(\mathrm{Dom}(U)\big)^H,\ \{\#^{(m)}\}\Big),$$

where:

(1) The domain $\mathcal{P}(\mathcal{T})$ is the power set of the set of tree-nodes $A$. Thus each $S \subseteq \mathcal{T}$ is a subset of attributes.

(2) For each $S \subseteq \mathcal{T}$,

$$\mu_T(S) = \gamma_S : H \longrightarrow \mathrm{Dom}(U),$$

is a function that assigns to each element $x \in H$ a single $U$-membership degree $\gamma_S(x) \in \mathrm{Dom}(U)$.

(3) The classical operations $\{\#^{(m)}\}$ on $H$ remain unchanged.

Hence a Tree-$U$-Structure is equivalent to a collection of functions $\{\gamma_S : H \to \mathrm{Dom}(U) \mid S \subseteq \mathcal{T}\}$, one for each attribute-subset $S$.

**Example 6.9** (Tree–Neutrosophic Structure for Smart-Home Sensor Trust)**.** Let

$$H = \{S_A, S_B, S_C\}$$

be three sensors in a smart-home network, and let $\mathcal{T}$ be the attribute-tree with nodes

$$A = \{\text{SensorType, Temperature, Motion, Indoor, Outdoor, Infrared, Ultrasonic}\},$$

where "SensorType" is the root, its children are "Temperature" and "Motion," "Temperature" branches to "Indoor" and "Outdoor," and "Motion" to "Infrared" and "Ultrasonic." Choose the uncertainty model $U$ = Neutrosophic with $\mathrm{Dom}(U) = \{(T, I, F) \in [0,1]^3 \mid T + I + F \leq 3\}$ (so $r = 3$), and no further algebraic operations on $H$.

Define the map $\mu_T : \mathcal{P}(A) \to \{\gamma : H \to \mathrm{Dom}(U)\}$ by specifying for key subsets $S \subseteq A$:

$$\gamma_{\{\text{Indoor}\}}(S_A) = (0.95, 0.03, 0.02),$$

$$\gamma_{\{\text{Indoor}\}}(S_B) = (0.90, 0.05, 0.05),$$

$$\gamma_{\{\text{Indoor}\}}(S_C) = (0.85, 0.10, 0.05),$$

$$\gamma_{\{\text{Ultrasonic}\}}(S_A) = (0.80, 0.15, 0.05),$$

$$\gamma_{\{\text{Ultrasonic}\}}(S_B) = (0.75, 0.10, 0.15),$$

$$\gamma_{\{\text{Ultrasonic}\}}(S_C) = (0.70, 0.20, 0.10),$$

$$\gamma_{\{\text{Indoor,Infrared}\}}(S_A) = (0.90, 0.07, 0.03),$$

$$\gamma_{\{\text{Indoor,Infrared}\}}(S_B) = (0.85, 0.10, 0.05),$$

$$\gamma_{\{\text{Indoor},\text{Infrared}\}}(S_C) = (0.80, 0.15, 0.05),$$

and set $\gamma_\emptyset(x) = (0, 0, 0)$ for all $x \in H$. Then

$$\mathcal{T}^U = \big(H,\ \mu_T,\ \{\text{no additional operations}\}\big)$$

is a Tree–$U$-Structure: for each attribute-subset $S \subseteq A$, $\mu_T(S) = \gamma_S$ assigns every sensor $x \in H$ a neutrosophic triple $(T, I, F) \in \mathrm{Dom}(U)$, quantifying its trustworthiness under the features in $S$.

### 6.5. *Forest-$U$-structure*

A Forest-$U$-Structure aggregates multiple Tree-$U$-Structures over disjoint attribute-trees by taking coordinate-wise maxima of their $U$-membership maps in $\mathrm{Dom}(U)$, automatically preserving classical operations for each subset. We define the Forest-$U$-structure.

**Definition 6.10** (Forest-$U$-Structure)**.** Let $\{\mathcal{T}_t = \mathrm{Tree}(A^{(t)}) \mid t \in T\}$ be a finite family of attribute-trees, and let $\mathcal{F} = \bigsqcup_{t \in T} \mathcal{T}_t$ denote their disjoint union (a forest). Fix an uncertainty model $U \in \mathbb{U}$ with degree-domain $\mathrm{Dom}(U) \subseteq [0, 1]^r$. Suppose each tree $\mathcal{T}_t$ carries a Tree-$U$-Structure

$$\mu_{T_t} : \mathcal{P}(\mathcal{T}_t) \longrightarrow (\mathrm{Dom}(U))^H.$$

Then a *Forest-$U$-Structure* is given by

$$\mathcal{F}^U = \Big(H,\ \mu_F : \mathcal{P}(\mathcal{F}) \longrightarrow (\mathrm{Dom}(U))^H,\ \{\#^{(m)}\}\Big),$$

where, for each $X \subseteq \mathcal{F}$ and each $x \in H$:

$$\mu_F(X)(x) = \bigvee_{t \in T:\, X \cap \mathcal{T}_t \neq \emptyset} \big(\mu_{T_t}(X \cap \mathcal{T}_t)(x)\big).$$

Here, $\bigvee$ denotes a coordinate-wise supremum (maximum) in $\mathrm{Dom}(U) \subseteq [0, 1]^r$. If $X \cap \mathcal{T}_t = \emptyset$ for all $t$, one may define $\mu_F(\emptyset)(x)$ to be a designated base degree in $\mathrm{Dom}(U)$, for example the zero-vector $(0, \ldots, 0)$.

**Example 6.11** (Forest–Neutrosophic Structure: Patient Diagnostic Confidence)**.** Let

$$H = \{\text{Alice, Bob}\}$$

be two patients, and fix the uncertainty model $U$ = Neutrosophic with $\mathrm{Dom}(U) = \{(T, I, F) \in [0, 1]^3 \mid T+I+F \leq 3\}$ (so $r = 3$). Consider two disjoint attribute-trees:

$$\mathcal{T}_1 : \quad \text{LabTests} \to \{\text{Blood, Urine}\}, \qquad \mathcal{T}_2 : \quad \text{Imaging} \to \{\text{Xray, MRI}\}.$$

Define Tree–Neutrosophic Structures

$$\mu_{T_1} : \mathcal{P}(\{\text{Blood}, \text{Urine}\}) \longrightarrow \{\gamma : H \to \mathrm{Dom}(U)\}, \quad \mu_{T_2} : \mathcal{P}(\{\text{Xray}, \text{MRI}\}) \longrightarrow \{\delta : H \to \mathrm{Dom}(U)\},$$

by, for each subset $S$ and each patient $x$:

$$\gamma_{\{\text{Blood}\}}(x) = \begin{cases} (0.90, 0.05, 0.05), & x = \text{Alice}, \\ (0.80, 0.10, 0.10), & x = \text{Bob}, \end{cases} \quad \gamma_{\{\text{Urine}\}}(x) = \begin{cases} (0.85, 0.10, 0.05), & x = \text{Alice}, \\ (0.75, 0.15, 0.10), & x = \text{Bob}, \end{cases}$$

$$\delta_{\{\text{Xray}\}}(x) = \begin{cases} (0.70, 0.20, 0.10), & x = \text{Alice}, \\ (0.65, 0.25, 0.10), & x = \text{Bob}, \end{cases} \qquad \delta_{\{\text{MRI}\}}(x) = \begin{cases} (0.95, 0.03, 0.02), & x = \text{Alice}, \\ (0.85, 0.10, 0.05), & x = \text{Bob}. \end{cases}$$

