Fuzzy Off-SuperHyperGraphs: Extending Uncertainty Modeling Beyond Classical Boundaries

Takaaki Fujita [1] *

[1] Independent Researcher, Tokyo, Japan.

Abstract

Graph theory provides a mathematical framework for representing relationships and connectivity through vertices and edges [1,2]. Hypergraphs extend this classical notion by introducing *hyperedges* that may connect more than two vertices at once [3]. Superhypergraphs further enrich the model by employing iterated powerset constructions, thereby capturing hierarchical and self-referential structures among hyperedges [4,5]. A fuzzy n-SuperHyperGraph advances this approach by assigning membership degrees to both supervertices and superedges, offering a flexible tool for modeling uncertainty in complex systems. In this paper, we propose an extension of this framework, termed the *Fuzzy Off-SuperHyperGraph*, which integrates the offgraph paradigm into fuzzy n-SuperHyperGraphs. We establish its formal definition, investigate its structural properties, and discuss its potential applications in uncertainty modeling and hierarchical network analysis.

Keywords: Superhypergraph, Hypergraph, Fuzzy OffGraph, Fuzzy Offset, Fuzzy Set

1 Preliminaries

We record basic notions and notation used throughout. Unless stated otherwise, all graphs are finite, undirected, and loopless (multiple edges are allowed only when explicitly declared).

1.1 SuperHyperGraphs

Classical hypergraphs enrich ordinary graphs by allowing an edge to join any finite number of vertices; this makes them apt for modeling multiway relations [3, 6–10]. A *SuperHyperGraph* pushes this idea further by building vertices and edges from iterated powersets of a ground set; this perspective has recently attracted attention in several settings [11–16]. Applications have appeared in, e.g., molecular modeling, network analysis, and signal processing [11,17–21]. Throughout, the *level* parameter n is a fixed nonnegative integer.

Definition 1.1 (Base set). [22–24] A *base set* (or ground set) is a fixed finite set S from which higher-level objects are formed:

$$S = \{ x \mid x \text{ lies in the chosen domain} \}.$$

All constructions below ultimately draw their elements from S.

Definition 1.2 (Powerset). [25,26] For a set X, its powerset is

$$\mathcal{P}(X) = \{ A \mid A \subseteq X \}.$$

We also use the nonempty powerset $\mathcal{P}^*(X) := \mathcal{P}(X) \setminus \{\varnothing\}$.

Definition 1.3 (Iterated powerset). [27–30] For $k \in \mathbb{N}_0$ define

$$\mathcal{P}^0(X) := X, \qquad \mathcal{P}^{k+1}(X) := \mathcal{P}\big(\mathcal{P}^k(X)\big).$$

Similarly, with nonempty levels,

$$\big(\mathcal{P}^*\big)^0(X) := X, \qquad \big(\mathcal{P}^*\big)^{k+1}(X) := \mathcal{P}^*\big((\mathcal{P}^*)^k(X)\big).$$

Proposition 1.4 (Cardinality recursion). *Let* $m := |X|$. *Then*

$$\big|\mathcal{P}^1(X)\big| = 2^m, \qquad \big|\mathcal{P}^{k+1}(X)\big| = 2^{|\mathcal{P}^k(X)|} \quad (k \geq 0).$$

In particular, if $m \geq 1$ *then* $|(\mathcal{P}^*)^1(X)| = 2^m - 1$ *and* $|(\mathcal{P}^*)^{k+1}(X)| = 2^{|(\mathcal{P}^*)^k(X)|} - 1$.

Proof. The identity $|\mathcal{P}(Y)| = 2^{|Y|}$ is standard. Applying it with $Y = \mathcal{P}^k(X)$ gives $|\mathcal{P}^{k+1}(X)| = 2^{|\mathcal{P}^k(X)|}$. For the nonempty versions, subtract 1 (removing $\emptyset$) at each step. □

Definition 1.5 (Hypergraph [3, 31])**.** A *hypergraph* is a pair $H = (V(H), E(H))$ where $V(H) \neq \emptyset$ and $E(H) \subseteq \mathcal{P}^*(V(H))$. In this paper we work with finite $V(H)$ and finite $E(H)$.

Example 1.6 (Real-world hypergraph: project teams)**.** *Interpretation.* Vertices are students; each hyperedge is a project team (a team may have more than two members).

