MetaHyperGraphs, MetaSuperHyperGraphs, and Iterated MetaGraphs: Modeling Graphs of Graphs, Hypergraphs of Hypergraphs, Superhypergraphs of Superhypergraphs, and Beyond

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Abstract

Graph theory studies mathematical structures composed of vertices and edges to model relationships and connectivity [1, 2]. Hypergraphs extend traditional graphs by allowing *hyperedges* that connect more than two vertices simultaneously [3]. Superhypergraphs further enrich this concept by introducing iterated powerset layers, enabling hierarchical and self-referential connections among hyperedges [4, 5]. A MetaGraph is a graph whose vertices are themselves graphs, with edges representing specified relations between those graphs. In this paper, we formally define the hypergraph analogue (MetaHyperGraph) and the superhypergraph analogue (MetaSuperHyperGraph) of MetaGraphs, and provide a concise discussion of their characteristics and illustrative applications. We also introduce iterative constructions such as the Iterated MetaGraph, representing a "graph of graphs of . . . of graphs," and briefly explore their properties and potential uses.

Keywords: Superhypergraph, Hypergraph, MetaHyperGraph, MetaSuperHyperGraph, MetaGraph

Contents in this paper

The remainder of this paper is organized as follows.

1 Preliminaries

This section collects the basic notions used throughout the paper. Unless stated otherwise, all graphs are assumed to be *finite*, *undirected*, and *simple*.

1.1 SuperHyperGraph

In classical graph theory, a hypergraph extends an ordinary graph by allowing *hyperedges* that may join more than two vertices, which makes it suitable for modeling multiway relations in many areas [3, 6–11]. A *SuperHyperGraph* augments this idea by organizing vertices and edges through iterated powersets, thereby enabling hierarchical and self-referential linkages; see, e.g., [12–17]. Beyond theory, SuperHyperGraphs have seen use in applications such as molecular and network modeling and signal processing [12, 18–22]. Throughout, the parameter n in the n-th (non-)empty powerset and in an n-SuperHyperGraph is a nonnegative integer.

Definition 1.1 (Base set)**.** A *base set* S is the underlying universe from which all objects are formed. Formally,

$$S = \{ x \mid x \text{ belongs to the specified domain} \}.$$

All elements of constructions such as $\mathcal{P}(S)$ and $\mathcal{P}_n(S)$ are ultimately drawn from S.

Definition 1.2 (Powerset)**.** For a set S, the *powerset* $\mathcal{P}(S)$ is the family of all subsets of S:

$$\mathcal{P}(S) = \{ A \mid A \subseteq S \}.$$

Definition 1.3 (Hypergraph [3,23])**.** A *hypergraph* is a pair $H = (V(H), E(H))$ where

- $V(H)$ is a nonempty set of vertices;
- $E(H) \subseteq \mathcal{P}^*(V(H))$ is a set of nonempty subsets of $V(H)$ (the hyperedges).

In this work we consider only finite hypergraphs.

Definition 1.4 (n-th powerset)**.** (cf. [24–26]) For a set H and $n \geq 1$, define recursively

$$\mathcal{P}_1(H) := \mathcal{P}(H), \qquad \mathcal{P}_{n+1}(H) := \mathcal{P}\big(\mathcal{P}_n(H)\big).$$

The *n-th nonempty powerset* $\mathcal{P}_n^*(H)$ is defined analogously by

$$\mathcal{P}_1^*(H) := \mathcal{P}^*(H), \qquad \mathcal{P}_{n+1}^*(H) := \mathcal{P}^*\big(\mathcal{P}_n^*(H)\big),$$

where $\mathcal{P}^*(H) := \mathcal{P}(H) \setminus \{\emptyset\}$.

Example 1.5 (A real-life reading of the n-th powerset: baskets → promotions → campaigns)**.** Let the base set of *products* be

$$H = \{\text{Milk, Bread, Eggs}\}.$$

Then the first powerset $\mathcal{P}_1(H) = \mathcal{P}(H)$ (all possible *shopping baskets*) is

$$\mathcal{P}_1(H) = \big\{ \emptyset, \{\text{Milk}\}, \{\text{Bread}\}, \{\text{Eggs}\}, \{\text{Milk, Bread}\}, \{\text{Milk, Eggs}\}, \{\text{Bread, Eggs}\}, \{\text{Milk, Bread, Eggs}\} \big\}.$$

An element of the second powerset $\mathcal{P}_2(H) = \mathcal{P}\big(\mathcal{P}_1(H)\big)$ is a *promotion*, i.e. a *set of baskets*. For instance,

$$\mathcal{F}_1 := \big\{ \{\text{Milk}\}, \{\text{Bread, Eggs}\} \big\}, \qquad \mathcal{F}_2 := \big\{ \emptyset, \{\text{Milk, Bread, Eggs}\} \big\}.$$

Since each listed basket belongs to $\mathcal{P}_1(H)$, we indeed have $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{P}_2(H)$.

An element of the third powerset $\mathcal{P}_3(H) = \mathcal{P}\big(\mathcal{P}_2(H)\big)$ is a *campaign*, i.e. a *set of promotions*. For example,

$$C := \{ \mathcal{F}_1, \mathcal{F}_2 \} \in \mathcal{P}_3(H),$$

because $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{P}_2(H)$.

Interpretation. $\mathcal{P}_1(H)$: all baskets a customer could buy. $\mathcal{P}_2(H)$: a retailer's promotion, specified as a collection of eligible baskets. $\mathcal{P}_3(H)$: a marketing campaign, specified as a collection of promotions. This realizes the recursion $\mathcal{P}_{n+1}(H) = \mathcal{P}(\mathcal{P}_n(H))$ in a concrete setting.

Definition 1.6 (n-SuperHyperGraph)**.** [4,27,28]
Let V_0 be a finite base vertex set. Define the iterated powerset by

$$\mathcal{P}^0(V_0) := V_0, \qquad \mathcal{P}^{k+1}(V_0) := \mathcal{P}\big(\mathcal{P}^k(V_0)\big) \quad (k \geq 0),$$

and write $\mathcal{P}^*(X) := \mathcal{P}(X) \setminus \{\emptyset\}$ for the family of all nonempty subsets of a set X. An (undirected, simple) *n-SuperHyperGraph* on V_0 is a pair

$$\mathrm{SHG}^{(n)} = (V, E)$$

such that

$$V \subseteq \mathcal{P}^n(V_0) \text{ is finite}, \qquad E \subseteq \mathcal{P}^*(V).$$

Elements of V are called *n-supervertices*, and each $e \in E$ is a (nonempty) *n-superedge*, i.e. a finite subset of V. Equivalently, $\mathrm{SHG}^{(n)}$ is a (simple) hypergraph whose vertex set is a finite subset of $\mathcal{P}^n(V_0)$; the incidence relation $\iota \subseteq V \times E$ is given by $v \,\iota\, e \iff v \in e$.

Example 1.7 (A 2-SuperHyperGraph)**.** Let the base set be $V_0 = \{a, b, c\}$. Then $\mathcal{P}(V_0)$ consists of

$$\emptyset,\ \{a\},\ \{b\},\ \{c\},\ \{a,b\},\ \{a,c\},\ \{b,c\},\ \{a,b,c\},$$

and $\mathcal{P}^2(V_0) = \mathcal{P}\big(\mathcal{P}(V_0)\big)$.

Define three 2-supervertices (each is a subset of $\mathcal{P}(V_0)$, hence an element of $\mathcal{P}^2(V_0)$):

$$S_1 := \big\{\{a\}, \{b,c\}\big\}, \qquad S_2 := \big\{\emptyset, \{b\}\big\}, \qquad S_3 := \big\{\{a,c\}\big\}.$$

Set the vertex set and hyperedge set as

$$V := \{S_1, S_2, S_3\} \subseteq \mathcal{P}^2(V_0), \qquad E := \big\{\, e_1, e_2 \,\big\} \subseteq \mathcal{P}^*(V),$$

with

$$e_1 := \{S_1, S_2\}, \qquad e_2 := \{S_2, S_3\}.$$

Then $\mathrm{SHG}^{(2)} := (V, E)$ is a (simple, undirected) 2-SuperHyperGraph.

Verification (universe membership):

$$\{a\}, \{b,c\}, \emptyset, \{b\}, \{a,c\} \in \mathcal{P}(V_0) \implies S_1, S_2, S_3 \in \mathcal{P}^2(V_0),$$

and e_1, e_2 are nonempty subsets of V, so $E \subseteq \mathcal{P}^*(V)$.

Example 1.8 (A 3-SuperHyperGraph)**.** Let $V_0 = \{a, b, c\}$ as above. First choose elements of $\mathcal{P}^2(V_0)$ (i.e., subsets of $\mathcal{P}(V_0)$):

$$A_1 := \big\{\{a\}, \{b\}\big\}, \qquad A_2 := \big\{\{a,b\}, \{c\}\big\}, \qquad A_3 := \big\{\emptyset\big\}.$$

Since each $A_i \subseteq \mathcal{P}(V_0)$, we have $A_1, A_2, A_3 \in \mathcal{P}^2(V_0)$.

Define three 3-supervertices (each is a subset of $\mathcal{P}^2(V_0)$, hence an element of $\mathcal{P}^3(V_0)$):

$$U_1 := \{A_1, A_3\}, \qquad U_2 := \{A_2\}, \qquad U_3 := \{A_1, A_2\}.$$

Set

$$V' := \{U_1, U_2, U_3\} \subseteq \mathcal{P}^3(V_0), \qquad E' := \big\{\, f_1, f_2 \,\big\} \subseteq \mathcal{P}^*(V'),$$

with

$$f_1 := \{U_1, U_2\}, \qquad f_2 := \{U_2, U_3\}.$$

Then $\mathrm{SHG}^{(3)} := (V', E')$ is a valid (simple, undirected) 3-SuperHyperGraph.

Verification (universe membership):

$$A_1, A_2, A_3 \in \mathcal{P}^2(V_0) \implies U_1, U_2, U_3 \in \mathcal{P}^3(V_0),$$

and f_1, f_2 are nonempty subsets of V', hence $E' \subseteq \mathcal{P}^*(V')$.

1.2 MetaGraph(Graph of Graph)

A MetaGraph is a graph whose vertices are themselves graphs, with edges representing specified relations between those graphs (cf. [29, 29–33]).

Definition 1.9 (Metagraph (graph of graphs))**.** (cf. [34]) Fix a nonempty universe $\mathfrak{G}$ of finite graphs (undirected, loopless by default) and a nonempty family of binary relations

$$\mathcal{R} \ \subseteq\ \mathcal{P}\big(\mathfrak{G} \times \mathfrak{G}\big).$$

A *metagraph over* $(\mathfrak{G}, \mathcal{R})$ is a directed, labelled multigraph

$$M \ = \ (V, E, s, t, \lambda)$$

with

$$V \subseteq \mathfrak{G}, \qquad s, t : E \to V, \qquad \lambda : E \to \mathcal{R},$$

satisfying the incidence constraint

$$\forall e \in E : \ \big(s(e),\ t(e)\big) \in \lambda(e).$$

Elements of V are *meta-vertices* (each is a graph $G \in \mathfrak{G}$). For $e \in E$ with $\lambda(e) = R$, we write $s(e) \xrightarrow{R} t(e)$ and call e a *meta-edge*. If $\mathcal{R} = \{R\}$ is a singleton, labels may be omitted. If every $R \in \mathcal{R}$ is symmetric, M can be viewed as an undirected labelled multigraph.

