Meta-Fuzzy Graph, Meta-Neutrosophic Graph, Meta-Digraph, and Meta-MultiGraph with some applications

Takaaki Fujita [1] *
[1] Independent Researcher, Tokyo, Japan.

Abstract

Graph theory investigates mathematical structures consisting of vertices and edges to model relationships and connectivity [1, 2]. A *MetaGraph* is a higher-level graph whose vertices are themselves graphs, with edges representing specified relations among those graphs. An *Iterated MetaGraph* extends this idea recursively: its vertices are MetaGraphs, yielding a hierarchy of graph-of-graphs structures across multiple levels.

Fuzzy graphs incorporate fuzzy membership functions on vertices and edges, thereby capturing uncertainty and graded strength of connectivity. Neutrosophic graphs generalize this further by assigning to each vertex and edge three independent membership values—truth, indeterminacy, and falsity—providing a more comprehensive framework for uncertainty. A weighted graph is a graph in which each edge is assigned a numerical value (weight), typically representing cost, distance, or intensity. Multigraphs, which allow multiple parallel edges and loops, appear naturally when such multiplicities are required. Bidirected graphs (bidigraphs) assign local orientations to each vertex-edge incidence, allowing edges to point independently at both ends.

In this paper, we extend the frameworks of fuzzy graphs, neutrosophic graphs, multigraphs, weighted graphs, digraphs, and bidirected graphs by embedding them into the unified setting of MetaGraphs and Iterated MetaGraphs.

Keywords: Fuzzy Graph, Neutrosophic Graph, MultiGraph, Bidirected Graph, Weighted Graph, MetaGraph, Iterated MetaGraph

Contents in this paper

The remainder of this paper is organized as follows.

1 Preliminaries

This section presents the fundamental concepts and definitions that underpin the discussions in this paper. Unless otherwise noted, all graphs considered here are *undirected*, *finite*, and *simple*.

1.1 MetaGraph(Graph of Graph)

Graph theory investigates mathematical structures consisting of vertices and edges to model relationships and connectivity [1, 2]. A MetaGraph is a graph whose vertices are themselves graphs, with edges representing specified relations between those graphs (cf. [3, 3–5]).

Definition 1.1 (Metagraph (graph of graphs))**.** (cf. [6]) Fix a nonempty universe $\mathfrak{G}$ of finite graphs (undirected, loopless by default) and a nonempty family of binary relations

$$\mathcal{R} \subseteq \mathcal{P}(\mathfrak{G} \times \mathfrak{G}).$$

A *metagraph over* $(\mathfrak{G}, \mathcal{R})$ is a directed, labelled multigraph

$$M = (V, E, s, t, \lambda)$$

with

$$V \subseteq \mathfrak{G}, \qquad s, t : E \to V, \qquad \lambda : E \to \mathcal{R},$$

satisfying the incidence constraint

$$\forall e \in E : \big(s(e),\, t(e)\big) \in \lambda(e).$$

Elements of V are *meta-vertices* (each is a graph $G \in \mathfrak{G}$). For $e \in E$ with $\lambda(e) = R$, we write $s(e) \xrightarrow{R} t(e)$ and call e a *meta-edge*. If $\mathcal{R} = \{R\}$ is a singleton, labels may be omitted. If every $R \in \mathcal{R}$ is symmetric, M can be viewed as an undirected labelled multigraph.

Example 1.2 (Urban Mobility as a Metagraph)**.** Let $\mathfrak{G}$ be the set of *city-level transit graphs* $G_c = (V_c, E_c)$, where V_c are stations in city c and E_c are intra-city rail/bus links. Define two binary relations on $\mathfrak{G}$:

$$\mathrm{HSR} := \{(G_{c_1}, G_{c_2}) \mid \text{there exists a direct high-speed rail service between } c_1 \text{ and } c_2\},$$

$$\mathrm{AIR}_\theta := \{(G_{c_1}, G_{c_2}) \mid \text{there exist } \geq \theta \text{ direct flights per week between } c_1 \text{ and } c_2\},$$

and set $\mathcal{R} := \{\mathrm{HSR}, \mathrm{AIR}_\theta\}$ for a fixed threshold $\theta \in \mathbb{N}$.

Consider the metagraph $M = (V, E, s, t, \lambda)$ with

$$V = \{G_{\text{Tokyo}},\, G_{\text{Osaka}},\, G_{\text{Nagoya}}\},$$

and meta-edges

$$e_1 : G_{\text{Tokyo}} \xrightarrow{\text{HSR}} G_{\text{Osaka}}, \qquad e_2 : G_{\text{Osaka}} \xrightarrow{\text{HSR}} G_{\text{Nagoya}}, \qquad e_3 : G_{\text{Tokyo}} \xrightarrow{\text{AIR}_\theta} G_{\text{Nagoya}}.$$

By construction, each e_i satisfies the incidence constraint $\big(s(e_i), t(e_i)\big) \in \lambda(e_i)$: there is high-speed rail between Tokyo–Osaka and Osaka–Nagoya (so $(G_{\text{Tokyo}}, G_{\text{Osaka}}) \in \mathrm{HSR}$ and $(G_{\text{Osaka}}, G_{\text{Nagoya}}) \in \mathrm{HSR}$), and the direct-flight relation AIR_θ holds for the chosen threshold on Tokyo–Nagoya. Thus M is a concrete *metagraph of cities*, where meta-vertices are city transit graphs and meta-edges encode inter-city mobility modes.

1.2 Iterated MetaGraph(Graph of Graph of ... of Graph)

An Iterated MetaGraph is a graph whose vertices are metagraphs, recursively extending graph-of-graphs structure to multiple hierarchical levels.

Definition 1.3 (Unit metagraph embedding)**.** For $X \in \mathfrak{G}$ define the *unit metagraph*

$$\mathsf{U}(X) := \big(\{X\},\, \emptyset,\, \_,\, \_,\, \_\big).$$

This gives an injective map $\mathsf{U} : \mathfrak{G} \hookrightarrow \mathrm{Obj}\big(\mathrm{Meta}(\mathfrak{G}, \mathcal{R})\big)$.

Definition 1.4 (Relation lifting)**.** Given $\mathcal{R}$ on $\mathfrak{G}$, define its *lift* $\mathcal{R}^{\uparrow}$ on finite metagraphs over $(\mathfrak{G}, \mathcal{R})$ by

$$\forall R \in \mathcal{R}, \quad (M_1, M_2) \in R^{\uparrow} \iff \exists x \in V(M_1),\ y \in V(M_2) :\ (x, y) \in R.$$

Set $\mathcal{R}^{\uparrow} := \{R^{\uparrow} : R \in \mathcal{R}\}$.

Definition 1.5 (Iterated object and relation universes)**.** Define recursively for $t \in \mathbb{N}_0$:

$$\mathfrak{G}^{(0)} := \mathfrak{G}, \qquad \mathcal{R}^{(0)} := \mathcal{R},$$

$$\mathfrak{G}^{(t+1)} := \Big\{ \text{finite metagraphs over } \big(\mathfrak{G}^{(t)}, \mathcal{R}^{(t)}\big) \Big\}, \qquad \mathcal{R}^{(t+1)} := \big(\mathcal{R}^{(t)}\big)^{\uparrow}.$$

Definition 1.6 (Iterated MetaGraph of depth t)**.** For $t \in \mathbb{N}_0$, an *iterated metagraph of depth* t is a metagraph

$$M^{(t)} = (V^{(t)}, E^{(t)}, s^{(t)}, t^{(t)}, \lambda^{(t)})$$

over $\big(\mathfrak{G}^{(t)}, \mathcal{R}^{(t)}\big)$, i.e., $V^{(t)} \subseteq \mathfrak{G}^{(t)}$, $\lambda^{(t)} : E^{(t)} \to \mathcal{R}^{(t)}$ and

$$\forall e \in E^{(t)} : \ \big(s^{(t)}(e),\, t^{(t)}(e)\big) \in \lambda^{(t)}(e).$$

Example 1.7 (Supply-Chain-of-Chains as an Iterated Metagraph)**.** Let $\mathfrak{G}^{(0)}$ be the set of *facility-level logistics graphs* $G = (V, E)$, where V are production/storage sites inside a firm and E are internal transport links. Fix a base relation family $\mathcal{R}^{(0)}$ containing

$$\mathrm{SHIP}_{\geq q} := \{(G_1, G_2) \mid \text{there exists a scheduled shipment from some site in } G_1 \text{ to some site in } G_2 \text{ at } \geq q \text{ trips/week}\},$$

for a chosen $q \in \mathbb{N}$.

Level 1 (metagraphs of firms). Firm A has two facilities with internal flows, modeled by $G_{A,1}, G_{A,2} \in \mathfrak{G}^{(0)}$; Firm B has one facility $G_{B,1} \in \mathfrak{G}^{(0)}$. Define metagraphs

$$M_A = (\{G_{A,1}, G_{A,2}\}, E_A, s_A, t_A, \lambda_A), \qquad M_B = (\{G_{B,1}\}, E_B, s_B, t_B, \lambda_B),$$

where edges of M_A record A's internal transfers (e.g. $G_{A,1} \xrightarrow{\mathrm{SHIP}_{\geq 5}} G_{A,2}$ if 5 trips/week are scheduled). Assume there is a *cross-firm* weekly shipment from A's facility $G_{A,2}$ to B's $G_{B,1}$ at rate 3/week. Then at base level,

$$(G_{A,2}, G_{B,1}) \in \mathrm{SHIP}_{\geq 1} \ \subseteq \ \mathcal{R}^{(0)}.$$

Level 2 (iterated metagraph of firms). By relation lifting, the lifted relation $\mathrm{SHIP}^{\uparrow}_{\geq 1} \in \mathcal{R}^{(1)}$ satisfies

$$\big(M_A, M_B\big) \in \mathrm{SHIP}^{\uparrow}_{\geq 1} \quad \Longleftrightarrow \quad \exists\, X \in V(M_A),\, Y \in V(M_B) : \ (X, Y) \in \mathrm{SHIP}_{\geq 1}.$$

Since $(G_{A,2}, G_{B,1}) \in \mathrm{SHIP}_{\geq 1}$, we obtain a meta-edge

$$M_A \xrightarrow{\mathrm{SHIP}^{\uparrow}_{\geq 1}} M_B$$

in the *iterated metagraph* whose vertices are firm-level metagraphs. Thus, real shipments between specific facilities induce (via lifting) contractual/supply relationships between the firms at the meta-level.

1.3 Fuzzy Graph

A fuzzy set assigns each element a membership degree between 0 and 1, modeling partial belonging and uncertainty in classification [7, 8]. A fuzzy graph combines fuzzy vertex and edge membership functions, representing relationships with uncertainty and graded connectivity among nodes [9–12].

Definition 1.8 (Fuzzy set)**.** [7, 13] Let Y be a non-empty universe. A *fuzzy set* τ on Y is a function

$$\tau : Y \longrightarrow [0, 1],$$

assigning to each $y \in Y$ a membership value $\tau(y)$. A *fuzzy relation* on Y is a fuzzy subset δ of $Y \times Y$. Given a fuzzy set τ on Y, the relation δ is said to be a *fuzzy relation on* τ whenever

$$\delta(y, z) \ \leq \ \min\{\tau(y), \tau(z)\}, \quad \forall\, y, z \in Y.$$

Definition 1.9 (Fuzzy graph)**.** [11] A *fuzzy graph* on a vertex set V is a pair $G = (\sigma, \mu)$ consisting of:

- A vertex membership function $\sigma : V \to [0,1]$, where $\sigma(x)$ gives the degree to which $x \in V$ belongs to the graph.
- An edge membership function $\mu : V \times V \to [0,1]$, which is a fuzzy relation on σ, satisfying

$$\mu(x, y) \le \sigma(x) \wedge \sigma(y), \quad \forall x, y \in V,$$

  where $\wedge$ denotes the minimum operator.

The associated *crisp graph* $G^* = (\sigma^*, \mu^*)$ is determined by

$$\sigma^* = \{ x \in V \mid \sigma(x) > 0 \}, \qquad \mu^* = \{ (x, y) \in V \times V \mid \mu(x, y) > 0 \}.$$

A *fuzzy subgraph* $H = (\sigma', \mu')$ of G is obtained by choosing a subset $X \subseteq V$ and defining

- a restricted vertex membership $\sigma' : X \to [0,1]$,
- an edge membership $\mu' : X \times X \to [0,1]$ such that

$$\mu'(x, y) \le \sigma'(x) \wedge \sigma'(y), \quad \forall x, y \in X.$$

A neutrosophic set assigns to each element three independent membership degrees: truth, indeterminacy, and falsity, enabling richer uncertainty representation [14, 15]. A neutrosophic graph assigns each vertex and edge three membership values: truth, indeterminacy, and falsity, capturing uncertainty comprehensively [16–21].

Definition 1.10 (Neutrosophic Graph)**.** [16] Let V be a (finite) vertex set. A *neutrosophic graph* on V is a pair

$$G = (\sigma, \mu),$$

where

- $\sigma : V \to [0,1]^3$, $v \mapsto \sigma(v) = \big(T_\sigma(v), I_\sigma(v), F_\sigma(v)\big)$ assigns to each vertex v its *truth*, *indeterminacy*, and *falsity* degrees, with $0 \le T_\sigma(v), I_\sigma(v), F_\sigma(v) \le 1$ and

$$0 \le T_\sigma(v) + I_\sigma(v) + F_\sigma(v) \le 3.$$

- $\mu : V \times V \to [0,1]^3$, $(u, v) \mapsto \mu(u, v) = \big(T_\mu(u, v), I_\mu(u, v), F_\mu(u, v)\big)$ assigns neutrosophic degrees to (unordered) pairs $\{u, v\}$ (take $\mu(u, u)$ for loops if allowed), with $0 \le T_\mu, I_\mu, F_\mu \le 1$ and

$$0 \le T_\mu(u, v) + I_\mu(u, v) + F_\mu(u, v) \le 3.$$

These maps satisfy, for all distinct $u, v \in V$,

$$T_\mu(u, v) \le \min\{T_\sigma(u), T_\sigma(v)\}, \qquad I_\mu(u, v) \ge \max\{I_\sigma(u), I_\sigma(v)\}, \qquad F_\mu(u, v) \ge \max\{F_\sigma(u), F_\sigma(v)\}.$$

(For a nonedge one may set $\mu(u, v) = (0, 0, 0)$.) The *underlying crisp graph* $G^* = (V, E^*)$ is given by

$$E^* := \big\{\{u, v\} \subseteq V : T_\mu(u, v) > 0 \text{ or } I_\mu(u, v) > 0 \text{ or } F_\mu(u, v) > 0\big\}.$$

Unless stated otherwise, neutrosophic graphs are taken to be undirected and loopless.

Example 1.11 (News–Source Credibility as a Neutrosophic Graph)**.** Consider three online sources $V = \{\mathrm{A}, \mathrm{B}, \mathrm{C}\}$. Assign neutrosophic vertex-memberships (credibility T, uncertainty I, falsity risk F):

$$\sigma(\mathrm{A}) = (0.90,\ 0.10,\ 0.00), \quad \sigma(\mathrm{B}) = (0.60,\ 0.30,\ 0.10), \quad \sigma(\mathrm{C}) = (0.40,\ 0.50,\ 0.20).$$

Define neutrosophic edge-memberships (pairwise agreement/compatibility):

$$\mu(\mathrm{A}, \mathrm{B}) = (0.55,\ 0.35,\ 0.15), \quad \mu(\mathrm{A}, \mathrm{C}) = (0.25,\ 0.55,\ 0.25), \quad \mu(\mathrm{B}, \mathrm{C}) = (0.35,\ 0.50,\ 0.20).$$

These values satisfy the neutrosophic constraints. For instance, for (A, B):

$$\min\{T_\sigma(\mathrm{A}), T_\sigma(\mathrm{B})\} = \min\{0.90, 0.60\} = 0.60 \ge T_\mu(\mathrm{A}, \mathrm{B}) = 0.55,$$
$$\max\{I_\sigma(\mathrm{A}), I_\sigma(\mathrm{B})\} = \max\{0.10, 0.30\} = 0.30 \le I_\mu(\mathrm{A}, \mathrm{B}) = 0.35,$$
$$\max\{F_\sigma(\mathrm{A}), F_\sigma(\mathrm{B})\} = \max\{0.00, 0.10\} = 0.10 \le F_\mu(\mathrm{A}, \mathrm{B}) = 0.15,$$

and similarly for (A, C) and (B, C). Interpretation: A is highly credible, C is uncertain; their edge (A, C) has modest truth degree 0.25 with higher indeterminacy 0.55, reflecting inconsistent or incomplete cross-corroboration between those sources.