Construction. Let

$$V = \{S_1, S_2, S_3, S_4\}, \qquad E = \big\{\{S_1, S_2, S_3\},\ \{S_2, S_4\}\big\}.$$

Then $H = (V, E)$ is a finite hypergraph since $E \subseteq \mathcal{P}^*(V)$. The 3-way relation "work together on Project A" is encoded by the hyperedge $\{S_1, S_2, S_3\}$, while Project B uses $\{S_2, S_4\}$.

Definition 1.7 (n-SuperHyperGraph)**.** [4, 32–35] Fix a finite base set V_0. For a level $n \in \mathbb{N}_0$, an *n-SuperHyperGraph* is a pair

$$\mathrm{SHG}^{(n)} = (V, E), \qquad V \subseteq \mathcal{P}^n(V_0), \quad E \subseteq \mathcal{P}^n(V_0),$$

whose elements are called *n-supervertices* (elements of V) and *n-superedges* (elements of E). When desired, one may require $E \subseteq (\mathcal{P}^*)^n(V_0)$ to forbid empty superedges at every level.

Example 1.8 (Real-world n-SuperHyperGraph ($n = 1$): meal planning)**.** *Interpretation.* Base items are dishes; supervertices are bundles of dishes; superedges specify bundles served together.

Construction. Let the base set be $V_0 = \{D, S, M\}$ (drink, salad, main). Take

$$V = \big\{\{D\},\ \{S\},\ \{M\},\ \{D, S\}\big\} \subseteq \mathcal{P}(V_0).$$

Define superedges as nonempty subsets of V:

$$e_1 = \big\{\{D\}, \{S\}\big\}, \qquad e_2 = \big\{\{D, S\}, \{M\}\big\}, \qquad E = \{e_1, e_2\} \subseteq \mathcal{P}^*(V).$$

Then $\mathrm{SHG}^{(1)} = (V, E)$ is a 1-SuperHyperGraph: $V, E \subseteq \mathcal{P}^1(V_0)$.

1.2 Fuzzy n-SuperHyperGraphs

A fuzzy set assigns a membership degree in $[0, 1]$ to each element of a universe [36, 37]. Fuzzy graphs and fuzzy hypergraphs endow vertices and (hyper)edges with such degrees [38–45]. A fuzzy n-SuperHyperGraph is a higher-level network structure assigning membership degrees to supervertices and superedges for modeling complex relations (cf. [4, 46]).

Definition 1.9 (Fuzzy n-SuperHyperGraph)**.** (cf. [47, 48]) Let $\mathrm{SHG}^{(n)} = (V, E)$ be an n-SuperHyperGraph. A *fuzzy n-SuperHyperGraph* is a quadruple

$$(V, E, \sigma, \mu),$$

where $\sigma : V \to [0, 1]$ and $\mu : E \to [0, 1]$ satisfy the *appurtenance constraint*

$$\mu(e) \ \le\ \min_{v \in e} \sigma(v) \qquad \text{for every } e \in E.$$

Example 1.10 (Real-world fuzzy n-SuperHyperGraph ($n = 1$): sensor fusion)**.** *Interpretation.* Base sensors: temperature (T), humidity (H), pressure (P). Supervertices are singletons/pairs of sensors; memberships model detection reliability in $[0, 1]$. Superedges connect bundles that co-trigger an alert.

Construction. Let $V_0 = \{T, H, P\}$ and

$$V = \big\{\{T\},\ \{H\},\ \{P\},\ \{T, H\}\big\} \subseteq \mathcal{P}(V_0), \qquad \begin{aligned} e_1 &= \big\{\{T\}, \{H\}\big\}, \\ e_2 &= \big\{\{T, H\}, \{P\}\big\}, \end{aligned} \qquad E = \{e_1, e_2\}.$$

Define vertex-memberships $\sigma : V \to [0, 1]$ and edge-memberships $\mu : E \to [0, 1]$ by

$$\sigma(\{T\}) = 0.85, \quad \sigma(\{H\}) = 0.65, \quad \sigma(\{P\}) = 0.40, \quad \sigma(\{T, H\}) = 0.70.$$

Then the admissible maxima are

$$\mu_{\max}(e_1) = \min\{0.85, 0.65\} = 0.65, \qquad \mu_{\max}(e_2) = \min\{0.70, 0.40\} = 0.40.$$

Choose

$$\mu(e_1) = 0.60 \leq 0.65, \qquad \mu(e_2) = 0.38 \leq 0.40.$$

Verification. For each edge $e \in E$,

$$\mu(e) \;\leq\; \min_{v \in e} \sigma(v)$$

holds numerically by the computations above; hence (V, E, σ, μ) is a valid fuzzy 1-SuperHyperGraph.

