Note for Line and Total SuperHyperGraphs: Connecting Vertices, Edges, Edges of Edges, Edges of Edges of Edges in Hierarchical Systems

Takaaki Fujita [1] *
[1] Independent Researcher, Shinjuku, Shinjuku-ku, Tokyo, Japan.
Email: Takaaki.fujita060@gmail.com

Abstract

Hypergraphs extend classical graphs by allowing *hyperedges* to connect any nonempty subset of vertices, thereby capturing complex group-level relationships. Superhypergraphs advance this framework by introducing recursively nested powerset layers, enabling the representation of hierarchical and self-referential links among hyperedges. A *line graph* encodes the adjacencies between edges of an original graph by transforming each edge into a vertex and connecting two vertices if their corresponding edges share a common endpoint. A *total graph* incorporates both the vertices and edges of the original graph as its own vertices, with edges representing adjacency or incidence between these entities. An *iterated line graph* arises from the repeated application of the line graph construction, where each iteration takes the previous line graph as its input. Similarly, an *iterated total graph* is generated by iteratively applying the total graph transformation a specified number of times. This paper investigates the hypergraph and superhypergraph analogues of these constructions, providing a foundation for further theoretical development.

Keywords: SuperHyperGraph, HyperGraph, Line Graph, Total Graph, Iterated line graph, Iterated total graph

Contents in this paper

The remainder of this paper is organized as follows.

1 Preliminaries

In this section, we review the key concepts and notation used throughout this paper. All graphs and HyperGraphs considered here are finite.

1.1 Hypergraphs and Superhypergraphs

In classical graphs, it can be difficult to represent certain real-world concepts, such as hierarchical network structures. For this reason, frameworks like HyperGraphs and SuperHyperGraphs have been developed and studied. A *hypergraph* extends the notion of a simple graph by allowing each edge—referred to as a *hyperedge*—to connect an arbitrary nonempty subset of vertices, thereby representing relationships that involve more than two elements [1–4]. A *superhypergraph* further enriches this structure by incorporating recursively applied powerset layers into the hypergraph, producing hierarchically nested connections among hyperedges. This generalization has been investigated in several recent studies [5–10].

Definition 1.1 (Base Set)**.** [11] Let S be a nonempty set, called the *base set*. All higher-order objects, such as powersets and supervertices, are constructed from S:

$$S = \{\, x \mid x \text{ is an element of the domain} \,\}.$$

Definition 1.2 (Powerset)**.** For any set S, its *powerset* $\mathrm{POWS}(S)$ is the collection of all subsets of S, including $\emptyset$ and S itself:

$$\mathrm{POWS}(S) = \{\, A \mid A \subseteq S \,\}.$$

Definition 1.3 (n-th Powerset)**.** [9,12–14]. Let H be a set. The *n-th powerset* $\mathrm{POWS}^n(H)$ is defined recursively by

$$\mathrm{POWS}^0(H) = H, \qquad \mathrm{POWS}^{k+1}(H) = \mathrm{POWS}\big(\mathrm{POWS}^k(H)\big), \quad k \geq 0.$$

The *nonempty n-th powerset* $\mathrm{POWS}^{*n}(H)$ is defined similarly but removes the empty set at each stage:

$$\mathrm{POWS}^{*0}(H) = H, \qquad \mathrm{POWS}^{*(k+1)}(H) = \mathrm{POWS}^*\big(\mathrm{POWS}^{*k}(H)\big),$$

where $\mathrm{POWS}^*(X) = \mathrm{POWS}(X) \setminus \{\emptyset\}$

Example 1.4 (n-th Powerset Example)**.** Let $H = \{a, b\}$. Then:

$$\begin{aligned}
\mathrm{POWS}^0(H) &= \{a, b\},\\
\mathrm{POWS}^1(H) &= \{\emptyset,\ \{a\},\ \{b\},\ \{a, b\}\},\\
\mathrm{POWS}^2(H) &= \{\ \emptyset,\ \{\emptyset\},\ \{\{a\}\},\ \{\{b\}\},\ \{\{a, b\}\},\ \{\emptyset, \{a\}\},\ \{\emptyset, \{b\}\},\ \{\emptyset, \{a, b\}\},\\
&\qquad \{\{a\}, \{b\}\},\ \{\{a\}, \{a, b\}\},\ \{\{b\}, \{a, b\}\},\ \{\emptyset, \{a\}, \{b\}\},\ \{\emptyset, \{a\}, \{a, b\}\},\\
&\qquad \{\emptyset, \{b\}, \{a, b\}\},\ \{\{a\}, \{b\}, \{a, b\}\},\ \{\emptyset, \{a\}, \{b\}, \{a, b\}\}\ \}.
\end{aligned}$$

This illustrates the recursive growth of $\mathrm{POWS}^n(H)$ with n.

Definition 1.5 (HyperGraph)**.** [15, 16] A finite *HyperGraph* $H = (V, E)$ is specified by:

- A nonempty vertex set V.
- A collection E of hyperedges, where each $e \in E$ is a nonempty subset of V.

HyperGraphs generalize ordinary graphs by permitting edges to join any number of vertices, making them ideal for modeling higher-order relationships. In this work, we assume both V and E are finite.

Definition 1.6 (Level-n SuperHyperGraph (incidence form))**.** (cf. [6, 9, 17, 18]) Fix a finite base set V_0 and an integer $n \geq 0$. Let $V_n \subseteq \mathrm{POWS}^n(V_0)$ be a finite set, whose elements are called *n-supervertices*. A *level-n SuperHyperGraph* is a pair

$$\mathcal{H}^{(n)} \;=\; \big(V_n,\, \mathcal{E}\big), \quad \text{with} \quad \emptyset \neq \mathcal{E} \subseteq \mathrm{POWS}(V_n) \setminus \{\emptyset\}.$$

Thus each *n-superedge* $E \in \mathcal{E}$ is a nonempty subset of the vertex set V_n. (When $n = 0$, this is an ordinary finite hypergraph; when, additionally, every $E \in \mathcal{E}$ has size 2, it is an ordinary graph.)

Notation 1.7 (Stars)**.** *For* $\mathcal{H}^{(n)} = (V_n, \mathcal{E})$ *and* $v \in V_n$*, the* star *of* v *is*

$$\mathsf{Star}_{\mathcal{H}}(v) \;:=\; \{\, E \in \mathcal{E} \;:\; v \in E \,\} \;\subseteq\; \mathcal{E}.$$

We also write $\mathcal{E}^{\neq\emptyset}(v) := \mathsf{Star}_{\mathcal{H}}(v)$ *and* $\mathcal{E}^{(\geq 2)}(v) := \{\, E \in \mathcal{E} : v \in E \text{ and } |E| \geq 2 \,\}$ *when we wish to exclude size-1 edges in the star.*

Example 1.8 (A Level-2 SuperHyperGraph and its Stars)**.** Let the base set be $V_0 = \{a, b, c\}$. The level-1 supervertices are

$$V_1 = \big\{\{a, b\}, \{b, c\}, \{a\}\big\}.$$

The level-2 supervertices are taken as

$$V_2 = \big\{\{\{a, b\}, \{b, c\}\},\ \{\{a\}, \{b, c\}\},\ \{\{a, b\}\}\big\}.$$

Define the superedges by

$$\mathcal{E} = \big\{\{\{\{a, b\}, \{b, c\}\},\ \{\{a\}, \{b, c\}\}\},\ \{\{\{a\}, \{b, c\}\},\ \{\{a, b\}\},\ \{\{a, b\}, \{b, c\}\}\}\big\}.$$

Then $\mathcal{H}^{(2)} = (V_2, \mathcal{E})$ is a level-2 SuperHyperGraph.

For example, the star of the vertex $v = \{\{a, b\}, \{b, c\}\}$ is

$$\mathsf{Star}_{\mathcal{H}}(v) = \big\{\{\{\{a, b\}, \{b, c\}\},\ \{\{a\}, \{b, c\}\}\},\ \{\{\{a\}, \{b, c\}\},\ \{\{a, b\}\},\ \{\{a, b\}, \{b, c\}\}\}\big\}.$$

2 Review and Result: Line Graph

2.1 Line Graph

A line graph represents edges of a graph as vertices, linking them if the original edges share a common endpoint (cf. [19–24]).

Definition 2.1 (Line graph)**.** Let $G = (V(G), E(G))$ be a (loopless) undirected simple graph. The *line graph* $L(G)$ is the (simple) graph defined by

$$V\big(L(G)\big) = E(G), \qquad \{e, f\} \in E\big(L(G)\big) \iff e \cap f \neq \emptyset,$$

i.e., two vertices of $L(G)$ are adjacent exactly when the corresponding two edges of G share an endpoint.

Example 2.2 (Line Graph — flight connections)**.** Consider a small airline network with airports $V = \{A, B, C, D\}$ and direct flights

$$E(G) = \{\ e_1 = AB,\ e_2 = AC,\ e_3 = BC,\ e_4 = BD\ \}.$$

The *line graph* $L(G)$ has one vertex for each flight e_i, and two vertices are adjacent iff the flights share an airport. Explicitly,

$$E\big(L(G)\big) = \big\{\ e_1e_2,\ e_1e_3,\ e_1e_4,\ e_2e_3,\ e_3e_4\ \big\},$$

so $L(G)$ is a K_4 missing the edge e_2e_4. Degrees are

$$\deg_{L(G)}(e_1) = 3, \quad \deg_{L(G)}(e_2) = 2, \quad \deg_{L(G)}(e_3) = 3, \quad \deg_{L(G)}(e_4) = 2.$$

(Verification by the standard formula: $\deg_G(A) = 2$, $\deg_G(B) = 3$, $\deg_G(C) = 2$, $\deg_G(D) = 1$ gives $|E(L(G))| = \sum_{v \in V} \binom{\deg_G(v)}{2} = 1 + 3 + 1 + 0 = 5$ as listed.) This models *connection possibilities*: a path e_i–e_j in $L(G)$ corresponds to a feasible one-stop transfer sharing an airport.

2.2 Line HyperGraph

A line hypergraph transforms each hyperedge into a vertex, connecting vertices via hyperedges that share at least one original vertex (cf. [25–29]).

Definition 2.3 (Clique of rank r)**.** Given a finite set X and an integer $r \geq 0$, the *clique of rank r* on X is the (simple) uniform hypergraph whose vertex set is X and whose edge set is $\binom{X}{r}$ (with the conventions that rank 2 is an ordinary complete graph on X; rank 1 has edges the singletons of X; rank 0 is a single isolated vertex).

Definition 2.4 (Line hypergraph (Tyshkevich–Zverovich))**.** Let $H = (V, E)$ be a hypergraph without isolated vertices, and list $V = \{v_1, \dots, v_n\}$. Define two integer vectors

$$\mathbf{1}_H = (\deg(v_1), \dots, \deg(v_n)), \qquad \mathbf{0}_H = (\,0_{v_i}\,)_{i=1}^{n} \text{ with } 0_{v_i} = \begin{cases} 0, & \deg(v_i) = 1, \\ 2, & \deg(v_i) \geq 2. \end{cases}$$

Let $\mathcal{D}_H := \{D = (d_{v_i})_{i=1}^{n} : \mathbf{0}_H \leq D \leq \mathbf{1}_H \text{ componentwise}\}$. For any $D \in \mathcal{D}_H$, define for each $v \in V$ the clique F_v of rank d_v on the vertex set $E(v)$, and set

$$L_D(H) := \bigcup_{v \in V} F_v.$$

The (multi-valued) *line hypergraph* of H is the set

$$L(H) := \{\, L_D(H) : D \in \mathcal{D}_H \,\}.$$

Example 2.5 (Line Hypergraph — meetings sharing attendees)**.** Let $H = (V, \mathcal{E})$ encode meetings with *people* as vertices

$$V = \{\text{Alice, Bob, Chloe, Dan}\},$$

and *meetings* (hyperedges)

$$\mathcal{E} = \{\, E_1 = \{\text{Alice, Bob, Chloe}\},\ E_2 = \{\text{Bob, Dan}\},\ E_3 = \{\text{Chloe, Dan}\}\,\}.$$

The *line hypergraph* $L(H)$ has vertex set $\mathcal{E}$, and for each person $v \in V$ we add a hyperedge collecting all meetings that v attends:

$$\text{Star}_H(\text{Alice}) = \{E_1\}, \quad \text{Star}_H(\text{Bob}) = \{E_1, E_2\}, \quad \text{Star}_H(\text{Chloe}) = \{E_1, E_3\}, \quad \text{Star}_H(\text{Dan}) = \{E_2, E_3\}.$$

Hence

$$L(H) = \Big(\ \{E_1, E_2, E_3\},\ \big\{\{E_1\}, \{E_1, E_2\}, \{E_1, E_3\}, \{E_2, E_3\}\big\}\ \Big).$$

Interpretation: vertices are meetings; a hyperedge $\{E_i, E_j\}$ says "the two meetings share at least one common attendee" (here, Bob or Dan or Chloe).

