MONTY-HALL PROBLEM : TO SWITCH OR NOT TO SWITCH PARADOX RESOLVED

PIPR:©: Dr.(Prof.) Keshava Prasad Halemane,
Professor - retired from
Department of Mathematical And Computational Sciences
National Institute of Technology Karnataka, Surathkal
Srinivasnagar, Mangaluru - 575025, India.
SASHESHA, 8-129/12 Sowjanya Road, Naigara Hills,
Bikarnakatte, Kulshekar Post, Mangaluru-575005. Karnataka State, India.
https://www.linkedin.com/in/keshavaprasadahalemane/
https://colab.ws/researchers/R-3D34E-09884-MI42Z
https://github.com/KpH8MACS4KREC2NITK
https://orcid.org/0000-0003-3483-3521
https://osf.io/xftv8/

ABSTRACT

This research report presents a deep re-look at the classical Monty-Hall Problem, refuting the widely accepted position held by the leading subject area experts, and establishing that there is no rational basis for a switched choice in the decision to be made by the guest of the game show.

Many a times, the additional knowledge gained, revealing a losing-chance, although leads to an updated smaller sample-space, may not be specific enough for refinement/update on the relative chances between/among the now-available alternatives in the resultant smaller sample-space.

Keywords: A-Priori Probability; A-Posteriori Probability; Mutually Independent Events; Mutually Exclusive Together Exhaustive Alternatives; Joint Probability; Restricted Probability, Conditional Probability. Marginal Probability.

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1. INTRODUCTION

The classical "Monty-Hall Problem", also referred to as the "Three-Door Problem" is based on a game show "Let's Make a Deal" wherein the host reveals a losing choice to the guest, who had earlier made an initial choice, and in turn offers the guest an enticing option to switch from the earlier choice to a second available choice with an aim to enhance the chances of winning the prize. The most prevalent position, as reported in literature, among the leading eminent mathematicians, statisticians, logicians, subject-area-experts and rational intellectuals, is that an appropriate detailed study & analysis of the scenario using the well accepted standard approach of Probability & Statistics, would lead to a recommendation to the guest to switch to the second available choice based on the knowledge obtained from the host revealing a losing choice.

It is argued that the default of sticking to the initial choice will result in a probability of success being only one-third whereas a switch to the alternative second available choice will result in a probability of success being two-third and hence a switched choice is recommended. However, it will be shown here that this approach itself cannot be justified, and therefore the resultant recommendation for switched choice is indeed baseless.

2. DESCRIPTION OF THE PROBLEM - BACKGROUND SCENARIO

We shall focus only on the so-called *classical* Monty-Hall Problem, for the purpose of this report.

For the sake of clarity, let us consider the standard Monty-Hall Problem as reported widely in the literature - with a prize hidden behind one of the three doors; a guest making a choice of the door to pick the prize; the host who knows the location of the prize as well as the choice made by the guest, now *reveals a distinctly different yet a losing choice*. The host also offers the guest, an option to switch from the earlier choice to the now available second choice, anticipating an enhanced chance of winning the prize, based on the knowledge obtained about a losing choice.

Let us represent the action/events associated with the three doors: (1) let $x \in \{1,2,3\}$ be the door behind which the prize is hidden (2) let $y \in \{1,2,3\}$ be the choice of the door pointed out by the guest and (3) let $z \in \{1,2,3\}$ be the door opened by the host to reveal a losing choice.

It is essential to note here that x and y being mutually independent of each other as well as z, whereas z itself is dependent on both x and y, as per the rules of the game. Also, note that the focus is on the decision-making process & the action to be taken by the guest. Therefore, the *problem formulation* (*modelling*) must necessarily be from the view-point of the guest.

Since the guest has absolutely zero knowledge about the *door x behind which the prize is hidden*, no *assumptions* need to be made, even about its possible probability distribution. Similarly, the choice of the *door y initially pointed out by the guest* is based on zero-knowledge without any strategy as such, and therefore at best a random (blind) guess. However, *to facilitate a concrete analysis of the problem scenario and to provide a framework towards a rigorous mathematical model*, it may be useful to make an *assumption* that these two events/actions are *equally probable among the three available mutually exclusive together exhaustive alternative possibilities, each having an a-priori probability of 1/3 thus adding up to one*.