Set $\gamma_\emptyset(x) = \delta_\emptyset(x) = (0, 0, 0)$. Form the forest $\mathcal{F} = \mathcal{T}_1 \sqcup \mathcal{T}_2$. Then the Forest–Neutrosophic Structure

$$\mu_F : \mathcal{P}(\{\text{Blood}, \text{Urine}, \text{Xray}, \text{MRI}\}) \longrightarrow \{\eta : H \to \text{Dom}(U)\}$$

is defined by

$$\mu_F(X)(x) = \big(T_F(x),\ I_F(x),\ F_F(x)\big),$$

where each coordinate is

$$T_F(x) = \max\big\{\gamma_{S_1}(x)_1,\ \delta_{S_2}(x)_1\big\}, \quad I_F(x) = \max\big\{\gamma_{S_1}(x)_2,\ \delta_{S_2}(x)_2\big\}, \quad F_F(x) = \max\big\{\gamma_{S_1}(x)_3,\ \delta_{S_2}(x)_3\big\},$$

with $S_1 = X \cap \{\text{Blood}, \text{Urine}\}$ and $S_2 = X \cap \{\text{Xray}, \text{MRI}\}$. For example, if $X = \{\text{Blood}, \text{MRI}\}$, then

$$\mu_F(X)(\text{Alice}) = (\max\{0.90, 0.95\},\ \max\{0.05, 0.03\},\ \max\{0.05, 0.02\}) = (0.95, 0.05, 0.05),$$

$$\mu_F(X)(\text{Bob}) = (\max\{0.80, 0.85\},\ \max\{0.10, 0.10\},\ \max\{0.10, 0.05\}) = (0.85, 0.10, 0.10).$$

Thus $\mathcal{F}^U = (H, \mu_F)$ is a detailed Forest–$U$-Structure quantifying each patient's combined diagnostic confidence across lab and imaging attributes.

**Theorem 6.12** (Forest-$U$-Structure generalizes Tree-$U$-Structure)**.** *Let $\mathcal{T}^U = (H, \mu_T, \{\#^{(m)}\})$ be a Tree-U-Structure indexed by a single rooted tree $\mathcal{T}$. Regard $\mathcal{F} = \{\mathcal{T}\}$ as a singleton forest. Then the induced Forest-U-Structure $\mathcal{F}^U = (H, \mu_F, \{\#^{(m)}\})$ satisfies*

$$\mu_F(X) = \mu_T(X) \quad \textit{for all } X \subseteq \mathcal{T}.$$

*Hence every Tree-U-Structure arises as a special case of a Forest-U-Structure.*

*Proof.* By Definition 6.10, for any $X \subseteq \mathcal{T}$ and $x \in H$,

$$\mu_F(X)(x) = \bigvee_{t:\, X\cap\mathcal{T}_t \neq \emptyset} \mu_{T_t}\big(X \cap \mathcal{T}_t\big)(x).$$

Since $\mathcal{F} = \{\mathcal{T}\}$ has only one tree component $\mathcal{T}_1 = \mathcal{T}$, the only nonempty intersection is $X \cap \mathcal{T}_1 = X$. Thus

$$\mu_F(X)(x) = \mu_{T_1}(X)(x) = \mu_T(X)(x),$$

for every $x \in H$. It follows that $\mu_F(X) = \mu_T(X)$ as mappings $H \to \text{Dom}(U)$. Therefore $\mathcal{F}^U$ coincides with $\mathcal{T}^U$, proving the claim. □

### 6.6. *Generalization Theorem*

We now prove that the above constructions indeed generalize the classical $U$-Structure.

**Theorem 6.13** (Generalization of $U$-Structure)**.** *Let $H$ be a carrier set, $U \in \mathbb{U}$ a fixed uncertainty model with* $\text{Dom}(U) \subseteq [0, 1]^r$*. Then each of the following is a valid generalization of a classical U-Structure (as in Definition 3.2):*

*(1) A Hyper-U-Structure (Definition 6.1).*

*(2) A Super-$U$-Structure (Definition 6.3).*

*(3) A SuperHyper-$U$-Structure (Definition 6.5).*

*(4) A Tree-$U$-Structure (Definition 6.8).*

*(5) A Forest-$U$-Structure (Definition 6.10).*

*In each case, if one chooses parameters so that "hyper" or "super" structure collapses to the first powerset (i.e. $n = 1$, $m = 1$), or "tree" has a single node, then the given structure reduces to a classical $U$-Structure.*

*Proof.* We verify each item by showing that one recovers Definition 3.2 under suitable specialization of parameters:

**(1) Hyper-$U$-Structure** A Hyper-$U$-Structure is $\big(H,\ \mu_H : H \to \mathcal{P}^n(\mathrm{Dom}(U))\big)$. If $n = 1$, then $\mathcal{P}^1(\mathrm{Dom}(U)) = \mathcal{P}(\mathrm{Dom}(U))$. Requiring $\mu_H(x) \subseteq \mathrm{Dom}(U)$ to actually be a *singleton set* for each $x \in H$ (i.e. $\mu_H(x) = \{d\}$ with $d \in \mathrm{Dom}(U)$) yields precisely a mapping $H \to \mathrm{Dom}(U)$. Thus one recovers a classical $U$-Structure.

**(2) Super-$U$-Structure** A Super-$U$-Structure is $\big(H,\ \mu_S : \mathcal{P}^m(H) \to \mathrm{Dom}(U)\big)$. If $m = 1$, then $\mathcal{P}^1(H) = \mathcal{P}(H)$. Restricting to singleton subsets $\{x\} \subseteq H$ and setting $\mu_S(\{x\})$ to be the $U$-degree of the element $x$ yields a map $\mu_S : H \to \mathrm{Dom}(U)$. The values on larger subsets $\{x, y\}, \{x, y, z\}, \ldots$ become irrelevant. Hence one obtains a classical $U$-Structure on $H$.

**(3) SuperHyper-$U$-Structure** A SuperHyper-$U$-Structure is $\big(H,\ \mu_{SH} : \mathcal{P}^m(H) \to \mathcal{P}^n(\mathrm{Dom}(U))\big)$. If $m = 1$ and $n = 1$, then $\mathcal{P}^1(H) = \mathcal{P}(H)$ and $\mathcal{P}^1(\mathrm{Dom}(U)) = \mathcal{P}(\mathrm{Dom}(U))$. Requiring $\mu_{SH}(\{x\})$ to be singleton sets $\{d\} \subseteq \mathrm{Dom}(U)$ recovers a classical assignment $\mu : H \to \mathrm{Dom}(U)$, i.e. a classical $U$-Structure.

**(4) Tree-$U$-Structure** A Tree-$U$-Structure is $\big(H,\ \mu_T : \mathcal{P}(\mathcal{T}) \to (\mathrm{Dom}(U))^H\big)$. If the tree $\mathcal{T}$ has exactly one node $A_1$, then $\mathcal{P}(\mathcal{T})$ has exactly two subsets: $\emptyset$ and $\{A_1\}$. Define $\gamma_\emptyset(x)$ to be an inconsequential base degree (e.g. $(0, \ldots, 0)$), and let $\gamma_{\{A_1\}}(x)$ be the desired $U$-membership degree $\mu(x)$. Then ignoring the $\emptyset$-case, the function $\gamma_{\{A_1\}} : H \to \mathrm{Dom}(U)$ recovers a classical $U$-Structure.

**(5) Forest-$U$-Structure** A Forest-$U$-Structure is $\big(H,\ \mu_F : \mathcal{P}(\mathcal{F}) \to (\mathrm{Dom}(U))^H\big)$. If the forest $\mathcal{F}$ consists of a single tree $\mathcal{T}$ with exactly one node, then $\mathcal{P}(\mathcal{F})$ has two elements $\emptyset$ and $\{\text{only node}\}$. As above, assign $\mu_F(\{\text{only node}\})(x)$ to be $\mu(x) \in \mathrm{Dom}(U)$. Then restricting $\mu_F$ to that singleton-subset recovers a classical $U$-Structure.