Example 1.10 (Real-life MetaGraph: cross-citing departments (graph of graphs))**.** Let $\mathfrak{G}$ be the class of finite directed acyclic *citation graphs* (vertices=papers, edges=citations). Consider three department graphs

$$\begin{aligned} G_{\mathrm{CS}}:\ & V=\{c_1,c_2,c_3\},\ E=\{c_2\to c_1,\ c_3\to c_2\},\\ G_{\mathrm{Bio}}:\ & V=\{b_1,b_2\},\ E=\{b_2\to b_1\},\\ G_{\mathrm{Math}}:\ & V=\{m_1,m_2\},\ E=\{m_2\to m_1\}. \end{aligned}$$

Let X be the set of observed *cross-department* citations:

$$X=\{\,c_3\to b_1,\ c_1\to m_1,\ b_2\to c_2,\ m_2\to c_1\,\}.$$

Define the directed relation R_τ on $\mathfrak{G}$ by the numeric threshold

$$c(G,H)\ :=\ \big|\{(p,q)\in V(G)\times V(H):\ p\to q\in X\}\big|,\qquad (G,H)\in R_\tau\iff c(G,H)\geq\tau.$$

With $\tau=1$, the counts are

$$c(G_{\mathrm{CS}},G_{\mathrm{Bio}})=1,\quad c(G_{\mathrm{CS}},G_{\mathrm{Math}})=1,\quad c(G_{\mathrm{Bio}},G_{\mathrm{CS}})=1,\quad c(G_{\mathrm{Math}},G_{\mathrm{CS}})=1,$$

all others = 0. Hence the metagraph over $(\mathfrak{G},\{R_1\})$ is

$$M=(V,E,s,t,\lambda),\quad V=\{G_{\mathrm{CS}},G_{\mathrm{Bio}},G_{\mathrm{Math}}\},$$

with (edge set and maps made explicit)

$$\begin{aligned} &E=\{e_1,e_2,e_3,e_4\},\quad \lambda(e_i)=R_1\ \ (i=1,\ldots,4),\\ s(e_1)=G_{\mathrm{CS}},\ t(e_1)=G_{\mathrm{Bio}},\qquad & s(e_2)=G_{\mathrm{CS}},\ t(e_2)=G_{\mathrm{Math}},\\ s(e_3)=G_{\mathrm{Bio}},\ t(e_3)=G_{\mathrm{CS}},\qquad & s(e_4)=G_{\mathrm{Math}},\ t(e_4)=G_{\mathrm{CS}}. \end{aligned}$$

By construction, each incidence satisfies $(s(e_i),t(e_i))\in R_1$ because the corresponding $c(\cdot,\cdot)=1$.

2 Review and Result

2.1 Iterated MetaGraph(Graph of Graph of ... of Graph)

An Iterated MetaGraph is a graph whose vertices are metagraphs, recursively extending graph-of-graphs structure to multiple hierarchical levels.

Definition 2.1 (Unit metagraph embedding)**.** For $X\in\mathfrak{G}$ define the *unit metagraph*

$$\mathsf{U}(X)\ :=\ \big(\{X\},\ \emptyset,\ \_,\ \_,\ \_\big).$$

This gives an injective map $\mathsf{U}:\mathfrak{G}\hookrightarrow\mathrm{Obj}\big(\mathrm{Meta}(\mathfrak{G},\mathcal{R})\big)$.

Definition 2.2 (Relation lifting)**.** Given $\mathcal{R}$ on $\mathfrak{G}$, define its *lift* $\mathcal{R}^{\uparrow}$ on finite metagraphs over $(\mathfrak{G},\mathcal{R})$ by

$$\forall R\in\mathcal{R},\quad (M_1,M_2)\in R^{\uparrow}\iff\exists x\in V(M_1),\ y\in V(M_2):\ (x,y)\in R.$$

Set $\mathcal{R}^{\uparrow}:=\{R^{\uparrow}:R\in\mathcal{R}\}$.

Definition 2.3 (Iterated object and relation universes)**.** Define recursively for $t\in\mathbb{N}_0$:

$$\mathfrak{G}^{(0)}:=\mathfrak{G},\qquad \mathcal{R}^{(0)}:=\mathcal{R},$$

$$\mathfrak{G}^{(t+1)}:=\Big\{\text{finite metagraphs over }\big(\mathfrak{G}^{(t)},\mathcal{R}^{(t)}\big)\Big\},\qquad \mathcal{R}^{(t+1)}:=\big(\mathcal{R}^{(t)}\big)^{\uparrow}.$$

Definition 2.4 (Iterated MetaGraph of depth t)**.** For $t\in\mathbb{N}_0$, an *iterated metagraph of depth* t is a metagraph

$$M^{(t)}=(V^{(t)},E^{(t)},s^{(t)},t^{(t)},\lambda^{(t)})$$

over $\big(\mathfrak{G}^{(t)},\mathcal{R}^{(t)}\big)$, i.e., $V^{(t)}\subseteq\mathfrak{G}^{(t)}$, $\lambda^{(t)}:E^{(t)}\to\mathcal{R}^{(t)}$ and

$$\forall e\in E^{(t)}:\ \big(s^{(t)}(e),\ t^{(t)}(e)\big)\in\lambda^{(t)}(e).$$

Remark 2.5. Depth 0 vertices are graphs; depth 1 vertices are metagraphs of graphs; depth 2 vertices are metagraphs whose vertices are metagraphs of graphs; etc. Edges are always certified by the corresponding lifted relation at that depth.

Example 2.6 (Real-life Iterated MetaGraph: universities built from departmental metagraphs)**.** Fix the same thresholded relation R_τ on departmental citation graphs as above.

University A has two departments with internal paper-level graphs

$$G^{\mathsf{A}}_{\mathrm{CS}} : \{a_c^2 \to a_c^1\}, \qquad G^{\mathsf{A}}_{\mathrm{Bio}} : \{a_b^2 \to a_b^1\},$$

and *within*-A cross-citations

$$X_{\mathsf{A}} = \{\, a_c^2 \to a_b^1 \,\} \;\Rightarrow\; c\big(G^{\mathsf{A}}_{\mathrm{CS}}, G^{\mathsf{A}}_{\mathrm{Bio}}\big) = 1.$$

Thus the (depth-0) metagraph for A is

$$M_{\mathsf{A}} = (\{G^{\mathsf{A}}_{\mathrm{CS}}, G^{\mathsf{A}}_{\mathrm{Bio}}\}, \{e_{\mathsf{A}}\}, s, t, \lambda), \quad s(e_{\mathsf{A}}) = G^{\mathsf{A}}_{\mathrm{CS}},\ t(e_{\mathsf{A}}) = G^{\mathsf{A}}_{\mathrm{Bio}},\ \lambda(e_{\mathsf{A}}) = R_1.$$

University B has departments

$$G^{\mathsf{B}}_{\mathrm{CS}} : \text{ no internal edge}, \qquad G^{\mathsf{B}}_{\mathrm{Math}} : \{b_m^2 \to b_m^1\},$$

and *within*-B cross-citations

$$X_{\mathsf{B}} = \{\, b_c^1 \to b_m^1 \,\} \;\Rightarrow\; c\big(G^{\mathsf{B}}_{\mathrm{CS}}, G^{\mathsf{B}}_{\mathrm{Math}}\big) = 1,$$

so

$$M_{\mathsf{B}} = (\{G^{\mathsf{B}}_{\mathrm{CS}}, G^{\mathsf{B}}_{\mathrm{Math}}\}, \{e_{\mathsf{B}}\}, s, t, \lambda), \quad s(e_{\mathsf{B}}) = G^{\mathsf{B}}_{\mathrm{CS}},\ t(e_{\mathsf{B}}) = G^{\mathsf{B}}_{\mathrm{Math}},\ \lambda(e_{\mathsf{B}}) = R_1.$$

Now define *cross-university* citations

$$X_{\mathsf{A}\to\mathsf{B}} = \{\, a_b^2 \to b_m^1 \,\}, \qquad X_{\mathsf{B}\to\mathsf{A}} = \{\, b_c^1 \to a_c^1 \,\}.$$

The lifted relation $R_1^{\uparrow}$ on metagraphs (as in the iterated construction) satisfies

$$\big(M_{\mathsf{A}}, M_{\mathsf{B}}\big) \in R_1^{\uparrow} \iff \exists\, G \in V(M_{\mathsf{A}}),\, H \in V(M_{\mathsf{B}}) : \ (G, H) \in R_1.$$

Numerically,

$$c\big(G^{\mathsf{A}}_{\mathrm{Bio}}, G^{\mathsf{B}}_{\mathrm{Math}}\big) = 1 \Rightarrow \big(M_{\mathsf{A}}, M_{\mathsf{B}}\big) \in R_1^{\uparrow},$$
$$c\big(G^{\mathsf{B}}_{\mathrm{CS}}, G^{\mathsf{A}}_{\mathrm{CS}}\big) = 1 \Rightarrow \big(M_{\mathsf{B}}, M_{\mathsf{A}}\big) \in R_1^{\uparrow}.$$

Hence the *iterated metagraph* (depth 1) of universities is

$$\mathbf{M} = (V^{(1)}, E^{(1)}, s^{(1)}, t^{(1)}, \lambda^{(1)}), \quad V^{(1)} = \{M_{\mathsf{A}}, M_{\mathsf{B}}\},$$
$$E^{(1)} = \{E_{\mathsf{A}\to\mathsf{B}}, E_{\mathsf{B}\to\mathsf{A}}\}, \quad s^{(1)}(E_{\mathsf{A}\to\mathsf{B}}) = M_{\mathsf{A}},\ t^{(1)}(E_{\mathsf{A}\to\mathsf{B}}) = M_{\mathsf{B}},$$
$$s^{(1)}(E_{\mathsf{B}\to\mathsf{A}}) = M_{\mathsf{B}},\ t^{(1)}(E_{\mathsf{B}\to\mathsf{A}}) = M_{\mathsf{A}}, \quad \lambda^{(1)}(\cdot) = R_1^{\uparrow}.$$

Each edge satisfies the incidence constraint by the explicit counts $c(\cdot,\cdot) = 1$ shown above.

Example 2.7 (Iterated MetaGraph of depth 3: airlines → alliances → consortia → clusters)**.** We build the structure level by level and verify every incidence numerically.