1.4 Weighted Graph

A weighted graph is a graph in which each edge is assigned a numerical value (weight), typically representing cost, distance, or intensity(cf. [22–27]).

Definition 1.12 (Weighted Graph)**.** (cf. [22, 23]) A *Weighted Graph* augments the structure of a graph by assigning a numerical weight to each edge. Formally, a weighted graph is defined as a triple $G = (V, E, w)$ where:

- V is a non-empty set of vertices.
- $E \subseteq \{\{u, v\} \mid u, v \in V,\ u \neq v\}$ is a set of edges.
- $w : E \to \mathbb{R}$ is a weight function that assigns a unique real number to each edge $e \in E$.

1.5 Directed Graph and Bidirected Graph

A directed graph consists of vertices connected by ordered edges, where each edge has a defined direction from source to target [28–30].

Definition 1.13 (Directed Graph)**.** [31] A *directed graph* (digraph) $G = (V, E)$ consists of:

- V: A finite set of vertices.
- $E \subseteq V \times V$: A set of directed edges, where each edge is an ordered pair (u, v) with $u, v \in V$.

The edge (u, v) indicates a directed connection from vertex u (source) to vertex v (target).

A Bidirected graph (Bidigraph) assigns local directions to each vertex-edge pair, enabling edges to point independently at both ends [32–35].

Definition 1.14 (Bidirected Graph (Bidigraph))**.** [33–35] A *bidirected graph* (also known as a *bigraph*) is a pair $B = (G, \tau)$, where:

- $G = (V, E)$ is a simple undirected graph, where V is a non-empty set of vertices and E is a set of edges (without parallel edges or loops).
- $\tau : V \times E \to \{-1, 0, 1\}$ is a function called the *bidirection function*, which assigns a *local orientation* to each vertex-edge pair (v, e) as follows:
  - $\tau(v, e) = 1$: Edge e is directed *towards* vertex v.
  - $\tau(v, e) = -1$: Edge e is directed *away from* vertex v.
  - $\tau(v, e) = 0$: Vertex v is not incident to edge e.

The graph G is referred to as the *underlying graph* of B, and the function τ provides the bidirected structure on G by assigning a direction at each endpoint of every edge in E.

Example 1.15 (Last-Mile Logistics Lane as a Bidirected Graph)**.** Let $G = (V, E)$ with facilities $V = \{\mathrm{F}, \mathrm{C}, \mathrm{R}\}$ for *Factory*, *Cross-dock*, and *Retail store*. Undirected transport links are

$$E = \{e_{\mathrm{FC}} = \{\mathrm{F}, \mathrm{C}\},\ e_{\mathrm{CR}} = \{\mathrm{C}, \mathrm{R}\},\ e_{\mathrm{FR}} = \{\mathrm{F}, \mathrm{R}\}\}.$$

Define a bidirection function $\tau : V \times E \to \{-1, 0, 1\}$ where $\tau(v, e) = +1$ means the lane is oriented *towards* v (arrivals at v), $\tau(v, e) = -1$ means *away from* v (departures from v), and $\tau(v, e) = 0$ if $v \notin e$. Set

$$(\tau(\mathrm{F}, e_{\mathrm{FC}}), \tau(\mathrm{C}, e_{\mathrm{FC}})) = (-1, +1) \quad \text{(factory ships to cross-dock)},$$

$$(\tau(\mathrm{C}, e_{\mathrm{CR}}), \tau(\mathrm{R}, e_{\mathrm{CR}})) = (-1, +1) \quad \text{(cross-dock dispatches to retail)},$$

$$(\tau(\mathrm{R}, e_{\mathrm{FR}}), \tau(\mathrm{F}, e_{\mathrm{FR}})) = (-1, +1) \quad \text{(empty containers return from retail to factory).}$$

For completeness, non-incidences have value 0. A compact table is:

	e_{FC}	e_{CR}	e_{FR}
$\tau(\mathrm{F}, \cdot)$	-1	0	$+1$
$\tau(\mathrm{C}, \cdot)$	$+1$	-1	0
$\tau(\mathrm{R}, \cdot)$	0	$+1$	-1

This bidirected model captures local endpoint behavior (depart/arrive) on each lane with a *single* undirected edge, which is natural for physical corridors where the admissible direction at one end can be specified independently of the other.

1.6 MultiGraph

A multigraph is a graph that allows multiple parallel edges between the same vertices and loops on vertices, used when necessary (cf. [36–40]). As a related concept, Iterative MultiStructure is also known [41].

Definition 1.16 (Multigraph)**.** A *multigraph* is a triple $G = (V, E, \varphi)$ where:

- V is a finite set of *vertices*.
- E is a finite *multiset* of *edges*.
- $\varphi : E \longrightarrow \big\{\{u, v\} \mid u, v \in V,\ u = v \text{ or } u \neq v\big\}$ is an *incidence function* that assigns to each edge $e \in E$ an unordered pair $\{u, v\}$ of one or two vertices. In particular:
  1. If $\varphi(e) = \{u, v\}$ with $u \neq v$, then e is a *(simple) edge* connecting u and v.
  2. If $\varphi(e) = \{v, v\}$ for some $v \in V$, then e is a *loop* at vertex v.
  3. If there exist distinct edges $e_1, e_2 \in E$ such that $\varphi(e_1) = \varphi(e_2) = \{u, v\}$, then e_1 and e_2 are *parallel edges* between u and v.

2 Reviews and Main Results

In this section, we present the main results of this paper along with related discussions.

2.1 Meta-Fuzzy Graph

A Meta-Fuzzy Graph models relationships among fuzzy graphs, where vertices represent fuzzy graphs and edges encode higher-level fuzzy relations.

Definition 2.1 (Meta-Fuzzy Graph over $(\mathsf{FG}, \mathcal{R})$)**.** A *Meta-Fuzzy Graph* is a triple

$$\mathbb{M} := \big(\sigma_M,\ \mu_M,\ L_M\big)$$

consisting of:

- a *meta-vertex membership* $\sigma_M : \mathsf{FG} \to [0, 1]$;
- a *meta-edge membership* $\mu_M : \mathsf{FG} \times \mathsf{FG} \to [0, 1]$ satisfying the fuzzy-graph constraint

$$\mu_M(F, G) \ \le\ \min\{\sigma_M(F), \sigma_M(G)\} \qquad (\forall F, G \in \mathsf{FG});$$

- a *label selector* $L_M : \mathsf{FG} \times \mathsf{FG} \to \mathcal{P}_{\mathrm{fin}}(\mathcal{R})$ assigning to each ordered pair (F, G) a finite set $L_M(F, G)$ of labels from $\mathcal{R}$.

These data must satisfy the *witnessing (incidence) constraint*

$$\mu_M(F,G) \;\le\; \sup_{R \in L_M(F,G)} R(F,G) \qquad (\forall F, G \in \mathsf{FG}), \tag{1}$$

with the convention that $\sup \emptyset := 0$. The *support* (meta-vertex set) is $V(\mathbb{M}) = \{\, F \in \mathsf{FG} \mid \sigma_M(F) > 0 \,\}$, and the *crisp underlying meta-graph* is the directed graph with vertex set $V(\mathbb{M})$ and arc set $A(\mathbb{M}) = \{\, (F,G) \mid \mu_M(F,G) > 0 \,\}$, optionally decorated by $L_M(F,G)$.

Definition 2.2 (Crisp embedding of graphs into fuzzy graphs)**.** Given a (finite) simple graph $G = (V, E)$, define its *crisp fuzzy realization* $\iota(G) = (\sigma_G, \mu_G)$ by

$$\sigma_G(x) \;:=\; \begin{cases} 1, & x \in V, \\ 0, & \text{otherwise}, \end{cases} \qquad \mu_G(x,y) \;:=\; \begin{cases} 1, & \{x,y\} \in E, \\ 0, & \text{otherwise}. \end{cases}$$

Example 2.3 (Regional Retail Analytics as a Meta-Fuzzy Graph)**.** Let FG contain three fuzzy graphs representing *product co-purchase networks* for Tokyo (F_T), Osaka (F_O), and Nagoya (F_N). Each $F_\bullet$ has fuzzy vertex/edge memberships (omitted here), e.g. higher membership on products and pairs that appear frequently.

Set the meta-vertex membership (data quality / recency weight)

$$\sigma_M(F_T) = 0.90, \qquad \sigma_M(F_O) = 0.70, \qquad \sigma_M(F_N) = 0.50.$$

Label family $\mathcal{R}$. Define two fuzzy relations on $\mathsf{FG} \times \mathsf{FG}$:

$$R_{\text{overlap}}(F_i, F_j) = \text{normalized catalog overlap}, \qquad R_{\text{api}}(F_i, F_j) = \text{API interoperability score}.$$

Assume the following values (symmetric):

	F_T	F_O	F_N
$R_{\text{overlap}}(F_T, \cdot)$	–	0.65	0.35
$R_{\text{overlap}}(F_O, \cdot)$	0.65	–	0.45
$R_{\text{overlap}}(F_N, \cdot)$	0.35	0.45	–

	F_T	F_O	F_N
$R_{\text{api}}(F_T, \cdot)$	–	0.55	0.25
$R_{\text{api}}(F_O, \cdot)$	0.55	–	0.30
$R_{\text{api}}(F_N, \cdot)$	0.25	0.30	–

Choose the label set $L_M(F_i, F_j) = \{R_{\text{overlap}}, R_{\text{api}}\}$ for all $i \neq j$ and define

$$\mu_M(F_T, F_O) = 0.60, \qquad \mu_M(F_O, F_N) = 0.40, \qquad \mu_M(F_T, F_N) = 0.30.$$

(i) Fuzzy-graph constraint:

$$\mu_M(F_T, F_O) = 0.60 \le \min\{0.90, 0.70\} = 0.70, \quad \mu_M(F_O, F_N) = 0.40 \le \min\{0.70, 0.50\} = 0.50,$$

$$\mu_M(F_T, F_N) = 0.30 \le \min\{0.90, 0.50\} = 0.50.$$

(ii) Witnessing:

$$\sup_{R \in L_M(F_T, F_O)} R(F_T, F_O) = \max\{0.65, 0.55\} = 0.65 \;\ge\; 0.60,$$

$$\sup_{R \in L_M(F_O, F_N)} R(F_O, F_N) = \max\{0.45, 0.30\} = 0.45 \;\ge\; 0.40,$$

$$\sup_{R \in L_M(F_T, F_N)} R(F_T, F_N) = \max\{0.35, 0.25\} = 0.35 \;\ge\; 0.30.$$

Hence $\mathbb{M} = (\sigma_M, \mu_M, L_M)$ is a valid *Meta-Fuzzy Graph* modeling cross-regional analytics integration.

Definition 2.4 (Crisp-to-fuzzy lifting of labels)**.** Let $\mathfrak{G}$ be a universe of (finite) graphs and $\mathcal{R}_{\text{cr}} \subseteq \mathcal{P}(\mathfrak{G} \times \mathfrak{G})$ a nonempty family of *crisp* binary relations (e.g. subgraph, homomorphism, minor, isomorphism). Define

$$\mathcal{R}^\flat \;:=\; \left\{\, R^\flat \;\middle|\; R \in \mathcal{R}_{\text{cr}} \right\} \;\subseteq\; [0,1]^{\,\mathsf{FG} \times \mathsf{FG}}$$

by

$$R^\flat\big(\iota(G_1), \iota(G_2)\big) \;:=\; \begin{cases} 1, & (G_1, G_2) \in R, \\ 0, & \text{otherwise}, \end{cases} \qquad R^\flat(F_1, F_2) := 0 \text{ if some } F_i \neq \iota(\,\cdot\,).$$

Theorem 2.5 (Meta-Fuzzy Graph generalizes MetaGraph)**.** *Let $M = (V, E, s, t, \lambda)$ be a (directed, labeled) MetaGraph over $(\mathfrak{G}, \mathcal{R}_{\mathrm{cr}})$, i.e., $V \subseteq \mathfrak{G}$, $\lambda : E \to \mathcal{R}_{\mathrm{cr}}$, and*

$$\forall e \in E : \quad \big(s(e),\, t(e)\big) \in \lambda(e).$$

Define a Meta-Fuzzy Graph $\mathbb{M} = (\sigma_M, \mu_M, L_M)$ over $(\mathsf{FG}, \mathcal{R}^{\flat})$ by

$$\sigma_M(F) \;:=\; \mathbf{1}_{\{\iota(v) \mid v \in V\}}(F),$$

$$\mu_M(F, G) \;:=\; \max_{e \in E} \mathbf{1}_{\{\iota(s(e))\}}(F) \cdot \mathbf{1}_{\{\iota(t(e))\}}(G),$$

$$L_M(F, G) \;:=\; \big\{ \lambda(e) \;\big|\; e \in E,\; F = \iota\big(s(e)\big),\; G = \iota\big(t(e)\big) \big\}.$$

Then $\mathbb{M}$ is a Meta-Fuzzy Graph, and its underlying crisp meta-graph is (canonically) isomorphic to M.

Proof. (1) Fuzzy-graph constraint.) Fix $F, G \in \mathsf{FG}$. By definition, $\sigma_M(F) \in \{0, 1\}$ and $\sigma_M(G) \in \{0, 1\}$. If $\mu_M(F, G) = 1$, then there exists $e \in E$ with $F = \iota\big(s(e)\big)$ and $G = \iota\big(t(e)\big)$. Hence $\sigma_M(F) = \sigma_M(G) = 1$, so

$$\mu_M(F, G) = 1 \;\le\; \min\{1, 1\} = \min\{\sigma_M(F), \sigma_M(G)\}.$$

If $\mu_M(F, G) = 0$, the inequality holds trivially. Thus the fuzzy-graph constraint is satisfied.

(2) Witnessing constraint.) Let $F, G \in \mathsf{FG}$. If $L_M(F, G) = \emptyset$, then by construction there is no $e \in E$ with $F = \iota(s(e))$, $G = \iota(t(e))$, hence $\mu_M(F, G) = 0$, and (1) reads $0 \le \sup \emptyset = 0$, which holds. If $L_M(F, G) \ne \emptyset$, then there exists $e \in E$ with $F = \iota(s(e))$, $G = \iota(t(e))$ and $R := \lambda(e) \in L_M(F, G)$. By the MetaGraph incidence, $(s(e), t(e)) \in R$, hence by the definition of $R^{\flat}$,

$$R^{\flat}(F, G) \;=\; R^{\flat}\big(\iota(s(e)), \iota(t(e))\big) \;=\; 1.$$

Therefore

$$\sup_{Q \in L_M(F,G)} Q^{\flat}(F, G) \;\ge\; R^{\flat}(F, G) \;=\; 1 \;\ge\; \mu_M(F, G),$$

since $\mu_M(F, G) \in \{0, 1\}$ and equals 1 exactly when such an e exists. Thus (1) holds.

(3) Underlying crisp meta-graph.) The support is $V(\mathbb{M}) = \{\iota(v) \mid v \in V\}$. The arc set is

$$A(\mathbb{M}) \;=\; \{\, (F, G) \mid \mu_M(F, G) > 0 \,\} \;=\; \big\{\, (\iota(s(e)), \iota(t(e))) \mid e \in E \,\big\}.$$

Hence the map $\iota : V \to V(\mathbb{M})$, $v \mapsto \iota(v)$, is a bijection that extends to a digraph isomorphism from M to the underlying crisp meta-graph of $\mathbb{M}$, preserving labels via L_M. □

2.2 Iterated Meta-Fuzzy Graph

An Iterated Meta-Fuzzy Graph repeatedly applies meta-fuzzy construction across levels, producing hierarchical fuzzy graph networks capturing multi-scale relational uncertainty.