2.3 Line SuperHyperGraph

A line superhypergraph maps each superedge to a vertex, linking them through superedges containing a common supervertex in the original structure.

Definition 2.6 (Line SuperHyperGraph)**.** Let $\mathcal{H}^{(n)} = (V_n, \mathcal{E})$ be a level-n SuperHyperGraph. Its *line SuperHyperGraph* is the pair

$$\mathsf{L}^{(n)}(\mathcal{H}) = (V'_{n+1}, \mathcal{E}'_{n+1})$$

defined by

$$V'_{n+1} := \mathcal{E} \qquad \text{and} \qquad \mathcal{E}'_{n+1} := \big\{\, \mathsf{Star}_{\mathcal{H}}(v) \subseteq \mathcal{E} \ :\ v \in V_n,\ \mathsf{Star}_{\mathcal{H}}(v) \neq \emptyset \,\big\}.$$

Intuitively, $\mathsf{L}^{(n)}(\mathcal{H})$ has one vertex for each superedge of $\mathcal{H}$; for every supervertex $v \in V_n$, we add a (hyper)edge collecting *all* superedges that contain v.

Remark 2.7 (Level bookkeeping and singletons)**.** Since $V_n \subseteq \text{POWS}^n(V_0)$, we have $V'_{n+1} = \mathcal{E} \subseteq \text{POWS}(V_n) \subseteq \text{POWS}\big(\text{POWS}^n(V_0)\big) = \text{POWS}^{n+1}(V_0)$. Thus vertices of $\mathsf{L}^{(n)}(\mathcal{H})$ naturally live one level up. Optionally replacing $\mathsf{Star}_{\mathcal{H}}(v)$ by $\mathcal{E}^{(\geq 2)}(v)$ removes singleton edges without changing the 2-section adjacency. We keep the nonempty stars for generality.

Example 2.8 (Line SuperHyperGraph — programs sharing teams)**.** Start with a level-1 SuperHyperGraph that models *teams of people* and *programs grouping teams*. Let the base set of individuals be $V_0 = \{a, b, c, d\}$. Take the level-1 supervertex set (teams)

$$V_1 = \{\, T_1 = \{a, b\},\ T_2 = \{b, c\},\ T_3 = \{c, d\}\,\} \subseteq \text{POW}(V_0),$$

and define superedges (programs) as sets of teams:

$$\mathcal{E} = \{\, P_1 = \{T_1, T_2\},\ P_2 = \{T_2, T_3\} \,\} \subseteq \mathrm{POW}(V_1).$$

The *line SuperHyperGraph* $\mathsf{L}^{(1)}(\mathcal{H})$ has vertex set $V'_2 = \mathcal{E} = \{P_1, P_2\}$. For each team $T \in V_1$, add one (super)hyperedge that collects all programs containing T:

$$\mathrm{Star}_{\mathcal{H}}(T_1) = \{P_1\}, \quad \mathrm{Star}_{\mathcal{H}}(T_2) = \{P_1, P_2\}, \quad \mathrm{Star}_{\mathcal{H}}(T_3) = \{P_2\}.$$

Therefore

$$\mathsf{L}^{(1)}(\mathcal{H}) = \Big(\, \{P_1, P_2\},\ \big\{\{P_1\}, \{P_1, P_2\}, \{P_2\}\big\} \,\Big).$$

Real-world reading: vertices are *programs*; a (super)hyperedge joins all programs that *share the same team.* Thus $\{P_1, P_2\}$ indicates that the two programs overlap operationally through team T_2.

Theorem 2.9 (Well-definedness and SuperHyperGraph property)**.** *For any level-n SuperHyperGraph $\mathcal{H}^{(n)} = (V_n, \mathcal{E})$, the line construction $\mathsf{L}^{(n)}(\mathcal{H}) = (V'_{n+1}, \mathcal{E}'_{n+1})$ of Definition 2.6 is a level-$(n+1)$ SuperHyperGraph. In particular,*

$$V'_{n+1} \subseteq \mathrm{POWS}^{n+1}(V_0) \quad and \quad \emptyset \neq \mathcal{E}'_{n+1} \subseteq \mathrm{POWS}(V'_{n+1}) \setminus \{\emptyset\}.$$

Proof. First, $V'_{n+1} = \mathcal{E} \subseteq \mathrm{POWS}(V_n) \subseteq \mathrm{POWS}^{n+1}(V_0)$ as noted in Remark 2.7; hence V'_{n+1} is a finite set of $(n+1)$-level objects. Second, by construction every $\mathsf{Star}_{\mathcal{H}}(v)$ is a subset of $\mathcal{E} = V'_{n+1}$. Nonemptiness of $\mathsf{Star}_{\mathcal{H}}(v)$ is part of the definition of $\mathcal{E}'_{n+1}$, so $\mathcal{E}'_{n+1} \subseteq \mathrm{POWS}(V'_{n+1}) \setminus \{\emptyset\}$. Finally, because $\mathcal{H}^{(n)}$ is finite, only finitely many $v \in V_n$ yield nonempty stars, so $\mathcal{E}'_{n+1}$ is finite and nonempty whenever $\mathcal{E} \neq \emptyset$. Therefore $\mathsf{L}^{(n)}(\mathcal{H})$ satisfies Definition 1.6 with level $n+1$. □

Definition 2.10 (2–section of a (Super)HyperGraph)**.** For a pair $(U, \mathcal{F})$ with U finite and $\emptyset \neq \mathcal{F} \subseteq \mathrm{POWS}(U) \setminus \{\emptyset\}$, its *2–section* $[\,(U, \mathcal{F})\,]_2$ is the (simple) graph on vertex set U in which distinct $x, y \in U$ are adjacent iff there exists $F \in \mathcal{F}$ with $\{x, y\} \subseteq F$.

Theorem 2.11 (Graph case)**.** *Let $G = (V, E)$ be a finite simple loopless graph (so $E \subseteq \mathrm{POWS}(V)$, $|e| = 2$ for all $e \in E$). Regard G as a level-0 SuperHyperGraph $\mathcal{H}^{(0)} = (V, \mathcal{E})$ with $\mathcal{E} = E$. Then the 2–section of its line SuperHyperGraph coincides with the classical line graph:*

$$\big[\, \mathsf{L}^{(0)}(\mathcal{H}) \,\big]_2 \;=\; L(G).$$

Proof. By Definition 2.6, the vertex set of $\mathsf{L}^{(0)}(\mathcal{H})$ is $V'_1 = \mathcal{E} = E$; we therefore identify vertices with edges of G. For $v \in V$, the star $\mathsf{Star}_{\mathcal{H}}(v) = \{\, e \in E : v \in e \,\}$ is precisely the set of all edges of G incident with v. In the 2–section $[\, \mathsf{L}^{(0)}(\mathcal{H}) \,]_2$, two distinct vertices $e, f \in E$ are adjacent iff there exists $v \in V$ with $\{e, f\} \subseteq \mathsf{Star}_{\mathcal{H}}(v)$, i.e. iff e and f are both incident with v, equivalently $e \cap f \neq \emptyset$. This is exactly the adjacency rule of the line graph $L(G)$. □

Theorem 2.12 (Hypergraph case: $\mathsf{L}^{(0)}(H)$ is the incidence line hypergraph)**.** *Let $H = (V, \mathcal{E})$ be a finite hypergraph (i.e. a level-0 SuperHyperGraph). Then $\mathsf{L}^{(0)}(H) = (\mathcal{E}, \mathcal{E}')$ with*

$$\mathcal{E}' \;=\; \big\{\, \{\, E \in \mathcal{E} : v \in E \,\} \;:\; v \in V,\ \{E \in \mathcal{E} : v \in E\} \neq \emptyset \,\big\}$$

is exactly the standard incidence *line hypergraph of H: its vertices are the edges of H, and for each $v \in V$ we add one hyperedge consisting of all edges incident with v.*

Proof. This is immediate from Definition 2.6 with $n = 0$. By construction, the vertex set is $\mathcal{E}$ (the original hyperedges), and for each $v \in V$ the star $\mathsf{Star}_{\mathcal{H}}(v)$ collects exactly those edges of H that contain v. □

Corollary 2.13 (Compatibility with the classical line graph via 2–section)**.** *For a graph G viewed as a rank-2 hypergraph $H_G = (V, \mathcal{E})$ with $\mathcal{E} = \{\{u, v\} : uv \in E(G)\}$, the 2–section $[\, \mathsf{L}^{(0)}(H_G) \,]_2$ equals the classical line graph $L(G)$.*

Proof. Apply Theorem 2.11 with $\mathcal{H}^{(0)} = H_G$. □

3 Review and Result: Total Graph

3.1 Total Graph

A total graph has vertices for both vertices and edges of a graph, joining them by adjacency or incidence relationships (cf. [30–37]).

Definition 3.1 (Total graph)**.** Let $G = (V(G), E(G))$ be a loopless simple undirected graph. The *total graph* $T(G)$ is the (simple) graph with

$$V\big(T(G)\big) = V(G) \,\dot\cup\, E(G),$$

where two distinct vertices $x, y \in V(G) \cup E(G)$ are adjacent in $T(G)$ iff one of the following holds:

1. (vertex–vertex) $x, y \in V(G)$ and $xy \in E(G)$;
2. (edge–edge) $x, y \in E(G)$ and $x \cap y \neq \emptyset$ in G;
3. (vertex–edge) $\{x, y\} = \{v, e\}$ with $v \in V(G)$, $e \in E(G)$, and $v \in e$.

Example 3.2 (Total Graph — a tiny road network)**.** Consider three intersections $V(G) = \{A, B, C\}$ with two roads $E(G) = \{AB,\ BC\}$. The *total graph* $T(G)$ has vertex set

$$V\big(T(G)\big) = \{A, B, C,\ AB,\ BC\}.$$

Edges in $T(G)$ encode: (i) road adjacency between intersections, (ii) two roads sharing an endpoint, and (iii) incidence between an intersection and a road passing through it:

$$\begin{array}{ll} \text{(vertex–vertex)} & \{A, B\},\ \{B, C\}; \\ \text{(edge–edge)} & \{AB, BC\} \quad \text{(they share } B\text{)}; \\ \text{(vertex–edge)} & \{A, AB\},\ \{B, AB\},\ \{B, BC\},\ \{C, BC\}. \end{array}$$

Interpretation: $T(G)$ simultaneously represents the places (intersections) and the links (roads), and how they relate (including transfer from a place onto a road, and between adjacent roads).

3.2 Total HyperGraph

A total hypergraph contains original vertices and hyperedges, connecting them through vertex–vertex, edge–edge, and vertex–edge incidence hyperedges.