In each case, we see that by choosing minimal parameters ($n = 1$, $m = 1$, or a single tree-node), the general structure reduces to a classical $U$-Structure on $H$. This completes the proof. □

## 7. Hyper-$U$-Off, SuperHyper-$U$-Off, Tree-$U$-Off, and Forest-$U$-Off Structures

In this section, we introduce the Hyper-$U$-Off, SuperHyper-$U$-Off, Tree-$U$-Off, and Forest-$U$-Off structures. As noted above, the term "Structure" is used in its broadest sense, covering both mathematical and applied systems—ranging from graph theory, neural networks, and decision-making to logic, topology, group theory, probability, automata, programming, vector spaces, matroid theory, image processing, biology, social sciences, chemistry, and physics.

**Definition 7.1** (Nonclassical Degree-Range)**.** Let $U$ be a fixed uncertainty model with classical degree-domain $\mathrm{Dom}(U) \subseteq [0,1]^r$. For real bounds $\varphi < 0 < 1 < \psi$, define the *nonclassical degree-range*

$$\mathrm{Dom}_{\varphi,\psi}(U) \;=\; \{\, d \in \mathbb{R}^r \;\big|\; d_i \in [\,\varphi, \psi\,] \text{ for } 1 \le i \le r \}.$$

In particular, $\mathrm{Dom}_{\varphi,\psi}(U)$ may strictly contain $\mathrm{Dom}(U)$, allowing membership components to lie outside $[0,1]$.

**Example 7.2** (Online Customer Satisfaction with Over-/Under-Rating)**.** Let $U = \{\mathrm{Alice}, \mathrm{Bob}, \mathrm{Carol}\}$ be customers. We choose $\varphi = -0.2$, $\psi = 1.2$, so

$$\mathrm{Dom}_{\varphi,\psi}(U) = [-0.2, 1.2].$$

Define a rating map $\mu : U \to [-0.2, 1.2]$ by

$$\mu(\mathrm{Alice}) = 1.1, \quad \mu(\mathrm{Bob}) = 0.8, \quad \mu(\mathrm{Carol}) = -0.1.$$

Here $\mu > 1$ models exceptional enthusiasm ("over-satisfaction"), while $\mu < 0$ captures active dissatisfaction beyond the normal scale.

### 7.1. *Hyper-U-Off Structure*

A Hyper-$U$-Off Structure associates each element of a classical structure with a set of off-range $U$-membership tuples in $[\varphi, \psi]^r$, strictly generalizing Hyper-$U$. We define the Hyper-$U$-Off Structure.

**Definition 7.3** (Hyper-$U$-Off Structure)**.** Let $H$ be a nonempty carrier set, let $U$ be a fixed uncertainty model with classical degree-domain $\mathrm{Dom}(U) \subseteq [0,1]^r$, and fix an integer $n \ge 1$. Let $\varphi < 0 < 1 < \psi$ be real bounds, and denote $\mathrm{Dom}_{\varphi,\psi}(U) \subseteq \mathbb{R}^r$ as in Definition 7.1. A *Hyper-U-Off Structure* is a triple

$$\mathcal{H}^{U,\mathrm{Off}} \;=\; \Big(H,\; \mu_H^{\mathrm{Off}} \colon H \;\longrightarrow\; \mathcal{P}^n\big(\mathrm{Dom}_{\varphi,\psi}(U)\big),\; \{\#^{(m)}\}\Big),$$

where

$$\mu_H^{\mathrm{Off}}(x) \;\subseteq\; \mathrm{Dom}_{\varphi,\psi}(U) \quad \text{for each } x \in H,$$

and $\mathcal{P}^n(\mathrm{Dom}_{\varphi,\psi}(U))$ denotes the $n$-th powerset of $\mathrm{Dom}_{\varphi,\psi}(U)$. Equivalently, each element $x \in H$ is assigned a set of possible "off-degree" tuples, whose components may lie in $[\varphi, \psi]$. The classical operations $\{\#^{(m)}\}$ on $H$ remain unchanged.

**Theorem 7.4** (Hyper-$U$-Off Structure Generalizes Hyper-$U$-Structure)**.** *Let H be a nonempty set, let U be an uncertainty model with classical degree-domain* $\mathrm{Dom}(U) \subseteq [0,1]^r$*, and fix* $n \ge 1$*. Let* $\varphi < 0 < 1 < \psi$ *and set* $\mathrm{Dom}_{\varphi,\psi}(U) = \{d \in \mathrm{R}^r \mid d_i \in [\varphi, \psi]\}$*. If* $\varphi = 0$ *and* $\psi = 1$*, then* $\mathrm{Dom}_{\varphi,\psi}(U) = \mathrm{Dom}(U)$*, and any Hyper-U-Off Structure*

$$\big(H,\; \mu_H^{\mathrm{Off}} \colon H \to \mathcal{P}^n(\mathrm{Dom}_{\varphi,\psi}(U))\big)$$

*becomes a Hyper-U-Structure* $\big(H,\; \mu_H \colon H \to \mathcal{P}^n(\mathrm{Dom}(U))\big)$ *by setting* $\mu_H(x) = \mu_H^{\mathrm{Off}}(x)$*. Conversely, any Hyper-U-Structure can be viewed as a Hyper-U-Off Structure with* $\varphi = 0$*,* $\psi = 1$*. Hence Hyper-U-Off Structures strictly generalize Hyper-U-Structures.*

*Proof.* By definition, a Hyper-$U$-Off Structure assigns to each $x \in H$ a subset $\mu_H^{\text{Off}}(x) \subseteq \text{Dom}_{\varphi,\psi}(U)$. When $\varphi = 0$ and $\psi = 1$, one has $\text{Dom}_{\varphi,\psi}(U) = \text{Dom}(U)$. Therefore

$$\mu_H^{\text{Off}} \ : \ H \ \longrightarrow \ \mathcal{P}^n\big(\text{Dom}(U)\big)$$

satisfies exactly the requirements of a Hyper-$U$-Structure. No further modification is needed: define $\mu_H = \mu_H^{\text{Off}}$.

Conversely, given any Hyper-$U$-Structure $\mu_H : H \to \mathcal{P}^n(\text{Dom}(U))$, one obtains a Hyper-$U$-Off Structure by choosing $\varphi = 0$, $\psi = 1$ and setting $\mu_H^{\text{Off}}(x) = \mu_H(x)$. Since $\text{Dom}(U) \subseteq \text{Dom}_{\varphi,\psi}(U)$, the off-structure axioms hold immediately.

Thus the Hyper-$U$-Off notion indeed encompasses all Hyper-$U$-Structures as the special case $\varphi = 0, \psi = 1$. □

7.2. *Super-U-Off Structure*

A Super-$U$-Off Structure assigns each $m$-subset of the carrier a single off-range $U$-membership degree tuple in $[\varphi, \psi]^r$, modeling both under- and over-confidence. We define the Super-$U$-Off Structure.

**Definition 7.5** (Super-$U$-Off Structure)**.** Let $H$ be a nonempty carrier set, let $U$ be a fixed uncertainty model with classical degree-domain $\text{Dom}(U) \subseteq [0,1]^r$, and fix an integer $m \geq 1$. Let $\varphi < 0 < 1 < \psi$ be real bounds and $\text{Dom}_{\varphi,\psi}(U)$ as in Definition 7.1. A *Super-U-Off Structure* is a triple

$$\mathcal{S}^{U,\text{Off}} \ = \ \Big(H, \ \mu_S^{\text{Off}} : \mathcal{P}^m(H) \ \longrightarrow \ \text{Dom}_{\varphi,\psi}(U), \ \{\#^{(m)}\}\Big),$$

where

$$\mu_S^{\text{Off}}(X) \ \in \ \text{Dom}_{\varphi,\psi}(U) \quad \text{for each } X \in \mathcal{P}^m(H),$$

and $\mathcal{P}^m(H)$ denotes the $m$-th powerset of $H$. In other words, each $m$-th-level subset $X \subseteq H$ is assigned a single off-degree tuple whose components lie in $[\varphi, \psi]$. The classical operations $\{\#^{(m)}\}$ on $H$ remain unchanged.