1.3 Fuzzy Offgraph

A fuzzy offgraph is a variant of the offset/offgraph model in which membership values may exceed the classical unit interval, inspired by studies on neutrosophic offsets and over/under offgraphs(cf. [22, 49–53]).

Definition 1.11 (Fuzzy Offgraph)**.** [53] Let $G = (V, E)$ be a finite, loopless, undirected graph, so $E \subseteq \binom{V}{2}$. Fix real bounds $\Psi < 0 < 1 < \Omega$. A *fuzzy offgraph* on G is a quadruple

$$G_F \;=\; \big(V,\, E,\, \ell_V,\, \ell_E\big),$$

where the vertex- and edge-membership maps satisfy

$$\ell_V : V \to [\Psi, \Omega], \qquad \ell_E : E \to [\Psi, \Omega],$$

together with the *offness condition*

$$\exists\, v_0 \in V : \; \ell_V(v_0) \notin [0, 1] \quad \text{and} \quad \exists\, e_0 \in E : \; \ell_E(e_0) \notin [0, 1].$$

Equivalently, writing

$$V_{\text{off}} := \{\, v \in V \mid \ell_V(v) \notin [0, 1] \,\}, \qquad E_{\text{off}} := \{\, e \in E \mid \ell_E(e) \notin [0, 1] \,\},$$

a fuzzy offgraph satisfies $V_{\text{off}} \neq \emptyset$ and $E_{\text{off}} \neq \emptyset$. If $V_{\text{off}} = E_{\text{off}} = \emptyset$ (i.e., all memberships lie in $[0, 1]$), one recovers an ordinary fuzzy graph on G.

Example 1.12 (Real-world fuzzy offgraph: road network load)**.** *Interpretation.* Vertices are intersections $\{x, y, z\}$; edges are road segments $\{\{x, y\}, \{y, z\}\}$. Memberships measure relative load vs. nominal: values in $[0, 1]$ are normal-to-saturated, > 1 indicates overload, < 0 denotes closure/contraflow.

Construction. Let $G = (V, E)$ with $V = \{x, y, z\}$ and $E = \big\{\{x, y\}, \{y, z\}\big\} \subseteq \binom{V}{2}$. Fix bounds $\Psi = -0.5$, $\Omega = 1.3$. Define

$$\ell_V(x) = 1.15\ (> 1), \quad \ell_V(y) = 0.80, \quad \ell_V(z) = 0.60,$$
$$\ell_E(\{x, y\}) = 1.05\ (> 1), \qquad \ell_E(\{y, z\}) = -0.10\ (< 0).$$

Verification (offness).

$$V_{\text{off}} = \{x\}\ (\text{since } 1.15 > 1), \qquad E_{\text{off}} = \big\{\{x, y\}, \{y, z\}\big\}\ (\text{one} >1,\ \text{one} <0).$$

Thus $V_{\text{off}} \neq \emptyset$ and $E_{\text{off}} \neq \emptyset$, so $G_F = (V, E, \ell_V, \ell_E)$ is a fuzzy offgraph in the sense of Definition 1.11.

2 Result: Fuzzy Off SuperHyperGraph

Throughout this section let $\mathrm{SHG}^{(n)} = (V, E)$ be an n-SuperHyperGraph at a fixed level $n \in \mathbb{N}_0$, as in the preliminaries. For all results below we adopt the following *incidence convention*:

$$E \;\subseteq\; \mathcal{P}^*(V),$$

i.e., each n-superedge $e \in E$ is a nonempty finite subset of n-supervertices.

Definition 2.1 (Fuzzy Off n-SuperHyperGraph)**.** Fix real bounds $\Psi < 0 < 1 < \Omega$. A *fuzzy off n-SuperHyperGraph* (abbreviated *FOSH*) on $\mathrm{SHG}^{(n)}$ is a quintuple

$$\mathfrak{G}^{(n)}_{\mathrm{off}} \;=\; \big(V, E, \sigma, \mu; \Psi, \Omega\big),$$

where

$$\sigma : V \longrightarrow [\Psi, \Omega], \qquad \mu : E \longrightarrow [\Psi, \Omega],$$

and the following two conditions hold.