Definition 3.3 (Total hypergraph)**.** Let $H = (V, E)$ be a hypergraph. Form the disjoint union of the ground objects

$$U \;:=\; V \,\dot\cup\, E,$$

where we regard vertices $v \in V$ and hyperedges $e \in E$ as distinct types of elements of U. Define three families of subsets of U:

$$\mathcal{A} \;:=\; \{\, e \subseteq V \;:\; e \in E \,\}, \qquad \mathcal{B} \;:=\; \{\, E_H(v) \subseteq E \;:\; v \in V,\ |E_H(v)| \geq 2 \,\},$$

$$C \;:=\; \{\, \{v\} \cup E_H(v) \subseteq U \;:\; v \in V \,\}.$$

The *total hypergraph* of H is the hypergraph

$$T(H) \;:=\; \big(U,\ \mathcal{A} \,\cup\, \mathcal{B} \,\cup\, C\big).$$

Intuitively: $\mathcal{A}$ records vertex–vertex co-membership (original hyperedges), $\mathcal{B}$ records edge–edge intersection at a common vertex, and C records vertex–edge incidence.

Example 3.4 (Total HyperGraph — meetings and attendees)**.** Let $H = (V, \mathcal{E})$ encode people and meetings:

$$V = \{\text{Alice}, \text{Bob}, \text{Chloe}\}, \qquad \mathcal{E} = \{E_1 = \{\text{Alice}, \text{Bob}\},\ E_2 = \{\text{Bob}, \text{Chloe}\}\}.$$

Form $U := V \dot\cup \mathcal{E} = \{\text{Alice}, \text{Bob}, \text{Chloe}, E_1, E_2\}$. The *total hypergraph* $T(H) = (U, \mathcal{A} \cup \mathcal{B} \cup C)$ has:

$$\mathcal{A} = \{E_1, E_2\}, \quad \mathcal{B} = \big\{\, \mathcal{E}_H(\text{Bob}) = \{E_1, E_2\} \,\big\}, \quad C = \big\{\, \{\text{Alice}, E_1\},\ \{\text{Bob}, E_1, E_2\},\ \{\text{Chloe}, E_2\} \,\big\}.$$

Here $\mathcal{A}$ records each original meeting (as a subset of people), $\mathcal{B}$ groups meetings sharing the same attendee (Bob attends both), and C couples each person with all meetings they attend. Thus $T(H)$ captures people, meetings, and their incidences within one hypergraph.

Theorem 3.5 (Well-definedness)**.** *For every hypergraph $H = (V, E)$, the pair $T(H) = \big(U, \mathcal{A} \cup \mathcal{B} \cup C\big)$ is a hypergraph.*

Proof. By construction $U = V \dot\cup E$ is a finite set. Each member of $\mathcal{A}$ is a nonempty subset of $V \subseteq U$ because $E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$. Each member of $\mathcal{B}$ is a subset of $E \subseteq U$ and is required to have size at least 2, hence nonempty. Each member of C has the form $\{v\} \cup E_H(v)$ and is therefore nonempty (it contains v). Thus $\mathcal{A} \cup \mathcal{B} \cup C \subseteq \mathcal{P}(U) \setminus \{\emptyset\}$, so $T(H)$ meets the definition of a hypergraph. □

Definition 3.6 (2–section of a hypergraph)**.** For a hypergraph $K = (U, \mathcal{F})$, its *2–section* $[K]_2$ is the (simple) graph with vertex set U in which distinct $x, y \in U$ are adjacent iff there exists $F \in \mathcal{F}$ with $\{x, y\} \subseteq F$.

Theorem 3.7 (Generalization)**.** *Let $G = (V(G), E(G))$ be a loopless simple graph and let H_G be the rank-2 hypergraph with*

$$V(H_G) = V(G), \qquad E(H_G) = \{\, \{u, v\} : uv \in E(G) \,\}.$$

Then the 2–section of the total hypergraph of H_G coincides with the total graph of G:

$$\big[\, T(H_G) \,\big]_2 \;=\; T(G).$$

Proof. Write $U = V(G) \dot\cup E(G)$. We must show that two distinct elements of U are adjacent in $[T(H_G)]_2$ iff they are adjacent in $T(G)$.

($\Rightarrow$) Suppose x, y are adjacent in $[T(H_G)]_2$. Then $\{x, y\} \subseteq F$ for some $F \in \mathcal{A} \cup \mathcal{B} \cup C$ as defined for H_G. We consider cases.

1) If $F \in \mathcal{A}$, then $F = e = \{u, v\}$ for some edge $e = uv \in E(G)$. Hence $x, y \in V(G)$ and $\{x, y\} = \{u, v\}$, so x and y are adjacent in $T(G)$ by rule (vertex–vertex).

2) If $F \in \mathcal{B}$, then $F = E_{H_G}(v) = \{\, e \in E(G) : v \in e \,\}$ for some $v \in V(G)$ with $|E_{H_G}(v)| \geq 2$. Thus $x, y \in E(G)$ and both are incident with v; hence x and y share an endpoint in G and are adjacent in $T(G)$ by rule (edge–edge).

3) If $F \in C$, then $F = \{v\} \cup E_{H_G}(v)$ for some $v \in V(G)$. If $x, y \in V(G)$, impossible since F contains exactly one vertex element v. If $x, y \in E(G)$, then both lie in $E_{H_G}(v)$ and we are back to case 2). Otherwise $\{x, y\} = \{v, e\}$ with $e \in E(G)$ incident to v; hence x and y are adjacent in $T(G)$ by rule (vertex–edge).

Thus in all cases, adjacency in $[T(H_G)]_2$ implies adjacency in $T(G)$.

($\Leftarrow$) Conversely, suppose x, y are adjacent in $T(G)$. Again split by rule.

1) If $x, y \in V(G)$ with $xy \in E(G)$, then $\{x, y\} \subseteq e$ for $e = \{x, y\} \in \mathcal{A}$, so x, y are adjacent in $[T(H_G)]_2$.

2) If $x, y \in E(G)$ share an endpoint v, then $\{x, y\} \subseteq E_{H_G}(v) \in \mathcal{B}$, hence adjacent in $[T(H_G)]_2$.

3) If $\{x, y\} = \{v, e\}$ with $v \in V(G)$, $e \in E(G)$, $v \in e$, then $\{x, y\} \subseteq \{v\} \cup E_{H_G}(v) \in C$, hence adjacent in $[T(H_G)]_2$.

Therefore adjacency in $T(G)$ implies adjacency in $[T(H_G)]_2$.

Combining both directions yields $[\, T(H_G) \,]_2 = T(G)$. □

3.3 Total SuperHyperGraph

A total superhypergraph extends total hypergraphs to hierarchical supervertex structures, linking supervertices and superedges by membership, intersection, or incidence.

Definition 3.8 (Total superhypergraph)**.** Let $\mathcal{H}^{(n)} = (V_n, \mathcal{E})$. Define three edge-families in U_{n+1}:

$$\mathcal{A} := \{\, \iota(E) \subseteq \iota(V_n) \ : \ E \in \mathcal{E} \,\}, \qquad \mathcal{B} := \{\, \mathrm{Star}_{\mathcal{H}}(v) \subseteq \mathcal{E} \ : \ v \in V_n,\ |\,\mathrm{Star}_{\mathcal{H}}(v)| \geq 2 \,\},$$

$$C := \{\, \{\iota(v)\} \cup \mathrm{Star}_{\mathcal{H}}(v) \ \subseteq U_{n+1} \ : \ v \in V_n \,\}.$$

The *total superhypergraph* of $\mathcal{H}^{(n)}$ is

$$\mathbf{T}(\mathcal{H}^{(n)}) \ := \ \big(U_{n+1},\ \mathcal{A} \cup \mathcal{B} \cup C\big).$$

(Respectively: vertex–vertex co-membership, edge–edge "meet at a vertex," and vertex–edge incidence.)

Example 3.9 (Total SuperHyperGraph — programs, teams, and members)**.** Start from a level-1 superhypergraph whose supervertices are *teams of people* and whose superedges are *programs made of teams*. Let the base set of individuals be $V_0 = \{a, b, c\}$. Take the team set

$$V_1 = \{T_1 = \{a, b\},\ T_2 = \{b, c\}\} \subseteq \mathcal{P}(V_0),$$

and the programs

$$\mathcal{E} = \{P_1 = \{T_1, T_2\},\ P_2 = \{T_2\}\} \subseteq \mathcal{P}(V_1).$$

Promote teams via $\iota(T) := \{T\}$ and form the level-2 universe

$$U_2 = \{\iota(T_1), \iota(T_2)\} \ \dot\cup \ \{P_1, P_2\}.$$

The *total superhypergraph* $\mathbf{T}(\mathcal{H}^{(1)}) = (U_2, \mathcal{A} \cup \mathcal{B} \cup C)$ has

$$\mathcal{A} = \{\ \iota(P_1) = \{\iota(T_1), \iota(T_2)\},\ \ \iota(P_2) = \{\iota(T_2)\}\ \},$$

$$\mathcal{B} = \big\{\ \mathrm{Star}_{\mathcal{H}}(T_2) = \{P_1, P_2\}\ \big\} \quad (\mathrm{Star}_{\mathcal{H}}(T_1) = \{P_1\} \text{ is size 1}),$$

$$C = \big\{\ \{\iota(T_1), P_1\},\ \{\iota(T_2), P_1, P_2\}\ \big\}.$$

Reading: vertices are "singleton teams" $\iota(T_i)$ and programs P_j. The family $\mathcal{A}$ encodes which teams each program contains; $\mathcal{B}$ links programs that share a team (here T_2 participates in two programs); and C records, for each team, the joint presence of that team with all programs using it.

Proposition 3.10 (Well-definedness and level shift)**.** *If $\mathcal{H}^{(n)}$ is a level-n SuperHyperGraph, then $\mathbf{T}(\mathcal{H}^{(n)})$ is a level-$(n{+}1)$ SuperHyperGraph.*

Proof. By the Definition, $U_{n+1} \subseteq \mathcal{P}(V_n) \subseteq \mathcal{P}^{n+1}(V_0)$. By the Definition, every member of $\mathcal{A} \cup \mathcal{B} \cup C$ is a nonempty subset of U_{n+1}. Finiteness is inherited from $\mathcal{H}^{(n)}$. Hence $\mathbf{T}(\mathcal{H}^{(n)})$ satisfies the Definition at level $n{+}1$. □

Theorem 3.11 (Reduction to the Total HyperGraph when $n = 0$)**.** *Let $H = (V, \mathcal{E})$ be a hypergraph (so level $n = 0$). Identify $\iota(V) \cong V$ via $\iota(v) \leftrightarrow v$. Then there is a canonical isomorphism*

$$\mathbf{T}(H) \ \cong \ T(H).$$

Proof. For $n = 0$, the universe $U_1 = \iota(V)\dot\cup\mathcal{E}$ of $\mathbf{T}(H)$ corresponds bijectively to $V\dot\cup\mathcal{E}$ of $T(H)$ by $\iota(v) \leftrightarrow v$ and $E \leftrightarrow E$. Under this identification:

$$\mathcal{A} \text{ of } \mathbf{T}(H) \ \leftrightarrow \ \{E : E \in \mathcal{E}\} \text{ of } T(H), \qquad \mathcal{B} \ \leftrightarrow \ \{\mathcal{E}_H(v) : v \in V,\ |\mathcal{E}_H(v)| \geq 2\},$$

$$C \ \leftrightarrow \ \{\{v\} \cup \mathcal{E}_H(v) : v \in V\}.$$

Thus vertex and edge sets correspond termwise, giving an isomorphism $\mathbf{T}(H) \cong T(H)$. □

Theorem 3.12 (Line SuperHyperGraph as an induced part of the Total SuperHyperGraph)**.** *For every level-n SuperHyperGraph $\mathcal{H}^{(n)} = (V_n, \mathcal{E})$, the Line SuperHyperGraph appears as the induced sub(super)hypergraph on the vertex set $\mathcal{E}$:*

$$\mathbf{L}(\mathcal{H}^{(n)}) \ = \ \mathbf{T}(\mathcal{H}^{(n)})\big[\,\mathcal{E}\,\big].$$

Proof. In $\mathbf{T}(\mathcal{H}^{(n)})$ the vertices are $\iota(V_n)\dot\cup\mathcal{E}$. Restricting to the vertex set $\mathcal{E}$ deletes all $\iota(v)$'s and keeps only hyperedges contained in $\mathcal{E}$. Among $\mathcal{A} \cup \mathcal{B} \cup C$, the only members contained in $\mathcal{E}$ are precisely the $\mathcal{B}$-family $\{\mathrm{Star}_{\mathcal{H}}(v) \ : \ v \in V_n\}$, which is exactly the edge set of $\mathbf{L}(\mathcal{H}^{(n)})$ by the Definition. Hence $\mathbf{T}(\mathcal{H}^{(n)})[\,\mathcal{E}\,] = \mathbf{L}(\mathcal{H}^{(n)})$. □

4 Review and Result: iterated line graphs

4.1 iterated line graphs

An iterated line graph is formed by repeatedly applying the line graph transformation to a graph, using each result as the next input [38–43].