Level 0 (base graphs: airline route graphs). Let the set of airports be $\mathcal{A} = \{X, Y, Z, W, V\}$. Define five finite (undirected, simple) graphs, each representing an airline's direct-flight network:

$$G_{\mathrm{A1}} : V = \{X, Y\},\ E = \big\{\{X, Y\}\big\}, \quad G_{\mathrm{A2}} : V = \{Y, Z\},\ E = \big\{\{Y, Z\}\big\},$$
$$G_{\mathrm{B1}} : V = \{W, X\},\ E = \big\{\{W, X\}\big\}, \quad G_{\mathrm{B2}} : V = \{Z, W\},\ E = \big\{\{Z, W\}\big\},$$
$$G_{\mathrm{C1}} : V = \{V, W\},\ E = \big\{\{V, W\}\big\}.$$

Let $\mathrm{name}(G) \in \{\mathrm{A1}, \mathrm{A2}, \mathrm{B1}, \mathrm{B2}, \mathrm{C1}\}$ be the airline code.

Codeshare relation at level 0. Let the symmetric "codeshare list" be

$$\mathcal{X} = \big\{\ (\mathrm{A1}, \mathrm{A2}),\ (\mathrm{B1}, \mathrm{B2}),\ (\mathrm{A1}, \mathrm{B1}),\ (\mathrm{A2}, \mathrm{B2}),\ (\mathrm{B1}, \mathrm{C1})\ \big\}.$$

Define

$$c(G, H) := \begin{cases} 1, & \big(\mathrm{name}(G), \mathrm{name}(H)\big) \in \mathcal{X} \text{ or its swap}, \\ 0, & \text{otherwise}, \end{cases} \qquad (G, H) \in R_{\mathrm{cs}} \iff c(G, H) = 1.$$

Thus $R_{\mathrm{cs}} \subseteq \{G_{\mathrm{A1}}, G_{\mathrm{A2}}, G_{\mathrm{B1}}, G_{\mathrm{B2}}, G_{\mathrm{C1}}\}^2$ is a well-defined symmetric binary relation.

Depth 1 (metagraphs of airlines: alliances). Form three metagraphs whose vertices are the airline graphs, with meta-edges labeled by R_{cs}:

$$\begin{aligned} M_A = (V_A, E_A, s, t, \lambda), \quad & V_A = \{G_{\mathrm{A1}}, G_{\mathrm{A2}}\}, \\ & E_A = \{e_A\},\ s(e_A) = G_{\mathrm{A1}},\ t(e_A) = G_{\mathrm{A2}},\ \lambda(e_A) = R_{\mathrm{cs}}; \\ M_B = (V_B, E_B, s, t, \lambda), \quad & V_B = \{G_{\mathrm{B1}}, G_{\mathrm{B2}}\}, \\ & E_B = \{e_B\},\ s(e_B) = G_{\mathrm{B1}},\ t(e_B) = G_{\mathrm{B2}},\ \lambda(e_B) = R_{\mathrm{cs}}; \\ M_\Gamma = (V_\Gamma, E_\Gamma, s, t, \lambda), \quad & V_\Gamma = \{G_{\mathrm{C1}}\},\ E_\Gamma = \emptyset. \end{aligned}$$

Incidence check at depth 1. We have

$$c(G_{\mathrm{A1}}, G_{\mathrm{A2}}) = 1, \quad c(G_{\mathrm{B1}}, G_{\mathrm{B2}}) = 1,$$

hence $\big(s(e), t(e)\big) \in R_{\mathrm{cs}}$ for $e \in \{e_A, e_B\}$, as required.

Depth 2 (metagraph of alliances: consortia). Lift R_{cs} to alliances by

$$(M, N) \in R_{\mathrm{cs}}^{\uparrow} \iff \exists\, x \in V(M),\ y \in V(N) :\ (x, y) \in R_{\mathrm{cs}}.$$

Define the "consortia" metagraph

$$\mathbf{M}^{(2)} = (\mathbf{V}^{(2)}, \mathbf{E}^{(2)}, s^{(2)}, t^{(2)}, \lambda^{(2)}), \quad \mathbf{V}^{(2)} = \{M_A, M_B, M_\Gamma\},$$

with edges

$$\mathbf{E}^{(2)} = \{E_{A\to B},\ E_{B\to\Gamma}\}, \quad \lambda^{(2)}(\cdot) = R_{\mathrm{cs}}^{\uparrow},$$

and

$$s^{(2)}(E_{A\to B}) = M_A,\ t^{(2)}(E_{A\to B}) = M_B, \quad s^{(2)}(E_{B\to\Gamma}) = M_B,\ t^{(2)}(E_{B\to\Gamma}) = M_\Gamma.$$

Incidence check at depth 2. Witness for $E_{A\to B}$: take $x = G_{\mathrm{A1}} \in V(M_A)$, $y = G_{\mathrm{B1}} \in V(M_B)$, then $c(x, y) = 1 \Rightarrow (x, y) \in R_{\mathrm{cs}}$, hence $(M_A, M_B) \in R_{\mathrm{cs}}^{\uparrow}$. Witness for $E_{B\to\Gamma}$: $x = G_{\mathrm{B1}} \in V(M_B)$, $y = G_{\mathrm{C1}} \in V(M_\Gamma)$, so $c(x, y) = 1$ and $(M_B, M_\Gamma) \in R_{\mathrm{cs}}^{\uparrow}$. By design there is *no* edge $M_A \to M_\Gamma$ since $c(G_{\mathrm{A1}}, G_{\mathrm{C1}}) = c(G_{\mathrm{A2}}, G_{\mathrm{C1}}) = 0$.

Depth 3 (metagraph of consortia: clusters). Form two "cluster" vertices whose elements are consortia (i.e., vertices of $\mathbf{M}^{(2)}$):

$$\mathcal{C}_{\text{North}} := \{M_A, M_B\}, \qquad \mathcal{C}_{\text{South}} := \{M_\Gamma\}.$$

Lift once more:

$$(\mathcal{C}, \mathcal{D}) \in R_{\text{cs}}^{\uparrow\uparrow} \iff \exists\, M \in \mathcal{C},\ N \in \mathcal{D} :\ (M, N) \in R_{\text{cs}}^{\uparrow}.$$

Define the depth-3 metagraph

$$\mathbf{M}^{(3)} = (\mathbf{V}^{(3)}, \mathbf{E}^{(3)}, s^{(3)}, t^{(3)}, \lambda^{(3)}), \quad \mathbf{V}^{(3)} = \{\mathcal{C}_{\text{North}}, \mathcal{C}_{\text{South}}\},$$

with a single edge $E_{\text{North}\to\text{South}}$ labeled $\lambda^{(3)} = R_{\text{cs}}^{\uparrow\uparrow}$ and

$$s^{(3)}(E_{\text{North}\to\text{South}}) = \mathcal{C}_{\text{North}}, \qquad t^{(3)}(E_{\text{North}\to\text{South}}) = \mathcal{C}_{\text{South}}.$$

Incidence check at depth 3 (explicit witnesses). Choose $M = M_B \in \mathcal{C}_{\text{North}}$ and $N = M_\Gamma \in \mathcal{C}_{\text{South}}$. From the depth-2 verification, $(M_B, M_\Gamma) \in R_{\text{cs}}^{\uparrow}$ via the pair $(G_{\text{B1}}, G_{\text{C1}})$ with $c = 1$. Therefore $(\mathcal{C}_{\text{North}}, \mathcal{C}_{\text{South}}) \in R_{\text{cs}}^{\uparrow\uparrow}$, and the incidence constraint for $E_{\text{North}\to\text{South}}$ holds.

In summary, we have explicitly constructed an Iterated MetaGraph of depth 3: airline route graphs (level 0) $\to$ alliances (depth 1) $\to$ consortia (depth 2) $\to$ clusters (depth 3), with all edges justified by concrete codeshare witnesses at each lift.

Theorem 2.8 (Iterated MetaGraphs generalize MetaGraphs)**.** *Every metagraph M over $(\mathfrak{G}, \mathcal{R})$ is (canonically) isomorphic to an induced sub-metagraph of some depth-1 iterated metagraph over $\big(\mathfrak{G}^{(1)}, \mathcal{R}^{(1)}\big)$. In particular, the depth-0 case is exactly Definition of metagraph.*

Proof. Let $M = (V, E, s, t, \lambda)$ be a metagraph over $(\mathfrak{G}, \mathcal{R})$. Define a vertex map

$$\Phi_V : V \longrightarrow \mathfrak{G}^{(1)}, \qquad \Phi_V(X) := \mathsf{U}(X).$$

For each $e \in E$ with $\lambda(e) = R \in \mathcal{R}$ and $(s(e), t(e)) \in R$, create a depth-1 edge

$$\Phi_E(e) :\ \mathsf{U}\big(s(e)\big) \xrightarrow{R^{\uparrow}} \mathsf{U}\big(t(e)\big).$$

This is well-defined because

$$\big(\mathsf{U}(s(e)),\, \mathsf{U}(t(e))\big) \in R^{\uparrow} \iff \exists x \in \{s(e)\},\ y \in \{t(e)\} :\ (x, y) \in R \iff \big(s(e), t(e)\big) \in R,$$

using the definition of U and of $R^{\uparrow}$. Let $M^{(1)}$ be the depth-1 metagraph with vertex set $\Phi_V(V)$ and edge set $\Phi_E(E)$. Then (Φ_V, Φ_E) is a label-preserving metagraph isomorphism from M onto the induced sub-metagraph of $M^{(1)}$ on $\Phi_V(V)$, since

$$\lambda(\,e\,) = R \quad \Longrightarrow \quad \lambda^{(1)}\big(\Phi_E(e)\big) = R^{\uparrow},$$

and incidence is preserved by the equivalence above. Hence any (depth-0) metagraph embeds canonically into a depth-1 iterated metagraph. The assertion that depth 0 recovers the Definition is immediate from Definitions 2.3–2.4. □

2.2 MetaHyperGraph(HyperGraph of HyperGraph)

A MetaHyperGraph is a hypergraph whose vertices are themselves hypergraphs, with hyperedges encoding relations among these component hypergraphs.

Notation 2.9. *For a set X, write $\mathcal{P}_{\text{fin}}(X)$ for the family of all finite (possibly empty) subsets of X and $\mathcal{P}^*_{\text{fin}}(X) := \mathcal{P}_{\text{fin}}(X) \setminus \{\emptyset\}$.*

Definition 2.10 (Directed hypergraph)**.** A *directed hypergraph* is a tuple $H = (V, E, T, Hd)$ with V a vertex set, E an edge set, and $T, Hd : E \to \mathcal{P}_{\text{fin}}(V)$ the tail/head maps such that $T(e) \cup Hd(e) \neq \emptyset$ for all $e \in E$.

Definition 2.11 (MetaHyperGraph over $(\mathfrak{U}, \mathcal{R})$)**.** Let $\mathfrak{U}$ be a nonempty universe of objects and let

$$\mathcal{R} \subseteq \mathcal{P}\Big(\mathcal{P}_{\text{fin}}(\mathfrak{U}) \times \mathcal{P}_{\text{fin}}(\mathfrak{U})\Big)$$

be a nonempty family of *admissible set–relations*. A *MetaHyperGraph* over $(\mathfrak{U}, \mathcal{R})$ is a labelled directed hypergraph

$$\mathsf{M} = (V, E, T, Hd, \lambda)$$

with $V \subseteq \mathfrak{U}$, $T, Hd : E \to \mathcal{P}_{\text{fin}}(V)$, $\lambda : E \to \mathcal{R}$, satisfying the incidence constraint

$$\forall e \in E : \quad \big(T(e),\ Hd(e)\big) \in \lambda(e).$$

Vertices in V are *meta-vertices*. If $\mathfrak{U}$ is the class of finite hypergraphs, we say that M is a *HyperGraph of HyperGraphs*.