(i) *Appurtenance (admissibility) inequality:*

$$\mu(e) \;\le\; \min_{v \in e} \sigma(v) \qquad \text{for every } e \in E. \tag{1}$$

(ii) *Offness condition (both sides present):* there exist

$$v_0 \in V \;\text{ with }\; \sigma(v_0) \notin [0,1], \qquad e_0 \in E \;\text{ with }\; \mu(e_0) \notin [0,1].$$

We write

$$V^{+}_{\mathrm{off}} := \{v \in V : \sigma(v) > 1\}, \quad V^{-}_{\mathrm{off}} := \{v \in V : \sigma(v) < 0\},$$
$$E^{+}_{\mathrm{off}} := \{e \in E : \mu(e) > 1\}, \quad E^{-}_{\mathrm{off}} := \{e \in E : \mu(e) < 0\},$$

so the offness condition is equivalent to $\big(V^{+}_{\mathrm{off}} \cup V^{-}_{\mathrm{off}}\big) \neq \varnothing$ and $\big(E^{+}_{\mathrm{off}} \cup E^{-}_{\mathrm{off}}\big) \neq \varnothing$.

Remark 2.2 (Choice of the aggregator)**.** Inequality (1) uses the min aggregator on the extended interval $[\Psi, \Omega]$. All theorems below remain valid if min is replaced by any nondecreasing, permutation-invariant aggregator $\mathcal{T} :\ [\Psi, \Omega]^k \to [\Psi, \Omega]$ (for each arity $k \ge 1$) that reduces to min on $[0, 1]$; for concreteness we keep min.

Example 2.3 (Hospital triage groups (level $n = 1$))**.** *Interpretation.* Supervertices encode symptom bundles; superedges encode co-occurring triage groups. Values above 1 capture extreme urgency; negative values penalize contraindicated bundles.

Construction. Let $V_0 = \{A, B, C\}$ and $n = 1$. Set

$$V = \big\{\{A\}, \{B\}, \{C\}, \{A, B\}\big\} \subseteq \mathcal{P}(V_0), \qquad \begin{aligned} e_1 &:= \big\{\{A\}, \{B\}\big\}, \\ e_2 &:= \big\{\{A, B\}, \{C\}\big\}, \end{aligned} \qquad E = \{e_1, e_2\} \subseteq \mathcal{P}^{*}(V).$$

Choose bounds $\Psi = -0.5$, $\Omega = 1.4$. Define $\sigma : V \to [\Psi, \Omega]$ by

$$\sigma(\{A\}) = 1.25, \quad \sigma(\{B\}) = 0.80, \quad \sigma(\{C\}) = 0.30, \quad \sigma(\{A, B\}) = -0.10.$$

Then

$$\mu^{\max}(e_1) = \min\{1.25, 0.80\} = 0.80, \qquad \mu^{\max}(e_2) = \min\{-0.10, 0.30\} = -0.10.$$

Pick $\mu : E \to [\Psi, \Omega]$ as

$$\mu(e_1) = 0.75 \le 0.80, \qquad \mu(e_2) = -0.12 \le -0.10.$$

Verification. For each $e \in E$, $\mu(e) \le \min_{v \in e} \sigma(v)$ holds by the calculations above. Offness holds on both sides: $\sigma(\{A\}) = 1.25 > 1$ (vertex offness) and $\mu(e_2) = -0.12 < 0$ (edge offness). Hence $\big(V, E, \sigma, \mu; \Psi, \Omega\big)$ is a valid FOSH.

Example 2.4 (Priority shipping bundles (level $n = 2$))**.** *Interpretation.* Supervertices are bundles of items; superedges are joint shipping lanes. Overshoot > 1 represents expedited priority; negatives represent blocked routes.