Definition 4.1 (Iterated line graphs)**.** Define $L^0(G) := G$. For each integer $n \geq 1$, the *n-th iterated line graph* of G is defined recursively by

$$L^n(G) \;:=\; L\big(L^{n-1}(G)\big).$$

Equivalently, many authors write $L_1(G) = L(G)$ and $L_n(G) = L\big(L_{n-1}(G)\big)$ for $n \geq 2$.

Example 4.2 (Iterated line graphs — subway transfers and transfer-of-transfers)**.** Consider a tiny subway map with stations $V(G) = \{A, B, C, D\}$ and tracks

$$E(G) = \{\, e_1 = AB,\ e_2 = BC,\ e_3 = CD \,\}.$$

The line graph $L(G)$ has one vertex per track and joins two tracks when they meet at a station:

$$V\big(L(G)\big) = \{e_1, e_2, e_3\}, \qquad E\big(L(G)\big) = \{\, e_1e_2,\ e_2e_3 \,\} \cong P_2.$$

Here $L(G)$ encodes *one-stop transfers*: e_1 connects to e_2 at B, and e_2 connects to e_3 at C. The iterated line graph $L^2(G) = L(L(G))$ has

$$V\big(L^2(G)\big) = E\big(L(G)\big) = \{\, e_1e_2,\ e_2e_3 \,\}, \qquad E\big(L^2(G)\big) = \{\, (e_1e_2)(e_2e_3) \,\} \cong K_2.$$

Thus $L^2(G)$ represents *transfer-of-transfer* possibilities (two consecutive transfers through the middle track e_2).

4.2 iterated line hypergraphs

An iterated line hypergraph applies the line hypergraph construction repeatedly to a hypergraph, capturing higher-order adjacency of hyperedges over iterations.

Definition 4.3 (Iterated line hypergraphs)**.** Define $L^0(H) := H$ and, for each $n \geq 1$, set

$$L^n(H) \;:=\; L\big(L^{n-1}(H)\big).$$

Example 4.4 (Iterated line hypergraphs — meetings sharing attendees, then "overlaps of overlaps")**.** Let a schedule be a hypergraph $H = (V, \mathcal{E})$ with people

$$V = \{\text{Alice}, \text{Bob}, \text{Chloe}, \text{Dan}\}$$

and meetings (hyperedges)

$$\mathcal{E} = \{\, E_1 = \{\text{Alice}, \text{Bob}\},\ E_2 = \{\text{Bob}, \text{Chloe}\},\ E_3 = \{\text{Chloe}, \text{Dan}\} \,\}.$$

The line hypergraph $L(H)$ has vertices $\{E_1, E_2, E_3\}$ and, for each person v, one hyperedge collecting all meetings that v attends:

$$\text{Star}_H(\text{Alice}) = \{E_1\}, \quad \text{Star}_H(\text{Bob}) = \{E_1, E_2\}, \quad \text{Star}_H(\text{Chloe}) = \{E_2, E_3\}, \quad \text{Star}_H(\text{Dan}) = \{E_3\}.$$

Hence

$$L(H) = \Big(\, \{E_1, E_2, E_3\},\ \big\{\{E_1\}, \{E_1, E_2\}, \{E_2, E_3\}, \{E_3\}\big\} \,\Big).$$

Iterating once more, $L^2(H) = L(L(H))$ has vertex set

$$V\big(L^2(H)\big) = \big\{\, \{E_1\},\ \{E_1, E_2\},\ \{E_2, E_3\},\ \{E_3\} \,\big\},$$

and its hyperedges correspond to each original meeting E_i: gather all vertices of $L^2(H)$ that contain E_i. Concretely,

$$\text{for } E_1 : \ \{\, \{E_1\}, \{E_1, E_2\} \,\}, \qquad \text{for } E_2 : \ \{\, \{E_1, E_2\}, \{E_2, E_3\} \,\}, \qquad \text{for } E_3 : \ \{\, \{E_2, E_3\}, \{E_3\} \,\}.$$

Thus $L^2(H)$ captures *overlaps of overlaps*: two "meeting-groups" are linked when they share a common meeting.

Theorem 4.5 (Trivial recovery of the line hypergraph)**.** *For every hypergraph H, the first iterate coincides with the line hypergraph:*

$$L^1(H) = L(H).$$

Proof. By the recursive definition $L^1(H) := L(L^0(H)) = L(H)$. □

Lemma 4.6 (2–section after two line-steps)**.** *For any hypergraph $K = (U, \mathcal{F})$,*

$$\big[\, L^2(K) \,\big]_2 \;\cong\; [K]_2.$$

Proof. By definition,

$$L(K) = \big(\, \mathcal{F}, \;\; \{\mathrm{Star}_K(u) : u \in U, \; \mathrm{Star}_K(u) \neq \emptyset\} \,\big).$$

Hence

$$L^2(K) \;=\; L\big(L(K)\big)$$

has vertex set $\{\mathrm{Star}_K(u) : u \in U, \; \mathrm{Star}_K(u) \neq \emptyset\}$. Two distinct vertices $\mathrm{Star}_K(u)$ and $\mathrm{Star}_K(v)$ of $L^2(K)$ are adjacent in the 2–section $[L^2(K)]_2$ iff there exists a hyperedge of $L(K)$ containing both of them, i.e. iff there exists $F \in \mathcal{F}$ such that

$$\mathrm{Star}_K(u) \ni F \quad \text{and} \quad \mathrm{Star}_K(v) \ni F,$$

which is equivalent to $u \in F$ and $v \in F$. This, in turn, is exactly the adjacency condition for u and v in $[K]_2$. The map $\mathrm{Star}_K(u) \mapsto u$ is therefore a graph isomorphism $[L^2(K)]_2 \to [K]_2$. □

Corollary 4.7 (All iterates, seen through 2–sections)**.** *For any hypergraph K and any integer $m \geq 0$,*

$$\big[\, L^{2m}(K) \,\big]_2 \;\cong\; [K]_2, \qquad \big[\, L^{2m+1}(K) \,\big]_2 \;\cong\; \big[\, L(K) \,\big]_2.$$

Proof. Apply Lemma 4.6 repeatedly: $[L^{2(m+1)}(K)]_2 \cong [L^{2m}(K)]_2 \cong \cdots \cong [K]_2$, and similarly $[L^{2m+3}(K)]_2 \cong [L^{2m+1}(K)]_2 \cong [L(K)]_2$. □

Definition 4.8 (Graphs as 2–uniform hypergraphs)**.** For a (finite, loopless, simple) graph $G = (V, E)$, write $H_G := (V, \mathcal{E})$ with $\mathcal{E} = \{\{u, v\} : uv \in E\}$. Then $[H_G]_2 = G$.

Lemma 4.9 (One step: line graph via 2–section)**.** *For every graph G,*

$$\big[\, L(H_G) \,\big]_2 \;=\; L(G).$$

Proof. Vertices of $L(H_G)$ are edges of G. Two such vertices (i.e. two edges of G) are adjacent in the 2–section iff they belong to a common star $\mathrm{Star}_{H_G}(v)$, that is, iff they are both incident with v in G. This is exactly the definition of adjacency in $L(G)$. □

Theorem 4.10 (Iterated line hypergraphs generalize line graphs and their iteration (up to the natural 2–section))**.** *Let G be a finite simple graph and H_G its associated 2–uniform hypergraph. For every integer $m \geq 0$,*

$$\big[\, L^{2m}(H_G) \,\big]_2 \;=\; G, \qquad \big[\, L^{2m+1}(H_G) \,\big]_2 \;=\; L(G).$$

Consequently:

- *taking $m = 0$ in the second equality recovers the classical line graph $L(G)$ from the (first) line hypergraph of H_G;*
- *taking $n = 1$ in Theorem 4.5 recovers the usual line hypergraph of an arbitrary hypergraph;*
- *the iterated line hypergraph sequence $\{L^n(H_G)\}_{n \geq 0}$ forms a* hypergraph lift *of the classical iterated line graph process, whose 2–sections alternate between G and $L(G)$.*

Proof. By Corollary 4.7 applied to $K = H_G$, $[L^{2m}(H_G)]_2 \cong [H_G]_2 = G$. Similarly, $[L^{2m+1}(H_G)]_2 \cong [L(H_G)]_2$, and Lemma 5.12 gives $[L(H_G)]_2 = L(G)$. □

4.3 iterated line superhypergraphs

An iterated line superhypergraph repeatedly applies the line superhypergraph transformation to a superhypergraph, modeling evolving hierarchical incidence patterns across multiple levels.

Definition 4.11 (Iterated line superhypergraphs)**.** For an initial level-n superhypergraph $\mathcal{H}^{(n)}$, define

$$\mathbf{L}^0(\mathcal{H}^{(n)}) := \mathcal{H}^{(n)}, \qquad \mathbf{L}^{t+1}(\mathcal{H}^{(n)}) := \mathbf{L}\big(\mathbf{L}^t(\mathcal{H}^{(n)})\big) \quad (t \geq 0).$$

By Proposition, $\mathbf{L}^t(\mathcal{H}^{(n)})$ is level $n+t$.

Example 4.12 (Iterated line superhypergraphs — programs (by teams) and shared-team structure across two iterations)**.** Start from a level-1 superhypergraph where supervertices are teams of people and superedges are programs (collections of teams). Let the individual set be $V_0 = \{a, b, c, d\}$. Take teams

$$V_1 = \{\, T_1 = \{a, b\},\ T_2 = \{b, c\},\ T_3 = \{c, d\} \,\} \subseteq \mathcal{P}(V_0),$$

and programs (superedges)

$$\mathcal{E} = \{\, P_1 = \{T_1, T_2\},\ P_2 = \{T_2, T_3\} \,\} \subseteq \mathcal{P}(V_1).$$

The line superhypergraph $\mathbf{L}^1(\mathcal{H})$ has vertex set $\{P_1, P_2\}$ and, for each team T, a (super)hyperedge collecting the programs that include T:

$$\mathrm{Star}_{\mathcal{H}}(T_1) = \{P_1\}, \quad \mathrm{Star}_{\mathcal{H}}(T_2) = \{P_1, P_2\}, \quad \mathrm{Star}_{\mathcal{H}}(T_3) = \{P_2\}.$$

Hence

$$\mathbf{L}^1(\mathcal{H}) = \Big(\ \{P_1, P_2\},\ \big\{\{P_1\}, \{P_1, P_2\}, \{P_2\}\big\}\ \Big).$$

Iterating again, $\mathbf{L}^2(\mathcal{H}) = \mathbf{L}\big(\mathbf{L}^1(\mathcal{H})\big)$ has vertices

$$V\big(\mathbf{L}^2(\mathcal{H})\big) = \big\{\ S_1 = \{P_1\},\ S_{12} = \{P_1, P_2\},\ S_2 = \{P_2\}\ \big\},$$

and for each program P_j we add a (super)hyperedge consisting of all S-vertices that contain P_j:

$$\text{for } P_1 :\ \{S_1, S_{12}\}, \qquad \text{for } P_2 :\ \{S_{12}, S_2\}.$$

Interpretation: $\mathbf{L}^1$ tells which programs share a team; $\mathbf{L}^2$ then records how those *program-groups* themselves are linked by sharing a common program.