Remark 2.12. Undirected MetaHyperGraphs are obtained by requiring each $R \in \mathcal{R}$ to be symmetric and identifying (T, Hd) with (Hd, T). Ordinary directed metagraphs (graphs of graphs) arise when every hyperedge is a pair of singletons; see Theorem 2.14.

Example 2.13 (Real-life MetaHyperGraph: hospital departments sharing patients)**.** Let each vertex be a finite (undirected) hypergraph whose vertices are *patient IDs* and whose hyperedges are *procedure sessions*. Consider three departmental hypergraphs:

$$\begin{aligned} H_{\text{Rad}} &: \ V = \{p_1, p_2, p_3\}, \quad E = \{\{p_1, p_2\},\ \{p_2, p_3\}\}, \\ H_{\text{Card}} &: \ V = \{p_2, p_4\}, \quad E = \{\{p_2\},\ \{p_2, p_4\}\}, \\ H_{\text{Onc}} &: \ V = \{p_1, p_2, p_5\}, \quad E = \{\{p_1, p_2\},\ \{p_2, p_5\}\}. \end{aligned}$$

Define the admissible set–relation R_{share} on finite families of departmental hypergraphs by

$$(S, T) \in R_{\text{share}} \iff \exists\, x \text{ (patient ID) such that } \forall\, H \in S \cup T,\ \exists\, e \in E(H) :\ x \in e.$$

Form a MetaHyperGraph $\mathsf{M} = (V, E, T, Hd, \lambda)$ over $(\{H_{\text{Rad}}, H_{\text{Card}}, H_{\text{Onc}}\}, \{R_{\text{share}}\})$ with

$$V = \{H_{\text{Rad}}, H_{\text{Card}}, H_{\text{Onc}}\}, \qquad E = \{e_1\}, \qquad T(e_1) = \{H_{\text{Rad}}, H_{\text{Card}}\}, \quad Hd(e_1) = \{H_{\text{Onc}}\},$$

and $\lambda(e_1) = R_{\text{share}}$. Verification of the incidence constraint:

$$(S, T) = \big(T(e_1), Hd(e_1)\big) = \big(\{H_{\text{Rad}}, H_{\text{Card}}\}, \{H_{\text{Onc}}\}\big).$$

Take $x = p_2$. Then

$$p_2 \in \{p_1, p_2\} \in E(H_{\text{Rad}}), \quad p_2 \in \{p_2\} \in E(H_{\text{Card}}), \quad p_2 \in \{p_1, p_2\} \in E(H_{\text{Onc}}),$$

so $(S, T) \in R_{\text{share}}$, hence $(T(e_1), Hd(e_1)) \in \lambda(e_1)$ as required.

Theorem 2.14 (MetaHyperGraphs generalize MetaGraphs)**.** *Fix a universe* $\mathfrak{U}$ *and a family of binary relations* $\mathcal{R}_2 \subseteq \mathcal{P}(\mathfrak{U} \times \mathfrak{U})$*. Embed* $\mathcal{R}_2$ *into set–relations by*

$$\iota :\ \mathcal{R}_2 \longrightarrow \mathcal{P}\Big(\mathcal{P}_{\text{fin}}(\mathfrak{U}) \times \mathcal{P}_{\text{fin}}(\mathfrak{U})\Big), \qquad \iota(R) \ := \ \Big\{ (\{x\}, \{y\}) \mid (x, y) \in R \Big\}.$$

Then the full subcategory of MetaHyperGraphs over $(\mathfrak{U}, \iota(\mathcal{R}_2))$ *with edges satisfying* $|T(e)| = |Hd(e)| = 1$ *is (canonically) equivalent to the category of labelled directed* metagraphs *over* $(\mathfrak{U}, \mathcal{R}_2)$.

Proof. Given a MetaHyperGraph $\mathsf{M} = (V, E, T, Hd, \lambda)$ with $|T(e)| = |Hd(e)| = 1$, define a metagraph

$$M = (V, E, s, t, \lambda_2)$$

by $s(e)$ the unique element of $T(e)$, $t(e)$ the unique element of $Hd(e)$, and $\lambda_2(e)$ the unique $R \in \mathcal{R}_2$ with $\lambda(e) = \iota(R)$. The incidence constraint gives

$$\big(T(e), Hd(e)\big) \in \lambda(e) = \iota(R) \iff \big(\{s(e)\}, \{t(e)\}\big) \in \iota(R) \iff \big(s(e), t(e)\big) \in R,$$

hence M is a metagraph over $(\mathfrak{U}, \mathcal{R}_2)$.

Conversely, given a metagraph $M = (V, E, s, t, \lambda_2)$ over $(\mathfrak{U}, \mathcal{R}_2)$, set

$$\mathsf{M} = (V, E, T, Hd, \lambda), \qquad T(e) := \{s(e)\},\ Hd(e) := \{t(e)\},\ \lambda(e) := \iota\big(\lambda_2(e)\big).$$

Then $(T(e), Hd(e)) \in \iota(\lambda_2(e))$ holds iff $(s(e), t(e)) \in \lambda_2(e)$, so M is a MetaHyperGraph. These two constructions are mutually inverse on objects and morphisms, giving the claimed equivalence. □

Theorem 2.15 (MetaHyperGraphs generalize HyperGraphs)**.** *Let X be a nonempty set and define the trivial admissible relation*

$$\mathbf{U} := \mathcal{P}^*_{\text{fin}}(X) \times \{\emptyset\} \subseteq \mathcal{P}_{\text{fin}}(X) \times \mathcal{P}_{\text{fin}}(X),$$

and $\mathcal{R} := \{\mathbf{U}\}$. The assignment

$$\Phi : \textit{MetaHyperGraphs over } (X, \mathcal{R}) \textit{ with } Hd(e) = \emptyset \longleftrightarrow \textit{(undirected) hypergraphs on } X$$

given by

$$\Phi\colon (V{=}X, E, T, Hd \equiv \emptyset, \lambda \equiv \mathbf{U}) \longmapsto H = (X, E_H), \ E_H := \{T(e) : e \in E\},$$

is a bijection on isomorphism classes. In particular, every undirected hypergraph arises as a specialization of a MetaHyperGraph.

Proof. For any specialized MetaHyperGraph as stated, the incidence constraint reads $(T(e), \emptyset) \in \mathbf{U}$, which is tautologically true for all nonempty $T(e) \in \mathcal{P}^*_{\text{fin}}(X)$. Thus $E_H := \{T(e) : e \in E\} \subseteq \mathcal{P}^*_{\text{fin}}(X)$ defines a hypergraph $H = (X, E_H)$.

Conversely, given any hypergraph $H = (X, E_H)$, define $E := E_H$, $T(e) := e$, $Hd(e) := \emptyset$ and $\lambda(e) := \mathbf{U}$; then $(T(e), \emptyset) \in \mathbf{U}$ holds by definition, so $\mathsf{M} = (X, E, T, Hd, \lambda)$ is a MetaHyperGraph over $(X, \mathcal{R})$. Isomorphisms clearly match under these maps, yielding a bijection on isomorphism classes. □

2.3 Iterated MetaHyperGraph(HyperGraph of HyperGraph of ... of HyperGraph)

An Iterated MetaHyperGraph is a hypergraph whose vertices are meta-hypergraphs, recursively building hypergraph-of-hypergraph structures over multiple hierarchical depths.

Definition 2.16 (Lift of set–relations for iteration)**.** Let $\mathfrak{U}$, $\mathcal{R}$ be as above. For $R \in \mathcal{R}$ and MetaHyperGraphs $\mathsf{M}_1 = (V_1, \ldots)$, $\mathsf{M}_2 = (V_2, \ldots)$ over $(\mathfrak{U}, \mathcal{R})$, define the *lift* $R^{\Uparrow}$ on $\mathcal{P}_{\text{fin}}(\{\mathsf{M}\}) \times \mathcal{P}_{\text{fin}}(\{\mathsf{M}\})$ by

$$(S, T) \in R^{\Uparrow} \iff \exists M \in S, \ N \in T, \ \exists A \in \mathcal{P}_{\text{fin}}(V(M)), \ B \in \mathcal{P}_{\text{fin}}(V(N)) : \ (A, B) \in R.$$

Set $\mathcal{R}^{\Uparrow} := \{R^{\Uparrow} : R \in \mathcal{R}\}$.

Definition 2.17 (Iterated universes)**.** Define recursively for $t \in \mathbb{N}_0$:

$$\mathfrak{U}^{(0)} := \mathfrak{U}, \quad \mathcal{R}^{(0)} := \mathcal{R},$$
$$\mathfrak{U}^{(t+1)} := \{\text{finite MetaHyperGraphs over } (\mathfrak{U}^{(t)}, \mathcal{R}^{(t)})\},$$
$$\mathcal{R}^{(t+1)} := (\mathcal{R}^{(t)})^{\Uparrow}.$$

Definition 2.18 (Iterated MetaHyperGraph of depth t)**.** For $t \in \mathbb{N}_0$, an *Iterated MetaHyperGraph (IMHG) of depth t* is a MetaHyperGraph

$$\mathsf{M}^{(t)} = (V^{(t)}, E^{(t)}, T^{(t)}, Hd^{(t)}, \lambda^{(t)})$$

over $(\mathfrak{U}^{(t)}, \mathcal{R}^{(t)})$, i.e., $V^{(t)} \subseteq \mathfrak{U}^{(t)}$, $\lambda^{(t)} : E^{(t)} \to \mathcal{R}^{(t)}$, and

$$\forall e \in E^{(t)} : \ (T^{(t)}(e), \ Hd^{(t)}(e)) \in \lambda^{(t)}(e).$$

Remark 2.19. Depth 0 vertices are base objects; depth 1 vertices are MetaHyperGraphs of base objects; depth 2 vertices are MetaHyperGraphs whose vertices are MetaHyperGraphs, etc. Edge labels are always drawn from the lifted family at the corresponding depth.

Example 2.20 (Real-life Iterated MetaHyperGraph: hospitals linked by transfers)**.** Within Hospital A, define departmental hypergraphs (patients $\{p_1, p_2, p_4, p_5\}$):

$$H^{\mathsf{A}}_{\text{Rad}} : \ V = \{p_1, p_2\}, \ E = \{\{p_1, p_2\}\},$$
$$H^{\mathsf{A}}_{\text{Card}} : \ V = \{p_2, p_4\}, \ E = \{\{p_2\}\},$$
$$H^{\mathsf{A}}_{\text{Onc}} : \ V = \{p_2, p_5\}, \ E = \{\{p_2, p_5\}\}.$$

As in the first example, with R_{share}, build the (level-0) MetaHyperGraph

$$\mathsf{M}_{\mathsf{A}} = (V_{\mathsf{A}}, E_{\mathsf{A}}, T_{\mathsf{A}}, Hd_{\mathsf{A}}, \lambda_{\mathsf{A}}), \qquad V_{\mathsf{A}} = \{H^{\mathsf{A}}_{\text{Rad}}, H^{\mathsf{A}}_{\text{Card}}, H^{\mathsf{A}}_{\text{Onc}}\},$$

with a meta-hyperedge e_{A}:

$$T_{\mathsf{A}}(e_{\mathsf{A}}) = \{H^{\mathsf{A}}_{\text{Rad}}, H^{\mathsf{A}}_{\text{Card}}\}, \quad Hd_{\mathsf{A}}(e_{\mathsf{A}}) = \{H^{\mathsf{A}}_{\text{Onc}}\},$$

$$\lambda_{\mathsf{A}}(e_{\mathsf{A}}) = R_{\text{share}},$$

witnessed by patient p_2.