**Theorem 7.6** (Super-$U$-Off Generalizes Super-$U$)**.** *Let $H$ be a nonempty set, let $U$ be an uncertainty model with classical degree-domain* $\text{Dom}(U) \subseteq [0,1]^r$*, and fix $m \geq 1$. Let $\varphi < 0 < 1 < \psi$ and set*

$$\text{Dom}_{\varphi,\psi}(U) = \{\, d \in \mathrm{R}^r \mid d_i \in [\varphi, \psi]\}.$$

*If $\varphi = 0$ and $\psi = 1$, then* $\text{Dom}_{\varphi,\psi}(U) = \text{Dom}(U)$*, and any Super-U-Off Structure*

$$\big(H, \ \mu_S^{\text{Off}} : \mathcal{P}^m(H) \to \text{Dom}_{\varphi,\psi}(U)\big)$$

*becomes a Super-U-Structure*

$$\big(H, \ \mu_S : \mathcal{P}^m(H) \to \text{Dom}(U)\big)$$

*by setting $\mu_S(X) = \mu_S^{\text{Off}}(X)$ for all $X$. Conversely, any Super-U-Structure extends to a Super-U-Off Structure by choosing $\varphi = 0$, $\psi = 1$. Hence Super-U-Off Structures strictly generalize Super-U-Structures.*

*Proof.* A Super-$U$-Off Structure assigns each $m$-subset $X \subseteq H$ a vector $\mu_S^{\text{Off}}(X) \in \text{Dom}_{\varphi,\psi}(U)$. When $\varphi = 0$ and $\psi = 1$, we have $\text{Dom}_{\varphi,\psi}(U) = \text{Dom}(U)$. Thus $\mu_S^{\text{Off}} : \mathcal{P}^m(H) \to \text{Dom}(U)$ satisfies exactly the Super-$U$-Structure definition, so setting $\mu_S = \mu_S^{\text{Off}}$ yields a Super-$U$-Structure.

Conversely, given any Super-$U$-Structure $\mu_S : \mathcal{P}^m(H) \to \mathrm{Dom}(U)$, choose $\varphi = 0, \psi = 1$ and define $\mu_S^{\mathrm{Off}}(X) = \mu_S(X)$. Since $\mathrm{Dom}(U) \subseteq \mathrm{Dom}_{\varphi,\psi}(U)$, this map meets the Super-$U$-Off requirements. Therefore every Super-$U$-Structure arises as a special case of a Super-$U$-Off Structure. □

### 7.3. *SuperHyper-U-Off Structure*

A SuperHyper-$U$-Off Structure assigns each $m$-subset of $H$ an $n$-th level subset of off-range membership vectors in $[\varphi, \psi]^r$ domain. We define the SuperHyper-$U$-Off Structure.

**Definition 7.7** (SuperHyper-$U$-Off Structure)**.** Let $H$ be a nonempty carrier set, let $U$ be a fixed uncertainty model with degree-domain $\mathrm{Dom}(U) \subseteq [0, 1]^r$, and fix integers $m, n \geq 1$. Let $\varphi < 0 < 1 < \psi$ be real bounds and $\mathrm{Dom}_{\varphi,\psi}(U)$ as above. A *SuperHyper-U-Off Structure* is a triple

$$\mathcal{SH}^{U,\mathrm{Off}} = \Big(H,\ \mu_{SH}^{\mathrm{Off}} : \mathcal{P}^m(H) \longrightarrow \mathcal{P}^n\big(\mathrm{Dom}_{\varphi,\psi}(U)\big),\ \{\#^{(m)}\}\Big),$$

where

$$\mu_{SH}^{\mathrm{Off}}(X) \subseteq \mathrm{Dom}_{\varphi,\psi}(U) \quad \text{for each } X \in \mathcal{P}^m(H).$$

That is, each $m$-th-level subset $X \subseteq H$ is assigned a collection (an $n$-th-level subset) of possible off-degree tuples. The classical operations $\{\#^{(m)}\}$ on $H$ remain unchanged.

**Theorem 7.8** (SuperHyper-$U$-Off Generalizes Hyper-, Super- and SuperHyper-$U$)**.** *Let*

$$\mathcal{SH}^{U,\mathrm{Off}} = \big(H,\ \mu_{SH}^{\mathrm{Off}} : \mathcal{P}^m(H) \to \mathcal{P}^n(\mathrm{Dom}_{\varphi,\psi}(U))\big)$$

*be a SuperHyper-U-Off Structure. Then:*

*(1)* (Hyper-$U$-Off) *Define $\mu_H^{\mathrm{Off}}(x) = \mu_{SH}^{\mathrm{Off}}(\{x\})$ for $x \in H$. Then $\mu_H^{\mathrm{Off}} : H \to \mathcal{P}^n(\mathrm{Dom}_{\varphi,\psi}(U))$ is a Hyper-U-Off Structure.*

*(2)* (Super-$U$-Off) *If each $\mu_{SH}^{\mathrm{Off}}(X)$ is a singleton set $\{d_X\} \subseteq \mathrm{Dom}_{\varphi,\psi}(U)$, then $\mu_S^{\mathrm{Off}}(X) = d_X$ defines a Super-U-Off Structure $\mu_S^{\mathrm{Off}} : \mathcal{P}^m(H) \to \mathrm{Dom}_{\varphi,\psi}(U)$.*

*(3)* (SuperHyper-$U$) *If $\varphi = 0$ and $\psi = 1$, then $\mathrm{Dom}_{\varphi,\psi}(U) = \mathrm{Dom}(U)$, and $\mu_{SH}^{\mathrm{Off}}$ becomes $\mu_{SH} : \mathcal{P}^m(H) \to \mathcal{P}^n(\mathrm{Dom}(U))$, i.e. a SuperHyper-U-Structure.*

*Hence every Hyper-U-Off, every Super-U-Off, and every SuperHyper-U Structure arises as a specialization of a SuperHyper-U-Off Structure.*

*Proof.* (1) Since $\{x\} \in \mathcal{P}^m(H)$ whenever $m \geq 1$, the assignment $\mu_H^{\mathrm{Off}}(x) = \mu_{SH}^{\mathrm{Off}}(\{x\})$ yields a map $H \to \mathcal{P}^n(\mathrm{Dom}_{\varphi,\psi}(U))$. This satisfies exactly the Hyper-$U$-Off definition.

(2) If each $\mu_{SH}^{\mathrm{Off}}(X)$ is a singleton $\{d_X\}$, define $\mu_S^{\mathrm{Off}}(X) = d_X$. Then $\mu_S^{\mathrm{Off}} : \mathcal{P}^m(H) \to \mathrm{Dom}_{\varphi,\psi}(U)$ meets the Super-$U$-Off requirements.

(3) When $\varphi = 0, \psi = 1$, we have $\mathrm{Dom}_{\varphi,\psi}(U) = \mathrm{Dom}(U)$. Hence $\mu_{SH}^{\mathrm{Off}} : \mathcal{P}^m(H) \to \mathcal{P}^n(\mathrm{Dom}(U))$ is exactly the data of a SuperHyper-$U$-Structure.

Each case is obtained by restricting parameters or imposing singleton-ness, proving that SuperHyper-$U$-Off indeed generalizes all three structures. □

### 7.4. *Tree-U-Off Structure*

A Tree-$U$-Off Structure assigns each node-subset of a rooted tree an off-range uncertainty function mapping elements to $[\varphi,\psi]^r$ tuples. We define the Tree-$U$-Off Structure.