Example 4.13 (An iterated line 2–superhypergraph (two iterations))**. Step 0: Build a level-2 superhypergraph.** Let the base set of individuals be $V_0 = \{a, b, c, d\}$. Form level-1 supervertices (teams)

$$V_1 = \big\{\, T_1 = \{a, b\},\ T_2 = \{b, c\},\ T_3 = \{c, d\} \,\big\} \subseteq \mathcal{P}(V_0),$$

and level-2 supervertices (collections of teams)

$$V_2 = \big\{\, X_1 = \{T_1, T_2\},\ \ X_2 = \{T_2, T_3\},\ \ X_3 = \{T_1\} \,\big\} \subseteq \mathcal{P}(V_1).$$

Define the level-2 superedges

$$\mathcal{E} = \big\{\, P_1 = \{X_1, X_2\},\ \ P_2 = \{X_2, X_3\} \,\big\} \subseteq \mathcal{P}(V_2),$$

and set $\mathcal{H}^{(2)} = (V_2, \mathcal{E})$.

Step 1: First line step $\mathbf{L}^1(\mathcal{H}^{(2)}) = \mathbf{L}(\mathcal{H}^{(2)})$. By definition, the vertex set is the superedge set:

$$V\big(\mathbf{L}^1\big) = \mathcal{E} = \{P_1, P_2\}.$$

For each $v \in V_2$, form the star $\mathrm{Star}_{\mathcal{H}}(v) = \{E \in \mathcal{E} : v \in E\}$:

$$\mathrm{Star}_{\mathcal{H}}(X_1) = \{P_1\}, \qquad \mathrm{Star}_{\mathcal{H}}(X_2) = \{P_1, P_2\}, \qquad \mathrm{Star}_{\mathcal{H}}(X_3) = \{P_2\}.$$

Thus the (super)edge set of $\mathbf{L}^1$ is

$$\mathcal{E}^{(1)} = \big\{ \{P_1\},\ \{P_1, P_2\},\ \{P_2\} \big\},$$

and

$$\mathbf{L}^1(\mathcal{H}^{(2)}) = \Big(\{P_1, P_2\},\ \big\{\{P_1\}, \{P_1, P_2\}, \{P_2\}\big\} \Big).$$

Step 2: Second line step $\mathbf{L}^2(\mathcal{H}^{(2)}) = \mathbf{L}\big(\mathbf{L}^1(\mathcal{H}^{(2)})\big)$. Now the vertices are the superedges of $\mathbf{L}^1$:

$$V\big(\mathbf{L}^2\big) = \big\{ S_1 = \{P_1\},\ S_{12} = \{P_1, P_2\},\ S_2 = \{P_2\} \big\}.$$

For each old vertex $u \in V(\mathbf{L}^1) = \{P_1, P_2\}$, compute

$$\mathrm{Star}_{\mathbf{L}^1}(P_1) = \{ S_1, S_{12} \}, \qquad \mathrm{Star}_{\mathbf{L}^1}(P_2) = \{ S_{12}, S_2 \}.$$

Hence the (super)edge set of $\mathbf{L}^2$ is

$$\mathcal{E}^{(2)} = \big\{ \{S_1, S_{12}\},\ \{S_{12}, S_2\} \big\},$$

and

$$\mathbf{L}^2(\mathcal{H}^{(2)}) = \Big(\{S_1, S_{12}, S_2\},\ \big\{\{S_1, S_{12}\}, \{S_{12}, S_2\}\big\} \Big).$$

Interpretation. $\mathbf{L}^1$ records which programs P_1, P_2 share a common level-2 supervertex (here X_2). $\mathbf{L}^2$ then links the "program-groups" S_1, S_{12}, S_2 when they share a common program, yielding a simple chain $\{S_1, S_{12}\}, \{S_{12}, S_2\}$ at the next level.

Theorem 4.14 (First iterate recovers the line superhypergraph)**.** *For every level-n superhypergraph $\mathcal{H}^{(n)}$, $\mathbf{L}^1(\mathcal{H}^{(n)}) = \mathbf{L}(\mathcal{H}^{(n)})$ as in the Definition.*

Proof. Directly from Definition 4.11 with $t = 0$. □

Theorem 4.15 (Hypergraph case: one step equals the incidence line hypergraph)**.** *Let $H = (V, \mathcal{E})$ be an ordinary hypergraph ($n = 0$). Then*

$$\mathbf{L}(H) \ = \ \big(\mathcal{E},\ \{ \{E \in \mathcal{E} : v \in E \} : v \in V,\ \mathrm{Star}_H(v) \neq \emptyset \} \big),$$

i.e., the standard incidence-based line hypergraph. Consequently, $\mathbf{L}^t(H)$ is the iterated line hypergraph *sequence.*

Proof. This is exactly Definition **??** with $n = 0$. □

Lemma 4.16 (2–section parity identity)**.** *For any (super)hypergraph $K = (U, \mathcal{F})$,*

$$\big[\mathbf{L}^2(K)\big]_2 \ \cong \ [K]_2.$$

Proof. Write $\mathbf{L}(K) = (\mathcal{F}, \{\mathrm{Star}_K(u) : u \in U,\ \mathrm{Star}_K(u) \neq \emptyset\})$. Then $\mathbf{L}^2(K) = \mathbf{L}(\mathbf{L}(K))$ has vertex set $\{\mathrm{Star}_K(u) : u \in U,\ \mathrm{Star}_K(u) \neq \emptyset\}$. Two distinct such vertices $\mathrm{Star}_K(u), \mathrm{Star}_K(v)$ are adjacent in the 2–section iff they lie together in some hyperedge of $\mathbf{L}(K)$, i.e., iff there exists $F \in \mathcal{F}$ with $F \in \mathrm{Star}_K(u) \cap \mathrm{Star}_K(v)$, equivalently $u, v \in F$. This is exactly the adjacency condition in $[K]_2$. The map $\mathrm{Star}_K(u) \mapsto u$ is the desired graph isomorphism. □

Corollary 4.17 (Alternation for all iterates)**.** *For any K and $m \geq 0$,*

$$\big[\mathbf{L}^{2m}(K)\big]_2 \ \cong \ [K]_2, \qquad \big[\mathbf{L}^{2m+1}(K)\big]_2 \ \cong \ \big[\mathbf{L}(K)\big]_2.$$

Theorem 4.18 (Graphs as 2–uniform hypergraphs: lift of iterated line graphs)**.** *Let $G = (V, E)$ be a finite simple loopless graph and encode it as a 2–uniform hypergraph $H_G = (V, \mathcal{E})$ with $\mathcal{E} = \{\{u, v\} : uv \in E\}$. Then, for all $m \geq 0$,*

$$\big[\mathbf{L}^{2m}(H_G)\big]_2 \ = \ G, \qquad \big[\mathbf{L}^{2m+1}(H_G)\big]_2 \ = \ L(G).$$

In particular, $\mathbf{L}^1(H_G)$ projects (via 2–section) to the classical line graph, and the superhypergraph chain $\{\mathbf{L}^t(H_G)\}_{t \geq 0}$ is a superhypergraph lift *of the iterated line–graph process, alternating between G and $L(G)$ under $[\,\cdot\,]_2$.*

Proof. By Corollary 4.17 with $K = H_G$, $[\mathbf{L}^{2m}(H_G)]_2 \cong [H_G]_2 = G$. Also $[\mathbf{L}^{2m+1}(H_G)]_2 \cong [\mathbf{L}(H_G)]_2$. But in $\mathbf{L}(H_G)$, vertices are edges of G, and two such vertices are adjacent in the 2–section iff they belong to a common star $\mathrm{Star}_{H_G}(v)$, i.e., iff the corresponding edges of G share an endpoint v. Thus $[\mathbf{L}(H_G)]_2 = L(G)$. □

5 Review and Result: iterated total graphs

5.1 iterated total graphs

An iterated total graph repeatedly applies the total graph operation to a graph, incorporating vertices, edges, and all incidence relationships at each stage [44–46].

Definition 5.1 (Iterated total graphs)**.** Define $T^0(G) := G$, and for each integer $k \geq 1$ set

$$T^k(G) \ := \ T\big(T^{k-1}(G)\big).$$

(Thus $T^1(G) = T(G)$, $T^2(G) = T(T(G))$, etc.) This notation is also used in the literature. :contentReference[oaicite:2]index=2 :contentReference[oaicite:3]index=3

Example 5.2 (Iterated total graphs — roads, then "roads-of-roads")**.** Let G be a tiny road map with intersections $V(G) = \{A, B, C\}$ and roads

$$E(G) = \{\, AB,\ BC \,\}.$$

The total graph $T(G)$ has

$$V\big(T(G)\big) = \{A, B, C,\ AB,\ BC\},$$

and edges encoding: (i) intersection–intersection $\{A, B\}, \{B, C\}$, (ii) road–road $\{AB, BC\}$ (they share B), and (iii) intersection–road $\{A, AB\}, \{B, AB\}, \{B, BC\}, \{C, BC\}$. Thus

$$E\big(T(G)\big) = \{\{A, B\}, \{B, C\}, \{AB, BC\}, \{A, AB\}, \{B, AB\}, \{B, BC\}, \{C, BC\}\}.$$

Iterating once more, $T^2(G) = T(T(G))$ has vertex set

$$V\big(T^2(G)\big) = V\big(T(G)\big) \,\dot{\cup}\, E\big(T(G)\big),$$

i.e., the five previous vertices together with the seven "edge-vertices"

$$\{A, B\}, \{B, C\}, \{AB, BC\}, \{A, AB\}, \{B, AB\}, \{B, BC\}, \{C, BC\}$$

. Now $T^2(G)$ records, in one graph, *places and roads* (from $T(G)$) *and* how any two of those items co-occur inside a single rule of $T(G)$—a compact way to encode two layers of incidence.

Lemma 5.3 (Line sits inside total)**.** *For every graph X, the line graph $L(X)$ is an induced subgraph of the total graph $T(X)$.*

Proof. Map each vertex $e \in V(L(X)) = E(X)$ to the vertex $e \in V(T(X))$ (the same edge, now viewed as a vertex of $T(X)$). Two vertices e, f are adjacent in $L(X)$ iff e and f share an endpoint in X, which is precisely the edge–edge adjacency rule in $T(X)$. No additional edges among $\{e, f\}$ appear in $T(X)$ beyond this rule, so the embedding is induced. □

Lemma 5.4 (Total preserves induced subgraphs)**.** *If H is an induced subgraph of G, then $T(H)$ is an induced subgraph of $T(G)$.*

Proof. Vertices of $T(H)$ are $V(H) \cup E(H) \subseteq V(G) \cup E(G) = V(T(G))$. Adjacency in $T(H)$ is determined by adjacency/incidence inside H, and because H is induced in G, the same adjacencies hold when viewed in G. Thus no extra edges among vertices from $V(H) \cup E(H)$ arise in $T(G)$; the inclusion is induced. □

Theorem 5.5 (Iterated total graphs generalize iterated line graphs)**.** *For every finite simple graph G and every $k \geq 1$, the iterated line graph $L^k(G)$ is an induced subgraph of the iterated total graph $T^k(G)$. Consequently, the sequence $\{T^k(G)\}_{k\geq 0}$* generalizes *$\{L^k(G)\}_{k\geq 0}$ in the sense that each $L^k(G)$ occurs canonically inside $T^k(G)$.*

Proof. We proceed by induction on k.

Base case $k = 1$. By Lemma 5.3, $L(G)$ is an induced subgraph of $T(G)$.

Inductive step. Assume $L^{k-1}(G)$ is an induced subgraph of $T^{k-1}(G)$. Apply Lemma 5.4 with $H = L^{k-1}(G)$ and $G' = T^{k-1}(G)$: then $T\big(L^{k-1}(G)\big)$ is an induced subgraph of $T\big(T^{k-1}(G)\big) = T^k(G)$. Finally, apply Lemma 5.3 to $X = L^{k-1}(G)$ to see that

$$L^k(G) \;=\; L\big(L^{k-1}(G)\big) \text{ is an induced subgraph of } T\big(L^{k-1}(G)\big).$$

By transitivity of "is an induced subgraph of," we conclude $L^k(G) \subseteq_{\text{ind}} T^k(G)$. □

5.2 iterated total hypergraphs

An iterated total hypergraph is obtained by successively applying the total hypergraph transformation, encoding vertices, hyperedges, and their incidences through multiple iteration layers.