Within Hospital B (patients $\{p_2, p_3\}$), define

$$H^{\mathsf{B}}_{\text{Rad}} : \ V = \{p_2, p_3\}, \ E = \{\{p_2\}\}, \qquad H^{\mathsf{B}}_{\text{Card}} : \ V = \{p_3\}, \ E = \{\{p_3\}\}.$$

Its MetaHyperGraph M_{B} has a meta-hyperedge e_{B} with

$$T_{\mathsf{B}}(e_{\mathsf{B}}) = \{H^{\mathsf{B}}_{\text{Card}}\}, \quad Hd_{\mathsf{B}}(e_{\mathsf{B}}) = \{H^{\mathsf{B}}_{\text{Rad}}\},$$

$$\lambda_{\mathsf{B}}(e_{\mathsf{B}}) = R_{\text{share}},$$

witnessed by patient p_3.

Now form the *Iterated MetaHyperGraph* at depth 1 whose vertices are the level-0 MetaHyperGraphs

$$\mathcal{V}^{(1)} = \{\mathsf{M}_{\mathsf{A}}, \ \mathsf{M}_{\mathsf{B}}\}.$$

Use the lifted relation $R^{\Uparrow}_{\text{share}}$:

$$(S, T) \in R^{\Uparrow}_{\text{share}} \iff \exists M \in S, \ N \in T, \ \exists A \subseteq V(M), \ B \subseteq V(N) : \ (A, B) \in R_{\text{share}}.$$

Define one meta-hyperedge $E_{\mathsf{A}\to\mathsf{B}}$ with

$$T^{(1)}(E_{\mathsf{A}\to\mathsf{B}}) = \{\mathsf{M}_{\mathsf{A}}\}, \qquad Hd^{(1)}(E_{\mathsf{A}\to\mathsf{B}}) = \{\mathsf{M}_{\mathsf{B}}\},$$

$$\lambda^{(1)}(E_{\mathsf{A}\to\mathsf{B}}) = R^{\Uparrow}_{\text{share}}.$$

Incidence check (explicit witnesses): choose

$$M = \mathsf{M}_{\mathsf{A}}, \ N = \mathsf{M}_{\mathsf{B}},$$

$$A = \{H^{\mathsf{A}}_{\text{Onc}}\} \subseteq V(\mathsf{M}_{\mathsf{A}}), \ B = \{H^{\mathsf{B}}_{\text{Rad}}\} \subseteq V(\mathsf{M}_{\mathsf{B}}).$$

Patient p_2 satisfies

$$p_2 \in \{p_2, p_5\} \in E\big(H^{\mathsf{A}}_{\text{Onc}}\big), \qquad p_2 \in \{p_2\} \in E\big(H^{\mathsf{B}}_{\text{Rad}}\big),$$

hence $(A, B) \in R_{\text{share}}$, so $\big(T^{(1)}(E_{\mathsf{A}\to\mathsf{B}}), \ Hd^{(1)}(E_{\mathsf{A}\to\mathsf{B}})\big) \in R^{\Uparrow}_{\text{share}}$. Thus the depth-1 Iterated MetaHyperGraph correctly captures an inter-hospital linkage induced by shared patient p_2.

Theorem 2.21 (IMHGs generalize MetaHyperGraphs)**.** *Depth* 0 *IMHGs are exactly MetaHyperGraphs over* $(\mathfrak{U}, \mathcal{R})$.

Proof. By Definitions 2.17–2.18, taking $t = 0$ yields $\mathfrak{U}^{(0)} = \mathfrak{U}$ and $\mathcal{R}^{(0)} = \mathcal{R}$, hence IMHGs of depth 0 are precisely MetaHyperGraphs over $(\mathfrak{U}, \mathcal{R})$. □

Definition 2.22 (Metagraph and its lift)**.** Let $\mathfrak{G}$ be a universe and $\mathcal{Q} \subseteq \mathcal{P}(\mathfrak{G} \times \mathfrak{G})$ a family of binary relations. A (labelled) *metagraph* over $(\mathfrak{G}, \mathcal{Q})$ is a directed, labelled multigraph $M = (V, E, s, t, \lambda)$ with $V \subseteq \mathfrak{G}$, $s, t : E \to V$, $\lambda : E \to \mathcal{Q}$, and $(s(e), t(e)) \in \lambda(e)$ for all e. Its standard lift is: for $R \in \mathcal{Q}$ and metagraphs M_1, M_2,

$$(M_1, M_2) \in R^{\uparrow} \iff \exists x \in V(M_1), \ y \in V(M_2) : \ (x, y) \in R.$$

Define iterated metagraph universes by $\mathfrak{G}^{(0)} := \mathfrak{G}$, $\mathcal{Q}^{(0)} := \mathcal{Q}$, and

$$\mathfrak{G}^{(t+1)} := \{\text{finite metagraphs over } (\mathfrak{G}^{(t)}, \mathcal{Q}^{(t)})\},$$

$$\mathcal{Q}^{(t+1)} := \big(\mathcal{Q}^{(t)}\big)^{\uparrow}.$$

Definition 2.23 (Singleton embedding of binary relations)**.** Embed Q into set–relations by

$$\iota : \; Q \longrightarrow \mathcal{P}\Big(\mathcal{P}_{\mathrm{fin}}(\mathfrak{G}) \times \mathcal{P}_{\mathrm{fin}}(\mathfrak{G})\Big), \qquad \iota(R) := \big\{ \, (\{x\}, \{y\}) \mid (x, y) \in R \, \big\}.$$

Lemma 2.24 (Lift commutes with ι on singletons)**.** *Let $t \geq 0$. View ι levelwise, i.e. as*

$$\iota : Q^{(t)} \to \mathcal{P}(\mathcal{P}_{\mathrm{fin}}(\mathfrak{G}^{(t)}) \times \mathcal{P}_{\mathrm{fin}}(\mathfrak{G}^{(t)}))$$

. Then

$$\big(\iota(R)\big)^{\Uparrow} \;\cap\; \big\{(\{M\}, \{N\})\big\} \;=\; \iota\big(R^{\uparrow}\big)$$

for all $R \in Q^{(t)}$.

Proof. By Definition 2.16, for $(\{M\}, \{N\})$ we have

$$
\begin{aligned}
(\{M\}, \{N\}) &\in \big(\iota(R)\big)^{\Uparrow} \\
\iff \;& \exists\, A \subseteq V(M),\; B \subseteq V(N) : \; (A, B) \in \iota(R) \\
\iff \;& \exists\, x \in V(M),\; y \in V(N) : \; (\{x\}, \{y\}) \in \iota(R) \\
\iff \;& \exists\, x \in V(M),\; y \in V(N) : \; (x, y) \in R \\
\iff \;& (M, N) \in R^{\uparrow}.
\end{aligned}
$$

This is exactly $(\{M\}, \{N\}) \in \iota(R^{\uparrow})$. □

Theorem 2.25 (IMHGs generalize Iterated MetaGraphs)**.** *Fix $t \geq 0$. Consider IMHGs over $\big(\mathfrak{G}^{(t)},\, \iota(Q^{(t)})\big)$ and restrict to the full subcategory $\mathbf{IMHG}_1^{(t)}$ in which every hyperedge e satisfies $|T^{(t)}(e)| = |Hd^{(t)}(e)| = 1$. Then $\mathbf{IMHG}_1^{(t)}$ is canonically equivalent to the category of iterated metagraphs of depth t over $\big(\mathfrak{G}^{(t)}, Q^{(t)}\big)$.*

Proof. Define functors F_t and G_t (mutually inverse on objects and morphisms).

(F_t : metagraph → IMHG) Given a depth-t metagraph

$$M^{(t)} = (V, E, s, t, \lambda) \quad \text{over } (\mathfrak{G}^{(t)}, Q^{(t)}),$$

set $\mathsf{M}^{(t)} := F_t(M^{(t)})$ with the same vertex set V, edge set E, and for each $e \in E$ put

$$T^{(t)}(e) := \{s(e)\}, \qquad Hd^{(t)}(e) := \{t(e)\}, \qquad \lambda^{(t)}(e) := \iota\big(\lambda(e)\big).$$

Incidence holds because $(s(e), t(e)) \in \lambda(e)$ iff $(\{s(e)\}, \{t(e)\}) \in \iota(\lambda(e))$.

(G_t : IMHG → metagraph) Given a depth-t IMHG in $\mathbf{IMHG}_1^{(t)}$,

$$\mathsf{M}^{(t)} = (V, E, T, Hd, \lambda) \quad \text{over } (\mathfrak{G}^{(t)}, \iota(Q^{(t)})),$$

define $M^{(t)} := G_t(\mathsf{M}^{(t)})$ with the same vertices V, edges E, and for each e take $s(e)$, $t(e)$ to be the unique elements of $T(e)$, $Hd(e)$, and set $\lambda(e) \in Q^{(t)}$ uniquely so that $\lambda^{(t)}(e) = \iota\big(\lambda(e)\big)$. Incidence is equivalent by the definition of ι.

Compatibility with depth-t labels (which are lifts of depth-$(t-1)$ labels) follows from Lemma 2.24: the lifted label families correspond under ι when restricting to singleton tails/heads. Functoriality on morphisms (label-preserving incidence-commuting maps) is inherited verbatim. Clearly $G_t \circ F_t = \mathrm{id}$ and $F_t \circ G_t = \mathrm{id}$ on the stated subcategory, yielding the claimed equivalence. □

2.4 MetaSuperHyperGraph(SuperHyperGraph of SuperHyperGraph)

A MetaSuperHyperGraph is a superhypergraph whose vertices are themselves superhypergraphs, with superhyperedges describing relations between these higher-order structures.

Notation 2.26. *For a set X, write $\mathcal{P}_{\mathrm{fin}}(X)$ for the family of all finite subsets of X and $\mathcal{P}^*_{\mathrm{fin}}(X) := \mathcal{P}_{\mathrm{fin}}(X) \setminus \{\emptyset\}$. For a function f and a set A, put $f[A] := \{f(a) : a \in A\}$.*

Definition 2.27 (Iterated singleton embedding)**.** Let U be any set. Define $j_0 : U \to U$ by $j_0(x) := x$ and inductively $j_{k+1} : U \to \mathcal{P}(\mathcal{P}^k(U))$ by $j_{k+1}(x) := \{ j_k(x) \}$. Thus $j_n : U \hookrightarrow \mathcal{P}^n(U)$ is the canonical depth-n embedding.