Definition 2.6 (Lifting of fuzzy relations to higher levels)**.** Suppose $\mathcal{U}$ is a class whose elements themselves carry fuzzy vertex-memberships (so each $X \in \mathcal{U}$ has a map $\Sigma_X : \cdot \to [0, 1]$ on its own vertex-universe). Given $S : \mathcal{U} \times \mathcal{U} \to [0, 1]$ and elements $X, Y \in \mathcal{U}$, define the *lifted* relation

$$S^{\uparrow}(X, Y) \;:=\; \sup_{u \in \mathrm{Vert}(X),\, v \in \mathrm{Vert}(Y)} \min\big(\Sigma_X(u),\, \Sigma_Y(v),\, S(u, v)\big),$$

where $\mathrm{Vert}(X)$ denotes the level-below vertex-universe on which Σ_X lives.

Definition 2.7 (Iterated universes and lifted families)**.** Define recursively for $t \in \mathbb{N}_0$:

$$\mathsf{FG}^{(0)} := \mathsf{FG}, \qquad \mathcal{R}^{(0)} \;\text{ as fixed above},$$

$$\mathsf{FG}^{(t+1)} := \Big\{\text{all Meta-Fuzzy Graphs on } \mathsf{FG}^{(t)} \text{ with witnessing family } \mathcal{R}^{(t)}\Big\},$$

$$\mathcal{R}^{(t+1)} \;:=\; \big\{\, R^{\uparrow} \mid R \in \mathcal{R}^{(t)} \,\big\}.$$

Definition 2.8 (Iterated Meta-Fuzzy Graph of depth t)**.** For $t \in \mathbb{N}_0$, an *Iterated Meta-Fuzzy Graph of depth t* is a Meta-Fuzzy Graph

$$\mathbb{M}^{(t)} = (\Sigma^{(t)}, \mathrm{M}^{(t)})$$

on the vertex universe $\mathsf{FG}^{(t)}$ with witnessing family $\mathcal{R}^{(t)}$, i.e., for all $X, Y \in \mathsf{FG}^{(t)}$:

$$\mathrm{M}^{(t)}(X,Y) \ \leq \ \min\{\Sigma^{(t)}(X), \Sigma^{(t)}(Y)\} \quad \text{and} \quad \mathrm{M}^{(t)}(X,Y) \ \leq \ \sup_{R \in \mathcal{R}^{(t)}} R(X,Y).$$

Example 2.9 (Division-Level Planning as an Iterated Meta-Fuzzy Graph)**.** Continue with $F_T, F_O, F_N \in \mathsf{FG}^{(0)}$ and the relations $R_{\text{overlap}}, R_{\text{api}} \in \mathcal{R}^{(0)}$ above. Construct two level-1 meta-fuzzy graphs (division views):

East division X on vertex set $\{F_T, F_N\}$ with internal meta-memberships

$$\Sigma_X(F_T) = 0.80, \qquad \Sigma_X(F_N) = 0.60,$$

and some internal meta-edges (omitted). *West division Y* on $\{F_O\}$ with

$$\Sigma_Y(F_O) = 0.90.$$

Lifted relations. By the lifting rule

$$R^{\uparrow}(X,Y) = \sup_{u \in V(X),\, v \in V(Y)} \min\bigl(\Sigma_X(u),\ \Sigma_Y(v),\ R(u,v)\bigr),$$

we compute

$$R^{\uparrow}_{\text{overlap}}(X,Y) = \max\bigl\{\min(0.80, 0.90, 0.65),\ \min(0.60, 0.90, 0.45)\bigr\} = \max\{0.65,\ 0.45\} = 0.65,$$

$$R^{\uparrow}_{\text{api}}(X,Y) = \max\bigl\{\min(0.80, 0.90, 0.55),\ \min(0.60, 0.90, 0.30)\bigr\} = \max\{0.55,\ 0.30\} = 0.55.$$

Thus in $\mathcal{R}^{(1)}$ we have two candidate labels between X and Y.

Level-1 meta-edge. Set the level-1 meta-vertex membership

$$\Sigma^{(1)}(X) = 0.85, \qquad \Sigma^{(1)}(Y) = 0.90,$$

and define a cross-division meta-edge

$$\mathrm{M}^{(1)}(X,Y) = 0.60, \qquad L^{(1)}(X,Y) = \{R^{\uparrow}_{\text{overlap}},\ R^{\uparrow}_{\text{api}}\}.$$

Verification. Fuzzy-graph bound at level 1:

$$\mathrm{M}^{(1)}(X,Y) = 0.60 \ \leq \ \min\{\Sigma^{(1)}(X), \Sigma^{(1)}(Y)\} = \min\{0.85, 0.90\} = 0.85.$$

Witnessing via lifted labels:

$$\sup_{R \in L^{(1)}(X,Y)} R(X,Y) = \max\{0.65, 0.55\} = 0.65 \ \geq \ \mathrm{M}^{(1)}(X,Y) = 0.60.$$

Therefore $\mathbb{M}^{(1)} = (\Sigma^{(1)}, \mathrm{M}^{(1)})$ is a valid *Iterated Meta-Fuzzy Graph* whose vertices are the division-level meta-fuzzy graphs and whose inter-division linkage is witnessed by lifted overlap/API relations derived from facility-level data.

Theorem 2.10 (Depth 1 recovers Meta-Fuzzy Graph)**.** *The class of Iterated Meta-Fuzzy Graphs of depth* 1 *coincides with the class of Meta-Fuzzy Graphs over* $(\mathsf{FG}, \mathcal{R}^{(0)})$.

Proof. By definition, $\mathsf{FG}^{(1)}$ is precisely the class of Meta-Fuzzy Graphs on $\mathsf{FG}^{(0)} = \mathsf{FG}$ with witnessing family $\mathcal{R}^{(0)}$. Unfolding the definition of an Iterated Meta-Fuzzy Graph at $t = 1$ gives exactly the same constraints, hence the two classes are identical. □

Theorem 2.11 (Iterated Meta-Fuzzy Graphs generalize MetaGraph and Iterated MetaGraph)**.** *Fix a universe $\mathfrak{G}$ of finite (crisp) graphs and a nonempty family $\mathcal{R}_{\mathrm{cr}}^{(0)} \subseteq \mathcal{P}(\mathfrak{G} \times \mathfrak{G})$ of crisp relations (e.g. subgraph, homomorphism, minor, isomorphism). Let $\mathrm{IMeta}^{(t)}(\mathfrak{G}, \mathcal{R}_{\mathrm{cr}}^{(0)})$ denote the class of iterated metagraphs of depth t (directed, labeled) built from $(\mathfrak{G}, \mathcal{R}_{\mathrm{cr}}^{(0)})$ using the usual existential lifting of relations. Then for every $t \geq 0$ there exists an injective map*

$$\mathsf{E}_t : \mathrm{IMeta}^{(t)}(\mathfrak{G}, \mathcal{R}_{\mathrm{cr}}^{(0)}) \hookrightarrow \mathsf{FG}^{(t)}$$

such that:

- *for $t = 1$, E_1 identifies each MetaGraph with a* crisp *Meta-Fuzzy Graph (all memberships in $\{0, 1\}$);*
- *for all t, the underlying crisp meta-structure (vertices with positive membership; edges with positive membership) of $\mathsf{E}_t(\cdot)$ is canonically isomorphic to the given iterated metagraph.*

Consequently, Iterated Meta-Fuzzy Graphs (allowing arbitrary $[0, 1]$-memberships) strictly extend both Meta-Fuzzy Graphs (by Theorem 2.10) and Iterated MetaGraphs (by the embeddings E_t).

Proof. We construct E_t by induction on t and verify the defining inequalities numerically via min/sup.

Step 0 (crisp embedding of graphs). For a simple graph $G = (V, E)$, define $\iota(G) = (\sigma_G, \mu_G) \in \mathsf{FG}$ by

$$\sigma_G(x) = \begin{cases} 1, & x \in V, \\ 0, & \text{else}, \end{cases} \qquad \mu_G(x, y) = \begin{cases} 1, & \{x, y\} \in E, \\ 0, & \text{else}. \end{cases}$$

This yields an injective map $\iota : \mathfrak{G} \hookrightarrow \mathsf{FG}^{(0)}$. For any crisp relation $R \in \mathcal{R}_{\mathrm{cr}}^{(0)}$ define the $0/1$-valued fuzzy relation

$$R^\flat(F_1, F_2) := \begin{cases} 1, & \exists\, G_1, G_2 \in \mathfrak{G} : \ F_i = \iota(G_i) \text{ and } (G_1, G_2) \in R, \\ 0, & \text{otherwise}. \end{cases}$$

Set $\mathcal{R}^{(0)} := \{R^\flat : R \in \mathcal{R}_{\mathrm{cr}}^{(0)}\}$.

Inductive hypothesis. Assume for some $t \geq 0$ there is an injective map

$$\mathsf{E}_t : \mathrm{IMeta}^{(t)}(\mathfrak{G}, \mathcal{R}_{\mathrm{cr}}^{(0)}) \hookrightarrow \mathsf{FG}^{(t)},$$

and that for every crisp lifted relation $S \in \mathcal{R}_{\mathrm{cr}}^{(t)}$ (obtained by existential lifting at the crisp level), its fuzzy indicator $S^\flat$ belongs to $\mathcal{R}^{(t)}$ and satisfies

$$S^\flat\big(\mathsf{E}_t(X),\, \mathsf{E}_t(Y)\big) = \begin{cases} 1, & (X, Y) \in S, \\ 0, & \text{otherwise}. \end{cases}$$

Step $t \to t+1$. Let $M = (V_M, A_M, s, t, \lambda)$ be an iterated metagraph of depth $t+1$, i.e. a labeled digraph with $V_M \subseteq \mathrm{IMeta}^{(t)}$ and

$$\forall e \in A_M : \quad (\, s(e),\, t(e)\,) \in \lambda(e) \in \mathcal{R}_{\mathrm{cr}}^{(t)}.$$

Define $\mathsf{E}_{t+1}(M) = (\Sigma, \mathrm{M}) \in \mathsf{FG}^{(t+1)}$ by

$$\Sigma(X) := \mathbf{1}_{\{\mathsf{E}_t(v) \mid v \in V_M\}}(X), \qquad \mathrm{M}(X, Y) := \max_{e \in A_M} \mathbf{1}_{\{\mathsf{E}_t(s(e))\}}(X) \cdot \mathbf{1}_{\{\mathsf{E}_t(t(e))\}}(Y).$$

Then $\Sigma, \mathrm{M} \in \{0, 1\}$ and for all X, Y one has the fuzzy-graph bound

$$\mathrm{M}(X, Y) \leq \min\{\Sigma(X), \Sigma(Y)\},$$

since $\mathrm{M}(X, Y) = 1$ implies both indicators are 1. For the witnessing bound, fix $X = \mathsf{E}_t(s(e))$, $Y = \mathsf{E}_t(t(e))$ and $R := \lambda(e) \in \mathcal{R}_{\mathrm{cr}}^{(t)}$. By the inductive hypothesis,

$$R^\flat\big(X, Y\big) = 1.$$

Therefore
$$\sup_{Q\in\mathcal{R}^{(t)}} Q(X,Y) \;\geq\; R^{\flat}(X,Y) \;=\; 1 \;\geq\; \mathrm{M}(X,Y).$$

If no e connects X to Y, then $\mathrm{M}(X,Y)=0$ and the inequality is trivial. Thus (Σ,M) satisfies both constraints and is a valid Meta-Fuzzy Graph on $\mathsf{FG}^{(t)}$, i.e. $\mathsf{E}_{t+1}(M)\in\mathsf{FG}^{(t+1)}$.

Injectivity and underlying crisp structure. By construction, the support

$$\{X : \Sigma(X)=1\} \;=\; \{\mathsf{E}_t(v) \mid v\in V_M\}$$

is in bijection with V_M, and (X,Y) has $\mathrm{M}(X,Y)=1$ iff there exists e with $s(e)=v$, $t(e)=w$ and $X=\mathsf{E}_t(v)$, $Y=\mathsf{E}_t(w)$. Hence the underlying crisp meta-digraph of $\mathsf{E}_{t+1}(M)$ is canonically isomorphic to M. Distinct M give distinct (Σ,M), so E_{t+1} is injective.

Base. For $t=0$, take $\mathsf{E}_0=\iota$ and note that all verifications above reduce to checking $0/1\leq\min(0/1,0/1)$ and $0/1\leq\sup\{0,1\}$, which hold.

By induction, E_t exists for all t, proving the claim. □

2.3 Meta-DiGraph

Throughout, a (finite) *digraph* is a quadruple

$$D=(V_D,A_D,s_D,t_D),$$

where V_D is a finite vertex set, A_D is a finite arc set, and $s_D,t_D : A_D\to V_D$ give the source/target of each arc. Loops ($s_D(a)=t_D(a)$) and multiple arcs are allowed iff explicitly stated.

Definition 2.12 (Universe and relation family for digraphs)**.** Fix a nonempty universe $\mathfrak{D}$ of finite digraphs and a nonempty family of binary relations
$$\mathcal{R} \;\subseteq\; \mathcal{P}\big(\mathfrak{D}\times\mathfrak{D}\big).$$

Typical choices include:

- Sub: the (induced) subdigraph relation,
- Hom: the homomorphic reachability relation $\big(D_1,D_2\big)\in$ Hom iff there exists a digraph homomorphism $D_1\to D_2$,
- Min: the (directed) minor relation (when defined),
- Iso: isomorphism.

Definition 2.13 (Meta digraph over $(\mathfrak{D},\mathcal{R})$)**.** A *meta digraph (digraph of digraphs)* over $(\mathfrak{D},\mathcal{R})$ is a labeled digraph
$$M=(V_M,A_M,s_M,t_M,\lambda_M)$$
with
$$V_M\subseteq\mathfrak{D},\qquad s_M,t_M : A_M\to V_M,\qquad \lambda_M : A_M\to\mathcal{R},$$
subject to the *incidence constraint*

$$\forall\, e\in A_M :\quad \big(s_M(e),\,t_M(e)\big) \;\in\; \lambda_M(e). \tag{2}$$

Each vertex of M is itself a (ground-level) digraph. For $e\in A_M$ with $\lambda_M(e)=R\in\mathcal{R}$, we write

$$s_M(e)\xrightarrow{R} t_M(e)$$

and call e a *meta-arc* (of type R).

Example 2.14 (Microservice Integration as a Meta Digraph)**.** Let $\mathfrak{D}$ contain the following ground-level *service call graphs* (digraphs):

$$\begin{aligned} D_{\mathrm{O}} &= (V_{\mathrm{O}}, A_{\mathrm{O}}), \quad V_{\mathrm{O}} = \{\mathrm{web}, \mathrm{pay}\}, \quad A_{\mathrm{O}} = \{(\mathrm{web}, \mathrm{pay})\}; \\ D_{\mathrm{P}} &= (V_{\mathrm{P}}, A_{\mathrm{P}}), \quad V_{\mathrm{P}} = \{\mathrm{gw}, \mathrm{auth}\}, \quad A_{\mathrm{P}} = \{(\mathrm{gw}, \mathrm{auth})\}; \\ D_{\mathrm{A}} &= (V_{\mathrm{A}}, A_{\mathrm{A}}), \quad V_{\mathrm{A}} = \{\mathrm{etl}, \mathrm{db}\}, \quad A_{\mathrm{A}} = \{(\mathrm{etl}, \mathrm{db})\}. \end{aligned}$$

Interpretation: Orders (D_{O}) calls Payments (D_{P}) which in turn feeds Analytics (D_{A}).

Let $\mathcal{R} := \{\mathrm{Hom}\}$, where $(D_1, D_2) \in \mathrm{Hom}$ iff there exists a digraph homomorphism $h : V(D_1) \to V(D_2)$ preserving arcs.

Witness homomorphisms. Define $h_1 : V_{\mathrm{O}} \to V_{\mathrm{P}}$ by $h_1(\mathrm{web}) = \mathrm{gw}$, $h_1(\mathrm{pay}) = \mathrm{auth}$. Then $(\mathrm{web}, \mathrm{pay}) \in A_{\mathrm{O}}$ maps to $(\mathrm{gw}, \mathrm{auth}) \in A_{\mathrm{P}}$, hence $(D_{\mathrm{O}}, D_{\mathrm{P}}) \in \mathrm{Hom}$. Define $h_2 : V_{\mathrm{P}} \to V_{\mathrm{A}}$ by $h_2(\mathrm{gw}) = \mathrm{etl}$, $h_2(\mathrm{auth}) = \mathrm{db}$. Then $(\mathrm{gw}, \mathrm{auth}) \in A_{\mathrm{P}}$ maps to $(\mathrm{etl}, \mathrm{db}) \in A_{\mathrm{A}}$, hence $(D_{\mathrm{P}}, D_{\mathrm{A}}) \in \mathrm{Hom}$.