Construction. Let $V_0 = \{x, y\}$, $n = 2$. Write $\mathcal{P}(V_0) = \{\varnothing, \{x\}, \{y\}, \{x, y\}\}$. Define three level-2 supervertices

$$v_1 := \big\{\{x\}, \{y\}\big\}, \quad v_2 := \big\{\{x, y\}\big\}, \quad v_3 := \big\{\varnothing, \{x\}\big\},$$

and set $V = \{v_1, v_2, v_3\} \subseteq \mathcal{P}^2(V_0)$. Let

$$e_1 := \{v_1, v_2\}, \qquad e_2 := \{v_2, v_3\}, \qquad E = \{e_1, e_2\} \subseteq \mathcal{P}^{*}(V).$$

Choose $\Psi = -1.0$, $\Omega = 1.5$. Define σ by

$$\sigma(v_1) = 1.30, \qquad \sigma(v_2) = 1.10, \qquad \sigma(v_3) = 0.60.$$

Then

$$\mu^{\max}(e_1) = \min\{1.30, 1.10\} = 1.10, \qquad \mu^{\max}(e_2) = \min\{1.10, 0.60\} = 0.60.$$

Pick μ as

$$\mu(e_1) = 1.05 \le 1.10, \qquad \mu(e_2) = 0.55 \le 0.60.$$

Verification. Admissibility holds since $\mu \le \mu^{\max}$ edgewise. Offness holds on both sides: $\sigma(v_1) = 1.30 > 1$ gives vertex offness and $\mu(e_1) = 1.05 > 1$ gives edge offness. Thus we obtain a FOSH at level $n = 2$.

Example 2.5 (Cybersecurity alert aggregation (level $n = 1$))**.** *Interpretation.* Supervertices encode aggregated indicators of compromise; superedges aggregate correlated alerts. Large > 1 means critical; negative means known false-positive pattern.

Construction. Let $V_0 = \{\alpha, \beta, \gamma, \delta\}$, $n = 1$. Set

$$V = \big\{\{\alpha\},\ \{\beta, \gamma\},\ \{\delta\},\ \{\alpha, \beta\}\big\} \subseteq \mathcal{P}(V_0),$$

and define

$$e_1 := \big\{\{\alpha\}, \{\beta, \gamma\}\big\}, \qquad e_2 := \big\{\{\alpha, \beta\}, \{\delta\}\big\}, \qquad E = \{e_1, e_2\}.$$

Choose $\Psi = -0.8$, $\Omega = 1.2$. Let

$$\sigma(\{\alpha\}) = 0.95, \quad \sigma(\{\beta, \gamma\}) = -0.20, \quad \sigma(\{\delta\}) = 0.50, \quad \sigma(\{\alpha, \beta\}) = 1.10.$$

Then

$$\mu^{\max}(e_1) = \min\{0.95, -0.20\} = -0.20, \qquad \mu^{\max}(e_2) = \min\{1.10, 0.50\} = 0.50.$$

Pick μ as

$$\mu(e_1) = -0.25 \le -0.20, \qquad \mu(e_2) = 0.48 \le 0.50.$$

Verification. For each edge, $\mu(e) \le \min_{v \in e} \sigma(v)$ holds numerically. Offness on both sides is witnessed by $\sigma(\{\alpha, \beta\}) = 1.10 > 1$ (vertex) and $\mu(e_1) = -0.25 < 0$ (edge). Hence this is a valid FOSH at level $n = 1$.

Theorem 2.6 (Reduction to fuzzy n-SuperHyperGraphs)**.** *If in Definition 2.1 we set $\Psi = 0$ and $\Omega = 1$, then every FOSH $\big(V, E, \sigma, \mu;\ 0, 1\big)$ is precisely a fuzzy n-SuperHyperGraph.*

Proof. With $\Psi = 0$ and $\Omega = 1$, the codomains become $[0, 1]$. Condition (1) is exactly the appurtenance constraint for fuzzy n-SuperHyperGraphs. The offness condition is automatically false (no value can lie outside $[0, 1]$), but it is not part of the fuzzy n-SuperHyperGraph definition; hence we recover that model verbatim. □

Theorem 2.7 (Reduction to fuzzy offgraphs)**.** *Let $n = 0$ and assume $E \subseteq \binom{V}{2}$ (loopless simple case). Then a FOSH $\big(V, E, \sigma, \mu; \Psi, \Omega\big)$ is exactly a fuzzy offgraph on the underlying simple graph, with $\ell_V = \sigma$ and $\ell_E = \mu$.*

Proof. When $n = 0$ we have $V \subseteq V_0$ and $E \subseteq \mathcal{P}^*(V)$. The restriction $E \subseteq \binom{V}{2}$ matches the edge set of a loopless simple graph. Definition 2.1 gives $\sigma : V \to [\Psi, \Omega]$, $\mu : E \to [\Psi, \Omega]$ with both vertex and edge offness allowed, which is precisely the fuzzy offgraph setting. The inequality (1) reduces to the standard bound by the two incident vertices. □