Definition 2.28 (Directed n-SuperHyperGraph)**.** Fix a base set Ω and $n \in \mathbb{N}_0$. A *directed n-SuperHyperGraph* is a tuple

$$\mathcal{S} = (V, E, T, Hd) \quad \text{with} \quad V \subseteq \mathcal{P}^n(\Omega), \;\; T, Hd : E \to \mathcal{P}_{\mathrm{fin}}(V),$$

such that $T(e) \cup Hd(e) \neq \emptyset$ for all $e \in E$. (Undirected variants arise by requiring $T(e) = Hd(e)$ for all e.)

Definition 2.29 (Universe of n-SuperHyperGraphs)**.** Let $\mathfrak{S}_n$
denote the class of all finite directed n-SuperHyperGraphs over arbitrary base sets. For

$$x \in \mathcal{P}^n(\Omega)$$

, the *unit n-SuperHyperGraph* at x is

$$\mathsf{U}_n(x) \;:=\; \big(\{x\}, \emptyset, \_, \_\big) \;\in\; \mathfrak{S}_n.$$

Definition 2.30 (MetaSuperHyperGraph over $(\mathfrak{S}_n, \mathcal{R})$)**.** Let $\mathfrak{S}_n$ be as above and let

$$\mathcal{R} \;\subseteq\; \mathcal{P}\Big(\mathcal{P}_{\mathrm{fin}}(\mathfrak{S}_n) \times \mathcal{P}_{\mathrm{fin}}(\mathfrak{S}_n)\Big)$$

be any nonempty family of admissible *set–relations on n-SuperHyperGraphs*. A *MetaSuperHyperGraph (MSHG)* over $(\mathfrak{S}_n, \mathcal{R})$ is a labelled directed hypergraph

$$\mathsf{M} = (V, E, T, Hd, \lambda)$$

with $V \subseteq \mathfrak{S}_n$, $T, Hd : E \to \mathcal{P}_{\mathrm{fin}}(V)$, and $\lambda : E \to \mathcal{R}$, satisfying the incidence constraint

$$\forall e \in E : \quad \big(T(e), Hd(e)\big) \;\in\; \lambda(e).$$

Vertices of M are (finite) n-SuperHyperGraphs; hyperedges relate finite families of such vertices.

Example 2.31 (Real-life MetaSuperHyperGraph: multi-department clinical cohorts)**.** Let the national patient-ID universe be $\Omega = \{u_1, u_2, u_3, u_4, u_5\}$ and consider depth-1 SuperHyperGraphs (vertices are subsets of Ω).

Department A (oncology) as a 1-SuperHyperGraph:

$$H_A = (V_A, E_A, T_A, Hd_A), \quad V_A = \{\{u_1, u_2\}, \{u_2, u_3\}\}, \quad E_A = \{e_A\},$$
$$T_A(e_A) = \big\{\{u_1, u_2\}\big\}, \qquad Hd_A(e_A) = \big\{\{u_2, u_3\}\big\}.$$

Department B (radiology):

$$H_B = (V_B, E_B, T_B, Hd_B), \quad V_B = \{\{u_2, u_4\}\}, \quad E_B = \emptyset.$$

Department C (cardiology):

$$H_C = (V_C, E_C, T_C, Hd_C), \quad V_C = \{\{u_1, u_2, u_5\}\}, \quad E_C = \emptyset.$$

Define the admissible set–relation on 1-SuperHyperGraphs

$$R_{\mathrm{shareV}} : \; (S, T) \in R_{\mathrm{shareV}} \iff \exists x \in \Omega \text{ s.t. } \forall H \in S \cup T \; \exists A_H \in V(H) \text{ with } x \in A_H.$$

Form the MetaSuperHyperGraph $\mathsf{M} = (V, E, T, Hd, \lambda)$ over the universe $V = \{H_A, H_B, H_C\}$ with one meta-hyperedge $e^\star$:

$$T(e^\star) = \{H_A, H_B\}, \qquad Hd(e^\star) = \{H_C\}, \qquad \lambda(e^\star) = R_{\mathrm{shareV}}.$$

Incidence check (explicit witness): pick $x = u_2$. Then $u_2 \in \{u_1, u_2\} \in V_A$, $u_2 \in \{u_2, u_4\} \in V_B$, and $u_2 \in \{u_1, u_2, u_5\} \in V_C$, so $(T(e^\star), Hd(e^\star)) \in R_{\mathrm{shareV}}$ as required.

Definition 2.32 (Singleton wrapping of relations)**.** For $n \geq 0$ define

$$\iota_n : Q \longrightarrow \mathcal{P}\Big(\mathcal{P}_{\text{fin}}(\mathfrak{S}_n) \times \mathcal{P}_{\text{fin}}(\mathfrak{S}_n)\Big), \quad \iota_n(R) := \Big\{ \big(\mathsf{U}_n[S], \mathsf{U}_n[T] \big) : (S,T) \in R \Big\}.$$

Theorem 2.33 (MSHG ⊃ MetaHyperGraph)**.** *Fix $n \geq 0$. Consider the full subcategory $\mathbf{MSHG}_n^{\text{unit}}$ of MSHGs over $(\mathfrak{S}_n, \iota_n(Q))$ whose vertex sets are contained in $\mathsf{U}_n[\mathfrak{U}]$. Then $\mathbf{MSHG}_n^{\text{unit}}$ is canonically equivalent to the category of MetaHyperGraphs over $(\mathfrak{U}, Q)$.*

Proof. Define mutually inverse functors on objects and morphisms.

(F : MHG → MSHG) Given $\mathsf{H} = (W, F, S, H, \mu)$, put

$$V := \mathsf{U}_n[W], \quad E := F, \quad T(e) := \mathsf{U}_n[S(e)], \quad Hd(e) := \mathsf{U}_n[H(e)], \quad \lambda(e) := \iota_n\big(\mu(e)\big).$$

Then $(T(e), Hd(e)) \in \iota_n(\mu(e))$ iff $(S(e), H(e)) \in \mu(e)$ by Definition 2.32, so $\mathsf{M} := F(\mathsf{H})$ is an MSHG in $\mathbf{MSHG}_n^{\text{unit}}$.

(G : MSHG → MHG) Conversely, let $\mathsf{M} = (V, E, T, Hd, \lambda)$ lie in $\mathbf{MSHG}_n^{\text{unit}}$. Write $W := \{ x \in \mathfrak{U} \mid \mathsf{U}_n(x) \in V \}$ and define

$$S(e) := \{ x \in W \mid \mathsf{U}_n(x) \in T(e) \}, \qquad H(e) := \{ y \in W \mid \mathsf{U}_n(y) \in Hd(e) \}.$$

For each e choose $\mu(e) \in Q$ with $\lambda(e) = \iota_n(\mu(e))$ (uniquely determined by Definition 2.32). Then $(S(e), H(e)) \in \mu(e)$ iff $(T(e), Hd(e)) \in \lambda(e)$, so $\mathsf{H} := G(\mathsf{M}) = (W, E, S, H, \mu)$ is an MHG. It is immediate that $G \circ F = \text{id}$ and $F \circ G = \text{id}$. □

Theorem 2.34 (MSHG ⊃ SuperHyperGraph)**.** *Fix a base set Ω and $n \geq 0$. Let $\mathbf{U}$ denote the* universal *set–relation on $\mathfrak{S}_n$:*

$$\mathbf{U} := \mathcal{P}_{\text{fin}}(\mathfrak{S}_n) \times \mathcal{P}_{\text{fin}}(\mathfrak{S}_n), \qquad \mathcal{R} := \{\mathbf{U}\}.$$

The assignment

$$\Phi : \{\textit{directed n-SuperHyperGraphs on } \Omega\} \longrightarrow \{\textit{MSHGs over } (\mathfrak{S}_n, \mathcal{R})\}$$

given by

$$\Phi(V, E, T, Hd) := \big(\mathsf{U}_n[V], E, \mathsf{U}_n[T(\cdot)], \mathsf{U}_n[Hd(\cdot)], \lambda \equiv \mathbf{U} \big)$$

is a bijection on isomorphism classes. In particular, every n-SuperHyperGraph is a specialization of an MSHG.

Proof. For any n-SuperHyperGraph (V, E, T, Hd), the image has vertex set $\mathsf{U}_n[V]$ and for each $e \in E$,

$$\big(T'(e), Hd'(e)\big) = \big(\mathsf{U}_n[T(e)], \mathsf{U}_n[Hd(e)]\big) \in \mathbf{U}$$

by definition of $\mathbf{U}$, hence $\Phi(V, E, T, Hd)$ is an MSHG. Conversely, given an MSHG $(V', E', T', Hd', \lambda \equiv \mathbf{U})$ whose vertices are all of the form $\mathsf{U}_n(x)$ with $x \in \mathcal{P}^n(\Omega)$, define

$$V := \{ x : \mathsf{U}_n(x) \in V' \}, \qquad T(e) := \{ x : \mathsf{U}_n(x) \in T'(e) \}, \qquad Hd(e) := \{ x : \mathsf{U}_n(x) \in Hd'(e) \}.$$

Then (V, E', T, Hd) is a directed n-SuperHyperGraph on Ω. Isomorphisms correspond by the obvious componentwise identification, proving the bijection on isomorphism classes. □

2.5 Iterated MetaSuperHyperGraph (SuperHyperGraph of SuperHyperGraph of ... of SuperHyperGraph)

An Iterated MetaSuperHyperGraph is a superhypergraph whose vertices are meta-superhypergraphs, recursively extending superhypergraph-of-superhypergraph structure across multiple hierarchical levels.

Definition 2.35 (Iterated universes for MetaSuperHyperGraphs)**.** Fix $n \in \mathbb{N}_0$. Let $\mathfrak{S}_n$ be the class of all finite directed n-SuperHyperGraphs (as used previously) and let $\mathcal{R} \subseteq \mathcal{P}(\mathcal{P}_{\text{fin}}(\mathfrak{S}_n) \times \mathcal{P}_{\text{fin}}(\mathfrak{S}_n))$ be a nonempty family of admissible set–relations (as in the MetaSuperHyperGraph section). Define, recursively for $t \in \mathbb{N}_0$,

$$\mathfrak{S}_n^{(0)} := \mathfrak{S}_n, \qquad \mathcal{R}^{(0)} := \mathcal{R},$$

$$\mathfrak{S}_n^{(t+1)} := \Big\{ \text{finite MetaSuperHyperGraphs over } \big(\mathfrak{S}_n^{(t)}, \mathcal{R}^{(t)}\big) \Big\},$$

$$\mathcal{R}^{(t+1)} := \big(\mathcal{R}^{(t)}\big)^{\Uparrow}, \quad \text{where } (S,T) \in R^{\Uparrow} \iff$$

$$\exists\, M \in S,\ N \in T,\ \exists\, A \in \mathcal{P}_{\text{fin}}\big(V(M)\big),\ B \in \mathcal{P}_{\text{fin}}\big(V(N)\big) :\ (A,B) \in R.$$

(Here $V(M)$ denotes the vertex set of M, and $\Uparrow$ is the same lifting operator used earlier.)