Meta digraph. Let

$$M = (V_M, A_M, s_M, t_M, \lambda_M), \qquad V_M = \{D_{\mathrm{O}}, D_{\mathrm{P}}, D_{\mathrm{A}}\},$$

with meta-arcs

$$e_1 : D_{\mathrm{O}} \xrightarrow{\mathrm{Hom}} D_{\mathrm{P}}, \qquad e_2 : D_{\mathrm{P}} \xrightarrow{\mathrm{Hom}} D_{\mathrm{A}},$$

i.e. $s_M(e_1) = D_{\mathrm{O}}$, $t_M(e_1) = D_{\mathrm{P}}$, $\lambda_M(e_1) = \mathrm{Hom}$, and similarly for e_2. Each e_i satisfies the incidence constraint (2) by the witnessed h_1, h_2. (Composition $h_2 \circ h_1$ also witnesses $(D_{\mathrm{O}}, D_{\mathrm{A}}) \in \mathrm{Hom}$, consistent with the meta-path $D_{\mathrm{O}} \to D_{\mathrm{P}} \to D_{\mathrm{A}}$.)

Proposition 2.15 (Symmetric relations induce undirected underlying meta structure)**.** *Suppose every $R \in \mathcal{R}$ is symmetric. Let M be a meta digraph over $(\mathfrak{D}, \mathcal{R})$. Then the underlying unlabeled multigraph obtained by identifying each pair of opposite meta-arcs is well-defined. In particular, if $e \in A_M$ with $\lambda_M(e) = R$ and R is symmetric, then*

$$\big(t_M(e), s_M(e)\big) \in R,$$

so there exists a meta-arc e' (possibly equal to e) with $s_M(e') = t_M(e)$ and $t_M(e') = s_M(e)$.

Proof. Fix $e \in A_M$. By (2), $\big(s_M(e), t_M(e)\big) \in R$ with $R = \lambda_M(e)$. Symmetry gives $\big(t_M(e), s_M(e)\big) \in R$, hence a valid reverse meta-arc with the same label exists (in a simple meta digraph it must be identical, in a multidigraph it may be distinct). □

Proposition 2.16 (Compositional soundness along labeled meta-paths)**.** *Assume $\mathcal{R}$ is closed under (relational) composition: for any $R_1, \dots, R_k \in \mathcal{R}$ there exists $R \in \mathcal{R}$ with $R \supseteq R_k \circ \cdots \circ R_1$. If $x = v_0 \xrightarrow{R_1} v_1 \xrightarrow{R_2} \cdots \xrightarrow{R_k} v_k = y$ is a labeled meta-path in M, then*

$$(x, y) \in R_k \circ \cdots \circ R_1 \subseteq R$$

for some $R \in \mathcal{R}$. In particular, reachability along meta-paths witnesses membership in (some) relation from $\mathcal{R}$ determined by composing the labels.

Proof. By (2), for each i one has $(v_{i-1}, v_i) \in R_i$. Hence $(x, y) \in R_k \circ \cdots \circ R_1$ by definition of relational composition. Closure provides $R \in \mathcal{R}$ with $R \supseteq R_k \circ \cdots \circ R_1$. □

2.4 Iterated Meta-DiGraph

An Iterated Meta-DiGraph recursively builds meta-level digraphs over directed graphs, enabling layered representations of directional structures and their relational hierarchies.

Definition 2.17 (Relation lifting for digraphs)**.** Given $\mathcal{R}$ on $\mathfrak{D}$, define its *lift* $\mathcal{R}^{\uparrow}$ on the class of finite meta digraphs over $(\mathfrak{D}, \mathcal{R})$ as

$$\mathcal{R}^{\uparrow} := \{\, R^{\uparrow} \mid R \in \mathcal{R} \,\}, \quad (M_1, M_2) \in R^{\uparrow} \iff \exists x \in V_{M_1},\ y \in V_{M_2} :\ (x, y) \in R.$$

Definition 2.18 (Iterated universes for digraphs)**.** Define recursively for $t \in \mathbb{N}_0$:

$$\mathfrak{D}^{(0)} := \mathfrak{D}, \qquad \mathcal{R}^{(0)} := \mathcal{R},$$

$$\mathfrak{D}^{(t+1)} := \Big\{ \text{all finite meta digraphs over } \big(\mathfrak{D}^{(t)}, \mathcal{R}^{(t)}\big) \Big\}, \qquad \mathcal{R}^{(t+1)} := \big(\mathcal{R}^{(t)}\big)^{\uparrow}.$$

Definition 2.19 (Iterated Meta Digraph of depth t)**.** For $t \in \mathbb{N}_0$, an *iterated meta digraph of depth t* is a labeled digraph

$$M^{(t)} = (V^{(t)}, A^{(t)}, s^{(t)}, t^{(t)}, \lambda^{(t)}),$$

with $V^{(t)} \subseteq \mathfrak{D}^{(t)}$, $\lambda^{(t)} : A^{(t)} \to \mathcal{R}^{(t)}$, and

$$\forall e \in A^{(t)} : \qquad \big(s^{(t)}(e),\, t^{(t)}(e)\big) \in \lambda^{(t)}(e). \tag{3}$$

Example 2.20 (Department-Level Adoption as an Iterated Meta Digraph)**.** Retain $\mathfrak{D}$ and $\mathcal{R} = \{\mathrm{Hom}\}$ as above. Form two level-1 *meta digraphs* representing departments:

Commerce M_{C} with vertex set $V(M_{\mathrm{C}}) = \{D_{\mathrm{O}}, D_{\mathrm{P}}\}$ and the meta-arc $D_{\mathrm{O}} \xrightarrow{\mathrm{Hom}} D_{\mathrm{P}}$ (witness h_1).

Data M_{D} with vertex set $V(M_{\mathrm{D}}) = \{D_{\mathrm{A}}\}$ (no internal meta-arc needed).

Define the lifted relation family $\mathcal{R}^{\uparrow}$ by

$$(M_1, M_2) \in \mathrm{Hom}^{\uparrow} \iff \exists x \in V(M_1),\ y \in V(M_2) :\ (x, y) \in \mathrm{Hom}.$$

Since $(D_{\mathrm{P}}, D_{\mathrm{A}}) \in \mathrm{Hom}$ (witness h_2) with $D_{\mathrm{P}} \in V(M_{\mathrm{C}})$ and $D_{\mathrm{A}} \in V(M_{\mathrm{D}})$, we have

$$(M_{\mathrm{C}}, M_{\mathrm{D}}) \in \mathrm{Hom}^{\uparrow}.$$

Iterated meta digraph (depth 1*).* Let

$$M^{(1)} = \big(V^{(1)}, A^{(1)}, s^{(1)}, t^{(1)}, \lambda^{(1)}\big), \qquad V^{(1)} = \{M_{\mathrm{C}}, M_{\mathrm{D}}\},$$

with the meta-arc

$$E :\ M_{\mathrm{C}} \xrightarrow{\mathrm{Hom}^{\uparrow}} M_{\mathrm{D}}.$$

By construction $\big(s^{(1)}(E), t^{(1)}(E)\big) = (M_{\mathrm{C}}, M_{\mathrm{D}}) \in \mathrm{Hom}^{\uparrow} = \lambda^{(1)}(E)$, so the incidence constraint (3) holds. Interpretation: because a concrete homomorphism maps Payments to Analytics at the service level, a *department-level* integration arrow exists from Commerce to Data at the iterated meta level.

Theorem 2.21 (Iterated Meta Digraph generalizes Meta Digraph)**.** *Depth* 1 *coincides with the meta level:*

$$\Big\{\textit{meta digraphs over } (\mathfrak{D}, \mathcal{R})\Big\} = \Big\{\textit{iterated meta digraphs of depth } 1\Big\}.$$

Hence the class of Iterated Meta Digraphs (depth $t \geq 1$) strictly extends the class of Meta Digraphs.

Proof. By Definition 2.18, $\mathfrak{D}^{(1)}$ is *by construction* the class of all finite meta digraphs over $(\mathfrak{D}, \mathcal{R})$, and $\mathcal{R}^{(1)} = \mathcal{R}^{\uparrow}$. Unfolding Definition 2.19 at $t = 1$ gives: an object

$$M^{(1)} = (V^{(1)}, A^{(1)}, s^{(1)}, t^{(1)}, \lambda^{(1)})$$

with $V^{(1)} \subseteq \mathfrak{D}^{(1)}$ and $\lambda^{(1)} : A^{(1)} \to \mathcal{R}^{(1)}$ satisfying (3). But this is *exactly* the definition of a meta digraph taken one level up (its vertices are digraphs-at-level 1, i.e., ordinary meta digraphs). Therefore "depth 1 iterated meta digraph" and "meta digraph" coincide after one expansion. Consequently, allowing $t \geq 1$ yields a hierarchy that contains the t=1 case as a special case, so Iterated Meta Digraphs generalize Meta Digraphs. □

2.5 Meta-MultiGraph

A Meta-MultiGraph treats multigraphs as vertices, with labeled meta-edges capturing relations like subgraph, isomorphism, or homomorphism among multigraphs.

Definition 2.22 (Universe and relation family for multigraphs)**.** Fix a nonempty universe $\mathfrak{M}$ of finite multigraphs and a nonempty family of binary relations

$$\mathcal{S} \subseteq \mathcal{P}(\mathfrak{M} \times \mathfrak{M}).$$

Typical relations include:

- $\mathrm{Sub}^{\times}$: multiplicity-respecting subgraph relation ($H \subseteq G$ and each edge multiplicity in H is $\leq$ that in G),
- $\mathrm{Hom}^{\times}$: existence of a multigraph homomorphism,
- $\mathrm{Iso}^{\times}$: multigraph isomorphism.

Definition 2.23 (Meta multigraph over $(\mathfrak{M}, \mathcal{S})$)**.** A *meta multigraph (multigraph of multigraphs)* over $(\mathfrak{M}, \mathcal{S})$ is a labeled multigraph

$$\mathbb{M} = (V_{\mathbb{M}}, E_{\mathbb{M}}, \varphi_{\mathbb{M}}, \lambda_{\mathbb{M}})$$

with

$$V_{\mathbb{M}} \subseteq \mathfrak{M}, \qquad \varphi_{\mathbb{M}} : E_{\mathbb{M}} \to \big\{\{X, Y\} \mid X, Y \in V_{\mathbb{M}}\big\}, \qquad \lambda_{\mathbb{M}} : E_{\mathbb{M}} \to \mathcal{S},$$

subject to the *incidence constraint*

$$\forall\, e \in E_{\mathbb{M}} : \quad \varphi_{\mathbb{M}}(e) = \{X, Y\} \;\Rightarrow\; (X, Y) \in \lambda_{\mathbb{M}}(e) \text{ and } (Y, X) \in \lambda_{\mathbb{M}}(e). \tag{4}$$

Thus each meta-edge e (possibly a loop when $X = Y$) connects two ground-level multigraphs $X, Y \in V_{\mathbb{M}}$ and is labeled by some $S = \lambda_{\mathbb{M}}(e) \in \mathcal{S}$ that relates X and Y. Parallel meta-edges $\{e_1, \dots\}$ between the same pair $\{X, Y\}$ are permitted and may carry distinct labels from $\mathcal{S}$.

Example 2.24 (Airline Route Networks as a Meta-Multigraph)**.** Let $\mathfrak{M}$ be multigraphs whose vertices are airports and whose parallel edges encode the *number of daily direct flights* on a city pair. Consider three carriers on the airport set {HND, KIX, CTS}:

Carrier J: G_J has 8 parallel edges on $\{\mathrm{HND}, \mathrm{CTS}\}$ and 6 on $\{\mathrm{HND}, \mathrm{KIX}\}$.
Carrier N: G_N has 5 on $\{\mathrm{HND}, \mathrm{KIX}\}$ and 3 on $\{\mathrm{KIX}, \mathrm{CTS}\}$.
Carrier F: G_F has 2 on $\{\mathrm{HND}, \mathrm{CTS}\}$.

Define two *symmetric* relations on $\mathfrak{M}$:

$$S^{\mathrm{Jacc}}_{\geq\theta}(X, Y) := \mathbf{1}\left\{\frac{|E^*_X \cap E^*_Y|}{|E^*_X \cup E^*_Y|} \geq \theta\right\}, \qquad S^{\mathrm{code}}(X, Y) := \mathbf{1}\{\text{codeshare agreement between the carriers of } X, Y\},$$

where E^* denotes the *underlying simple* edge set (multiplicities ignored). Fix $\theta = \frac{1}{3}$ and assume

$$S^{\mathrm{code}}(G_J, G_N) = 1, \qquad S^{\mathrm{code}}(G_J, G_F) = 0, \qquad S^{\mathrm{code}}(G_N, G_F) = 0.$$

Compute Jaccard indices:

$$E^*_{G_J} = \{\{\mathrm{HND}, \mathrm{KIX}\}, \{\mathrm{HND}, \mathrm{CTS}\}\}, \quad E^*_{G_N} = \{\{\mathrm{HND}, \mathrm{KIX}\}, \{\mathrm{KIX}, \mathrm{CTS}\}\}, \quad E^*_{G_F} = \{\{\mathrm{HND}, \mathrm{CTS}\}\}.$$

Hence

$$J(G_J, G_N) = \frac{1}{3}\ (\Rightarrow S^{\mathrm{Jacc}}_{\geq 1/3} = 1), \quad J(G_J, G_F) = \frac{1}{2}\ (\Rightarrow 1), \quad J(G_N, G_F) = 0\ (\Rightarrow 0).$$

Form the meta multigraph

$$\mathbb{M} = (V_{\mathbb{M}}, E_{\mathbb{M}}, \varphi_{\mathbb{M}}, \lambda_{\mathbb{M}}), \qquad V_{\mathbb{M}} = \{G_J, G_N, G_F\},$$

with (parallel) meta-edges over the same pair:

$$e_1 : \varphi_{\mathbb{M}}(e_1) = \{G_J, G_N\},\ \lambda_{\mathbb{M}}(e_1) = S^{\mathrm{Jacc}}_{\geq 1/3},$$
$$e_2 : \varphi_{\mathbb{M}}(e_2) = \{G_J, G_N\},\ \lambda_{\mathbb{M}}(e_2) = S^{\mathrm{code}},$$
$$e_3 : \varphi_{\mathbb{M}}(e_3) = \{G_J, G_F\},\ \lambda_{\mathbb{M}}(e_3) = S^{\mathrm{Jacc}}_{\geq 1/3}.$$

All labels are symmetric, so the incidence constraint (4) holds: for each e_i, $(X,Y) \in \lambda_{\mathbb{M}}(e_i)$ and $(Y,X) \in \lambda_{\mathbb{M}}(e_i)$. This concrete model exhibits the *multigraph* nature at the meta-level via the two parallel meta-edges e_1, e_2 between G_J and G_N, capturing distinct reasons (overlap vs. codeshare) for relating the two carriers.

Remark 2.25 (Directed variant)**.** A *directed* meta multigraph is obtained by replacing unordered pairs $\{X,Y\}$ with ordered pairs (X,Y) and dropping the symmetry requirement in (4).