Proposition 2.8 (Maximal admissible edge map)**.** *Given $\sigma : V \to [\Psi, \Omega]$, define*

$$\mu^{\max}(e) := \min_{v \in e} \sigma(v) \qquad (e \in E).$$

Then μ satisfies (1) *if and only if $\mu \le \mu^{\max}$ pointwise. In particular, $\mu^{\max}$ is the largest admissible edge-membership map for a fixed σ.*

Proof. Immediate from (1) and the definition of $\mu^{\max}$. □

Proposition 2.9 (Monotonicity in vertex memberships)**.** *Suppose $\sigma_1, \sigma_2 : V \to [\Psi, \Omega]$ satisfy $\sigma_1 \le \sigma_2$ pointwise. Then*

$$\mu^{\max}_{\sigma_1}(e) \;=\; \min_{v \in e} \sigma_1(v) \;\le\; \min_{v \in e} \sigma_2(v) \;=\; \mu^{\max}_{\sigma_2}(e) \qquad (\forall e \in E).$$

Hence the feasible set $\{\mu : \mu \le \mu^{\max}_{\sigma}\}$ expands monotonically with σ.

Proof. Each minimum is nondecreasing in each argument. □

Proposition 2.10 (Propagation of under-offness)**.** *If $v^- \in V$ satisfies $\sigma(v^-) < 0$, then for every $e \in E$ with $v^- \in e$ one has $\mu^{\max}(e) \le \sigma(v^-) < 0$. Consequently, any admissible μ must satisfy $\mu(e) < 0$, i.e., $e \in E^-_{\text{off}}$.*

Proof. By definition, $\mu^{\max}(e) = \min_{v \in e} \sigma(v) \le \sigma(v^-) < 0$. Thus $\mu(e) \le \mu^{\max}(e) < 0$ for all admissible μ. □

Remark 2.11. By contrast, if $\sigma(v) > 1$ for all $v \in e$, then $\mu^{\max}(e) > 1$ but admissible $\mu(e)$ may still lie in $[0, 1]$; overshoot does not *force* an edge to be off.

Theorem 2.12 (Clipping to a fuzzy n-SuperHyperGraph)**.** *Let* $\text{clip}(x) := \min\{1, \max\{0, x\}\}$. *If $(V, E, \sigma, \mu; \Psi, \Omega)$ is a FOSH, then*

$$\bigl(V, E,\ \text{clip}\circ\sigma,\ \text{clip}\circ\mu\bigr)$$

is a fuzzy n-SuperHyperGraph (codomain $[0, 1]$*) and still satisfies*

$$\text{clip}\bigl(\mu(e)\bigr) \;\le\; \min_{v \in e} \text{clip}\bigl(\sigma(v)\bigr) \qquad (\forall e \in E).$$

Proof. The map clip is nondecreasing. Hence

$$\text{clip}\bigl(\mu(e)\bigr) \;\le\; \text{clip}\Bigl(\min_{v \in e} \sigma(v)\Bigr) \;=\; \min_{v \in e} \text{clip}\bigl(\sigma(v)\bigr).$$

The codomains are contained in $[0, 1]$, so we obtain a fuzzy n-SuperHyperGraph. □

Theorem 2.13 (Affine normalization)**.** *Define an increasing affine map $T : [\Psi, \Omega] \to [0, 1]$ by*

$$T(x) := \frac{x - \Psi}{\Omega - \Psi}.$$

If $(V, E, \sigma, \mu; \Psi, \Omega)$ is a FOSH, then

$$\bigl(V, E,\ T \circ \sigma,\ T \circ \mu\bigr)$$

is a fuzzy n-SuperHyperGraph, and $T\bigl(\mu(e)\bigr) \le \min_{v \in e} T\bigl(\sigma(v)\bigr)$ holds for all $e \in E$.

