MONTY-HALL PROBLEM SOLVED : PARADOX RESOLVED

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ABSTRACT

This research report presents a deep re-look at the classical Monty-Hall Problem, refuting the widely accepted position held by the leading subject-area-experts, and establishing that there is no rational basis for a switched choice in the decision to be made by the guest of the game show.

Many a times, the additional knowledge gained, revealing a losing-chance, although leads to an updated smaller sample-space, may not be specific enough for refinement/update on the relative chances between/among the now-available alternatives in the resultant smaller sample-space.

Keywords: A-Priori Probability; A-Posteriori Probability;
Mutually Independent Events; Joint Probability;
Mutually Exclusive Together Exhaustive Alternatives;
Restricted Probability, Conditional Probability. Marginal Probability.

AMS MSC Mathematics Subject Classification: 60A99; 60C99; 62A99; 62C99.

1. INTRODUCTION

The classical "Monty-Hall Problem", also referred to as the "Three-Door Problem" is based on a game show "Let's Make a Deal" wherein the host reveals a losing choice to the guest, who had earlier made an initial choice, and in turn offers the guest an enticing option to switch from the earlier choice to a second available choice with an aim to enhance the chances of winning the prize. The most prevalent position, as reported in literature, among the leading eminent mathematicians, statisticians, logicians, subject-area-experts and rational intellectuals, is that an appropriate detailed study & analysis of the scenario using the well accepted standard approach of Probability & Statistics, would lead to a recommendation to the guest to switch to the second available choice based on the knowledge obtained from the host revealing a losing choice.

It is argued that the default of sticking to the initial choice will result in a probability of success being only one-third whereas a switch to the alternative second available choice will result in a probability of success being one-half, and therefore a switched choice is recommended. However, it will be shown here that this argument seems to have been based on some assumptions that cannot be justified, and therefore the resultant recommendation for switched choice is indeed baseless.

2. DESCRIPTION OF THE PROBLEM SCENARIO

We shall focus only on the so-called classical Monty-Hall Problem, for the purpose of this report.

For the sake of clarity, let us consider the standard Monty-Hall Problem as reported widely in the literature - with a prize hidden behind one of the three doors; a guest making a choice of the door to pick the prize; the host who knows the location of the prize as well as the choice made by the guest, now *reveals a distinctly different yet a losing choice*. The host also offers the guest, an option to switch from the earlier choice to the now available second choice, anticipating an enhanced chance of winning the prize, based on the knowledge obtained about a losing choice.

Let us represent the action/events associated with the three doors: (1) let r be the door behind which the prize is hidden (2) let p be the choice of the door pointed out by the guest and (3) let q be the door opened by the host to reveal the losing choice.

It is essential to note here that r and p being mutually independent of each other as well as q, whereas q itself is dependent on both r and p, as per the rules of the game. Also, note that the focus is on the decision-making process & the action to be taken by the guest. Therefore, the problem formulation must necessarily be from the view-point of the guest.

The door $r \in \{1, 2, 3\}$ behind which the prize-reward is hidden, because of complete lack of knowledge thereof, is assumed by the guest, to be equally probable, mutually exclusive together exhaustive event possibilities; each has a-priori probability of 1/3 thus adding up to one.

Similarly, the choice of the door $p \in \{1, 2, 3\}$ pointed out by the guest is based on zero-knowledge and therefore at best a random-guess, that it can be any one of the three doors, each possibility being assumed to be equally probable, mutually exclusive together exhaustive event possibilities; therefore, each has a probability of 1/3 thus adding up to one.

Now, because the two events $[r \in \{1, 2, 3\}]$ and $[p \in \{1, 2, 3\}]$ are mutually independent of each other, the joint probability of the combinations of these two events can be obtained as the product of the probabilities of the two component events. Therefore, the joint probability of each of the combined-duplet-events $[r \in \{1, 2, 3\}][p \in \{1, 2, 3\}]$ is 1/9 and the sum total of these nine joint probabilities is one.

Note that q the action of the host is indeed dependent on both p and r – why/how? Read further.