Proposition 2.26 (Projection and forgetful functors)**.** *There is a natural forgetful projection*

$$\Pi :\ \mathbb{M} \longmapsto \big(V_{\mathbb{M}},\ \{\{X,Y\} \mid \exists e \in E_{\mathbb{M}} \text{ with } \varphi_{\mathbb{M}}(e) = \{X,Y\}\}\big),$$

which sends a meta multigraph to its unlabeled, underlying simple graph on meta-vertices. Moreover, the label map $\lambda_{\mathbb{M}}$ factors through Π by aggregating labels on parallel meta-edges:

$$\Lambda :\ \{X,Y\} \longmapsto \{\ \lambda_{\mathbb{M}}(e)\ :\ e \in E_{\mathbb{M}},\ \varphi_{\mathbb{M}}(e) = \{X,Y\}\ \} \ \subseteq\ \mathcal{S}.$$

Proof. By definition of $\varphi_{\mathbb{M}}$, each meta-edge determines an unordered pair $\{X,Y\}$, giving the edge set of $\Pi(\mathbb{M})$. Parallel meta-edges over the same pair are collapsed; collecting their labels defines Λ. □

2.6 Iterated Meta-MultiGraph

Fix a nonempty universe $\mathfrak{M}$ of finite multigraphs and a nonempty family $\mathcal{S}$ of *symmetric* binary relations on $\mathfrak{M}$ (e.g. multigraph isomorphism, "mutual" similarity, etc.).

Definition 2.27 (Relation lifting for multigraphs)**.** Given $\mathcal{S}$ on $\mathfrak{M}$, define its *lift* $\mathcal{S}^{\uparrow}$ on the class of finite meta multigraphs over $(\mathfrak{M}, \mathcal{S})$ by

$$\mathcal{S}^{\uparrow} := \{\, S^{\uparrow} \mid S \in \mathcal{S}\,\}, \quad (\mathbb{M}_1, \mathbb{M}_2) \in S^{\uparrow} \iff \exists\, X \in V_{\mathbb{M}_1},\, Y \in V_{\mathbb{M}_2} :\ (X,Y) \in S.$$

If S is symmetric, then $S^{\uparrow}$ is symmetric as well.

Definition 2.28 (Iterated universes for multigraphs)**.** Define recursively for $t \in \mathbb{N}_0$:

$$\mathfrak{M}^{(0)} := \mathfrak{M}, \qquad \mathcal{S}^{(0)} := \mathcal{S},$$
$$\mathfrak{M}^{(t+1)} := \Big\{ \text{all finite meta multigraphs over } \big(\mathfrak{M}^{(t)}, \mathcal{S}^{(t)}\big) \Big\}, \qquad \mathcal{S}^{(t+1)} := \big(\mathcal{S}^{(t)}\big)^{\uparrow}.$$

Definition 2.29 (Iterated Meta Multigraph of depth t)**.** For $t \in \mathbb{N}_0$, an *iterated meta multigraph of depth t* is a labeled multigraph

$$\mathbb{M}^{(t)} = (V^{(t)}, E^{(t)}, \varphi^{(t)}, \lambda^{(t)}),$$

with $V^{(t)} \subseteq \mathfrak{M}^{(t)}$, $\lambda^{(t)} : E^{(t)} \to \mathcal{S}^{(t)}$, and

$$\forall e \in E^{(t)} : \qquad \varphi^{(t)}(e) = \{X,Y\} \implies (X,Y) \in \lambda^{(t)}(e). \tag{5}$$

Example 2.30 (University Pathways as an Iterated Meta Digraph)**.** Let $\mathfrak{D}$ be *course-prerequisite digraphs*. Consider three curricula, each a single prerequisite arrow:

$$D_{\mathrm{Calc}} = \big(\{C_1, C_2\}, \{(C_1, C_2)\}\big), \quad D_{\mathrm{Math}} = \big(\{M_1, M_2\}, \{(M_1, M_2)\}\big),$$
$$D_{\mathrm{Phys}} = \big(\{P_1, P_2\}, \{(P_1, P_2)\}\big).$$

Let $\mathcal{R} = \{\mathrm{Hom}\}$, where $(D_1, D_2) \in \mathrm{Hom}$ iff there exists a digraph homomorphism $h : V(D_1) \to V(D_2)$ preserving arcs. Define

$$h_{CM}(C_1) = M_1,\ h_{CM}(C_2) = M_2 \quad \Rightarrow \quad (D_{\mathrm{Calc}}, D_{\mathrm{Math}}) \in \mathrm{Hom},$$

$$h_{MP}(M_1) = P_1,\ h_{MP}(M_2) = P_2 \quad \Rightarrow \quad (D_{\text{Math}}, D_{\text{Phys}}) \in \text{Hom}.$$

Level 1 (Meta digraphs of departments). Let

$$M_{\text{Sci}} = (V_{\text{Sci}}, A_{\text{Sci}}, s, t, \lambda), \quad V_{\text{Sci}} = \{D_{\text{Calc}}, D_{\text{Math}}\}, \quad D_{\text{Calc}} \xrightarrow{\text{Hom}} D_{\text{Math}},$$

and

$$M_{\text{Eng}} = (V_{\text{Eng}}, A_{\text{Eng}}, s, t, \lambda), \quad V_{\text{Eng}} = \{D_{\text{Phys}}\} \text{ (no internal meta-arc)}.$$

Lifted relation. Define $\text{Hom}^{\uparrow}$ on level-1 meta objects by

$$(M_1, M_2) \in \text{Hom}^{\uparrow} \iff \exists x \in V_{M_1},\ y \in V_{M_2} :\ (x, y) \in \text{Hom}.$$

Because $(D_{\text{Math}}, D_{\text{Phys}}) \in \text{Hom}$ with $D_{\text{Math}} \in V_{\text{Sci}}$ and $D_{\text{Phys}} \in V_{\text{Eng}}$, we have

$$(M_{\text{Sci}}, M_{\text{Eng}}) \in \text{Hom}^{\uparrow}.$$

Iterated meta digraph (depth 1). Set

$$M^{(1)} = \big(\{M_{\text{Sci}}, M_{\text{Eng}}\}, \{E\}, s^{(1)}, t^{(1)}, \lambda^{(1)}\big), \qquad E :\ M_{\text{Sci}} \xrightarrow{\text{Hom}^{\uparrow}} M_{\text{Eng}}.$$

The incidence condition (3) holds since $\big(s^{(1)}(E), t^{(1)}(E)\big) = (M_{\text{Sci}}, M_{\text{Eng}}) \in \text{Hom}^{\uparrow}$. Interpretation: a concrete prerequisite mapping from Mathematics to Physics at the course level induces a *department-to-department* articulation arrow at the iterated meta level.

Theorem 2.31 (Iterated Meta Multigraph generalizes Meta Multigraph)**.** *Depth* 1 *coincides with the meta level:*

$$\Big\{\textit{meta multigraphs over } (\mathfrak{M}, \mathcal{S})\Big\} = \Big\{\textit{iterated meta multigraphs of depth } 1\Big\}.$$

Hence the class of Iterated Meta Multigraphs (depth $t \geq 1$) strictly extends the class of Meta Multigraphs.

Proof. By Definition 2.28, $\mathfrak{M}^{(1)}$ equals the class of all finite meta multigraphs over $(\mathfrak{M}, \mathcal{S})$, and $\mathcal{S}^{(1)} = \mathcal{S}^{\uparrow}$. Unfolding Definition 2.29 at $t = 1$ gives: an object

$$\mathbb{M}^{(1)} = (V^{(1)}, E^{(1)}, \varphi^{(1)}, \lambda^{(1)})$$

with $V^{(1)} \subseteq \mathfrak{M}^{(1)}$ and $\lambda^{(1)} : E^{(1)} \to \mathcal{S}^{(1)}$ satisfying (5). This is precisely the notion of a meta multigraph one level up (its vertices are multigraphs-at-level 1, i.e., ordinary meta multigraphs). Therefore depth 1 iterated meta multigraphs coincide (after one expansion) with meta multigraphs. Allowing $t \geq 1$ yields a hierarchy that contains the t=1 case, so iterated meta multigraphs generalize meta multigraphs. □

2.7 Meta-Bidigraph

A Meta-Bidigraph models relations among bidigraphs, with meta-edges labeled by corridor relations, ensuring endpoint orientations match operational requirements.

Definition 2.32 (Meta-Bidigraph over $(\mathfrak{B}, \mathcal{R})$)**.** Let $\mathcal{R} \subseteq \mathcal{P}(\mathfrak{B} \times \mathfrak{B})$ be a nonempty family of binary relations on $\mathfrak{B}$. A *Meta-Bidigraph* is a labeled bidigraph

$$\mathbb{M} := (\, V_{\mathbb{M}},\, E_{\mathbb{M}},\, \partial_{\mathbb{M}},\, \tau_{\mathbb{M}},\, \lambda_{\mathbb{M}} \,)$$

consisting of:

- $V_{\mathbb{M}} \subseteq \mathfrak{B}$ (meta-vertices are bidigraphs),
- an incidence map $\partial_{\mathbb{M}} : E_{\mathbb{M}} \to \big\{\{X, Y\} \subseteq V_{\mathbb{M}} \mid X \neq Y\big\}$,

- a meta bidirection $\tau_{\mathbb{M}} : V_{\mathbb{M}} \times E_{\mathbb{M}} \to \{-1, 0, 1\}$ with $\tau_{\mathbb{M}}(v, e) = 0$ iff $v \notin \partial_{\mathbb{M}}(e)$ and otherwise $\tau_{\mathbb{M}}(v, e) \in \{-1, +1\}$,
- a label map $\lambda_{\mathbb{M}} : E_{\mathbb{M}} \to \mathcal{R}$,

subject to the *witnessing constraint*: for each $e \in E_{\mathbb{M}}$ with $\partial_{\mathbb{M}}(e) = \{X, Y\}$, define the *demand set*

$$\mathrm{Dem}(e) := \begin{cases} \{(X, Y)\}, & \text{if } \big(\tau_{\mathbb{M}}(X, e), \tau_{\mathbb{M}}(Y, e)\big) = (-1, +1), \\ \{(Y, X)\}, & \text{if } \big(\tau_{\mathbb{M}}(X, e), \tau_{\mathbb{M}}(Y, e)\big) = (+1, -1), \\ \{(X, Y), (Y, X)\}, & \text{if } \big(\tau_{\mathbb{M}}(X, e), \tau_{\mathbb{M}}(Y, e)\big) = (+1, +1), \\ \emptyset, & \text{if } \big(\tau_{\mathbb{M}}(X, e), \tau_{\mathbb{M}}(Y, e)\big) = (-1, -1). \end{cases}$$

Then one requires $\mathrm{Dem}(e) \subseteq \lambda_{\mathbb{M}}(e)$ (i.e. every oriented requirement encoded by the local signs is witnessed by the chosen relation label).

Example 2.33 (City Logistics Corridors as a Meta-Bidigraph)**. Ground level (bidigraphs).** Let $\mathfrak{B}$ contain district-level curbside logistics networks modeled as bidigraphs B_T, B_O, B_N for *Tokyo*, *Osaka*, *Nagoya*. Each $B = (G, \tau)$ has intersections as vertices and street segments as edges. At each incident pair (v, e), the local sign $\tau(v, e) \in \{-1, +1\}$ indicates whether the segment e is configured *away from* v (-1, dispatch/loading outflow) or *towards* v ($+1$, arrival/unloading inflow). (Non-incidences get 0.)

Assume there is a long-haul corridor between the two districts B_T and B_O realized by a street segment e_{TO} whose endpoints lie on the district boundaries (abstracted at the meta level). Operationally, freight travels from B_T to B_O, so we record at the meta level the local signs

$$\big(\tau_{\mathbb{M}}(B_T, e_{TO}),\ \tau_{\mathbb{M}}(B_O, e_{TO})\big) = (-1, +1).$$

Similarly, suppose B_O and B_N operate a *balanced exchange* corridor (temporary shuttle both ways), so for a meta edge e_{ON} we set

$$\big(\tau_{\mathbb{M}}(B_O, e_{ON}),\ \tau_{\mathbb{M}}(B_N, e_{ON})\big) = (+1, +1).$$

Relation family. Let $\mathcal{R} = \{\mathrm{Ship}, \mathrm{Sync}\}$, where

$$(X, Y) \in \mathrm{Ship} \iff \text{there exists at least one operational outbound lane in } X \text{ and inbound lane in } Y \text{ along the corridor},$$

$$(X, Y) \in \mathrm{Sync} \iff (X, Y) \in \mathrm{Ship} \text{ and } (Y, X) \in \mathrm{Ship} \quad (\text{hence Sync is symmetric}).$$

Meta-Bidigraph. Define the Meta-Bidigraph

$$\mathbb{M} = \big(V_{\mathbb{M}}, E_{\mathbb{M}}, \partial_{\mathbb{M}}, \tau_{\mathbb{M}}, \lambda_{\mathbb{M}}\big), \qquad V_{\mathbb{M}} = \{B_T, B_O, B_N\},$$

with meta-edges and labels

$$\partial_{\mathbb{M}}(e_{TO}) = \{B_T, B_O\},\ \lambda_{\mathbb{M}}(e_{TO}) = \mathrm{Ship}, \qquad \partial_{\mathbb{M}}(e_{ON}) = \{B_O, B_N\},\ \lambda_{\mathbb{M}}(e_{ON}) = \mathrm{Sync}.$$

Witnessing holds:

$$\mathrm{Dem}(e_{TO}) = \{(B_T, B_O)\} \subseteq \mathrm{Ship} = \lambda_{\mathbb{M}}(e_{TO}), \qquad \mathrm{Dem}(e_{ON}) = \{(B_O, B_N), (B_N, B_O)\} \subseteq \mathrm{Sync} = \lambda_{\mathbb{M}}(e_{ON}).$$

Thus $\mathbb{M}$ is a concrete *Meta-Bidigraph* of district curb networks: meta-vertices are bidigraphs of local streets, and meta-edges encode corridor policies with endpoint-local orientations.

Remark 2.34 (Directed shadow)**.** Given a Meta-Bidigraph $\mathbb{M}$, its *directed shadow* is the labeled digraph

$$\overrightarrow{\mathbb{M}} = (V_{\mathbb{M}}, A, s, t, \Lambda), \quad A := \{\, e \in E_{\mathbb{M}} \mid \tau_{\mathbb{M}}(X, e) = -1,\ \tau_{\mathbb{M}}(Y, e) = +1 \,\},$$

with $s(e) = X$, $t(e) = Y$, and $\Lambda(e) := \lambda_{\mathbb{M}}(e)$. Thus $(-1, +1)$ at the endpoints realizes a directed meta-arc $X \to Y$.

Definition 2.35 (Simple digraph and Meta DiGraph (recall, restricted))**.** Let $\mathfrak{D}^\circ$ be the class of finite *simple* digraphs $D = (V_D, A_D)$: no loops; for distinct $u \neq v$ there is *at most one* of (u, v) or (v, u) in A_D (i.e. no 2-cycles and no parallel arcs). Let $Q \subseteq \mathcal{P}(\mathfrak{D}^\circ \times \mathfrak{D}^\circ)$ be a nonempty family of binary relations. A (labeled) *Meta DiGraph* over $(\mathfrak{D}^\circ, Q)$ is a quintuple

$$M = (V_M, A_M, s_M, t_M, \lambda_M),$$

with $V_M \subseteq \mathfrak{D}^\circ$, $s_M, t_M : A_M \to V_M$, $\lambda_M : A_M \to Q$ and the incidence constraint

$$\forall a \in A_M : \quad (s_M(a), t_M(a)) \in \lambda_M(a).$$

Definition 2.36 (Embedding $\Phi : \mathfrak{D}^\circ \hookrightarrow \mathfrak{B}$)**.** For $D = (V_D, A_D) \in \mathfrak{D}^\circ$, define the bidigraph $\Phi(D) = (G, \tau)$ as follows:

$$G = (V_D, E), \qquad E = \big\{\{u, v\} \subseteq V_D \mid (u, v) \in A_D \text{ or } (v, u) \in A_D\big\}.$$

For $e = \{u, v\} \in E$ set

$$(\tau(u, e), \tau(v, e)) := \begin{cases} (-1, +1), & \text{if } (u, v) \in A_D, \\ (+1, -1), & \text{if } (v, u) \in A_D. \end{cases}$$

This is well-defined because $\mathfrak{D}^\circ$ forbids 2-cycles, so at most one of (u, v), (v, u) exists.

Definition 2.37 (Relation transfer $Q \to \mathcal{R}$)**.** Given $Q \subseteq \mathcal{P}(\mathfrak{D}^\circ \times \mathfrak{D}^\circ)$, define

$$\mathcal{R} := \Big\{ R^\flat \subseteq \mathfrak{B} \times \mathfrak{B} \;\Big|\; R \in Q \text{ and } R^\flat\big(\Phi(D_1), \Phi(D_2)\big) \iff (D_1, D_2) \in R \Big\}.$$

Thus each crisp relation R on digraphs is transported to a relation $R^\flat$ on the image bidigraphs.