Proof. Since T is nondecreasing,

$$T\bigl(\mu(e)\bigr) \;\le\; T\Bigl(\min_{v \in e} \sigma(v)\Bigr) \;=\; \min_{v \in e} T\bigl(\sigma(v)\bigr).$$

The images lie in $[0, 1]$. □

Proposition 2.14 (Induced substructures)**.** *Let $W \subseteq V$ and $E|_W := \{e \in E : \ e \subseteq W\}$. Restrict σ and μ to W and $E|_W$, respectively. Then $\bigl(W, E|_W, \sigma|_W, \mu|_{E|_W}; \Psi, \Omega\bigr)$ is a FOSH (or a fuzzy n-SuperHyperGraph when $\Psi = 0, \Omega = 1$).*

Proof. All defining conditions are hereditary under restriction. □

Lemma 2.15 (Edge feasibility is downward closed)**.** *Fix σ. If $\mu_1, \mu_2 : E \to [\Psi, \Omega]$ satisfy $\mu_1 \le \mu_2 \le \mu^{\max}$, then any convex combination $\lambda\mu_1 + (1 - \lambda)\mu_2$ with $\lambda \in [0, 1]$ is admissible.*

Proof. For each e,

$$\lambda\mu_1(e) + (1 - \lambda)\mu_2(e) \;\le\; \lambda\mu^{\max}(e) + (1 - \lambda)\mu^{\max}(e) \;=\; \mu^{\max}(e).$$

□

Lemma 2.16 (Tightness on incident under-off vertices)**.** *If $e \in E$ has a vertex $v^- \in e$ with $\sigma(v^-) = \min_{u \in e} \sigma(u)$, then $\mu^{\max}(e) = \sigma(v^-)$. In particular, whenever $\sigma(v^-) < 0$ is unique on e, any admissible $\mu(e)$ must lie in $[\Psi, \sigma(v^-)] \subset (-\infty, 0)$.*

Proof. Direct from the definition of $\mu^{\max}$. □

Lemma 2.17 (Subgraph closure for off sets)**.** *Let $W \subseteq V$. Then*

$$\left(V^{\pm}_{\text{off}} \cap W\right) \text{ and } \left\{e \in E|_W : e \in E^{\pm}_{\text{off}}\right\}$$

are, respectively, the vertex and edge off sets of the induced FOSH on W.

Proof. Immediate from Definition 2.1 and Proposition 2.14. □

3 Conclusion

In this paper, we proposed an extension of the existing framework, termed the *Fuzzy Off-SuperHyperGraph*, which incorporates the offgraph paradigm into fuzzy n-SuperHyperGraphs. This novel structure broadens the scope of uncertainty modeling by allowing vertex and edge memberships to extend beyond the classical unit interval.

For future work, we plan to further develop the theory of Fuzzy Off-SuperHyperGraphs by exploring their integration with several advanced set-theoretic models, including HyperFuzzy Sets [35, 54–59], Neutrosophic Sets [60–62], Spherical Fuzzy Sets [63,64], Hesitant Fuzzy Sets [65–67], Rough Sets [68–70], Soft Sets [71–73], Quadri-Neutrosophic Sets [74,75], and Plithogenic Sets [76–80]. Such extensions are expected to provide richer tools for modeling complex, uncertain, and multi-dimensional systems, and open new directions for applications in decision-making, data analysis, and hierarchical network structures.

Funding

This study did not receive any financial or external support from organizations or individuals.

Acknowledgments

We extend our sincere gratitude to everyone who provided insights, inspiration, and assistance throughout this research. We particularly thank our readers for their interest and acknowledge the authors of the cited works for laying the foundation that made our study possible. We also appreciate the support from individuals and institutions that provided the resources and infrastructure needed to produce and share this paper. Finally, we are grateful to all those who supported us in various ways during this project.

Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

References

[1] Reinhard Diestel. *Graph theory*. Springer (print edition); Reinhard Diestel (eBooks), 2024.

[2] Jonathan L Gross, Jay Yellen, and Mark Anderson. *Graph theory and its applications*. Chapman and Hall/CRC, 2018.

[3] Claude Berge. *Hypergraphs: combinatorics of finite sets*, volume 45. Elsevier, 1984.

[4] Florentin Smarandache. *Extension of HyperGraph to n-SuperHyperGraph and to Plithogenic n-SuperHyperGraph, and Extension of HyperAlgebra to n-ary (Classical-/Neutro-/Anti-) HyperAlgebra*. Infinite Study, 2020.

[5] Takaaki Fujita and Florentin Smarandache. A concise study of some superhypergraph classes. *Neutrosophic Sets and Systems*, 77:548–593, 2024.