Now, because the two events/actions $[x \in \{1,2,3\}]$ and $[y \in \{1,2,3\}]$ are mutually independent of each other, the joint probability of the combinations of these two events can be obtained as the product of the probabilities of the two component events. Therefore, the joint probability of each of the combined-duplet-events $P[E\{(x \in \{1,2,3\}) \; \& \; (y \in \{1,2,3\})\}]$ is 1/9 and the sum total of these nine joint probabilities is one.

Note that the event/action of the host opening door $z \in \{1,2,3\}$ to show a losing choice, is indeed dependent on both y and x, as per the rules of the game show, that is $z \neq y$ and also $z \neq x$. Although the host has full & complete knowledge of the problem scenario, this dependency of z on y and x does indeed limit his options. It turns out that when $y \neq x$ the host doesn't have any option except to turn to the one and only one remaining door $z \neq (y \neq x)$; whereas when $y = x$ the host has the option of choosing between the two doors, that is, $z \neq (y = x)$. Because the host has this option, at least in a restricted sense, of choosing his door z, it introduces an uncertainty for the guest to predict/expect/anticipate the host's decision/action in this regard. However, as earlier, for the same very same reasons as stated above, it may be useful to make an *assumption* that the host's choice between the two options, whenever available, in a restricted sense, is *equally probable among the two available mutually exclusive together exhaustive alternative possibilities, each having an a-priori probability of 1/2 thus adding up to one*.

3. PROBLEM FORMULATION

With the above understanding of the background scenario of the classical Monty-Hall Problem, one can derive that there are exactly 12 possibilities for the combined-triplet-event $E(x, y, z)$ as represented in Table-1, listing each of the 12 triplets along with their associated joint probabilities of occurrence. Note that the event-space is of size 12 and not 27 which would have been the case if each of the three component-events were indeed mutually independent. Because the first two are independent giving rise to a combined-duplet-event-space $E(x, y)$ of size nine. When it is then combined with the third component-event $E(z)$, there results a splitting, in three cases. In the three cases when $x = y$, that is, $\{E11z, E22z, E33z\}$ the third component-event $E(z) \in \{Z1 \vee Z2 \vee Z3\}$ gets two alternative possibilities, namely, $Z1 \in \{2, 3\}$ and $Z2 \in \{1, 3\}$ and $Z3 \in \{1, 2\}$. Whereas in the remaining six of them, that is, $\{E12z, E13z, E21z, E23z, E31z, E32z\}$ wherein $x \neq y$, the third component-event $E(z)$ has a 'fixed choice' since there is one and only one single fixed choice, no splitting into multiple alternative possibilities.

Sl.No.	E(x)	E(y)	E(x,y)	E(z)	E(x,y,z)		P[E(x)]	P[E(y)]	P[E(z) | E(x,y)]	P[E(x,y,z)]
01	1	1	11	2	112		1/3	1/3	1/2	1/18
02	1	1	11	3	113		1/3	1/3	1/2	1/18
03	1	2	12	3	123		1/3	1/3	1	1/9
04	1	3	13	2	132		1/3	1/3	1	1/9
05	2	1	21	3	213		1/3	1/3	1	1/9
06	2	2	22	1	221		1/3	1/3	1/2	1/18
07	2	2	22	3	223		1/3	1/3	1/2	1/18
08	2	3	23	1	231		1/3	1/3	1	1/9
09	3	1	31	2	312		1/3	1/3	1	1/9
10	3	2	32	1	321		1/3	1/3	1	1/9
11	3	3	33	1	331		1/3	1/3	1/2	1/18
12	3	3	33	2	332		1/3	1/3	1/2	1/18

Table-1: 12 combined-triplet-event possibilities along with its joint-probabilities.

E(x): prize-behind-door-x; E(y): guest-choses-door-y; E(z): host-reveals-door-z

Mutually-Exclusive Together-Exhaustive Equi-Probable Alternative-Possibilities

4. ANALYSIS OF THE DECISION-MAKING SCENARIO

A general analysis of the decision-making scenario (modelling, from the viewpoint of the guest) is presented first, without being constrained by the three assumptions mentioned earlier, that the events/actions represented by x, y, z, are assumed to be mutually-exclusive together-exhaustive equi-probable among the available alternative-possibilities – just to accommodate for other possible specific alternative scenarios, especially if & when someone wishes to try out computer simulation and/or Monte-Carlo type of studies, etc. Specific results pertaining to the data entries given in Table-1 above, may always be easily worked out by plugging the corresponding data to each of the concerned parameters as needed.