**Definition 7.9** (Tree-$U$-Off Structure)**.** Let $H$ be a nonempty universe of elements, and let $\mathcal{T} = \mathrm{Tree}(A)$ be a finite rooted tree whose node set is $A = \{A_1, A_2, \ldots, A_k\}$. Fix an uncertainty model $U$ with $\mathrm{Dom}(U) \subseteq [0,1]^r$, and let $\varphi < 0 < 1 < \psi$ with $\mathrm{Dom}_{\varphi,\psi}(U)$ as in Definition 7.1. A *Tree-U-Off Structure* is a triple

$$\mathcal{T}^{U,\mathrm{Off}} = \Big(H,\ \mu_T^{\mathrm{Off}} : \mathcal{P}(\mathcal{T}) \longrightarrow \big(\mathrm{Dom}_{\varphi,\psi}(U)\big)^H,\ \{\#^{(m)}\}\Big),$$

where:

(1) $\mathcal{P}(\mathcal{T})$ is the power set of the tree's node set $A$.

(2) For each $S \subseteq \mathcal{T}$,

$$\mu_T^{\mathrm{Off}}(S) = \gamma_S^{\mathrm{Off}} : H \longrightarrow \mathrm{Dom}_{\varphi,\psi}(U),$$

assigns to each element $x \in H$ an off-degree tuple $\gamma_S^{\mathrm{Off}}(x) \in \mathrm{Dom}_{\varphi,\psi}(U)$.

(3) The classical operations $\{\#^{(m)}\}$ on $H$ remain unchanged.

Hence a Tree-$U$-Off Structure is equivalent to a collection of functions $\{\gamma_S^{\mathrm{Off}} : H \to \mathrm{Dom}_{\varphi,\psi}(U) \mid S \subseteq \mathcal{T}\}$.

**Theorem 7.10** (Tree-$U$-Off Generalizes Tree-$U$)**.** *Let $H$ be a nonempty set, let $\mathcal{T} = (A, E)$ be a finite rooted tree with node set $A$, and fix an uncertainty model $U$ with classical domain $\mathrm{Dom}(U) \subseteq [0,1]^r$. Let $\varphi < 0 < 1 < \psi$ and define*

$$\mathrm{Dom}_{\varphi,\psi}(U) = \{\, d \in \mathrm{R}^r \mid d_i \in [\varphi,\psi]\}.$$

*Suppose*

$$\mathcal{T}^{U,\mathrm{Off}} = \big(H,\ \mu_T^{\mathrm{Off}} : \mathcal{P}(A) \to \{\, \gamma : H \to \mathrm{Dom}_{\varphi,\psi}(U)\}\big)$$

*is a Tree-U-Off Structure. Then:*

*(1) If $\varphi = 0$ and $\psi = 1$, so that $\mathrm{Dom}_{\varphi,\psi}(U) = \mathrm{Dom}(U)$, the map*

$$\mu_T(S) = \mu_T^{\mathrm{Off}}(S) \quad : \quad H \longrightarrow \mathrm{Dom}(U)$$

*defines a classical Tree-U-Structure $\big(H,\ \mu_T : \mathcal{P}(A) \to \{\gamma : H \to \mathrm{Dom}(U)\}\big)$.*

*(2) Conversely, any classical Tree-U-Structure $\mu_T : \mathcal{P}(A) \to \{\gamma : H \to \mathrm{Dom}(U)\}$ extends to a Tree-U-Off Structure by choosing $\varphi = 0, \psi = 1$ and setting $\mu_T^{\mathrm{Off}}(S) = \mu_T(S)$.*

*Thus the Tree-U-Off notion strictly generalizes the classical Tree-U-Structure.*

*Proof.* By definition, a Tree-$U$-Off Structure assigns each $S \subseteq A$ a function $\gamma_S^{\mathrm{Off}}\colon H \to \mathrm{Dom}_{\varphi,\psi}(U)$. If $\varphi = 0$ and $\psi = 1$, then $\mathrm{Dom}_{\varphi,\psi}(U) = \mathrm{Dom}(U) \subseteq [0,1]^r$. Hence

$$\gamma_S = \gamma_S^{\mathrm{Off}} \quad : \quad H \longrightarrow \mathrm{Dom}(U)$$

meets exactly the requirements of a classical Tree-$U$-Structure. No further change is needed: set $\mu_T(S) = \mu_T^{\mathrm{Off}}(S)$.

Conversely, given a classical Tree-$U$-Structure $\mu_T$, define $\mu_T^{\mathrm{Off}}(S) = \mu_T(S)$ and choose $\varphi = 0, \psi = 1$. Then each $\mu_T^{\mathrm{Off}}(S)\colon H \to \mathrm{Dom}(U) = \mathrm{Dom}_{\varphi,\psi}(U)$ satisfies the Tree-$U$-Off axioms. This completes the proof that Tree-$U$-Off Structures generalize Tree-$U$-Structures. □

### 7.5. *Forest-U-Off Structure*

A Forest-$U$-Off Structure aggregates Tree-$U$-Off Structures by taking coordinatewise maxima of off-range membership degrees across tree components. We define the Forest-$U$-Off Structure.

**Definition 7.11** (Forest-$U$-Off Structure)**.** Let $\{\mathcal{T}_t = \mathrm{Tree}(A^{(t)}) \mid t \in T\}$ be a finite family of attribute-trees, and let $\mathcal{F} = \bigsqcup_{t\in T} \mathcal{T}_t$ denote their disjoint union (a forest). Fix an uncertainty model $U$ with $\mathrm{Dom}(U) \subseteq [0,1]^r$, and let $\varphi < 0 < 1 < \psi$ with $\mathrm{Dom}_{\varphi,\psi}(U)$ as in Definition 7.1. Suppose each tree $\mathcal{T}_t$ carries a Tree-$U$-Off Structure

$$\mu_{T_t}^{\mathrm{Off}} : \mathcal{P}(\mathcal{T}_t) \longrightarrow \big(\mathrm{Dom}_{\varphi,\psi}(U)\big)^H.$$

Then a *Forest-U-Off Structure* is given by

$$\mathcal{F}^{U,\mathrm{Off}} = \Big(H,\ \mu_F^{\mathrm{Off}} : \mathcal{P}(\mathcal{F}) \longrightarrow \big(\mathrm{Dom}_{\varphi,\psi}(U)\big)^H,\ \{\#^{(m)}\}\Big),$$

where, for each $X \subseteq \mathcal{F}$ and each $x \in H$:

$$\mu_F^{\mathrm{Off}}(X)(x) = \bigvee_{t\in T:\, X\cap\mathcal{T}_t\neq\varnothing} \Big(\mu_{T_t}^{\mathrm{Off}}\big(X \cap \mathcal{T}_t\big)(x)\Big),$$

with $\bigvee$ denoting a coordinate-wise supremum in $\mathrm{Dom}_{\varphi,\psi}(U)$. If $X \cap \mathcal{T}_t = \varnothing$ for all $t$, one may define $\mu_F^{\mathrm{Off}}(\varnothing)(x)$ to be a designated base off-degree, e.g. $(\varphi, \dots, \varphi)$.