3. ANALYSIS OF THE PROBLEM SCENARIO

The host takes note of the choice p made by the guest. Being already aware of the actual location r of the prize, a losing choice q that is other than p ($q \neq p$) and also different from r ($q \neq r$) can be shown to the guest, while also offering the guest with an enticing option to switch from the earlier selected choice and go for a distinctly different available alternative second choice.

In such a situation, the host may or may not have many alternative choices, and his action q is indeed dependent on both p and r; and therefore, we need to apply the rule of *restricted probabilities* – that is, such event-possibilities can be assumed to be mutually exclusive together exhaustive equally probable alternatives, at most only within such restricted domain in order to satisfy these restrictions.

It turns out that there are exactly 12 possibilities for the combined-triplet-event E(p, r, q) as represented in Table-1, listing each of the 12 triplets along with their associated joint probabilities of occurrence. Note that size of this event-space is 12 and not 27 which would have been the case if each of the three component-events were indeed mutually independent. Rather, the first two are independent giving rise to a combined-duplet-event-space E(p, r) of size nine. When it is then combined with the third component-event E(q), there results a splitting, in three cases: in the three cases {E11x, E22y, E33z} wherein p = r, the third component-event E(q) $\in$ [{x}$\vee${y}$\vee${z}] gets two alternative possibilities, namely, x $\in$ {2, 3} and y $\in$ {1, 3} and z $\in$ {1, 2}; whereas in the remaining six of them, wherein p $\neq$ r, the third component-event E(q) has a 'fixed choice' since there is no splitting into multiple alternatives.

Sl.No.	**E(p)**	**E(r)**	**E(p,r)**	**E(q)**	**E(p,r,q)**		**p**	**r**	**q**		**P[E(p)]**	**P[E(r)]**	**P[E(q)]**	**P[E(p,r,q)]**
01	1	1	11	2	112		1	1	2		1/3	1/3	1/2	1/18
02	1	1	11	3	113		1	1	3		1/3	1/3	1/2	1/18
03	1	2	12	3	123		1	2	3		1/3	1/3	1	1/9
04	1	3	13	2	132		1	3	2		1/3	1/3	1	1/9
05	2	1	21	3	213		2	1	3		1/3	1/3	1	1/9
06	2	2	22	1	221		2	2	1		1/3	1/3	1/2	1/18
07	2	2	22	3	223		2	2	3		1/3	1/3	1/2	1/18
08	2	3	23	1	231		2	3	1		1/3	1/3	1	1/9
09	3	1	31	2	312		3	1	2		1/3	1/3	1	1/9
10	3	2	32	1	321		3	2	1		1/3	1/3	1	1/9
11	3	3	33	1	331		3	3	1		1/3	1/3	1/2	1/18
12	3	3	33	2	332		3	3	2		1/3	1/3	1/2	1/18

Table-1: 12 combined-triplet-event possibilities along with its joint-probabilities.

E(p):guest-choses-door-p; E(r):prize-behind-door-r; E(q):host-reveals-door-q

4. WHAT IS WRONG WITH THE EXISTING APPROACH

The enticement offered by the host to the guest with an option to switch from the earlier choice to a distinctly different alternative available second choice, with a motivation to enhance the chances of winning the prize, is indeed made with a short-sighted view of the event-space scenario, focusing on a sub-space, for example, consisting of the guest's first choice of door-1 and without considering the fact that the host would have made an exact same offer even if the guest were to make the first choice to be door-2.

Also, from the event-space scenario presented in Table-1 it is intriguing to find out that – given that the host has opened door-3 to show a losing choice, the *a-posteriori probability* (revised & updated based upon whatever the additional knowledge gained) of winning the prize can be appropriately determined by focusing upon the table entries with q = 3.

For this, we need to fix q = 3 and collect terms E(113) E(123) E(213) & E(223) from Table-1, from which it is clear that the *a-posteriori conditional marginal probability* of r = 1 given q = 3 is $$P[E(.1|3)] = P[E(113)] + P[E(213)] \qquad \text{(Eqn.1)}$$

which is exactly the same as the *a-posteriori conditional marginal probability* of r = 2 given q = 3

that is $$P[E(.2|3)] = P[E(223)] + P[E(123)] \qquad \text{(Eqn.2)}$$

However, by the very nature of the rules of the game, and as listed in Table-1,

$$P[E(113)] \leq P[E(123)] \qquad \text{(Eqn.3)}$$

and also

$$P[E(223)] \leq P[E(213)] \qquad \text{(Eqn.4)}$$

which indicates that the same *paradoxical enticement* exists for a switched choice, whatever might have been the earlier choice.