Definition 2.36 (Iterated MetaSuperHyperGraph (IMSHG) of depth t)**.** For $t \in \mathbb{N}_0$, an *Iterated MetaSuperHyperGraph of depth t* is a labelled directed hypergraph

$$\mathsf{M}^{(t)} = (V^{(t)}, E^{(t)}, T^{(t)}, Hd^{(t)}, \lambda^{(t)})$$

such that

$$V^{(t)} \subseteq \mathfrak{S}_n^{(t)}, \qquad T^{(t)}, Hd^{(t)} : E^{(t)} \to \mathcal{P}_{\text{fin}}\big(V^{(t)}\big),$$

$$\lambda^{(t)} : E^{(t)} \to \mathcal{R}^{(t)},$$

and the incidence constraint holds:

$$\forall e \in E^{(t)} :\ \big(T^{(t)}(e),\ Hd^{(t)}(e)\big) \in \lambda^{(t)}(e).$$

Example 2.37 (Real-life Iterated MetaSuperHyperGraph: hospital networks via shared cohorts)**.** Extend the patient-ID universe to $\Omega' = \{u_1, u_2, u_3, u_4, u_5, u_6\}$.

Hospital A uses the three 1-SuperHyperGraphs above and forms the MetaSuperHyperGraph

$$\mathsf{M}_{\mathsf{A}} = (V_{\mathsf{A}}, E_{\mathsf{A}}, T_{\mathsf{A}}, Hd_{\mathsf{A}}, \lambda_{\mathsf{A}}),$$

$$V_{\mathsf{A}} = \{H_A, H_B, H_C\},$$

with the meta-hyperedge e_{A} already verified by $x = u_2$:

$$T_{\mathsf{A}}(e_{\mathsf{A}}) = \{H_A, H_B\}, \quad Hd_{\mathsf{A}}(e_{\mathsf{A}}) = \{H_C\},$$

$$\lambda_{\mathsf{A}}(e_{\mathsf{A}}) = R_{\text{shareV}}.$$

Hospital B builds two 1-SuperHyperGraphs using Ω':

$$H_D = (V_D, E_D, T_D, Hd_D), \quad V_D = \{\{u_2, u_6\}\},\ E_D = \emptyset;$$

$$H_E = (V_E, E_E, T_E, Hd_E), \quad V_E = \{\{u_2\}\},\ E_E = \emptyset.$$

Its MetaSuperHyperGraph is

$$\mathsf{M}_{\mathsf{B}} = (V_{\mathsf{B}}, E_{\mathsf{B}}, T_{\mathsf{B}}, Hd_{\mathsf{B}}, \lambda_{\mathsf{B}}),$$

$$V_{\mathsf{B}} = \{H_D, H_E\},$$

with one meta-hyperedge e_{B} witnessed by $x = u_2$:

$$T_{\mathsf{B}}(e_{\mathsf{B}}) = \{H_E\}, \quad Hd_{\mathsf{B}}(e_{\mathsf{B}}) = \{H_D\},$$

$$\lambda_{\mathsf{B}}(e_{\mathsf{B}}) = R_{\text{shareV}}.$$

Lift the relation to MetaSuperHyperGraphs (depth-1 lifting):

$$(S,T) \in R_{\text{shareV}}^{\Uparrow} \iff \exists\, M \in S,\ N \in T,\ \exists\, A \subseteq V(M),\ B \subseteq V(N) :\ (A,B) \in R_{\text{shareV}}.$$

Construct the depth-1 Iterated MetaSuperHyperGraph

$$\mathbf{M}^{(1)} = \big(V^{(1)}, E^{(1)}, T^{(1)}, Hd^{(1)}, \lambda^{(1)}\big),$$

$$V^{(1)} = \{\mathsf{M}_{\mathsf{A}}, \mathsf{M}_{\mathsf{B}}\},$$

with one meta-hyperedge $E_{\mathsf{A}\to\mathsf{B}}$:

$$T^{(1)}(E_{\mathsf{A}\to\mathsf{B}}) = \{\mathsf{M}_{\mathsf{A}}\},$$
$$Hd^{(1)}(E_{\mathsf{A}\to\mathsf{B}}) = \{\mathsf{M}_{\mathsf{B}}\},$$
$$\lambda^{(1)}(E_{\mathsf{A}\to\mathsf{B}}) = R^{\Uparrow}_{\text{shareV}}.$$

Incidence verification (explicit witnesses): choose $M = \mathsf{M}_{\mathsf{A}}$, $N = \mathsf{M}_{\mathsf{B}}$,

$$A = \{H_A\} \subseteq V_{\mathsf{A}}, \qquad B = \{H_D\} \subseteq V_{\mathsf{B}}.$$

Take $x = u_2$. Then $u_2 \in \{u_1, u_2\} \in V_A$ and $u_2 \in \{u_2, u_6\} \in V_D$, so $(A, B) \in R_{\text{shareV}}$, hence

$$\big(T^{(1)}(E_{\mathsf{A}\to\mathsf{B}}),\ Hd^{(1)}(E_{\mathsf{A}\to\mathsf{B}})\big) \in R^{\Uparrow}_{\text{shareV}}.$$

Therefore $\mathbf{M}^{(1)}$ is a valid depth-1 Iterated MetaSuperHyperGraph linking hospitals by shared patient cohorts.

Theorem 2.38 (IMSHG generalizes MetaSuperHyperGraph)**.** *Depth* 0 *IMSHGs are exactly MetaSuperHyperGraphs over* $(\mathfrak{S}_n, \mathcal{R})$.

Proof. By construction, $\mathfrak{S}_n^{(0)} = \mathfrak{S}_n$ and $\mathcal{R}^{(0)} = \mathcal{R}$. The definition of IMSHG with $t = 0$ is identical to the definition of a MetaSuperHyperGraph over $(\mathfrak{S}_n, \mathcal{R})$. □

Definition 2.39 (Unit 0-SuperHyperGraph and singleton embedding of relations)**.** Let $\mathsf{U}_0(x)$ denote the *unit* 0*-SuperHyperGraph* on a base object x (single vertex x, no edges). For a binary relation R on a universe $\mathfrak{U}$, define the singleton embedding

$$\iota_0(R) \;:=\; \big\{\, (\{x\}, \{y\}) \,\big|\, (x, y) \in R \,\big\},$$

viewed as a set–relation on $\mathcal{P}_{\text{fin}}(\mathfrak{U}) \times \mathcal{P}_{\text{fin}}(\mathfrak{U})$.

Lemma 2.40 (Lift commutes with singleton embedding on singletons)**.** *For any depth* $t \geq 0$*, any relation* R *on the depth-*t *universe, and any metalevel objects* M, N*,*

$$(\{M\}, \{N\}) \in \big(\iota_0(R)\big)^{\Uparrow} \quad\Longleftrightarrow\quad (M, N) \in R^{\uparrow}.$$

Proof. By the definition of $\Uparrow$ and ι_0,

$$(\{M\}, \{N\}) \in (\iota_0(R))^{\Uparrow} \iff \exists x \in V(M),\ y \in V(N):\ (\{x\}, \{y\}) \in \iota_0(R)$$
$$\iff \exists x \in V(M),\ y \in V(N):\ (x, y) \in R \iff (M, N) \in R^{\uparrow}.$$

□

Theorem 2.41 (IMSHG generalizes Iterated MetaHyperGraph)**.** *Fix any depth* $t \geq 0$*. Let an Iterated MetaHyperGraph (IMHG) of depth* t *over* $(\mathfrak{U}^{(t)}, Q^{(t)})$ *be given (as defined earlier). Consider IMSHGs of depth* t *with* $n = 0$ *over* $\big(\mathfrak{S}_0^{(t)},\ \iota_0(Q^{(t)})\big)$*, and restrict to the full subcategory* $\mathbf{IMSHG}_1^{(t)}$ *in which every hyperedge* e *satisfies* $|T^{(t)}(e)| = |Hd^{(t)}(e)| = 1$ *and every vertex is of the form* $\mathsf{U}_0(\cdot)$*. Then* $\mathbf{IMSHG_1^{(t)}}$ *is canonically equivalent to the category of depth-*t *IMHGs over* $(\mathfrak{U}^{(t)}, Q^{(t)})$*.*

Proof. Define mutually inverse functors F_t and G_t.

(F_t: IMHG $\to$ IMSHG) Let

$$\mathsf{H}^{(t)} = (V, E, S, H, \mu) \quad \text{(depth-}t\text{ IMHG over } (\mathfrak{U}^{(t)}, Q^{(t)})\text{)}.$$

Set

$$V' := \{\mathsf{U}_0(v) : v \in V\}, \quad E' := E,$$

$$T'(e) := \{\mathsf{U}_0(S(e))\}, \qquad Hd'(e) := \{\mathsf{U}_0(H(e))\}, \qquad \lambda'(e) := \iota_0\big(\mu(e)\big).$$

Incidence is preserved since

$$\big(T'(e), Hd'(e)\big) \in \iota_0(\mu(e)) \iff \big(\{S(e)\}, \{H(e)\}\big) \in \iota_0(\mu(e)) \iff (S(e), H(e)) \in \mu(e).$$

Thus $F_t(\mathsf{H}^{(t)}) = \mathsf{M}^{(t)} := (V', E', T', Hd', \lambda')$ is a depth-t IMSHG in $\mathbf{IMSHG}_1^{(t)}$.

(G_t: IMSHG $\to$ IMHG) Conversely, take

$$\mathsf{M}^{(t)} = (V', E', T', Hd', \lambda') \in \mathbf{IMSHG}_1^{(t)} \quad \text{over } (\mathfrak{S}_0^{(t)}, \iota_0(Q^{(t)})).$$

Write $V := \{v \in \mathfrak{U}^{(t)} : \mathsf{U}_0(v) \in V'\}$, $E := E'$, and for each $e \in E$ let $S(e), H(e)$ be the unique elements with

$$T'(e) = \{\mathsf{U}_0(S(e))\}, \qquad Hd'(e) = \{\mathsf{U}_0(H(e))\}.$$

Since $\lambda'(e) \in \iota_0(Q^{(t)})$, choose the unique $\mu(e) \in Q^{(t)}$ with $\lambda'(e) = \iota_0(\mu(e))$. Incidence is equivalent by the same calculation:

$$(S(e), H(e)) \in \mu(e) \iff \big(\{S(e)\}, \{H(e)\}\big) \in \iota_0(\mu(e)) \iff \big(T'(e), Hd'(e)\big) \in \lambda'(e).$$

Hence $G_t(\mathsf{M}^{(t)}) = \mathsf{H}^{(t)} := (V, E, S, H, \mu)$ is a depth-t IMHG.