Definition 5.6 (Iterated total hypergraphs)**.** Let $T^0(H) := H$ and, for $t \geq 1$, define recursively

$$T^t(H) \;:=\; T\big(T^{t-1}(H)\big).$$

Example 5.7 (Iterated total HyperGraph — meetings, people, and two layers of incidence)**.** Let $H = (V, \mathcal{E})$ encode meetings:

$$V = \{\text{Alice}, \text{Bob}, \text{Chloe}\}, \qquad \mathcal{E} = \{\, E_1 = \{\text{Alice}, \text{Bob}\},\ E_2 = \{\text{Bob}, \text{Chloe}\}\,\}.$$

The total hypergraph $T(H) = (U, \mathcal{A} \cup \mathcal{B} \cup C)$ has

$$U = \{\text{Alice}, \text{Bob}, \text{Chloe}, E_1, E_2\},$$

$$\mathcal{A} = \{E_1, E_2\}, \quad \mathcal{B} = \big\{\, \mathcal{E}_H(\text{Bob}) = \{E_1, E_2\}\,\big\}, \quad C = \big\{\, \{\text{Alice}, E_1\},\ \{\text{Bob}, E_1, E_2\},\ \{\text{Chloe}, E_2\}\,\big\}.$$

Iterating, $T^2(H) = T(T(H))$ has vertex set $U \,\dot\cup\, (\mathcal{A} \cup \mathcal{B} \cup C)$; e.g., the six previous hyperedges now also appear as vertices:

$$E_1,\ E_2,\ \{E_1, E_2\},\ \{\text{Alice}, E_1\},\ \{\text{Bob}, E_1, E_2\},\ \{\text{Chloe}, E_2\}.$$

Some illustrative hyperedges of $T^2(H)$ are:

$$\underbrace{\{E_1\},}_{\text{from } \mathcal{A}} \qquad \underbrace{\big\{\{E_1, E_2\}\big\},}_{\text{from } \mathcal{B} \text{ at Bob}} \qquad \underbrace{\big\{\{\text{Bob}\}, \{\text{Bob}, E_1, E_2\}\big\}}_{\text{from } C \text{ at Bob}}.$$

Interpretation: $T(H)$ mixes *people*, *meetings*, and *who attends what*; $T^2(H)$ adds a second layer that also relates these "attendance patterns" to one another inside a single object.

Theorem 5.8 (Well-definedness for all iterates)**.** *For every finite hypergraph H and every $t \geq 1$, $T^t(H)$ is a finite hypergraph.*

Proof. From Definition **??**, U is finite and each member of $\mathcal{A} \cup \mathcal{B} \cup C$ is a nonempty subset of U. Thus $T(H)$ is a finite hypergraph. Inductively $T^t(H) = T\big(T^{t-1}(H)\big)$ is finite for all t. □

Theorem 5.9. *For every hypergraph H, $T^1(H) = T(H)$.*

Proof. Immediate from Definition 5.6. □

Lemma 5.10 (Line sits inside total, and T preserves induced subhypergraphs)**.** *For every hypergraph H, $L(H)$ is the induced subhypergraph $T(H)[\,\mathcal{E}\,]$ on the vertex set $\mathcal{E}$. Moreover, if K is an induced subhypergraph of H, then $T(K)$ is an induced subhypergraph of $T(H)$.*

Proof. In $T(H)$, the vertices in $\mathcal{E}$ support exactly the hyperedge family $\mathcal{B} = \{\mathcal{E}_H(v)\}$, which equals the edge set of $L(H)$; hence $L(H) = T(H)[\mathcal{E}]$. For the second claim, let $K = H[W]$ with $W \subseteq V(H)$. Then $V\big(T(K)\big) = W \,\dot\cup\, \mathcal{E}_K$ is a subset of $V\big(T(H)\big) = V(H) \,\dot\cup\, \mathcal{E}_H$. Each hyperedge of $T(K)$ is obtained from the corresponding one in $T(H)$ by intersecting with $W \,\dot\cup\, \mathcal{E}_K$:

$$\mathcal{A}_K = \mathcal{A}_H \cap \mathcal{P}(W), \quad \mathcal{B}_K = \{\, \mathcal{E}_H(v) \cap \mathcal{E}_K : v \in W \,\}, \quad C_K = \{\, (\{v\} \cup \mathcal{E}_H(v)) \cap (W \cup \mathcal{E}_K) : v \in W \,\}.$$

Thus $T(K)$ is induced in $T(H)$. □

Theorem 5.11 (Iterated line hypergraphs inside iterated total hypergraphs)**.** *For every hypergraph H and every $t \geq 1$, $L^t(H)$ is an induced subhypergraph of $T^t(H)$.*

Proof. $t = 1$ follows from Lemma 5.10. Assume $L^{t-1}(H) \subseteq_{\text{ind}} T^{t-1}(H)$. By Lemma 5.10, $L\big(L^{t-1}(H)\big) \subseteq_{\text{ind}} T\big(L^{t-1}(H)\big) \subseteq_{\text{ind}} T\big(T^{t-1}(H)\big) = T^t(H)$. Since $L^t(H) = L\big(L^{t-1}(H)\big)$, the claim follows. □

Lemma 5.12 (One step: total graph as 2–section of total hypergraph)**.** $[\,T(H_G)\,]_2 \;=\; T(G)$.

Proof. Vertices of $[T(H_G)]_2$ are $V \,\dot\cup\, E$. Two original vertices are adjacent in $[T(H_G)]_2$ iff they lie together in some $e \in \mathcal{A}$, i.e. iff they are adjacent in G (vertex–vertex rule). Two edge-vertices $e, f \in E$ are adjacent in $[T(H_G)]_2$ iff they lie together in some $\mathcal{E}_{H_G}(v) \in \mathcal{B}$, i.e. iff e, f share an endpoint v in G (edge–edge rule). Finally $v \in V$ and $e \in E$ are adjacent in $[T(H_G)]_2$ iff $\{v, e\} \subseteq \{v\} \cup \mathcal{E}_{H_G}(v) \in C$, i.e. iff $v \in e$ (vertex–edge rule). These are exactly the adjacency rules of $T(G)$. □

Lemma 5.13 (Edge–hyperedge correspondence inside the total tower)**.** *For each $s \geq 0$ there exists a natural bijection*

$$\Phi_s : \; E\big(T^s(G)\big) \;\longleftrightarrow\; \mathcal{E}\big(T^s(H_G)\big)$$

such that, for any distinct $x, y \in V\big(T^s(G)\big)$,

$$\{x, y\} \in E\big(T^s(G)\big) \quad\Longleftrightarrow\quad \Phi_s(\{x, y\}) \text{ is a hyperedge of } T^s(H_G) \text{ containing } x \text{ and } y.$$

Proof. By induction on s. For $s = 0$, $E(T^0(G)) = E(G)$ and $\mathcal{E}(T^0(H_G)) = \mathcal{E}(H_G)$, with the identification $\{u, v\} \leftrightarrow \{u, v\}$. Assume Φ_s exists. In $T^{s+1}(G) = T(T^s(G))$, each edge arises uniquely from one of the three total-graph rules on $T^s(G)$: (vertex–vertex via $\mathcal{A}$, edge–edge via $\mathcal{B}$, vertex–edge via C). In $T^{s+1}(H_G) = T(T^s(H_G))$, the corresponding witnessing hyperedge is exactly the member of $\mathcal{A} \cup \mathcal{B} \cup C$ that contains the same pair. Uniqueness holds because a pair of vertices of $T^s(H_G)$ belongs to at most one of the three disjoint types: two "vertex-type", two "edge-type", or one of each. Define Φ_{s+1} by mapping each edge of $T^{s+1}(G)$ to this unique witnessing hyperedge. This extends the induction and preserves containment of endpoints. □

Theorem 5.14 (Iterated total hypergraphs lift iterated total graphs)**.** *For every finite simple graph G and every $t \geq 1$,*

$$\big[\, T^t(H_G)\,\big]_2 \;=\; T^t(G).$$

Proof. By induction on t. For $t = 1$, Lemma 5.12. Assume $[\,T^{t-1}(H_G)\,]_2 = T^{t-1}(G)$. Vertices of $[T^t(H_G)]_2$ are $V\big(T^{t-1}(G)\big) \,\dot\cup\, \mathcal{E}\big(T^{t-1}(H_G)\big)$, while vertices of $T^t(G)$ are $V\big(T^{t-1}(G)\big) \,\dot\cup\, E\big(T^{t-1}(G)\big)$. By Lemma 5.13 (with $s = t-1$), the bijection Φ_{t-1} identifies $\mathcal{E}\big(T^{t-1}(H_G)\big)$ with $E\big(T^{t-1}(G)\big)$ so that a pair of vertices is adjacent in $[T^t(H_G)]_2$ iff it is adjacent in $T^t(G)$ (the three total rules are matched one-to-one by Φ_{t-1}). Hence the graphs are equal. □

5.3 iterated total superhypergraphs

An iterated total superhypergraph results from repeatedly performing the total superhypergraph construction, capturing hierarchical vertices, superedges, and complex incidence structures across increasing levels.

Definition 5.15 (Iterated total superhypergraphs)**.** For $t \geq 0$ define

$$\mathbf{T}^0\big(\mathcal{H}^{(n)}\big) := \mathcal{H}^{(n)}, \qquad \mathbf{T}^{t+1}\big(\mathcal{H}^{(n)}\big) := \mathbf{T}\Big(\mathbf{T}^t\big(\mathcal{H}^{(n)}\big)\Big).$$

So $\mathbf{T}^t(\mathcal{H}^{(n)})$ is level $n+t$.

Example 5.16 (Iterated total SuperHyperGraph — programs (of teams) with two incidence layers)**.** Start with a level-1 superhypergraph: supervertices are teams of people, superedges are programs (sets of teams). Let $V_0 = \{a, b, c\}$ (people), teams

$$V_1 = \{\, T_1 = \{a, b\},\ T_2 = \{b, c\}\,\} \subseteq \mathcal{P}(V_0),$$

and programs

$$\mathcal{E} = \{\, P_1 = \{T_1, T_2\},\ P_2 = \{T_2\}\,\} \subseteq \mathcal{P}(V_1).$$

The total superhypergraph $\mathbf{T}(\mathcal{H}^{(1)}) = (U_2, \mathcal{A} \cup \mathcal{B} \cup C)$ has

$$U_2 = \{\{T_1\}, \{T_2\},\ P_1,\ P_2\},$$

$$\mathcal{A} = \Big\{\ \{T_1\}, \{T_2\} \text{ (as the image of } P_1),\ \ \{T_2\} \text{ (as the image of } P_2)\ \Big\},$$

$$\mathcal{B} = \Big\{\ \{P_1, P_2\} \text{ (the programs meeting at team } T_2)\ \Big\},$$

$$C = \Big\{\ \{\{T_1\}, P_1\},\ \{\{T_2\}, P_1, P_2\}\ \Big\}.$$

Iterating once more, $\mathbf{T}^2(\mathcal{H}^{(1)}) = \mathbf{T}\big(\mathbf{T}(\mathcal{H}^{(1)})\big)$ has vertex set $U_2 \,\dot\cup\, (\mathcal{A} \cup \mathcal{B} \cup C)$; for instance, $\{P_1, P_2\}$ and $\{\{T_2\}, P_1, P_2\}$ become vertices. A sample hyperedge of $\mathbf{T}^2$ arising from the "incidence at $\{T_2\}$" rule is

$$\Big\{\ \{\{T_2\}\},\ \{\{T_2\}, P_1, P_2\}\ \Big\},$$

which ties the singleton vertex $\{T_2\}$ to the pattern "$\{T_2\}$ participates in P_1 and P_2". Thus $\mathbf{T}$ captures programs/teams incidence in one level, while $\mathbf{T}^2$ additionally relates those incidence *patterns* to each other.