Theorem 2.38 (Meta-Bidigraph generalizes Meta DiGraph)**.** *Let $M = (V_M, A_M, s_M, t_M, \lambda_M)$ be a Meta DiGraph over $(\mathfrak{D}^\circ, Q)$. Define a Meta-Bidigraph*

$$\mathbb{M} := \big(V_{\mathbb{M}}, E_{\mathbb{M}}, \partial_{\mathbb{M}}, \tau_{\mathbb{M}}, \lambda_{\mathbb{M}}\big)$$

over $(\mathfrak{B}, \mathcal{R})$ by

$$V_{\mathbb{M}} := \{\Phi(D) \mid D \in V_M\}, \qquad E_{\mathbb{M}} := A_M, \qquad \partial_{\mathbb{M}}(a) := \big\{\Phi(s_M(a)),\, \Phi(t_M(a))\big\},$$

$$\tau_{\mathbb{M}}\big(\Phi(s_M(a)), a\big) = -1, \quad \tau_{\mathbb{M}}\big(\Phi(t_M(a)), a\big) = +1, \quad \lambda_{\mathbb{M}}(a) := \lambda_M(a)^\flat.$$

Then $\mathbb{M}$ is a Meta-Bidigraph, and its directed shadow $\overrightarrow{\mathbb{M}}$ is canonically isomorphic to M.

Proof. (1) *Well-defined bidirection.* For each $a \in A_M$, the edge a is incident exactly with the two meta-vertices $\Phi(s_M(a))$ and $\Phi(t_M(a))$. By definition $\tau_{\mathbb{M}}(\cdot, a) \in \{-1, +1\}$ on these two vertices and 0 elsewhere, hence it is a valid bidirection.

(2) *Witnessing constraint.* Let $a \in E_{\mathbb{M}}$ and write $X = \Phi(s_M(a)), Y = \Phi(t_M(a))$. Then $\big(\tau_{\mathbb{M}}(X, a), \tau_{\mathbb{M}}(Y, a)\big) = (-1, +1)$, so $\mathrm{Dem}(a) = \{(X, Y)\}$, which must be contained in $\lambda_{\mathbb{M}}(a)$. Since $\lambda_{\mathbb{M}}(a) = \lambda_M(a)^\flat$ and $(s_M(a), t_M(a)) \in \lambda_M(a)$ by the Meta DiGraph incidence, we get $(X, Y) \in \lambda_M(a)^\flat = \lambda_{\mathbb{M}}(a)$ by the definition of $R^\flat$. Thus the witnessing constraint holds for every a.

(3) *Directed shadow.* By construction, $a \in A$ (the shadow arcs) iff $\big(\tau_{\mathbb{M}}(X, a), \tau_{\mathbb{M}}(Y, a)\big) = (-1, +1)$, with $s(a) = X$ and $t(a) = Y$. The map

$$\varphi : V_M \to V_{\mathbb{M}}, \quad \varphi(D) := \Phi(D),$$

is a bijection (because Φ is injective); and $a \mapsto a$ identifies A_M with A preserving source, target, and labels ($\lambda_M(a)$ corresponds to $\lambda_{\mathbb{M}}(a) = \lambda_M(a)^\flat$). Hence $\overrightarrow{\mathbb{M}}$ is canonically isomorphic to M.

Therefore $\mathbb{M}$ is a Meta-Bidigraph whose directed shadow recovers the given Meta DiGraph, proving that Meta-Bidigraphs generalize Meta DiGraphs (on the natural simple-digraph domain $\mathfrak{D}^\circ$). $\square$

2.8 Iterated Meta-Bidigraph

An Iterated Meta-Bidigraph recursively builds meta-levels of bidigraphs, capturing hierarchical corridor structures where alliances of bidigraphs interact through lifted relations.

Definition 2.39 (Iterated universes and lifted relations)**.** Fix a nonempty universe $\mathfrak{B}$ of finite bidigraphs and a nonempty family $\mathcal{R}^{(0)} \subseteq \mathcal{P}(\mathfrak{B} \times \mathfrak{B})$ of binary relations on $\mathfrak{B}$. Define recursively for $t \in \mathbb{N}_0$:

$$\mathfrak{B}^{(0)} := \mathfrak{B}, \qquad \mathfrak{B}^{(t+1)} := \Big\{ \text{all Meta-Bidigraphs over } \big(\mathfrak{B}^{(t)}, \mathcal{R}^{(t)}\big) \Big\},$$

$$\mathcal{R}^{(t+1)} \;:=\; \big\{ R^{\uparrow} \;\big|\; R \in \mathcal{R}^{(t)} \big\}, \quad \text{where } \; (X, Y) \in R^{\uparrow} \iff \exists x \in V_X,\; y \in V_Y :\; (x, y) \in R.$$

(Here V_X denotes the meta-vertex set of $X \in \mathfrak{B}^{(t+1)}$.)

Definition 2.40 (Iterated Meta-Bidigraph of depth t)**.** For $t \in \mathbb{N}_0$, any object $\mathbb{M}^{(t)} \in \mathfrak{B}^{(t)}$ is called an *Iterated Meta-Bidigraph of depth t*. Equivalently, $\mathbb{M}^{(t+1)}$ is a Meta-Bidigraph whose meta-vertices are elements of $\mathfrak{B}^{(t)}$ and whose labels lie in $\mathcal{R}^{(t)}$, with the (previously fixed) Meta-Bidigraph witnessing constraint enforced at level $t{+}1$.

Example 2.41 (Regional Alliances as an Iterated Meta-Bidigraph)**. Level** 0 **(bidigraphs and relations).** Keep B_T, B_O, B_N and $\mathcal{R}^{(0)} = \{\text{Ship}, \text{Sync}\}$ as in the previous example, so that, in particular, $(B_T, B_O) \in \text{Ship}$ and $(B_O, B_N), (B_N, B_O) \in \text{Ship}$ (hence $(B_O, B_N) \in \text{Sync}$).

Level 1 **(Meta-Bidigraphs of alliances).** Form two *Meta-Bidigraph* alliances:

$$\mathbb{E} :\; V_{\mathbb{E}} = \{B_T, B_O\} \quad \text{(East Alliance)}, \qquad \mathbb{W} :\; V_{\mathbb{W}} = \{B_N\} \quad \text{(West Alliance)},$$

with their internal meta-edges inherited (e.g., $\{B_T, B_O\}$ labeled Ship, signs $(-1, +1)$ as above).

Lifted relations to level 1**.** By the iterated construction, define

$$\mathcal{R}^{(1)} = \{R^{\uparrow} \mid R \in \mathcal{R}^{(0)}\}, \qquad (X, Y) \in R^{\uparrow} \iff \exists x \in V_X,\; y \in V_Y :\; (x, y) \in R.$$

Because $(B_O, B_N) \in \text{Ship}$ with $B_O \in V_{\mathbb{E}}$ and $B_N \in V_{\mathbb{W}}$, we have

$$(\mathbb{E}, \mathbb{W}) \in \text{Ship}^{\uparrow}.$$

Iterated Meta-Bidigraph (depth 1**).** Create a level-2 object whose meta-vertices are the *level-1 Meta-Bidigraphs* $\mathbb{E}$ and $\mathbb{W}$:

$$\mathbb{M}^{(1)} = \big(V^{(1)}, E^{(1)}, \partial^{(1)}, \tau^{(1)}, \lambda^{(1)}\big), \qquad V^{(1)} = \{\mathbb{E}, \mathbb{W}\}.$$

Add a meta-edge $e^{(1)}$ with

$$\partial^{(1)}(e^{(1)}) = \{\mathbb{E}, \mathbb{W}\}, \quad \big(\tau^{(1)}(\mathbb{E}, e^{(1)}), \tau^{(1)}(\mathbb{W}, e^{(1)})\big) = (-1, +1), \quad \lambda^{(1)}(e^{(1)}) = \text{Ship}^{\uparrow}.$$

Witnessing at level 1 holds since

$$\text{Dem}(e^{(1)}) = \{(\mathbb{E}, \mathbb{W})\} \subseteq \text{Ship}^{\uparrow} = \lambda^{(1)}(e^{(1)}),$$

because it is witnessed by the base pair $(B_O, B_N) \in \text{Ship}$. Interpretation: a concrete corridor from Osaka to Nagoya at the district level induces an *alliance-level* oriented supply relation from the East to the West, encoded as a meta-edge in the *Iterated Meta-Bidigraph*.

Theorem 2.42 (Depth 1 recovers Meta-Bidigraphs)**.** *$\mathfrak{B}^{(1)}$ is precisely the class of Meta-Bidigraphs over $(\mathfrak{B}^{(0)}, \mathcal{R}^{(0)})$. Hence Iterated Meta-Bidigraphs (with arbitrary depth $t \geq 1$) generalize Meta-Bidigraphs.*

Proof. By definition, $\mathfrak{B}^{(1)} := \{\text{all Meta-Bidigraphs over } (\mathfrak{B}^{(0)}, \mathcal{R}^{(0)})\}$. Thus the statement is immediate from the recursive construction. □

Theorem 2.43 (Iterated Meta-Bidigraphs generalize Iterated Meta DiGraphs)**.** *Let $\mathfrak{D}^{(0)}$ be a universe of finite simple digraphs and $Q^{(0)} \subseteq \mathcal{P}\big(\mathfrak{D}^{(0)} \times \mathfrak{D}^{(0)}\big)$ a nonempty family of binary relations. Let* IMetaD$^{(t)}$ *denote the class of Iterated Meta DiGraphs of depth t over $\big(\mathfrak{D}^{(0)}, Q^{(0)}\big)$, with the same existential vertex-witness lifting used above to define $Q^{(t+1)}$ from $Q^{(t)}$. Assume $\mathcal{R}^{(0)}$ is chosen as the transport of $Q^{(0)}$ along a fixed injective base embedding $\Phi : \mathfrak{D}^{(0)} \hookrightarrow \mathfrak{B}^{(0)}$ (the standard "arc $\leftrightarrow$ signed end" translation), i.e.*

$$R^{\flat}\big(\Phi(D_1), \Phi(D_2)\big) \iff (D_1, D_2) \in R \quad \textit{for } R \in Q^{(0)},$$

and set $\mathcal{R}^{(0)} := \{R^{\flat} : R \in Q^{(0)}\}$. Then, for every $t \geq 0$, there exists an injective map

$$\mathsf{E}_t : \text{IMetaD}^{(t)} \hookrightarrow \mathfrak{B}^{(t)}$$

such that the directed shadow of $\mathsf{E}_t(M)$ is canonically isomorphic to M for all $M \in$ IMetaD$^{(t)}$*. Consequently, Iterated Meta-Bidigraphs strictly extend Iterated Meta DiGraphs.*

Proof. We proceed by induction on t.

Base case $t = 0$. Set $\mathsf{E}_0 := \Phi : \mathfrak{D}^{(0)} \hookrightarrow \mathfrak{B}^{(0)}$, which is injective by construction.

Inductive step $t \to t{+}1$. Assume E_t : IMetaD$^{(t)} \hookrightarrow \mathfrak{B}^{(t)}$ injective and that for every $S \in Q^{(t)}$ its transport $S^{\flat} \in \mathcal{R}^{(t)}$ satisfies

$$S^{\flat}\big(\mathsf{E}_t(X),\ \mathsf{E}_t(Y)\big) \iff (X, Y) \in S \qquad \text{for all } X, Y \in \text{IMetaD}^{(t)}.$$

Let $M = (V_M, A_M, s_M, t_M, \lambda_M) \in$ IMetaD$^{(t+1)}$. Define $\mathsf{E}_{t+1}(M)$ to be the Meta-Bidigraph

$$\big(V_{\mathbb{M}}, E_{\mathbb{M}}, \partial_{\mathbb{M}}, \tau_{\mathbb{M}}, \lambda_{\mathbb{M}}\big)$$

with

$$V_{\mathbb{M}} := \{\ \mathsf{E}_t(v) \mid v \in V_M\ \} \ \subseteq\ \mathfrak{B}^{(t)}, \quad E_{\mathbb{M}} := A_M, \quad \partial_{\mathbb{M}}(a) := \big\{\mathsf{E}_t(s_M(a)),\ \mathsf{E}_t(t_M(a))\big\},$$
$$\tau_{\mathbb{M}}\big(\mathsf{E}_t(s_M(a)), a\big) = -1, \qquad \tau_{\mathbb{M}}\big(\mathsf{E}_t(t_M(a)), a\big) = +1, \qquad \lambda_{\mathbb{M}}(a) := \lambda_M(a)^{\flat} \in \mathcal{R}^{(t)}.$$

This is well-defined: the local signs are $\{-1, +1\}$ at the two incident meta-vertices and 0 elsewhere. For the witnessing constraint at level $t{+}1$, fix $a \in E_{\mathbb{M}}$ and write $X := \mathsf{E}_t(s_M(a))$, $Y := \mathsf{E}_t(t_M(a))$. By construction, the demand set is $\text{Dem}(a) = \{(X, Y)\}$. Since $(s_M(a), t_M(a)) \in \lambda_M(a) \in Q^{(t)}$, the transport property gives

$$(X, Y) \in \lambda_M(a)^{\flat} = \lambda_{\mathbb{M}}(a),$$

so $\text{Dem}(a) \subseteq \lambda_{\mathbb{M}}(a)$, as required.

Directed shadow and injectivity. The directed shadow $\overrightarrow{\mathbb{M}}$ of $\mathsf{E}_{t+1}(M)$ has vertex set $V_{\mathbb{M}}$ and arc set in bijection with A_M, preserving sources/targets and labels; hence $\overrightarrow{\mathbb{M}} \cong M$. Distinct M yield distinct $\mathsf{E}_{t+1}(M)$, so E_{t+1} is injective.

Compatibility of lifted relations. By construction of $\mathcal{R}^{(t+1)}$ from $\mathcal{R}^{(t)}$ and of $Q^{(t+1)}$ from $Q^{(t)}$ via the same existential vertex-witness rule, the transport $(\cdot)^{\flat}$ commutes with lifting:

$$\big(S^{\uparrow}\big)^{\flat} \ =\ \big(S^{\flat}\big)^{\uparrow}.$$

Thus the label family at level $t{+}1$ matches the embedding above.

This completes the induction and proves the claim. $\square$

2.9 Meta-Neutrosophic Graph

A Meta-Neutrosophic Graph organizes neutrosophic graphs as vertices, with edges reflecting truth, indeterminacy, and falsity components of higher-level relational uncertainty.

Let NG be a fixed universe of (finite, undirected, loopless unless stated) *neutrosophic graphs* G with vertex-membership $\sigma_G : V(G) \to [0,1]^3$ and edge-membership $\mu_G : V(G) \times V(G) \to [0,1]^3$, written

$$\sigma_G(v) = (T_\sigma(v), I_\sigma(v), F_\sigma(v)), \qquad \mu_G(u,v) = (T_\mu(u,v), I_\mu(u,v), F_\mu(u,v)),$$

satisfying for all distinct u, v:

$$T_\mu(u,v) \le \min\{T_\sigma(u), T_\sigma(v)\}, \quad I_\mu(u,v) \ge \max\{I_\sigma(u), I_\sigma(v)\}, \quad F_\mu(u,v) \ge \max\{F_\sigma(u), F_\sigma(v)\}.$$

Let $\mathcal{R} \subseteq ([0,1]^3)^{\mathsf{NG}\times\mathsf{NG}}$ be a nonempty family of *neutrosophic relations* $R(X,Y) = (T_R, I_R, F_R)$.