**Theorem 7.12** (Forest-$U$-Off Generalizes Forest-$U$ and Tree-$U$-Off)**.** *Let $\{\mathcal{T}_t = (A^{(t)}, E^{(t)}) \mid t \in T\}$ be a finite family of rooted attribute-trees, and form the forest $\mathcal{F} = \bigsqcup_{t\in T} \mathcal{T}_t$. Fix an uncertainty model U with classical domain* $\mathrm{Dom}(U) \subseteq [0,1]^r$ *and real bounds $\varphi < 0 < 1 < \psi$. Suppose each $\mathcal{T}_t$ carries a Tree-U-Off Structure*

$$\mu_{T_t}^{\mathrm{Off}} : \mathcal{P}(A^{(t)}) \longrightarrow \{\gamma : H \to \mathrm{Dom}_{\varphi,\psi}(U)\},$$

*and let*

$$\mu_F^{\mathrm{Off}} : \mathcal{P}\big(\cup_t A^{(t)}\big) \longrightarrow \{\eta : H \to \mathrm{Dom}_{\varphi,\psi}(U)\}$$

*be the induced Forest-U-Off Structure (Definition 7.11). Then:*

*(1) (*Reduction to Tree-$U$-Off*) If $T = \{t_0\}$ is a singleton, so that $\mathcal{F} = \mathcal{T}_{t_0}$, then for every $X \subseteq A^{(t_0)}$,*

$$\mu_F^{\mathrm{Off}}(X) = \mu_{T_{t_0}}^{\mathrm{Off}}(X).$$

*Thus $\mu_F^{\mathrm{Off}}$ coincides with the given Tree-U-Off Structure on $\mathcal{T}_{t_0}$.*

*(2) (*Reduction to Forest-$U$*) If $\varphi = 0$ and $\psi = 1$, then $\mathrm{Dom}_{\varphi,\psi}(U) = \mathrm{Dom}(U)$. In this case each $\mu^{\mathrm{Off}}_{T_t}$ is a classical Tree-U-Structure and $\mu^{\mathrm{Off}}_{F}$ becomes exactly the classical Forest-U-Structure $\mu_F : \mathcal{P}(\cup_t A^{(t)}) \to \{\eta : H \to \mathrm{Dom}(U)\}$.*

*Hence the Forest-U-Off Structure simultaneously generalizes both the Tree-U-Off and the classical Forest-U-Structure.*

*Proof.* **(1) Singleton forest.** If $T = \{t_0\}$, then $\mathcal{F} = A^{(t_0)}$. By definition of $\mu^{\mathrm{Off}}_{F}$,

$$\mu^{\mathrm{Off}}_{F}(X)(x) = \bigvee_{t:X\cap A^{(t)}\neq\emptyset} \mu^{\mathrm{Off}}_{T_t}(X \cap A^{(t)})(x) = \mu^{\mathrm{Off}}_{T_{t_0}}(X)(x), \quad x \in H,$$

since the only nonempty intersection is with $\mathcal{T}_{t_0}$. Therefore $\mu^{\mathrm{Off}}_{F} = \mu^{\mathrm{Off}}_{T_{t_0}}$.

**(2) Restriction to** $[0, 1]$**.** Setting $\varphi = 0$ and $\psi = 1$ yields $\mathrm{Dom}_{\varphi,\psi}(U) = \mathrm{Dom}(U)$. Hence each $\mu^{\mathrm{Off}}_{T_t} : \mathcal{P}(A^{(t)}) \to \{\gamma : H \to \mathrm{Dom}(U)\}$ is a classical Tree-$U$-Structure, and the construction of $\mu^{\mathrm{Off}}_{F}$ coincides with that of the classical Forest-$U$-Structure $\mu_F : \mathcal{P}(\cup_t A^{(t)}) \to \{\eta : H \to \mathrm{Dom}(U)\}$. No further modification is needed. □

7.6. *Further Generalization Theorem*

The following theorem also holds.

**Theorem 7.13** (Generalization to Off Structures)**.** *Let $H$ be a carrier set, let $U$ be an uncertainty model with classical degree-domain $\mathrm{Dom}(U) \subseteq [0, 1]^r$, and fix real bounds $\varphi < 0 < 1 < \psi$. Then each of the following is a generalization of the corresponding "on" structure:*

*(1) Hyper-U-Off Structure generalizes Hyper-U-Structure.*

*(2) Super-U-Off Structure generalizes Super-U-Structure.*

*(3) SuperHyper-U-Off Structure generalizes SuperHyper-U-Structure.*

*(4) Tree-U-Off Structure generalizes Tree-U-Structure.*

*(5) Forest-U-Off Structure generalizes Forest-U-Structure.*

*In each case, one recovers the "on" structure by restricting all mappings to $\mathrm{Dom}(U) \subseteq \mathrm{Dom}_{\varphi,\psi}(U)$.*

*Proof.* We verify each item by showing that, under restriction to the classical domain $\mathrm{Dom}(U) \subseteq [0, 1]^r$, the off-structure collapses to the on-structure:

**(1) Hyper-$U$-Off vs. Hyper-$U$.** A Hyper-$U$-Off Structure is $\big(H,\ \mu^{\mathrm{Off}}_{H} : H \to \mathcal{P}^n(\mathrm{Dom}_{\varphi,\psi}(U))\big)$. If we require that, for each $x \in H$,

$$\mu^{\mathrm{Off}}_{H}(x) \ \subseteq\ \mathrm{Dom}(U),$$

then $\mu^{\mathrm{Off}}_{H}$ becomes a mapping $H \to \mathcal{P}^n(\mathrm{Dom}(U))$. That is exactly a Hyper-$U$-Structure (Definition 6.1).

**(2) Super-$U$-Off vs. Super-$U$.** A Super-$U$-Off Structure is $\big(H,\ \mu^{\mathrm{Off}}_{S} : \mathcal{P}^m(H) \to \mathrm{Dom}_{\varphi,\psi}(U)\big)$. If we further impose

$$\mu^{\mathrm{Off}}_{S}(X) \in \mathrm{Dom}(U) \quad \text{for all } X \in \mathcal{P}^m(H),$$

then $\mu_S^{\text{Off}}$ becomes a mapping $\mathcal{P}^m(H) \to \text{Dom}(U)$, which is precisely a Super-$U$-Structure (Definition 6.3).

**(3) SuperHyper-$U$-Off vs. SuperHyper-$U$.** A SuperHyper-$U$-Off Structure is $\big(H,\ \mu_{SH}^{\text{Off}} : \mathcal{P}^m(H) \to \mathcal{P}^n(\text{Dom}_{\varphi,\psi}(U))\big)$. Restricting

$$\mu_{SH}^{\text{Off}}(X) \subseteq \text{Dom}(U) \quad \text{for all } X \in \mathcal{P}^m(H),$$

yields $\mu_{SH}^{\text{Off}} : \mathcal{P}^m(H) \to \mathcal{P}^n(\text{Dom}(U))$, which is a SuperHyper-$U$-Structure (Definition 6.5).

**(4) Tree-$U$-Off vs. Tree-$U$.** A Tree-$U$-Off Structure is $\big(H,\ \mu_T^{\text{Off}} : \mathcal{P}(\mathcal{T}) \to (\text{Dom}_{\varphi,\psi}(U))^H\big)$. Imposing

$$\mu_T^{\text{Off}}(S)(x) \in \text{Dom}(U) \quad \text{for all } S \subseteq \mathcal{T},\ x \in H,$$

reduces $\mu_T^{\text{Off}}$ to $\mu_T : \mathcal{P}(\mathcal{T}) \to (\text{Dom}(U))^H$, i.e. a Tree-$U$-Structure (Definition 6.8).

**(5) Forest-$U$-Off vs. Forest-$U$.** A Forest-$U$-Off Structure is $\big(H,\ \mu_F^{\text{Off}} : \mathcal{P}(\mathcal{F}) \to (\text{Dom}_{\varphi,\psi}(U))^H\big)$. If we require

$$\mu_F^{\text{Off}}(X)(x) \in \text{Dom}(U) \quad \text{for all } X \subseteq \mathcal{F},\ x \in H,$$

then $\mu_F^{\text{Off}}$ becomes $\mu_F : \mathcal{P}(\mathcal{F}) \to (\text{Dom}(U))^H$, which is exactly a Forest-$U$-Structure (Definition 6.10).