The decision/choice/action of the host, represented by z∈{1,2,3} being dependent on x∈{1,2,3} and y∈{1,2,3}; implying, that the joint probability of the combined-triplet-event referred therein be determined by the corresponding conditional probability:

That is, in general, for any (z=q) and (x=r) and (y=p) we have,

P[E{(z=q)&(x=r)&(y=p)}] = P[E{(z=q)} | E{(x=r)&(y=p)}] * P[E{(x=r)&(y=p)}]; (Eqn.1)

From the rules of the game, when z ≠ (y ≠ x) we have the *conditional probability*,
P[E{(z=q) | (x=r) & (y=p)}] = 1; (Eqn.2)
and therefore, we get the joint probability,
P[E{(z=q) & (x=r) & (y=p)}] = P[E{(x=r) & (y=p)}]; (Eqn.3)

whereas, when z ≠ (y = x) we cannot make any stronger statement, except the general condition for the *restricted (conditional) probability*:
0 ≤ P[E{(z=q) | (x=r) & (y=r)}] ≤ 1. (Eqn.4)

That is, when z ≠ (y ≠ x) we have, the *joint probability* of the combined-triplet-event is given by the joint probability of the corresponding combined-duplet-event, as given by -
P[E123] = P[E12.]; P[E132] = P[E13.]; P[E213] = P[E21.];
P[E231] = P[E23.]; P[E312] = P[E31.]; P[E321] = P[E32.]; (Eqn.5)

whereas, when z ≠ (y = x) we cannot make any stronger statement except the general condition that the sum of the joint probabilities of the related combined-triplet-events will give the *marginal probability* (*joint probability* of the corresponding combined-duplet-event) as given by -
P[E112]+P[E113]=P[E11.]; P[E221]+P[E223]=P[E22.]; P[E331]+P[E332]=P[E33.] (Eqn.6)

With this information, we may proceed to determine the *a-posteriori-conditional* (for z = q, say) *marginal-probability* of the prize being hidden behind door x = r, as follows, with (z = q) ≠ (x = r):
P[E{(x=r) | (z=q)}] = P[E{(x=r)&(y=p)&(z=q)}] + P[E{(x=r)&(y=r)&(z=q)}] (Eqn.7)

Using Eqn.5 in specific instances of Eqn.7, we get -
P[E{(x=1) | (z=3)}] = P[E123] + P[E113]; P[E{(x=2) | (z=3)}] = P[E213] + P[E223];
P[E{(x=1) | (z=2)}] = P[E132] + P[E112]; P[E{(x=3) | (z=2)}] = P[E312] + P[E332];
P[E{(x=2) | (z=1)}] = P[E231] + P[E221]; P[E{(x=3) | (z=1)}] = P[E321] + P[E331]; (Eqn.8)

Notice in Eqn.8 above, that the terms P[E123], P[E213], P[E132], P[E312], P[E231], P[E321] are not under the control of the host, as can be confirmed from Eqn.3 & Eqn.5 above; whereas, the terms P[E113], P[E223], P[E112], P[E332], P[E221], P[E331] are indeed under the direct control of the host (of course, within certain limits as per Eqn.4 for the *restricted probability*) based on whatever strategy that one decides and acts accordingly, while following the rules of the game.

The decision of the guest as to whether to avail the offer of the host to opt for a switched choice, say, from door-1 to door-2 after knowing the losing choice behind door-3 as revealed by the host, must be based on a comparison between the two *a-posteriori(conditional) marginal probabilities* P[E{(x=1) | (z=3)}] and P[E{(x=2) | (z=3)}] for which Eqn.8 above provides a clear guideline.

However, any specific answer needs to be derived based on the relative magnitudes of the four joint probabilities involved therein, namely, P[E123], P[E213], P[E113] and P[E223]. That is where the need arises to pin down certain uncertainties (at least the ones that are not under the control of the host) by *assuming* certain probability distribution, as for example, P[E(x)] and also P[E(y)] to be uniformly distributed among the available (in this case, three) alternatives. If the decision & action of the host can also be *assumed* to adhere to certain probability distribution or certain strategy, it can be used to make specific comparisons that will lead to firm recommendation to the guest as to whether it is worth at all to consider a switched choice. If indeed the host adheres and follows a uniform distribution, in the sense that whenever faced with multiple alternatives, the specific choice of any one is equally probable and together exhaustive, then and only then, can we for sure, recommend to the guest that there is indeed no need to go for a switched choice and that the relative outcomes are exactly the same in either case - that was the *assumption* in preparing Table-1.