Definition 2.44 (Meta-Neutrosophic Graph)**.** A *Meta-Neutrosophic Graph* over $(\mathsf{NG}, \mathcal{R})$ is a quadruple

$$\mathbb{N} = (V, \Sigma_M, M_N, L_M)$$

with $V \subseteq \mathsf{NG}$, $\Sigma_M : V \to [0,1]^3$, $M_N : V \times V \to [0,1]^3$, and $L_M : V \times V \to \mathcal{P}_{\text{fin}}(\mathcal{R})$, such that for all $X, Y \in V$:

$$T_{M_N}(X,Y) \le \min\{T_{\Sigma_M}(X), T_{\Sigma_M}(Y)\}, \tag{6}$$
$$I_{M_N}(X,Y) \ge \max\{I_{\Sigma_M}(X), I_{\Sigma_M}(Y)\}, \tag{7}$$
$$F_{M_N}(X,Y) \ge \max\{F_{\Sigma_M}(X), F_{\Sigma_M}(Y)\}, \tag{8}$$

and the *label witnessing* bounds

$$T_{M_N}(X,Y) \le \sup_{R\in L_M(X,Y)} T_R(X,Y), \tag{9}$$
$$I_{M_N}(X,Y) \ge \inf_{R\in L_M(X,Y)} I_R(X,Y), \tag{10}$$
$$F_{M_N}(X,Y) \ge \inf_{R\in L_M(X,Y)} F_R(X,Y), \tag{11}$$

with conventions $\sup \emptyset := 0$ and $\inf \emptyset := 1$ (componentwise).

Example 2.45 (Hospital Data-Sharing under Uncertainty as a Meta-Neutrosophic Graph)**.** Let NG contain three hospital-level neutrosophic graphs A = (Tokyo General), B = (Osaka Central), C = (Nagoya West), each internally describing clinical entities (labs, encounters) with neutrosophic memberships (omitted here). Define the meta-vertex neutrosophic memberships (institution-level reliability/uncertainty):

$$\Sigma_M(A) = (0.85,\ 0.10,\ 0.05), \qquad \Sigma_M(B) = (0.75,\ 0.20,\ 0.10), \qquad \Sigma_M(C) = (0.65,\ 0.25,\ 0.15).$$

Let $\mathcal{R}$ contain two neutrosophic relations on $\mathsf{NG} \times \mathsf{NG}$:

$$R_{\text{align}}(X,Y) = (T_{\text{al}}(X,Y), I_{\text{al}}(X,Y), F_{\text{al}}(X,Y)) \quad \text{(policy \& coding alignment)},$$
$$R_{\text{share}}(X,Y) = (T_{\text{sh}}(X,Y), I_{\text{sh}}(X,Y), F_{\text{sh}}(X,Y)) \quad \text{(data-sharing readiness)}.$$

For (A, B) assume (symmetric values suffice here):

$$R_{\text{align}}(A,B) = (0.70,\ 0.25,\ 0.10), \qquad R_{\text{share}}(A,B) = (0.60,\ 0.15,\ 0.20).$$

Set the label set $L_M(A,B) = \{R_{\text{align}}, R_{\text{share}}\}$ and define the meta-edge membership

$$M_N(A,B) = (0.68,\ 0.22,\ 0.12).$$

Verification of the Meta-Neutrosophic constraints. First, vertex bounds (6)–(8):

$$\min\{T_{\Sigma_M}(A), T_{\Sigma_M}(B)\} = \min\{0.85, 0.75\} = 0.75 \ \ge\ T_{M_N}(A,B) = 0.68,$$

$$\max\{I_{\Sigma_M}(A), I_{\Sigma_M}(B)\} = \max\{0.10, 0.20\} = 0.20 \le I_{M_N}(A,B) = 0.22,$$
$$\max\{F_{\Sigma_M}(A), F_{\Sigma_M}(B)\} = \max\{0.05, 0.10\} = 0.10 \le F_{M_N}(A,B) = 0.12.$$

Next, witnessing bounds (9)–(11) with $L_M(A,B)$:

$$\sup_{R\in L_M} T_R(A,B) = \max\{0.70, 0.60\} = 0.70 \ge 0.68 = T_{M_N}(A,B),$$
$$\inf_{R\in L_M} I_R(A,B) = \min\{0.25, 0.15\} = 0.15 \le 0.22 = I_{M_N}(A,B),$$
$$\inf_{R\in L_M} F_R(A,B) = \min\{0.10, 0.20\} = 0.10 \le 0.12 = F_{M_N}(A,B).$$

All inequalities hold, so with $V = \{A, B, C\}$ and L_M defined likewise on other pairs, $\mathbb{N} = (V, \Sigma_M, M_N, L_M)$ is a concrete *Meta-Neutrosophic Graph* modeling hospital-to-hospital data-sharing under uncertainty.

Theorem 2.46 (Generalization)**.** *(i)* Neutrosophic graph $\Rightarrow$ meta. *Given $G \in \mathsf{NG}$ with vertex set $V(G)$, for each $v \in V(G)$ let A_v be the one-vertex neutrosophic graph with $\sigma_{A_v}(v) = \sigma_G(v)$ and no edges. Define*

$$V := \{A_v : v \in V(G)\}, \quad \Sigma_M(A_v) := \sigma_G(v), \quad M_N(A_u, A_v) := \mu_G(u,v),$$

and $L_M(A_u, A_v) := \{R_{u,v}\}$ where $R_{u,v}(A_u, A_v) := \mu_G(u,v)$ and $R_{u,v}(\cdot,\cdot) = (0,0,0)$ otherwise.

(ii) Meta-Fuzzy $\Rightarrow$ meta-neutrosophic. *For a Meta-Fuzzy Graph $\mathbb{M} = (\sigma_M, \mu_M, L)$ on a fuzzy-graph universe* FG*, define the lift*

$$a \mapsto a^\sharp := (a, 0, 1-a) \ \textit{on}\ [0,1].$$

Set $V := \mathrm{supp}(\sigma_M)$, $\Sigma_M(F) := \sigma_M(F)^\sharp$, $M_N(F,G) := \mu_M(F,G)^\sharp$, and $L_M(F,G) := \{R^\sharp : R \in L(F,G)\}$ where $(R^\sharp)_T = R$, $(R^\sharp)_I = 0$, $(R^\sharp)_F = 1 - R$.

The constructions in (i) and (ii) embed, respectively, neutrosophic graphs and Meta-Fuzzy Graphs into Meta-Neutrosophic Graphs. Hence Meta-Neutrosophic Graphs generalize both classes.

Proof. (i) For A_u, A_v,

$$T_{M_N} = T_{\mu_G}(u,v) \le \min\{T_{\sigma_G}(u), T_{\sigma_G}(v)\} = \min\{T_{\Sigma_M}(A_u), T_{\Sigma_M}(A_v)\},$$

and similarly (7)–(8). For witnessing, $R_{u,v}(A_u, A_v) = \mu_G(u,v)$ gives (9)–(11) with equality.

(ii) Since $\mu_M \le \min\{\sigma_M, \sigma_M\}$, we obtain

$$T_{M_N} = \mu_M \le \min\{\sigma_M, \sigma_M\} = \min\{T_{\Sigma_M}, T_{\Sigma_M}\}, \quad I_{M_N} = 0 \ge \max\{0,0\}, \quad F_{M_N} = 1-\mu_M \ge \max\{1-\sigma_M, 1-\sigma_M\}.$$

For witnessing, with $Q = R^\sharp$,

$$T_{M_N} \le \sup_{R\in L} R = \sup_{Q\in L_M} T_Q, \quad I_{M_N} = 0 \ge \inf_{Q\in L_M} I_Q = 0, \quad F_{M_N} = 1-\mu_M \ge \inf_{R\in L}(1-R) = \inf_{Q\in L_M} F_Q.$$

□

2.10 Iterated Meta-Neutrosophic Graph

An Iterated Meta-Neutrosophic Graph recursively applies neutrosophic meta-construction, forming multi-level structures representing layered uncertain interactions among complex neutrosophic networks.

Definition 2.47 (Neutrosophic lifting of relations)**.** Let each level-t object X have a vertex-set V_X and $\Sigma_X : V_X \to [0,1]^3$. For R on the previous level define, for X, Y,

$$T_{R^\uparrow}(X,Y) := \sup_{u\in V_X,\ v\in V_Y} \min\{T_{\Sigma_X}(u), T_{\Sigma_Y}(v), T_R(u,v)\},$$
$$I_{R^\uparrow}(X,Y) := \inf_{u\in V_X,\ v\in V_Y} \max\{I_{\Sigma_X}(u), I_{\Sigma_Y}(v), I_R(u,v)\},$$
$$F_{R^\uparrow}(X,Y) := \inf_{u\in V_X,\ v\in V_Y} \max\{F_{\Sigma_X}(u), F_{\Sigma_Y}(v), F_R(u,v)\}.$$

Definition 2.48 (Iterated universes and objects)**.** Fix $\mathsf{NG}^{(0)} := \mathsf{NG}$ and $\mathcal{R}^{(0)} \subseteq ([0,1]^3)^{\mathsf{NG}^{(0)} \times \mathsf{NG}^{(0)}}$. Recursively for $t \geq 0$,

$$\mathsf{NG}^{(t+1)} := \{ \text{ Meta-Neutrosophic Graphs over } (\mathsf{NG}^{(t)}, \mathcal{R}^{(t)}) \}, \qquad \mathcal{R}^{(t+1)} := \{R^{\uparrow} : R \in \mathcal{R}^{(t)}\}.$$

Any element of $\mathsf{NG}^{(t+1)}$ is an *Iterated Meta-Neutrosophic Graph of depth t*.

Example 2.49 (Regional Health Consortia as an Iterated Meta-Neutrosophic Graph)**.** Form two level-1 meta-neutrosophic vertices (consortia):

$$X = \text{East Consortium on } \{A, C\}, \qquad Y = \text{West Consortium on } \{B\}.$$

Give level-1 vertex memberships (institutional fitness at the consortium level):

$$\Sigma^{(1)}(X) = (0.82,\ 0.18,\ 0.12), \qquad \Sigma^{(1)}(Y) = (0.90,\ 0.15,\ 0.10).$$

For lifting (Definition 2.47), we need the *level-0* memberships of the inner vertices. Specify (componentwise) for X and Y:

$$\Sigma_X(A) = (0.80,\ 0.15,\ 0.10), \quad \Sigma_X(C) = (0.65,\ 0.25,\ 0.15), \quad \Sigma_Y(B) = (0.88,\ 0.20,\ 0.10).$$

Assume base-level relation values on pairs with B:

$$R_{\text{align}}(A, B) = (0.60,\ 0.30,\ 0.20), \qquad R_{\text{align}}(C, B) = (0.50,\ 0.35,\ 0.25),$$
$$R_{\text{share}}(A, B) = (0.55,\ 0.25,\ 0.25), \qquad R_{\text{share}}(C, B) = (0.40,\ 0.40,\ 0.30).$$

Lifted relations to level 1. Using Definition 2.47, compute for $R^{\uparrow}_{\text{align}}$:

$$\begin{aligned}
T_{R^{\uparrow}_{\text{align}}}(X, Y) &= \sup_{u \in \{A,C\},\, v=B} \min\{T_{\Sigma_X}(u), T_{\Sigma_Y}(v), T_{R_{\text{align}}}(u, v)\} \\
&= \max\{\min(0.80, 0.88, 0.60),\ \min(0.65, 0.88, 0.50)\} = \max\{0.60,\ 0.50\} = 0.60, \\
I_{R^{\uparrow}_{\text{align}}}(X, Y) &= \inf_{u,v} \max\{I_{\Sigma_X}(u), I_{\Sigma_Y}(v), I_{R_{\text{align}}}(u, v)\} \\
&= \min\{\max(0.15, 0.20, 0.30),\ \max(0.25, 0.20, 0.35)\} = \min\{0.30,\ 0.35\} = 0.30, \\
F_{R^{\uparrow}_{\text{align}}}(X, Y) &= \inf_{u,v} \max\{F_{\Sigma_X}(u), F_{\Sigma_Y}(v), F_{R_{\text{align}}}(u, v)\} \\
&= \min\{\max(0.10, 0.10, 0.20),\ \max(0.15, 0.10, 0.25)\} = \min\{0.20,\ 0.25\} = 0.20.
\end{aligned}$$

Hence $R^{\uparrow}_{\text{align}}(X, Y) = (0.60,\ 0.30,\ 0.20)$.

Similarly for $R^{\uparrow}_{\text{share}}$:

$$\begin{aligned}
T_{R^{\uparrow}_{\text{share}}}(X, Y) &= \max\{\min(0.80, 0.88, 0.55),\ \min(0.65, 0.88, 0.40)\} = \max\{0.55,\ 0.40\} = 0.55, \\
I_{R^{\uparrow}_{\text{share}}}(X, Y) &= \min\{\max(0.15, 0.20, 0.25),\ \max(0.25, 0.20, 0.40)\} = \min\{0.25,\ 0.40\} = 0.25, \\
F_{R^{\uparrow}_{\text{share}}}(X, Y) &= \min\{\max(0.10, 0.10, 0.25),\ \max(0.15, 0.10, 0.30)\} = \min\{0.25,\ 0.30\} = 0.25.
\end{aligned}$$

Thus $R^{\uparrow}_{\text{share}}(X, Y) = (0.55,\ 0.25,\ 0.25)$.

Level-1 meta-edge and verification. Take the meta-label set $L^{(1)}(X, Y) = \{R^{\uparrow}_{\text{align}}, R^{\uparrow}_{\text{share}}\}$ and define

$$M_N^{(1)}(X, Y) = (0.58,\ 0.26,\ 0.21).$$

Vertex bounds:

$$\begin{aligned}
\min\{T_{\Sigma^{(1)}}(X), T_{\Sigma^{(1)}}(Y)\} &= \min\{0.82, 0.90\} = 0.82 \ \geq\ 0.58, \\
\max\{I_{\Sigma^{(1)}}(X), I_{\Sigma^{(1)}}(Y)\} &= \max\{0.18, 0.15\} = 0.18 \ \leq\ 0.26, \\
\max\{F_{\Sigma^{(1)}}(X), F_{\Sigma^{(1)}}(Y)\} &= \max\{0.12, 0.10\} = 0.12 \ \leq\ 0.21.
\end{aligned}$$

Witnessing with lifted labels:

$$\sup_{R\in L^{(1)}} T_R(X,Y) = \max\{0.60, 0.55\} = 0.60 \;\geq\; 0.58,$$
$$\inf_{R\in L^{(1)}} I_R(X,Y) = \min\{0.30, 0.25\} = 0.25 \;\leq\; 0.26,$$
$$\inf_{R\in L^{(1)}} F_R(X,Y) = \min\{0.20, 0.25\} = 0.20 \;\leq\; 0.21.$$

Therefore $\mathbb{N}^{(1)} = (\{X,Y\}, \Sigma^{(1)}, M_N^{(1)}, L^{(1)})$ is a valid *Iterated Meta-Neutrosophic Graph*: a concrete hospital-pair evidence at level 0 lifts to a consortium-to-consortium relation at level 1 with all neutrosophic constraints satisfied componentwise.

Theorem 2.50 (Depth 1)**.** $\mathsf{NG}^{(1)}$ *equals the class of Meta-Neutrosophic Graphs over* $(\mathsf{NG}^{(0)}, \mathcal{R}^{(0)})$.

Proof. Unfold Definition 2.48 at $t = 0$. □

Theorem 2.51 (IMNG generalizes IMFG and MNG)**.** *Let* $\mathsf{FG}^{(0)}$ *be a fuzzy-graph universe with fuzzy relations* $\mathcal{S}^{(0)}$*. Define the levelwise lift* $\sharp$ *by* $a \mapsto (a, 0, 1-a)$ *on memberships and relations, and extend recursively:*

$$\sharp : \mathsf{FG}^{(t)} \hookrightarrow \mathsf{NG}^{(t)}, \qquad \sharp : \mathcal{S}^{(t)} \hookrightarrow \mathcal{R}^{(t)}.$$

Then for all $t \geq 0$*:*

1. $\sharp$ *embeds Iterated Meta-Fuzzy Graphs (depth* t*) into Iterated Meta-Neutrosophic Graphs (depth* t*).*
2. *At* $t = 0$ *this reduces to Theorem 2.46; at* $t = 1$ *to the Theorem.*

Proof. For any $R \in \mathcal{S}^{(t)}$ and $X, Y \in \mathsf{FG}^{(t)}$, using $a^\sharp = (a, 0, 1-a)$:

$$T_{(R^\sharp)^\uparrow}(\sharp X, \sharp Y) = \sup_{u,v} \min\{\Sigma_X(u), \Sigma_Y(v), R(u,v)\} = T_{R^\uparrow}(X,Y),$$
$$I_{(R^\sharp)^\uparrow}(\sharp X, \sharp Y) = \inf_{u,v} \max\{0,0,0\} = 0,$$
$$F_{(R^\sharp)^\uparrow}(\sharp X, \sharp Y) = \inf_{u,v} \max\{1-\Sigma_X(u), 1-\Sigma_Y(v), 1-R(u,v)\} = 1 - T_{R^\uparrow}(X,Y),$$

hence $(R^\sharp)^\uparrow = (R^\uparrow)^\sharp$ and the label families match levelwise. Edge/vertex bounds are preserved exactly as in Theorem 2.46(ii), now at level t. □

2.11 Meta-Weighted Graph

A meta-weighted graph has vertices that are graphs, edges labeled by relations, and real weights encoding costs, capacities, or strengths.