In each case, restricting the off-degree range to the classical domain $\text{Dom}(U)$ recovers the corresponding on-degree structure. This completes the proof. □

## 8. Additional Result: Functorial Set

We also introduce the notion of a *Functorial Set*, which generalizes concepts beyond Uncertain Sets. The formal definition is given below.

**Definition 8.1** (Functorial Set)**.** Let $C$ be a category and

$$F : C \longrightarrow \textbf{Set}$$

be a (covariant) endofunctor. For any object $X \in \text{Ob}(C)$, an *$F$-set over $X$* is an element

$$s \in F(X).$$

We denote the collection of all $F$-sets over $X$ simply by $F(X)$. A morphism $f : X \to Y$ in $C$ induces a *pushforward*

$$F(f) : F(X) \longrightarrow F(Y), \qquad s \mapsto F(f)(s).$$

**Example 8.2** (Powerset Functor: Classical Subsets)**.** Let $C = \textbf{Set}$ and $F = \mathcal{P} : \textbf{Set} \to \textbf{Set}$ with

$$\mathcal{P}(X) = \{ A \mid A \subseteq X \}, \qquad \mathcal{P}(f)(A) = \{ f(a) \mid a \in A \}.$$

An $F$-set over $X$ is just a subset $A \subseteq X$, and any function $f : X \to Y$ carries $A$ to its image $f(A) \subseteq Y$.

**Example 8.3** (Fuzzy-Powerset Functor: Fuzzy Sets)**.** Let $C = \textbf{Set}$ and $F = \mathcal{F} : \textbf{Set} \to \textbf{Set}$ with

$$\mathcal{F}(X) = \{\mu : X \to [0,1]\}, \qquad \big(\mathcal{F}(f)(\mu)\big)(y) = \sup\{\mu(x) \mid f(x) = y\}.$$

An $F$-set over $X$ is a fuzzy membership function $\mu$, and $f$ pushes it forward by taking the supremum over each fibre.

**Example 8.4** (Multiset Functor)**.** Let $C = \mathbf{Set}$ and $F = \mathrm{MSet}\colon \mathbf{Set} \to \mathbf{Set}$ with

$$\mathrm{MSet}(X) = \{\text{finite multisets } m\colon X \to \mathbb{N}\}, \qquad \big(\mathrm{MSet}(f)(m)\big)(y) = \sum_{x\,:\,f(x)=y} m(x).$$

An $F$-set over $X$ is a finite multiset of elements of $X$, and $f$ pushes it forward by summing multiplicities along fibres.

**Remark 8.5.** Since $F$ is arbitrary, this single definition simultaneously captures:

- **Classical subsets:** take $F = \mathcal{P}$, the ordinary powerset functor $\mathcal{P}\colon \mathbf{Set} \to \mathbf{Set}$. Then $\mathcal{P}(X)$ is the set of all crisp subsets of $X$.
- **Fuzzy sets:** take $F = \mathcal{F}\colon \mathbf{Set} \to \mathbf{Set}$ with $\mathcal{F}(X) = \{\,\mu\colon X \to [0,1]\}$.
- **Soft sets:** take $F = \mathrm{Soft}_A\colon \mathbf{Set} \to \mathbf{Set}$ for a fixed parameter set $A$, defined by $\mathrm{Soft}_A(X) = \{\,\alpha\colon A \to \mathcal{P}(X)\}$.
- **Multisets:** take $F = \mathrm{MSet}\colon \mathbf{Set} \to \mathbf{Set}$, the finite multiset functor.
- **Probabilistic sets:** take $F = \mathrm{Dist}\colon \mathbf{Set} \to \mathbf{Set}$, the discrete distribution functor.

**Theorem 8.6** (Instances of Functorial Sets)**.** *The following classical constructions are all Functorial Sets:*

*(1)* $F = \mathrm{Id}$ *(the identity functor) yields* ordinary sets*:* $\mathrm{Id}(X) = X$.

*(2)* $F = \mathcal{P}$ *yields* crisp subsets*:* $\mathcal{P}(X) = \{\,A \subseteq X\}$.

*(3)* $F = \mathcal{F}$ *with* $\mathcal{F}(X) = \{\,\mu\colon X \to [0,1]\}$ *yields* fuzzy sets.

*(4)* $F = \mathrm{Soft}_A$ *with* $\mathrm{Soft}_A(X) = \{\,\alpha\colon A \to \mathcal{P}(X)\}$ *yields* soft sets *parameterized by* $A$.

*(5)* $F = \mathrm{MSet}$ *yields* multisets.

*(6)* $F = \mathrm{Dist}$ *with* $\mathrm{Dist}(X) = \{\,p\colon X \to [0,1] \mid \sum p = 1\}$ *yields* probabilistic sets.

*Proof.* Each construction above is plainly a functor $F\colon \mathbf{Set} \to \mathbf{Set}$. By Definition 8.1, an $F$-set over $X$ is precisely an element of $F(X)$, matching the usual description:

$$\mathrm{Id}(X) = X, \quad \mathcal{P}(X) = \{A \subseteq X\}, \quad \mathcal{F}(X) = \{\,\mu : X \to [0,1]\}, \ \ldots$$

Functoriality of each $F$ guarantees well-defined pushforwards $F(f)$. Hence all listed constructions are special cases of Functorial Sets. □

**Corollary 8.7** (Generalization of Uncertain Sets)**.** *All previously studied "uncertain" set-models (hyper-uncertain, super-uncertain, tree-uncertain, etc.) arise by choosing $F$ to be a suitable composite or iterated powerset–style functor, so that their degree-domains become $F(X)$ for a base $X$.*

*Proof.* For instance, a $(m,n)$-SuperHyperUncertain Set on $X$ was a map $\tau\colon \mathcal{P}^m(X) \to \mathcal{P}^n([0,1]^r)$. Equivalently, let

$$F(Y) = \mathcal{P}^n\big([0,1]^r\big) \quad \text{for } Y = \mathcal{P}^m(X).$$

Then $\tau \in F\big(\mathcal{P}^m(X)\big)$, an $F$-set over $\mathcal{P}^m(X)$. Similar arguments apply to all other variants. □

**Remark 8.8.** Definition 8.1 provides a single, elegant abstraction: *any* way of assigning to each set $X$ a collection $F(X)$ of "structured subsets" (and to each function $f\colon X \to Y$ a natural pushforward $F(f)$) is simply an endofunctor $F$. One then studies the elements of $F(X)$ uniformly as "Functorial Sets."

We now proceed to define the Functorial Structure, inspired by the above definition of the Functorial Set. As noted above, the term "Structure" is used in its broadest sense, covering both mathematical and applied systems—ranging from graph theory, neural networks, and decision-making to logic, topology, group theory, probability, automata, programming, vector spaces, matroid theory, image processing, biology, social sciences, chemistry, and physics.

**Definition 8.9** (Functorial Structure)**.** Let $C$ be a category. A *Functorial Structure* on $C$ is simply a covariant functor

$$F \;:\; C \;\longrightarrow\; \mathbf{Set}.$$

For each object $X \in \mathrm{Ob}(C)$, an element

$$s \;\in\; F(X)$$

is called an *$F$-structure on $X$*. Every morphism $f\colon X \to Y$ in $C$ induces a *pushforward*

$$F(f)\colon F(X) \;\longrightarrow\; F(Y), \qquad s \;\longmapsto\; F(f)(s),$$

and the usual functoriality conditions $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ and $F(g \circ f) = F(g) \circ F(f)$ hold.

**Example 8.10** (Hospital Ward Patient Tracking as a Functorial Structure)**.** Consider the category $C_{\mathrm{Ward}}$ whose:

- **Objects** are the hospital wards: e.g. $\mathrm{Ward}_A, \mathrm{Ward}_B, \mathrm{Ward}_C, \ldots$.
- **Morphisms** $f\colon \mathrm{Ward}_X \to \mathrm{Ward}_Y$ represent patient-transfer events from ward $X$ to ward $Y$. Each such $f$ is a concrete record of one or more patients moving.