5. WHAT IS WRONG WITH THE EXISTING APPROACH

The enticement offered by the host to the guest with an option to switch from the earlier choice to a distinctly different alternative available second choice, needs to be well considered for objective evaluation with a motivation to enhance the chances of winning the prize, However, the recommendation for switched-choice is indeed the result of a short-sighted view of the event-space scenario, focusing on a sub-space, for example, consisting of the guest's first choice of door-1 and without considering the other possibilities under which the host would have made an exact same choice of opening door-3 even if the guest were to make the first choice to be door-2.

Also, from the event-space scenario presented in Table-1 it is intriguing to find out that – given that the host has opened door-3 to show a losing choice, the *a-posteriori probability* (revised & updated based upon whatever the additional knowledge gained) of winning the prize can be appropriately determined by focusing upon the table entries with z = 3.

For this, we need to fix z = 3 and collect terms E(113) E(123) E(213) & E(223) from Table-1, from which it is clear that the *a-posteriori(conditional) marginal probability* of x=1 given z=3; is

$$P[E(1y \mid 3)] = P[E(113)] + P[E(123)] \qquad \text{(Eqn.9)}$$

which is exactly the same as the *a-posteriori(conditional) marginal probability* of x=2 given z=3; that is

$$P[E(2y \mid 3)] = P[E(223)] + P[E(213)] \qquad \text{(Eqn.10)}$$

However, by the very nature of the rules of the game, and as listed in Table-1,

$$P[E(113)] \leq P[E(123)] \qquad \text{(Eqn.11)}$$

and also

$$P[E(223)] \leq P[E(213)] \qquad \text{(Eqn.12)}$$

which indicates that the same *counter-intuitively paradoxical enticement* exists for a switched choice, whatever might have been the earlier choice – justifying that MHP is indeed a paradox.

The entries in Table-1 establishes that the event-space scenario is symmetrical in terms of exchaning x and y for any given fixed z; that is,

$$P[E(x,y,z)] = P[E(y,x,z)] \text{ and } P[E(x,x,z)] = P[E(y,y,z)]; \text{ for any x, y, z.} \qquad \text{(Eqn.13)}$$

This *symmetry property*, even-if with the *seemingly paradoxical* *counter-intuitive* feature that,

$$P[E(y,y,z)] = P[E(x,x,z)] \quad \leq \quad P[E(y,x,z)] = P[E(x,y,z)] \text{ for any x, y, z;} \qquad \text{(Eqn.14)}$$

justifies the reasoning of the guest against going for a switched choice from an earlier selected choice to a distinctly different alternative available second choice in anticipation of enhanced chances to win the prize.

Note that Eqn.9, Eqn.10 & Eqn.13 are the basis for our refutation of the widely accepted position held by *Leading Subject-Area-Experts* who seem to derive their arguments from Eqn.11, Eqn.12 & Eqn.14. They seem to have considered the comparison $P[E(113)] \leq P[E(123)]$ as in Eqn.11 instead of $P[E(113)]+P[E(123)]=P[E(1y|3)]=P[E(2y|3)]=P[E(223)]+P[E(213)]$ from Eqns. 9, 10 & 13.

The approach adopted by the *Leading Subject-Area-Experts* seems to correspond to an attempt to model the situation wherein the guest holds on to his initial choice of door-1 while looking at the alternative second choice of door-2; rather than a first step to withdraw/cancel the initial choice of door-1 followed by a second step of detailed re-evaluation that would naturally be captured only by the approach recommended here.

This is an example problem, illustrating the fact that a change in the relative order in a sequence of operations as modelled in the problem formulation step, can lead to a huge difference in the outcome - erroneous problem formulation & erroneous model naturally leading to the erroneous results, that seems to have been confirmed through some erroneous computer simulation studies.

The clearly partitioned event-space, with 12 mutually-exclusive and together-exhaustive possible alternatives, represented in Table-1, as a framework to study, analyze & solve the problem, so that there is no possibility of missing any relevant (and/or including any irrelevant) component terms while going through the required calculations in order to derive whatever desired results.