Example 5.17 (An iterated total 3–superhypergraph (two iterations))**. Step 0: A level-3 superhypergraph.** Let the base set be $V_0 = \{a, b\}$. Choose level-1 supervertices (subsets of V_0)

$$V_1 = \Big\{\, A = \{a\},\ B = \{b\}\,\Big\},$$

level-2 supervertices (subsets of V_1)

$$V_2 = \Big\{\, X_1 = \{A\},\ X_2 = \{A, B\}\,\Big\},$$

and level-3 supervertices (subsets of V_2)

$$V_3 = \Big\{\, Y_1 = \{X_1, X_2\},\ \ Y_2 = \{X_2\}\,\Big\} \subseteq \mathrm{POWS}^3(V_0).$$

Define level-3 superedges as subsets of V_3:

$$\mathcal{E} = \Big\{\, E_1 = \{Y_1, Y_2\},\ \ E_2 = \{Y_2\}\,\Big\} \subseteq \mathrm{POWS}(V_3) \setminus \{\emptyset\}.$$

Then $\mathcal{H}^{(3)} = (V_3, \mathcal{E})$ is a level-3 SuperHyperGraph. For stars (incidence neighborhoods) in $\mathcal{H}^{(3)}$:

$$\mathrm{Star}_{\mathcal{H}}(Y_1) = \{E_1\}, \qquad \mathrm{Star}_{\mathcal{H}}(Y_2) = \{E_1, E_2\}.$$

Step 1: First total step $\mathbf{T}^1(\mathcal{H}^{(3)}) = \mathbf{T}(\mathcal{H}^{(3)})$. Embed V_3 into $\mathrm{POWS}(V_3)$ via $\iota(v) := \{v\}$ and set

$$U_4 \ := \ \iota(V_3) \,\dot\cup\, \mathcal{E} \ = \ \{\iota(Y_1), \iota(Y_2)\} \,\dot\cup\, \{E_1, E_2\}.$$

The three total families (all nonempty subsets of U_4) are:

$$\mathcal{A} = \{\, \iota(E_1) = \{\iota(Y_1), \iota(Y_2)\},\;\; \iota(E_2) = \{\iota(Y_2)\} \,\},$$

$$\mathcal{B} = \{\, \mathrm{Star}_{\mathcal{H}}(Y_2) = \{E_1, E_2\} \,\} \quad (\mathrm{Star}_{\mathcal{H}}(Y_1) = \{E_1\} \text{ has size 1}),$$

$$C = \{\, \{\iota(Y_1), E_1\},\; \{\iota(Y_2), E_1, E_2\} \,\}.$$

Thus

$$\mathbf{T}(\mathcal{H}^{(3)}) = \Big(U_4,\; \underbrace{\mathcal{A} \cup \mathcal{B} \cup C}_{=: \mathcal{F}_1} \Big),$$

which is a level-4 SuperHyperGraph.

Step 2: Second total step $\mathbf{T}^2(\mathcal{H}^{(3)}) = \mathbf{T}\big(\mathbf{T}(\mathcal{H}^{(3)})\big)$. Label the hyperedges of $\mathcal{F}_1$ for clarity:

$$A_1 := \{\iota(Y_1), \iota(Y_2)\},\;\; A_2 := \{\iota(Y_2)\},\;\; B_1 := \{E_1, E_2\},\;\; C_1 := \{\iota(Y_1), E_1\},\;\; C_2 := \{\iota(Y_2), E_1, E_2\}.$$

The level-5 vertex set is the disjoint union

$$U_5 \;:=\; \iota(U_4) \,\dot\cup\, \mathcal{F}_1 \;=\; \{\iota(\iota(Y_1)), \iota(\iota(Y_2)), \iota(E_1), \iota(E_2)\} \,\dot\cup\, \{A_1, A_2, B_1, C_1, C_2\}.$$

The three total families at this stage are obtained exactly as before:

$$\mathcal{A}^{(2)} = \{\, \iota(A_1),\; \iota(A_2),\; \iota(B_1),\; \iota(C_1),\; \iota(C_2) \,\},$$

$$\mathcal{B}^{(2)} = \{\, \mathrm{Star}_{\mathbf{T}}(\iota(Y_1)) = \{A_1, C_1\},\; \mathrm{Star}_{\mathbf{T}}(\iota(Y_2)) = \{A_1, A_2, C_2\},\; \mathrm{Star}_{\mathbf{T}}(E_1) = \{B_1, C_1, C_2\},\; \mathrm{Star}_{\mathbf{T}}(E_2) = \{B_1, C_2\} \,\},$$

$$C^{(2)} = \{\, \{\iota(\iota(Y_1)), A_1, C_1\},\; \{\iota(\iota(Y_2)), A_1, A_2, C_2\},\; \{\iota(E_1), B_1, C_1, C_2\},\; \{\iota(E_2), B_1, C_2\} \,\}.$$

Hence

$$\mathbf{T}^2(\mathcal{H}^{(3)}) \;=\; \Big(U_5,\; \mathcal{A}^{(2)} \cup \mathcal{B}^{(2)} \cup C^{(2)} \Big),$$

which is a level-5 SuperHyperGraph.

Interpretation. The first total step $\mathbf{T}$ combines the level-3 supervertices $\iota(Y_i)$ with superedges E_j and records: (i) original co-membership ($\mathcal{A}$), (ii) edge–edge intersection at a common supervertex ($\mathcal{B}$), and (iii) vertex–edge incidence (C). The second step $\mathbf{T}^2$ then treats these incidence patterns themselves as vertices and re-applies the same three rules, yielding a concrete instance of an *iterated total 3–superhypergraph*.

Theorem 5.18. *For every $\mathcal{H}^{(n)}$, $\mathbf{T}^1(\mathcal{H}^{(n)}) = \mathbf{T}(\mathcal{H}^{(n)})$.*

Proof. Immediate from Definition 5.15. □

Theorem 5.19 ($\mathbf{T}^t$ reduces to T^t at $n = 0$ (canonical isomorphism))**.** *Let $H = (V, \mathcal{E})$ be a hypergraph. Identify $\iota(V) \cong V$ via $\iota(v) \leftrightarrow v$. Then for all $t \geq 1$,*

$$\mathbf{T}^t(H) \;\cong\; T^t(H).$$

Proof. For $t = 1$, Definition 3.8 with $n = 0$ and the identification $\iota(v) \leftrightarrow v$ yields exactly $T(H)$. Inductively, the promotion at each step replaces V by $\iota(V)$, which we re-identify with V; the three edge-families correspond termwise. Hence $\mathbf{T}^t(H) \cong T^t(H)$ for all t. □

Lemma 5.20 (One step: total graph from total superhypergraph)**.** $[\, \mathbf{T}(H_G) \,]_2 \;=\; T(G)$.

Proof. Vertices: $V([\, \mathbf{T}(H_G) \,]_2) = \iota(V) \dot\cup E \cong V \dot\cup E$. Adjacency in the 2–section occurs exactly when a pair lies together in some member of $\mathcal{A}$ (vertex–vertex), $\mathcal{B}$ (edge–edge through a shared endpoint), or C (vertex–edge incidence), which reproduces the three adjacency rules of $T(G)$. □

Theorem 5.21 (Iterated lift of total graphs)**.** *For every finite simple graph G and every $t \geq 1$,*

$$\big[\, \mathbf{T}^t(H_G) \,\big]_2 \;=\; T^t(G).$$

Proof. By induction on t. The case $t = 1$ is Lemma 5.20. Assume $[\,\mathbf{T}^{t-1}(H_G)\,]_2 = T^{t-1}(G)$. Then the vertices of $[\,\mathbf{T}^t(H_G)\,]_2$ are $\iota\big(V(T^{t-1}(G))\big)\dot\cup\mathcal{E}\big(\mathbf{T}^{t-1}(H_G)\big)$, which correspond naturally to $V(T^{t-1}(G))\dot\cup E(T^{t-1}(G))$. As in the base step, the three families $\mathcal{A}, \mathcal{B}, C$ at stage t encode precisely the three total-graph adjacencies at stage t; hence $[\,\mathbf{T}^t(H_G)\,]_2 = T^t(G)$. □

Lemma 5.22 (Line is induced inside total; total preserves induced substructures)**.** *For every* $\mathcal{H}^{(n)}$*,* $\mathbf{L}(\mathcal{H}^{(n)}) = \mathbf{T}(\mathcal{H}^{(n)})[\,\mathcal{E}\,]$. *Moreover, if* $\mathcal{K}$ *is induced in* $\mathcal{H}$*, then* $\mathbf{T}(\mathcal{K})$ *is induced in* $\mathbf{T}(\mathcal{H})$*.*

Proof. The induced substructure on the vertex set $\mathcal{E} \subseteq U_{n+1}$ keeps exactly the $\mathcal{B}$-family, which equals the edge-set of $\mathbf{L}(\mathcal{H}^{(n)})$. For preservation, note that the three families $\mathcal{A}, \mathcal{B}, C$ are defined by local incidence relations; restricting vertices and intersecting hyperedges commute with the construction. □

Theorem 5.23 (Iterated line superhypergraphs inside iterated total superhypergraphs)**.** *For every* $\mathcal{H}^{(n)}$ *and* $t \geq 1$,

$$\mathbf{L}^t(\mathcal{H}^{(n)}) \text{ is an induced sub(super)hypergraph of } \mathbf{T}^t(\mathcal{H}^{(n)}).$$

Proof. For $t = 1$, Lemma 5.22. Assume the claim holds for $t - 1$. Applying Lemma 5.22 to $\mathbf{L}^{t-1}(\mathcal{H}^{(n)}) \subseteq_{\text{ind}} \mathbf{T}^{t-1}(\mathcal{H}^{(n)})$ and then the functoriality of $\mathbf{T}$ on induced substructures yields

$$\mathbf{L}\big(\mathbf{L}^{t-1}\big) \subseteq_{\text{ind}} \mathbf{T}\big(\mathbf{L}^{t-1}\big) \subseteq_{\text{ind}} \mathbf{T}\big(\mathbf{T}^{t-1}\big) = \mathbf{T}^t,$$

i.e. $\mathbf{L}^t \subseteq_{\text{ind}} \mathbf{T}^t$. □

6 Conclusion

This paper developed hypergraph and superhypergraph analogues of line, total, iterated line, and iterated total graph constructions, establishing a theoretical basis for modeling hierarchical, incidence-rich, and self-referential network structures. It is anticipated that future research will advance the study of extended models based on various graph-theoretic frameworks, including Fuzzy Graphs [47], Intuitionistic Fuzzy Graphs [48–50], Neutrosophic Graphs [51–54], Quadripartitioned Neutrosophic Graphs [55, 56], Fuzzy HyperGraphs [57, 58], Plithogenic Graphs [59, 60], Directed Graphs [61–63], and Multidirected Graphs [64]. Moreover, further progress in the development of algorithms for these structures is also expected.

Funding

No external funding was received for this work.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this work.

Acknowledgments

We thank all colleagues, reviewers, and readers whose comments and questions have greatly improved this manuscript. We are also grateful to the authors of the works cited herein for providing the theoretical foundations that underpin our study. Finally, we appreciate the institutional and technical support that enabled this research.

Data Availability

This paper is theoretical and did not generate or analyze any empirical data. We welcome future studies that apply and test these concepts in practical settings.

Ethical Approval

This research did not involve human participants or animals, and therefore did not require ethical approval.

Disclaimer

The ideas presented here are theoretical and have not yet been validated through empirical testing. While we have strived for accuracy and proper citation, inadvertent errors may remain. Readers should verify any referenced material independently. The opinions expressed are those of the authors and do not necessarily reflect the views of their institutions.

References

[1] M Amin Bahmanian and Mateja Šajna. Hypergraphs: connection and separation. *arXiv preprint arXiv:1504.04274*, 2015.

[2] Mohammad A Bahmanian and Mateja Sajna. Connection and separation in hypergraphs. *Theory and Applications of Graphs*, 2(2):5, 2015.

[3] Yifan Feng, Haoxuan You, Zizhao Zhang, Rongrong Ji, and Yue Gao. Hypergraph neural networks. In *Proceedings of the AAAI conference on artificial intelligence*, volume 33, pages 3558–3565, 2019.