Finally, $G_t \circ F_t = \mathrm{id}$ and $F_t \circ G_t = \mathrm{id}$ on objects and morphisms by construction. Label compatibility across depths follows from the lemma: for any $R \in Q^{(k)}$ and metalevel objects M, N at depth k,

$$(\{M\}, \{N\}) \in \big(\iota_0(R)\big)^{\Uparrow} \overset{\text{Lemma}}{\iff} (M, N) \in R^{\uparrow},$$

so lifted labels match at every level. Thus the categories are equivalent. □

3 Conclusion

In this paper, we formally defined the hypergraph analogue (MetaHyperGraph) and the superhypergraph analogue (MetaSuperHyperGraph) of MetaGraphs, and provided a concise discussion of their characteristics and illustrative applications. We also introduced iterative constructions such as the Iterated MetaGraph, representing a "graph of graphs of . . . of graphs," and briefly examined their properties and potential uses. It is hoped that future work will explore extensions incorporating frameworks such as Fuzzy Sets [35–38], Vague Sets [39–41], Neutrosophic Sets [20, 42–44], and Plithogenic Sets [45–47].

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Conflicts of Interest

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Data Availability

This paper is theoretical and did not generate or analyze any empirical data. We welcome future studies that apply and test these concepts in practical settings.

Research Integrity

The author confirms that this manuscript is original, has not been published elsewhere, and is not under consideration by any other journal.

Use of Computational Tools

All proofs and derivations were performed manually; no computational software (e.g., Mathematica, SageMath, Coq) was used.

Code Availability

No code or software was developed for this study.

Ethical Approval

This research did not involve human participants or animals, and therefore did not require ethical approval.

Use of Generative AI and AI-Assisted Tools

We use generative AI and AI-assisted tools for tasks such as English grammar checking, and We do not employ them in any way that violates ethical standards.

Supplementary Information

No supplementary materials accompany this paper.

Disclaimer

The ideas presented here are theoretical and have not yet been validated through empirical testing. While we have strived for accuracy and proper citation, inadvertent errors may remain. Readers should verify any referenced material independently. The opinions expressed are those of the authors and do not necessarily reflect the views of their institutions.

References

[1] Reinhard Diestel. *Graph theory*. Springer (print edition); Reinhard Diestel (eBooks), 2024.

[2] Jonathan L Gross, Jay Yellen, and Mark Anderson. *Graph theory and its applications*. Chapman and Hall/CRC, 2018.

[3] Claude Berge. *Hypergraphs: combinatorics of finite sets*, volume 45. Elsevier, 1984.

[4] Florentin Smarandache. *Extension of HyperGraph to n-SuperHyperGraph and to Plithogenic n-SuperHyperGraph, and Extension of HyperAlgebra to n-ary (Classical-/Neutro-/Anti-) HyperAlgebra*. Infinite Study, 2020.

[5] Takaaki Fujita and Florentin Smarandache. A concise study of some superhypergraph classes. *Neutrosophic Sets and Systems*, 77:548–593, 2024.

[6] Muhammad Akram, A Nagoor Gani, and A Borumand Saeid. Vague hypergraphs. *Journal of Intelligent & Fuzzy Systems*, 26(2):647–653, 2014.

[7] Yue Gao, Zizhao Zhang, Haojie Lin, Xibin Zhao, Shaoyi Du, and Changqing Zou. Hypergraph learning: Methods and practices. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 44(5):2548–2566, 2020.

[8] Yifan Feng, Haoxuan You, Zizhao Zhang, Rongrong Ji, and Yue Gao. Hypergraph neural networks. In *Proceedings of the AAAI conference on artificial intelligence*, volume 33, pages 3558–3565, 2019.

[9] Georg Gottlob, Nicola Leone, and Francesco Scarcello. Hypertree decompositions and tractable queries. In *Proceedings of the eighteenth ACM SIGMOD-SIGACT-SIGART symposium on Principles of database systems*, pages 21–32, 1999.

[10] Yue Gao, Yifan Feng, Shuyi Ji, and Rongrong Ji. Hgnn+: General hypergraph neural networks. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 45(3):3181–3199, 2022.

[11] Muhammad Akram and Gulfam Shahzadi. Hypergraphs in m-polar fuzzy environment. *Mathematics*, 6(2):28, 2018.

[12] Takaaki Fujita. Hypergraph and superhypergraph approaches in electronics: A hierarchical framework for modeling power-grid hypernetworks and superhypernetworks. *Journal of Energy Research and Reviews*, 17(6):102–136, 2025.

[13] Eduardo Martín Campoverde Valencia, Jessica Paola Chuisaca Vásquez, and Francisco Ángel Becerra Lois. Multineutrosophic analysis of the relationship between survival and business growth in the manufacturing sector of azuay province, 2020–2023, using plithogenic n-superhypergraphs. *Neutrosophic Sets and Systems*, 84:341–355, 2025.

[14] Masoud Ghods, Zahra Rostami, and Florentin Smarandache. Introduction to neutrosophic restricted superhypergraphs and neutrosophic restricted superhypertrees and several of their properties. *Neutrosophic Sets and Systems*, 50:480–487, 2022.

[15] Mohammad Hamidi, Florentin Smarandache, and Mohadeseh Taghinezhad. *Decision Making Based on Valued Fuzzy Superhypergraphs*. Infinite Study, 2023.

[16] Julio Cesar Méndez Bravo, Claudia Jeaneth Bolanos Piedrahita, Manuel Alberto Méndez Bravo, and Luis Manuel Pilacuan-Bonete. Integrating smed and industry 4.0 to optimize processes with plithogenic n-superhypergraphs. *Neutrosophic Sets and Systems*, 84:328–340, 2025.

[17] N. B. Nalawade, M. S. Bapat, S. G. Jakkewad, G. A. Dhanorkar, and D. J. Bhosale. Structural properties of zero-divisor hypergraph and superhypergraph over $\mathbb{Z}_n$: Girth and helly property. *Panamerican Mathematical Journal*, 35(4S):485, 2025.

[18] Takaaki Fujita. An introduction and reexamination of molecular hypergraph and molecular n-superhypergraph. *Asian Journal of Physical and Chemical Sciences*, 13(3):1–38, 2025.

[19] Berrocal Villegas Salomón Marcos, Montalvo Fritas Willner, Berrocal Villegas Carmen Rosa, Flores Fuentes Rivera María Yissel, Espejo Rivera Roberto, Laura Daysi Bautista Puma, and Dante Manuel Macazana Fernández. Using plithogenic n-superhypergraphs to assess the degree of relationship between information skills and digital competencies. *Neutrosophic Sets and Systems*, 84:513–524, 2025.

[20] Shouxian Zhu. Neutrosophic n-superhypernetwork: A new approach for evaluating short video communication effectiveness in media convergence. *Neutrosophic Sets and Systems*, 85:1004–1017, 2025.

[21] Takaaki Fujita. Modeling hierarchical systems in graph signal processing, electric circuits, and bond graphs via hypergraphs and superhypergraphs. *Journal of Engineering Research and Reports*, 27(5):542, 2025.

[22] E. J. Mogro, J. R. Molina, G. J. S. Canas, and P. H. Soria. Tree tobacco extract (*Nicotiana glauca*) as a plithogenic bioinsecticide alternative for controlling fruit fly (*Drosophila immigrans*) using n-superhypergraphs. *Neutrosophic Sets and Systems*, 74:57–65, 2024.

[23] Alain Bretto. Hypergraph theory. *An introduction. Mathematical Engineering. Cham: Springer*, 1, 2013.

[24] Florentin Smarandache. Foundation of superhyperstructure & neutrosophic superhyperstructure. *Neutrosophic Sets and Systems*, 63(1):21, 2024.

[25] Florentin Smarandache. The cardinal of the m-powerset of a set of n elements used in the superhyperstructures and neutrosophic superhyperstructures. *Systems Assessment and Engineering Management*, 2:19–22, 2024.

[26] F. Smarandache. Introduction to superhyperalgebra and neutrosophic superhyperalgebra. *Journal of Algebraic Hyperstructures and Logical Algebras*, 2022.

[27] Florentin Smarandache. n-superhypergraph and plithogenic n-superhypergraph. *Nidus Idearum*, 7:107–113, 2019.

[28] Takaaki Fujita. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*. Biblio Publishing, 2025.

[29] RB Azevedo, Rolf Lohaus, and Tiago Paixão. Networking networks. *Evol Dev*, 10:514–515, 2008.

[30] Claire Donnat and Susan Holmes. Tracking network dynamics: A survey using graph distances. *The Annals of Applied Statistics*, 12(2):971–1012, 2018.

[31] Claire Donnat and Susan Holmes. Tracking network dynamics: A survey of distances and similarity metrics. *arXiv preprint arXiv:1801.07351*, 2018.

[32] Stefano Ciliberti, Olivier C. Martin, and Andreas Wagner. Innovation and robustness in complex regulatory gene networks. *Proceedings of the National Academy of Sciences*, 104:13591 – 13596, 2007.

[33] Olivier C Martin and Andreas Wagner. Multifunctionality and robustness trade-offs in model genetic circuits. *Biophysical journal*, 94(8):2927–2937, 2008.

[34] Jiaqi Cao, Shengli Zhang, Qingxia Chen, Houtian Wang, Mingzhe Wang, and Naijin Liu. Network-wide task offloading with leo satellites: A computation and transmission fusion approach. *arXiv preprint arXiv:2211.09672*, 2022.

[35] Talal Al-Hawary. Complete fuzzy graphs. *International Journal of Mathematical Combinatorics*, 4:26, 2011.

[36] Lotfi A Zadeh. Fuzzy sets. *Information and control*, 8(3):338–353, 1965.

[37] Hans-Jürgen Zimmermann. *Fuzzy set theory—and its applications*. Springer Science & Business Media, 2011.

[38] Azriel Rosenfeld. Fuzzy graphs. In *Fuzzy sets and their applications to cognitive and decision processes*, pages 77–95. Elsevier, 1975.

[39] An Lu and Wilfred Ng. Vague sets or intuitionistic fuzzy sets for handling vague data: which one is better? In *International conference on conceptual modeling*, pages 401–416. Springer, 2005.

[40] W-L Gau and Daniel J Buehrer. Vague sets. *IEEE transactions on systems, man, and cybernetics*, 23(2):610–614, 1993.

[41] Humberto Bustince and P Burillo. Vague sets are intuitionistic fuzzy sets. *Fuzzy sets and systems*, 79(3):403–405, 1996.

[42] Florentin Smarandache. A unifying field in logics: Neutrosophic logic. In *Philosophy*, pages 1–141. American Research Press, 1999.

[43] Said Broumi, Mohamed Talea, Assia Bakali, and Florentin Smarandache. Interval valued neutrosophic graphs. *Critical Review, XII*, 2016:5–33, 2016.

[44] Haibin Wang, Florentin Smarandache, Yanqing Zhang, and Rajshekhar Sunderraman. *Single valued neutrosophic sets*. Infinite study, 2010.

[45] Florentin Smarandache. *Plithogenic set, an extension of crisp, fuzzy, intuitionistic fuzzy, and neutrosophic sets-revisited*. Infinite study, 2018.

[46] Fazeelat Sultana, Muhammad Gulistan, Mumtaz Ali, Naveed Yaqoob, Muhammad Khan, Tabasam Rashid, and Tauseef Ahmed. A study of plithogenic graphs: applications in spreading coronavirus disease (covid-19) globally. *Journal of ambient intelligence and humanized computing*, 14(10):13139–13159, 2023.

[47] WB Vasantha Kandasamy, K Ilanthenral, and Florentin Smarandache. *Plithogenic Graphs*. Infinite Study, 2020.