Define a covariant functor

$$F \;:\; C_{\mathrm{Ward}} \;\longrightarrow\; \mathbf{Set}$$

as follows:

- For each ward $W \in \mathrm{Ob}(C_{\mathrm{Ward}})$, let

$$F(W) \;=\; \{\text{IDs of patients currently assigned to } W\},$$

  a finite set of patient identifiers.
- For each transfer morphism $f\colon W_1 \to W_2$, define

$$F(f)\colon F(W_1) \;\longrightarrow\; F(W_2), \quad p \;\mapsto\; p,$$

  that is, each patient $p$ moved from $W_1$ becomes a member of the set of patients in $W_2$.

Functoriality checks.

- **Identity:** The identity morphism $\mathrm{id}_W : W \to W$ induces

$$F(\mathrm{id}_W) \;=\; \mathrm{id}_{F(W)},$$

since no patient changes wards.

- **Composition:** Given transfers $f : W_1 \to W_2$ and $g : W_2 \to W_3$, the composition $g \circ f$ sends patients directly from $W_1$ to $W_3$. On sets,

$$F(g \circ f) \;=\; F(g) \circ F(f),$$

because each patient's identifier is carried along exactly once, whether by two successive moves or by their composite.

Thus $\big(C_{\mathrm{Ward}}, F\big)$ is a concrete Functorial Structure: it assigns to each ward the set of its current patients, and to each transfer the natural "pushforward" of those patient IDs.

**Theorem 8.11** (Classical Structures as Functorial Structures)**.** *Take $C = \mathbf{Set}$. Then each of the following assignments defines a functor* $\mathbf{Set} \to \mathbf{Set}$*, hence a Functorial Structure in the sense of Definition 8.9:*

*(1)* ***Graphs.*** *For any set $X$,*

$$F_{\mathrm{Graph}}(X) = \mathcal{P}\big(X \times X\big) = \{\, E \subseteq X \times X\},$$

*and for $f : X \to Y$,*

$$F_{\mathrm{Graph}}(f)(E) = \{\, \big(f(x), f(x')\big) \mid (x, x') \in E\}.$$

*(2)* ***Topologies.*** *For any set $X$,*

$$F_{\mathrm{Top}}(X) = \big\{\tau \subseteq \mathcal{P}(X) \mid \tau \text{ is a topology on } X\big\},$$

*and for $f : X \to Y$,*

$$F_{\mathrm{Top}}(f)(\tau) = \big\{\, B \subseteq Y \mid f^{-1}(B) \in \tau\},$$

*which is readily checked to be a topology on $Y$.*

*(3)* ***Soft Sets.*** *Fix a parameter set $P$. For any $X$,*

$$F_{\mathrm{Soft}}(X) = \big\{\, g : P \to \mathcal{P}(X)\big\},$$

*and for $f : X \to Y$,*

$$F_{\mathrm{Soft}}(f)(g)(p) = \{\, f(x) \mid x \in g(p)\}, \quad p \in P.$$

*In each case one checks directly that $F(\mathrm{id}_X) = \mathrm{id}$ and $F(g \circ f) = F(g) \circ F(f)$. Hence every one of these classical "structures on a set" arises as a Functorial Structure.*

*Proof.* We outline the graph case; the other two are analogous.

**Identity.** If $f = \mathrm{id}_X$, then

$$F_{\mathrm{Graph}}(\mathrm{id}_X)(E) = \{(\mathrm{id}_X(x), \mathrm{id}_X(x')) \mid (x, x') \in E\} = E.$$

**Composition.** Given $f : X \to Y$ and $g : Y \to Z$,

$$F_{\mathrm{Graph}}(g \circ f)(E) = \{(g(f(x)), g(f(x'))) \mid (x, x') \in E\} = F_{\mathrm{Graph}}(g)\big(F_{\mathrm{Graph}}(f)(E)\big).$$

This verifies functoriality. The topology and soft-set constructions satisfy the same two checks, using $f^{-1}$ for topologies and direct image for soft-sets. □

**Theorem 8.12** (Functorial Sets are Functorial Structures)**.** *Let $C$ be any category and let*

$$F : C \longrightarrow \mathbf{Set}$$

*be a covariant functor. Then $F$ is by definition a Functorial Structure on $C$. In particular, Definition 8.1 is exactly the special case of Definition 8.9 when the target category is* **Set***.*

*Proof.* Immediate from the definitions: a functor $F : C \to \mathbf{Set}$ assigns to each object $X$ a set $F(X)$ (the "$F$-sets over $X$") and to each morphism $f : X \to Y$ a function $F(f) : F(X) \to F(Y)$, satisfying $F(\mathrm{id}) = \mathrm{id}$ and $F(g \circ f) = F(g) \circ F(f)$. This is precisely the data of a Functorial Structure. □

## 9. **Conclusion**

In this work, we have introduced the notions of the *Uncertain Model*, the *$U$-Structure*, and the *$U$-Off Structure*, and we have systematically revisited and extended the families of HyperUncertain, SuperUncertain, SuperHyper-Uncertain, TreeUncertain, and ForestUncertain structures—together with their corresponding Off-variants. This unified framework provides a flexible foundation for representing and reasoning about uncertainty across a vast spectrum of domains.

Here, the term "Structure" is used in the broadest sense, encompassing mathematical and applied systems such as graph theory, neural networks, decision-making, logic, topology, group theory, probability, automata, programming, vector spaces, matroid theory, image processing, biology, the social sciences, chemistry, and physics. By leveraging the $U$-Structure paradigm, properties of fuzzy, neutrosophic, and related uncertainty models can be studied and applied in a coherent, uniform manner. We anticipate that this framework will spur further developments in the mathematical theory of $U$-Structures as well as novel applications across diverse scientific and engineering disciplines.


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**Author Contributions**

Conceptualization, Takaaki Fujita and Florentin Smarandache; Investigation, Takaaki Fujita; Methodology, Takaaki Fujita; Writing – original draft, Takaaki Fujita; Writing – review & editing, Takaaki Fujita and Florentin Smarandache.

**Ethical Approval**

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

**Data Availability**

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

**Research Integrity**

The authors hereby confirm that, to the best of their knowledge, this manuscript is their original work, has not been published in any other journal, and is not currently under consideration for publication elsewhere at this stage.

**Use of Generative AI and AI-Assisted Tools**

I use generative AI and AI-assisted tools for tasks such as English grammar checking, and I do not employ them in any way that violates ethical standards.

**Disclaimer (Note on Computational Tools)**

No computer-assisted proof, symbolic computation, or automated theorem proving tools (e.g., Mathematica, SageMath, Coq, etc.) were used in the development or verification of the results presented in this paper. All proofs and derivations were carried out manually and analytically by the authors.

**Code Availability**

No code or software was developed for this study.

**Clinical Trial**

This study did not involve any clinical trials.

**Consent to Participate**

Not applicable.

### Disclaimer (Limitations and Claims)

The theoretical concepts presented in this paper have not yet been subject to practical implementation or empirical validation. Future researchers are invited to explore these ideas in applied or experimental settings. Although every effort has been made to ensure the accuracy of the content and the proper citation of sources, unintentional errors or omissions may persist. Readers should independently verify any referenced materials.

To the best of the authors' knowledge, all mathematical statements and proofs contained herein are correct and have been thoroughly vetted. Should you identify any potential errors or ambiguities, please feel free to contact the authors for clarification.

The results presented are valid only under the specific assumptions and conditions detailed in the manuscript. Extending these findings to broader mathematical structures may require additional research. The opinions and conclusions expressed in this work are those of the authors alone and do not necessarily reflect the official positions of their affiliated institutions.

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Author has declared that no competing interests exist.

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