[4] Yuxin Wang, Quan Gan, Xipeng Qiu, Xuanjing Huang, and David Wipf. From hypergraph energy functions to hypergraph neural networks. In *International Conference on Machine Learning*, pages 35605–35623. PMLR, 2023.

[5] Takaaki Fujita and Florentin Smarandache. Fundamental computational problems and algorithms for superhypergraphs. *HyperSoft Set Methods in Engineering*, 3:32–61, 2025.

[6] Florentin Smarandache. *Extension of HyperGraph to n-SuperHyperGraph and to Plithogenic n-SuperHyperGraph, and Extension of HyperAlgebra to n-ary (Classical-/Neutro-/Anti-) HyperAlgebra*. Infinite Study, 2020.

[7] Mohammad Hamidi, Florentin Smarandache, and Elham Davneshvar. Spectrum of superhypergraphs via flows. *Journal of Mathematics*, 2022(1):9158912, 2022.

[8] Mohammad Hamidi, Florentin Smarandache, and Mohadeseh Taghinezhad. *Decision Making Based on Valued Fuzzy Superhypergraphs*. Infinite Study, 2023.

[9] Takaaki Fujita. Exploration of graph classes and concepts for superhypergraphs and n-th power mathematical structures. 2025.

[10] Takaaki Fujita. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*. Biblio Publishing, 2025.

[11] Takaaki Fujita. Rethinking strategic perception: Foundations and advancements in hypergame theory and superhypergame theory. *Prospects for Applied Mathematics and Data Analysis*, 4(2):01–14, 2024.

[12] Florentin Smarandache. Foundation of superhyperstructure & neutrosophic superhyperstructure. *Neutrosophic Sets and Systems*, 63(1):21, 2024.

[13] Florentin Smarandache. *SuperHyperFunction, SuperHyperStructure, Neutrosophic SuperHyperFunction and Neutrosophic SuperHyperStructure: Current understanding and future directions*. Infinite Study, 2023.

[14] Florentin Smarandache. Extension of hyperalgebra to superhyperalgebra and neutrosophic superhyperalgebra (revisited). In *International Conference on Computers Communications and Control*, pages 427–432. Springer, 2022.

[15] Alain Bretto. Hypergraph theory. *An introduction. Mathematical Engineering. Cham: Springer*, 1, 2013.

[16] Claude Berge. *Hypergraphs: combinatorics of finite sets*, volume 45. Elsevier, 1984.

[17] Takaaki Fujita. Superhypergraph neural networks and plithogenic graph neural networks: Theoretical foundations. *arXiv preprint arXiv:2412.01176*, 2024.

[18] Takaaki Fujita. Review of some superhypergraph classes: Directed, bidirected, soft, and rough. In *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond (Second Volume)*. Biblio Publishing, 2024.

[19] Chen Yan. Properties of spectra of graphs and line graphs. *Applied Mathematics-A Journal of Chinese Universities*, 17(3):371–376, 2002.

[20] Robert Kincaid and Heidi Lam. Line graph explorer: scalable display of line graphs using focus+ context. In *Proceedings of the working conference on Advanced visual interfaces*, pages 404–411, 2006.

[21] Gary Chartrand. On hamiltonian line-graphs. *Transactions of the American Mathematical Society*, 134(3):559–566, 1968.

[22] Arti Sharma and Atul Gaur. Line graphs associated to the maximal graph. *Journal of Algebra and Related Topics*, 3(1):1–11, 2015.

[23] Alan J Hoffman. On the line graph of the complete bipartite graph. *The Annals of Mathematical Statistics*, 35(2):883–885, 1964.

[24] AJ Hoffman. On the line graph of a projective plane. *Proceedings of the American Mathematical Society*, 16(2):297–302, 1965.

[25] RI Tyshkevich and Vadim E Zverovich. Line hypergraphs. *Discrete Mathematics*, 161(1-3):265–283, 1996.

[26] Regina I Tyshkevich and Vadim E Zverovich. Line hypergraphs: A survey. *Acta Applicandae Mathematica*, 52(1-3):209–222, 1998.

[27] V Zverovich, Yu Metelsky, and P Skums. Line graphs and hypergraphs. *Methods of Graph Decompositions*, page 169, 2024.

[28] AG Levin and RI Tyshkevich. Line hypergraphs. *Discrete Mathematics and Applications*, 3(4):407–428, 1993.

[29] Jean-Claude Bermond, Fahir Ergincan, and Michel Syska. Line directed hypergraphs. In *Cryptography and Security: From Theory to Applications: Essays Dedicated to Jean-Jacques Quisquater on the Occasion of His 65th Birthday*, pages 25–34. Springer, 2012.

[30] Mehdi Behzad. A criterion for the planarity of the total graph of a graph. In *Mathematical proceedings of the Cambridge philosophical society*, volume 63, pages 679–681. Cambridge University Press, 1967.

[31] Yinkui Li, Ruijuan Gu, and Hui Lei. The generalized connectivity of the line graph and the total graph for the complete bipartite graph. *Applied Mathematics and Computation*, 347:645–652, 2019.

[32] DVS Sastry and B Syam Prasad Raju. Graph equations for line graphs, total graphs, middle graphs and quasi-total graphs. *Discrete Mathematics*, 48(1):113–119, 1984.

[33] Moytri Sarmah and Kuntala Patra. Line graph associated to total graph of idealization. *Afrika Matematika*, 27(3):485–490, 2016.

[34] Arthur M Hobbs. Powers of graphs, line graphs, and total graphs. In *Theory and Applications of Graphs: Proceedings, Michigan May 11–15, 1976*, pages 271–285. Springer, 2006.

[35] Mukti Acharya, Atul Gaur, Amit Kumar, et al. Line signed graph of a signed total graph. *Electronic Notes in Discrete Mathematics*, 63:389–397, 2017.

[36] Bhavanari Satyanarayana, Devanaboina Srinivasulu, and Kuncham Syam Prasad. Line graphs and quasi-total graphs. *International Journal of Computer Applications*, 105(3), 2014.

[37] Dragos M Cvetkovic. Spectrum of the total graph of a graph. *Publ. Inst. Math.(Beograd)*, 16(30):49–52, 1973.

[38] Martin Knor and Ludovit Niepel. Connectivity of iterated line graphs. *Discrete applied mathematics*, 125(2-3):255–266, 2003.

[39] Harishchandra S Ramane, Hanumappa B Walikar, Siddani Bhaskara Rao, B Devadas Acharya, Prabhakar Ramrao Hampiholi, Sudhir R Jog, and Ivan Gutman. Spectra and energies of iterated line graphs of regular graphs. *Applied mathematics letters*, 18(6):679–682, 2005.

[40] Liming Xiong and Zhanhong Liu. Hamiltonian iterated line graphs. *Discrete mathematics*, 256(1-2):407–422, 2002.

[41] Yehong Shao. A bound on connectivity of iterated line graphs. *Electronic Journal of Graph Theory & Applications*, 10(2), 2022.

[42] Martin Knor and L'udovít Niepel. Iterated line graphs are maximally ordered. *Journal of Graph Theory*, 52(2):171–180, 2006.

[43] Harishchandra Ramane, Ivan Gutman, and Mahadevappa Gundloor. Seidel energy of iterated line graphs of regular graphs. 2015.

[44] Gui-Xian Tian. The asymptotic behavior of (degree-) kirchhoff indices of iterated total graphs of regular graphs. *Discrete Applied Mathematics*, 233:224–230, 2017.

[45] Derek A Holton, Dingjun Lou, and KL Mcavaney. n-extendability of line graphs, power graphs, and total graphs. *Australas. J Comb.*, 11:215–222, 1995.

[46] Eber Lenes, Exequiel Mallea-Zepeda, María Robbiano, and Jonnathan Rodríguez. On the diameter and incidence energy of iterated total graphs. *Symmetry*, 10(7):252, 2018.

[47] Azriel Rosenfeld. Fuzzy graphs. In *Fuzzy sets and their applications to cognitive and decision processes*, pages 77–95. Elsevier, 1975.

[48] Sankar Sahoo and Madhumangal Pal. Product of intuitionistic fuzzy graphs and degree. *Journal of Intelligent & Fuzzy Systems*, 32(1):1059–1067, 2017.

[49] M. G. Karunambigai, R. Parvathi, and R. Buvaneswari. Arc in intuitionistic fuzzy graphs. *Notes on Intuitionistic Fuzzy Sets*, 17:37–47, 2011.

[50] P. K. Kishore Kumar, S. Lavanya, Hossein Rashmanlou, and Mostafa Nouri Jouybari. Magic labeling on interval-valued intuitionistic fuzzy graphs. *J. Intell. Fuzzy Syst.*, 33:3999–4006, 2017.

[51] Said Broumi, Mohamed Talea, Assia Bakali, and Florentin Smarandache. Single valued neutrosophic graphs. *Journal of New theory*, (10):86–101, 2016.

[52] Sumera Naz, Hossein Rashmanlou, and M Aslam Malik. Operations on single valued neutrosophic graphs with application. *Journal of Intelligent & Fuzzy Systems*, 32(3):2137–2151, 2017.

[53] Faruk Karaaslan and Bijan Davvaz. Properties of single-valued neutrosophic graphs. *Journal of Intelligent & Fuzzy Systems*, 34(1):57–79, 2018.

[54] Said Broumi, Assia Bakali, Mohamed Talea, and Florentin Smarandache. An isolated bipolar single-valued neutrosophic graphs. In *Information Systems Design and Intelligent Applications: Proceedings of Fourth International Conference INDIA 2017*, pages 816–822. Springer, 2018.

[55] S Satham Hussain, Hossein Rashmonlou, R Jahir Hussain, Sankar Sahoo, Said Broumi, et al. Quadripartitioned neutrosophic graph structures. *Neutrosophic Sets and Systems*, 51(1):17, 2022.

[56] Xiaolong Shi, Saeed Kosari, Hossein Rashmanlou, Said Broumi, and S Satham Hussain. Properties of interval-valued quadripartitioned neutrosophic graphs with real-life application. *Journal of Intelligent & Fuzzy Systems*, 44(5):7683–7697, 2023.

[57] Muhammad Akram and Anam Luqman. *Fuzzy hypergraphs and related extensions*. Springer, 2020.

[58] Leonid S Bershtein and Alexander V Bozhenyuk. Fuzzy graphs and fuzzy hypergraphs. In *Encyclopedia of Artificial Intelligence*, pages 704–709. IGI Global, 2009.

[59] Takaaki Fujita and Florentin Smarandache. A review of the hierarchy of plithogenic, neutrosophic, and fuzzy graphs: Survey and applications. In *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond (Second Volume)*. Biblio Publishing, 2024.

[60] Fazeelat Sultana, Muhammad Gulistan, Mumtaz Ali, Naveed Yaqoob, Muhammad Khan, Tabasam Rashid, and Tauseef Ahmed. A study of plithogenic graphs: applications in spreading coronavirus disease (covid-19) globally. *Journal of ambient intelligence and humanized computing*, 14(10):13139–13159, 2023.

[61] Yixuan He, Xitong Zhang, Junjie Huang, Benedek Rozemberczki, Mihai Cucuringu, and Gesine Reinert. Pytorch geometric signed directed: a software package on graph neural networks for signed and directed graphs. In *Learning on Graphs Conference*, pages 12–1. PMLR, 2024.

[62] Tahira Batool and Uzma Ahmad. A new approximation technique based on central bodies of rough fuzzy directed graphs for agricultural development. *Journal of Applied Mathematics and Computing*, 70(2):1673–1705, 2024.

[63] Enrico L. Enriquez, Grace M. Estrada, Carmelita Loquias, Reuella J. Bacalso, and Lanndon A. Ocampo. Domination in fuzzy directed graphs. *Mathematics*, 2021.

[64] Takaaki Fujita. Extensions of multidirected graphs: Fuzzy, neutrosophic, plithogenic, rough, soft, hypergraph, and superhypergraph variants. *International Journal of Topology*, 2(3):11, 2025.