Definition 2.52 (Universe and relation family)**.** Fix a nonempty universe $\mathfrak{G}$ of finite (undirected, loopless by default) graphs and a nonempty family

$$\mathcal{R} \;\subseteq\; \mathcal{P}(\mathfrak{G} \times \mathfrak{G})$$

of binary relations on $\mathfrak{G}$ (e.g. subgraph, homomorphism, minor, isomorphism).

Definition 2.53 (Meta-Weighted Graph over $(\mathfrak{G}, \mathcal{R})$)**.** A *Meta-Weighted Graph* is a directed, labeled multidigraph with edge weights

$$\mathbb{M} \;=\; (V, E, s, t, \lambda, w),$$

where $V \subseteq \mathfrak{G}$, $s, t : E \to V$, $\lambda : E \to \mathcal{R}$, and $w : E \to \mathbb{R}$, such that

$$\forall e \in E : \qquad (\, s(e),\, t(e)\,) \in \lambda(e) \quad \text{(incidence witnessing).}$$

Elements of V are *meta-vertices* (each is itself a graph $G \in \mathfrak{G}$). For $e \in E$ we write

$$s(e) \xrightarrow[w(e)]{\lambda(e)} t(e),$$

and call $w(e)$ the *meta-weight* of e. If every $R \in \mathcal{R}$ is symmetric and for each unordered pair $\{x, y\}$ the multiset of arcs $x \to y$ equals that of $y \to x$ with identical weights, $\mathbb{M}$ can be regarded as an undirected weighted, labeled meta-multigraph.

Remark 2.54 (Forgetting weights and labels)**.** There are natural forgetful maps:

$$\mathrm{U_w} : (V, E, s, t, \lambda, w) \longmapsto (V, E, s, t, \lambda), \qquad \mathrm{U_{lab}} : (V, E, s, t, \lambda, w) \longmapsto (V, E, s, t, w),$$

which discard, respectively, weights and labels.

Example 2.55 (Inter–city Transport Corridors as a Meta–Weighted Graph)**.** Let $\mathfrak{G}$ be the universe of finite *city–level* transportation graphs, each encoding local stations/airports and routes. Fix a family of binary relations $\mathcal{R} = \{\text{Rail}, \text{Air}, \text{Hwy}\}$ on $\mathfrak{G}$, where

$$(X, Y) \in \text{Rail} \iff \text{there exists a direct rail corridor between cities represented by } X \text{ and } Y,$$

and analogously for Air (direct scheduled flight) and Hwy (limited–access highway).

Consider three city graphs G_T (Tokyo), G_O (Osaka), G_N (Nagoya) in $\mathfrak{G}$. Define the meta–weighted graph

$$M = (V, E, s, t, \lambda, w), \qquad V = \{G_T, G_O, G_N\},$$

as a directed, labeled multigraph whose meta–edges record the type of corridor and whose weights are *door–to–door median travel times (hours)*:

$$\begin{aligned}
e^{\text{rail}}_{TO} &: s(e^{\text{rail}}_{TO}) = G_T,\ t(e^{\text{rail}}_{TO}) = G_O,\ \lambda(e^{\text{rail}}_{TO}) = \text{Rail},\ w(e^{\text{rail}}_{TO}) = 2.5,\\
e^{\text{air}}_{TO} &: s(e^{\text{air}}_{TO}) = G_T,\ t(e^{\text{air}}_{TO}) = G_O,\ \lambda(e^{\text{air}}_{TO}) = \text{Air},\ w(e^{\text{air}}_{TO}) = 1.0,\\
e^{\text{rail}}_{ON} &: s(e^{\text{rail}}_{ON}) = G_O,\ t(e^{\text{rail}}_{ON}) = G_N,\ \lambda(e^{\text{rail}}_{ON}) = \text{Rail},\ w(e^{\text{rail}}_{ON}) = 1.8,\\
e^{\text{hwy}}_{TN} &: s(e^{\text{hwy}}_{TN}) = G_T,\ t(e^{\text{hwy}}_{TN}) = G_N,\ \lambda(e^{\text{hwy}}_{TN}) = \text{Hwy},\ w(e^{\text{hwy}}_{TN}) = 3.1.
\end{aligned}$$

Each meta–edge e satisfies the incidence constraint $\big(s(e), t(e)\big) \in \lambda(e)$ by construction (e.g. $(G_T, G_O) \in \text{Air}$ and $(G_T, G_O) \in \text{Rail}$), while the weight $w(e)$ numerically encodes the operational cost/time of that corridor. Multiple edges between a pair (e.g. rail and air) capture alternative modalities.

Theorem 2.56 (Meta-Weighted Graphs generalize MetaGraphs)**.** *Let $M_0 = (V_0, E_0, s_0, t_0, \lambda_0)$ be a MetaGraph over $(\mathfrak{G}, \mathcal{R})$. Define*

$$\iota_{\mathrm{MG}}(M_0) \ :=\ (V_0, E_0, s_0, t_0, \lambda_0, w_1), \qquad w_1(e) \equiv 1 \quad (\forall e \in E_0).$$

Then $\iota_{\mathrm{MG}}(M_0)$ is a Meta-Weighted Graph and $\mathrm{U_w}\big(\iota_{\mathrm{MG}}(M_0)\big) = M_0$. Hence Meta-Weighted Graphs strictly extend MetaGraphs.

Proof. Incidence witnessing for $\iota_{\mathrm{MG}}(M_0)$ is identical to that of M_0, and constant weights pose no additional constraint. The equality under $\mathrm{U_w}$ is immediate. Strictness follows since different weightings of the same metagraph yield distinct Meta-Weighted Graphs. □

Definition 2.57 (Singleton-graph embedding)**.** For a set X, write $\mathsf{pt}(x)$ for the one-vertex graph with vertex x and no edges.

Theorem 2.58 (Meta-Weighted Graphs generalize Weighted Graphs)**.** *Let $G = (V_G, E_G, w_G)$ be a (finite, undirected, loopless) weighted graph with $w_G : E_G \to \mathbb{R}$. Fix any nonempty $\mathcal{R}$ and, for each unordered pair $\{u, v\} \subseteq V_G$, introduce a label*

$$R_{u,v} \ \in\ \mathcal{R} \qquad \textit{with} \qquad (\mathsf{pt}(u), \mathsf{pt}(v)) \in R_{u,v}.$$

Define a Meta-Weighted Graph over $(\mathfrak{G}, \mathcal{R})$ by

$$V := \{\mathsf{pt}(v) \mid v \in V_G\}, \quad E := \{e_{u\to v},\ e_{v\to u} \mid \{u, v\} \in E_G\},$$

$$s(e_{u\to v}) = \mathsf{pt}(u),\ \ t(e_{u\to v}) = \mathsf{pt}(v),\ \ \lambda(e_{u\to v}) = R_{u,v},\ \ w(e_{u\to v}) = w_G(\{u, v\}),$$

and symmetrically for $e_{v\to u}$. Then:

1. *Incidence is witnessed by construction, so (V, E, s, t, λ, w) is a Meta-Weighted Graph.*

2. *Collapsing opposite arcs into a single undirected edge and keeping the common weight recovers G.*

Thus weighted graphs embed into Meta-Weighted Graphs.

Proof. (1) By definition $(\mathsf{pt}(u), \mathsf{pt}(v)) \in R_{u,v} = \lambda(e_{u\to v})$; similarly for the reverse arc. (2) The map $\Psi : V_G \to V,\ v \mapsto \mathsf{pt}(v)$ is a bijection. For each $\{u, v\} \in E_G$, the two arcs $e_{u\to v}$ and $e_{v\to u}$ have identical weight $w_G(\{u, v\})$, so the undirected identification yields an edge of the same weight between $\Psi(u)$ and $\Psi(v)$. This gives a canonical isomorphism between G and the undirected quotient of (V, E, s, t, λ, w). □

2.12 Iterated Meta-Weighted Graphs

An iterated meta-weighted graph uses metagraphs as vertices, recursively repeating labeling and weighting to form multi-level graph-of-graphs structures across hierarchies.

Definition 2.59 (Relation lifting)**.** Given $\mathcal{R}$ on $\mathfrak{G}$, define its *lift* $\mathcal{R}^{\uparrow}$ on finite Meta-Weighted Graphs over $(\mathfrak{G}, \mathcal{R})$ by

$$\mathcal{R}^{\uparrow} := \{R^{\uparrow} \mid R \in \mathcal{R}\}, \qquad (M_1, M_2) \in R^{\uparrow} \iff \exists\, x \in V(M_1),\ y \in V(M_2) :\ (x, y) \in R.$$

(Weights play no role in witnessing; they are additional structure on edges.)

Definition 2.60 (Iterated universes)**.** Define recursively for $t \in \mathbb{N}_0$:

$$\mathfrak{G}^{(0)} := \mathfrak{G}, \qquad \mathcal{R}^{(0)} := \mathcal{R},$$

$$\mathfrak{G}^{(t+1)} := \Big\{\text{all finite Meta-Weighted Graphs over } \big(\mathfrak{G}^{(t)}, \mathcal{R}^{(t)}\big)\Big\}, \qquad \mathcal{R}^{(t+1)} := \big(\mathcal{R}^{(t)}\big)^{\uparrow}.$$

Definition 2.61 (Iterated Meta-Weighted Graph of depth t)**.** For $t \in \mathbb{N}_0$, an *Iterated Meta-Weighted Graph of depth t* is a tuple

$$\mathbb{M}^{(t)} = (V^{(t)}, E^{(t)}, s^{(t)}, t^{(t)}, \lambda^{(t)}, w^{(t)})$$

with $V^{(t)} \subseteq \mathfrak{G}^{(t)}$, $\lambda^{(t)} : E^{(t)} \to \mathcal{R}^{(t)}$, $w^{(t)} : E^{(t)} \to \mathbb{R}$, and

$$\forall e \in E^{(t)} : \qquad \big(s^{(t)}(e),\ t^{(t)}(e)\big) \in \lambda^{(t)}(e).$$

Example 2.62 (Division–Level Planning as an Iterated Meta–Weighted Graph)**.** Start from the meta–weighted graph M above. Form two *level–1 meta–vertices*, each of which is itself a meta–weighted graph:

$$X := \text{“East” corridor view on } \{G_T, G_N\}, \qquad Y := \text{“West” corridor view on } \{G_O\}.$$

Lift the relation family $\mathcal{R}$ to $\mathcal{R}^{\uparrow}$ by the usual existential witness rule:

$$(X, Y) \in R^{\uparrow} \iff \exists\, x \in V(X),\ y \in V(Y) :\ (x, y) \in R, \quad R \in \mathcal{R}.$$

Define the *iterated meta–weighted graph of depth* 1

$$M^{(1)} = \big(V^{(1)}, E^{(1)}, s^{(1)}, t^{(1)}, \lambda^{(1)}, w^{(1)}\big), \qquad V^{(1)} = \{X, Y\},$$

with two meta–edges whose labels are lifted relations and whose weights aggregate underlying corridor weights (minimum over witnessing pairs, i.e. best available time):

$$\begin{aligned}
e^{(1)}_{\text{air}} :\ s^{(1)} = X,\ t^{(1)} = Y,\ \lambda^{(1)}(e^{(1)}_{\text{air}}) = \text{Air}^{\uparrow},& \\
w^{(1)}(e^{(1)}_{\text{air}}) = \min\{\, w(e^{\text{air}}_{TO}) \,\} = 1.0,& \\
e^{(1)}_{\text{rail}} :\ s^{(1)} = X,\ t^{(1)} = Y,\ \lambda^{(1)}(e^{(1)}_{\text{rail}}) = \text{Rail}^{\uparrow},& \\
w^{(1)}(e^{(1)}_{\text{rail}}) = \min\{\, w(e^{\text{rail}}_{TO}),\ w(e^{\text{rail}}_{ON}) \,\} = 1.8.&
\end{aligned}$$

Incidence holds: $\text{Air}^{\uparrow}$ is witnessed by the pair (G_T, G_O) with a direct flight, and $\text{Rail}^{\uparrow}$ by (G_O, G_N) (or (G_T, G_O)). The weights $w^{(1)}$ summarize lower–level options into division–level best travel times, enabling hierarchical planning across meta–vertices X and Y.

Theorem 2.63 (Depth 1 recovers the meta level)**.**

$$\Big\{\textit{Meta-Weighted Graphs over } (\mathfrak{G}, \mathcal{R})\Big\} \;=\; \mathfrak{G}^{(1)}.$$

Proof. By Definition, $\mathfrak{G}^{(1)}$ is exactly the class of Meta-Weighted Graphs over $(\mathfrak{G}, \mathcal{R})$. □

Theorem 2.64 (Iterated Meta-Weighted Graphs generalize Iterated MetaGraphs and Meta-Weighted Graphs)**.** *For every $t \geq 0$:*

1. *The assignment*
$$(V^{(t)}, E^{(t)}, s^{(t)}, t^{(t)}, \lambda^{(t)}) \;\longmapsto\; (V^{(t)}, E^{(t)}, s^{(t)}, t^{(t)}, \lambda^{(t)}, \mathbf{1})$$
(where $\mathbf{1}$ is the constant-1 weight on $E^{(t)}$) embeds Iterated MetaGraphs of depth t into Iterated Meta-Weighted Graphs of depth t, and forgetting weights inverts this embedding.

2. *The case $t = 1$ reduces to the Theorem; hence Iterated Meta-Weighted Graphs strictly extend Meta-Weighted Graphs.*

Proof. (1) Incidence witnessing is unchanged; the extra constant weight satisfies the definition of an Iterated Meta-Weighted Graph. The forgetful map $\mathrm{U_w}$ applied levelwise recovers the original Iterated MetaGraph, giving a left inverse and hence an embedding.

(2) Immediate from Definition 2.61 and the Theorem. □

3 Conclusion

In this paper, we extended the frameworks of fuzzy graphs, neutrosophic graphs, multigraphs, digraphs, and bidirected graphs by embedding them into the unified setting of MetaGraphs and Iterated MetaGraphs. In the future, we intend to investigate extended systems of the concepts defined in this paper by employing Plithogenic Sets [21, 42–44], Intuitionstic Fuzzy Sets [45, 46], Bipolar Fuzzy Sets [47, 48], Hesitant Fuzzy Sets [49, 50], HyperFuzzy Sets [51–56], HyperGraphs [57–59], and SuperHyperGraphs [60–63].

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this work.

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Data Availability

This paper is theoretical and did not generate or analyze any empirical data. We welcome future studies that apply and test these concepts in practical settings.

Research Integrity

The author confirms that this manuscript is original, has not been published elsewhere, and is not under consideration by any other journal.

Use of Computational Tools

All proofs and derivations were performed manually; no computational software (e.g., Mathematica, SageMath, Coq) was used.

Code Availability

No code or software was developed for this study.

Ethical Approval

This research did not involve human participants or animals, and therefore did not require ethical approval.

Use of Generative AI and AI-Assisted Tools

We use generative AI and AI-assisted tools for tasks such as English grammar checking, and We do not employ them in any way that violates ethical standards.

Supplementary Information

No supplementary materials accompany this paper.

Disclaimer

The ideas presented here are theoretical and have not yet been validated through empirical testing. While we have strived for accuracy and proper citation, inadvertent errors may remain. Readers should verify any referenced material independently. The opinions expressed are those of the authors and do not necessarily reflect the views of their institutions.

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