6. A CHALLENGE TO THE SUBJECT AREA EXPERTS

Let us rephrase the Monty-Hall Problem, now adorned with a *jewel-on-the-crown* as below:

MONTY-HALL-PROBLEM (MHP) TO-SWITCH-OR-NOT-TO-SWITCH : THAT IS THE QUESTION
(1.1) The prize is hidden behind one of the three doors.
(1.2) I the guest make an initial choice of which door it could be, say door-1, to claim my prize.
(1.3) Then Monty the host opens a different door, say door-3, revealing a losing choice.
(2.1) I am given an option to withdraw/cancel the earlier choice of door-1 and switch to door-2.
(2.2) I appreciate the knowledge of a losing choice and also Monty's offer of the option to switch.
(3.1) I grab Monty's offer, withdraw/cancel my earlier choice of door-1.
(3.2) Then I re-evaluate the two choices available for me now, namely door-1 or door-2.
(3.3) I find that the chances of winning are exactly the same between the two available choices;
(4.1) Now, YOU the SUBJECT AREA EXPERT enter the Hall – unaware of my initial choice!
(4.2) I turn towards YOU seeking YOUr expert recommendation. What is YOUr recommendation?
(4.3) TO SWITCH OR NOT TO SWITCH : THAT IS THE QUESTION!

Note that your answer must necessarily be independent of my initial-choice (door-1); although the Monty's choice of door-3 revealing a losing choice was dependent on my initial choice (door-1) which he had to avoid as per the rules of the game. Hope YOUr expert advice is not an exemplification of the proverb "*the grass is always greener on the other side*"!

7. CONCLUSION

This research report presents a novel intriguing analysis of the Monty-Hall Problem, refuting the most widely held position - and advocating against acting on any enticing offers made by the host to the guest for an optional switch from the already selected choice to a possibly distinct alternative available choice - why, because there is indeed no advantage in terms of any enhanced chances to win the prize, unlike what has been widely accepted till today.

All subject-area-experts, please note that the *additional knowledge* gained from the host, revealing a losing-chance, does indeed *shrink the sample-space* for the guest, from one of size 3 to one of size 2, but that knowledge is *not specific enough to distinguish between the updated/now-available 2 alternatives* in the resultant updated/smaller sample-space.

In general, it is essential to note that *any additional knowledge gained*, revealing a losing (undesirable) possibility, although may lead to an *updated/smaller sample-space*, may not and/or *need not necessarily be specific enough for a refinement/update in the distinction between/among the probabilities of the updated/now-available alternatives* in the resultant updated sample-space.

This is the most critical KEY CONTRIBUTION of this author to the Science of Decision-Making in and through this problem-solving exercise.

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9. ACKNOWLEDGEMENT

I must necessarily confess here that the *core idea behind this analysis is so stunningly & elusively simple*, that one may simply be taken aback in a profound wonder-struck jaw-drop-silence, maybe with an after-thought: "*oh my goodness, how could it be that it never flashed on me any time earlier*"! as was also the case in an earlier research work reported in [13] by this author.

10. DEDICATION

To my ಅಜ್ಜ(ajja) Karinja Halemane Keshava Bhat & ಅಜ್ಜಿ(ajji) Thirumaleshwari, ಅಪ್ಪ(appa) Shama Bhat & ಅಮ್ಮ(amma) Thirumaleshwari, for their *teachings through love*, that *quality matters more than quantity*; to my wife Vijayalakshmi for her *ever consistent love & support*; to my daughter Sriwidya.Bharati and my twin sons Sriwidya.Ramana & Sriwidya.Prawina for their *love & affection*.

Whereas *this Original Author-Creator* holds the (PIPR:©:) Perpetual Intellectual Property Rights, it is but natural that his legal heirs (three children mentioned above) may avail the same for perpetuity.

To all the *cool-headed brave-hearts*, eagerly awaited but probably yet to be visible among the world professionals, especially the *subject-area-experts*, who would be attracted to and certainly capable of effectively understanding without any prejudice and appreciating the deeper insights enshrined in this short research report, who may opt for innate rational-&-intellectual common-sense and simple creativity over any sophisticated and/or complex theory in problem-solving to resolve any seemingly paradoxical scenario.

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