The entries in Table-1 establishes that the event-space scenario is symmetrical in terms of exchaning p and r for any given fixed q; that is,

$$P[E(x,y,z)] = P[E(y,x,z)] \text{ and } P[E(x,x,z)] = P[E(y,y,z)] \text{ for any } p=x, r=y, q=z. \qquad \text{(Eqn.5)}$$

This symmetry property, along with the *seemingly paradoxical* feature that –

$$P[E(y,y,z)] = P[E(x,x,z)] \quad \leq \quad P[E(y,x,z)] = P[E(x,y,z)] \text{ for any } p=x, r=y, q=z; \qquad \text{(Eqn.6)}$$

justifies the reasoning of the guest against going for a switched choice from an earlier selected choice to a distinctly different alternative available second choice in anticipation of enhanced chances to win the prize.

Note that Eqn.1 and Eqn.2 are the basis for our refutation of the widely accepted position held by leading subject-area-experts who seem to derive their arguments from Eqn.3 and Eqn.4.

5. CONCLUSION

This research report presents a novel intriguing analysis of the Monty-Hall Problem, refuting the most widely held position - and advocating against acting on any enticing offers made by the host to the guest for an optional switch from the already selected choice to a possibly distinct alternative available choice - why, because there is indeed no advantage in terms of any enhanced chances to win the prize, unlike what has been widely accepted till today.

It seems as if there has been indeed a widespread confusion in terms of the appropriate use of a-priori probabilities along with a-posteriori probabilities, especially in the context of joint probabilities & restricted probabilities, and appropriate comparison between/among the marginal probabilities and conditional probabilities; whereas, what is required here is the determination of the *a-posteriori conditional marginal probabilities* as in Eqn-1 & Eqn-2 above.

All subject-area-experts, please note that the *additional knowledge* gained from the host, revealing a losing-chance, does indeed *shrink the sample-space* for the guest, from one of size 3 to one of size 2, but that knowledge is *not specific enough to distinguish between the updated/now-available 2 alternatives* in the resultant updated/smaller sample-space.

In general, it is essential to note that *any additional knowledge gained*, revealing a losing (undesirable) possibility, although may lead to an *updated/smaller sample-space*, may not and/or *need not necessarily be specific enough for a refinement/update in the distinction between/among the probabilities of the updated/now-available alternatives* in the resultant updated sample-space.

This is the most critical KEY CONTRIBUTION of this author to the Science of Decision-Making in and through this problem-solving exercise.

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7. ACKNOWLEDGEMENT

I must necessarily confess here that the *core idea behind this analysis is so stunningly & elusively simple*, that one may simply be taken aback in a profound wonder-struck jaw-drop-silence, maybe with an after-thought: "*oh my goodness, how could it be that it never flashed on me any time earlier*"! as was also the case in an earlier research work reported in [13] by this author.

8. DEDICATION

To my ಅಜ್ಜ(ajja) Karinja Halemane Keshava Bhat & ಅಜ್ಜಿ(ajji) Thirumaleshwari, ಅಪ್ಪ(appa) Shama Bhat & ಅಮ್ಮ(amma) Thirumaleshwari, for their *teachings through love*, that *quality matters more than quantity*; to my wife Vijayalakshmi for her *ever consistent love & support*; to my daughter Sriwidya.Bharati and my twin sons Sriwidya.Ramana & Sriwidya.Prawina for their *love & affection*.

Whereas *this Original Author-Creator* holds the (PIPR:©:) Perpetual Intellectual Property Rights, it is but natural that his legal heirs (three children mentioned above) may avail the same for perpetuity.

To all the *cool-headed brave-hearts*, eagerly awaited but probably yet to be visible among the world professionals, especially the *subject-area-experts*, who would be attracted to and certainly capable of effectively understanding without any prejudice and appreciating the deeper insights enshrined in this short research report, who may opt for innate rational-&-intellectual common-sense and simple creativity over any sophisticated and/or complex theory in problem-solving to resolve any seemingly paradoxical scenario.

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