[6] Muhammad Akram, A Nagoor Gani, and A Borumand Saeid. Vague hypergraphs. *Journal of Intelligent & Fuzzy Systems*, 26(2):647–653, 2014.

[7] Yue Gao, Zizhao Zhang, Haojie Lin, Xibin Zhao, Shaoyi Du, and Changqing Zou. Hypergraph learning: Methods and practices. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 44(5):2548–2566, 2020.

[8] Yue Gao, Yifan Feng, Shuyi Ji, and Rongrong Ji. Hgnn+: General hypergraph neural networks. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 45(3):3181–3199, 2022.

[9] Yifan Feng, Haoxuan You, Zizhao Zhang, Rongrong Ji, and Yue Gao. Hypergraph neural networks. In *Proceedings of the AAAI conference on artificial intelligence*, volume 33, pages 3558–3565, 2019.

[10] Georg Gottlob, Nicola Leone, and Francesco Scarcello. Hypertree decompositions and tractable queries. In *Proceedings of the eighteenth ACM SIGMOD-SIGACT-SIGART symposium on Principles of database systems*, pages 21–32, 1999.

[11] Takaaki Fujita. Hypergraph and superhypergraph approaches in electronics: A hierarchical framework for modeling power-grid hypernetworks and superhypernetworks. *Journal of Energy Research and Reviews*, 17(6):102–136, 2025.

[12] Eduardo Martín Campoverde Valencia, Jessica Paola Chuisaca Vásquez, and Francisco Ángel Becerra Lois. Multineutrosophic analysis of the relationship between survival and business growth in the manufacturing sector of azuay province, 2020–2023, using plithogenic n-superhypergraphs. *Neutrosophic Sets and Systems*, 84:341–355, 2025.

[13] Masoud Ghods, Zahra Rostami, and Florentin Smarandache. Introduction to neutrosophic restricted superhypergraphs and neutrosophic restricted superhypertrees and several of their properties. *Neutrosophic Sets and Systems*, 50:480–487, 2022.

[14] Mohammad Hamidi, Florentin Smarandache, and Mohadeseh Taghinezhad. *Decision Making Based on Valued Fuzzy Superhypergraphs*. Infinite Study, 2023.

[15] Julio Cesar Méndez Bravo, Claudia Jeaneth Bolanos Piedrahita, Manuel Alberto Méndez Bravo, and Luis Manuel Pilacuan-Bonete. Integrating smed and industry 4.0 to optimize processes with plithogenic n-superhypergraphs. *Neutrosophic Sets and Systems*, 84:328–340, 2025.

[16] N. B. Nalawade, M. S. Bapat, S. G. Jakkewad, G. A. Dhanorkar, and D. J. Bhosale. Structural properties of zero-divisor hypergraph and superhypergraph over $\mathbb{Z}_n$: Girth and helly property. *Panamerican Mathematical Journal*, 35(4S):485, 2025.

[17] Takaaki Fujita. An introduction and reexamination of molecular hypergraph and molecular n-superhypergraph. *Asian Journal of Physical and Chemical Sciences*, 13(3):1–38, 2025.

[18] Berrocal Villegas Salomón Marcos, Montalvo Fritas Willner, Berrocal Villegas Carmen Rosa, Flores Fuentes Rivera María Yissel, Espejo Rivera Roberto, Laura Daysi Bautista Puma, and Dante Manuel Macazana Fernández. Using plithogenic n-superhypergraphs to assess the degree of relationship between information skills and digital competencies. *Neutrosophic Sets and Systems*, 84:513–524, 2025.

[19] Shouxian Zhu. Neutrosophic n-superhypernetwork: A new approach for evaluating short video communication effectiveness in media convergence. *Neutrosophic Sets and Systems*, 85:1004–1017, 2025.

[20] Takaaki Fujita. Review of probabilistic hypergraph and probabilistic superhypergraph. *Spectrum of Operational Research*, 3(1):319–338, 2026.

[21] Takaaki Fujita. Modeling hierarchical systems in graph signal processing, electric circuits, and bond graphs via hypergraphs and superhypergraphs. *Journal of Engineering Research and Reports*, 27(5):542, 2025.

[22] Takaaki Fujita and Florentin Smarandache. A unified framework for u-structures and functorial structure: Managing super, hyper, superhyper, tree, and forest uncertain over/under/off models. *Neutrosophic Sets and Systems*, 91:337–380, 2025.

[23] Takaaki Fujita, Atiqe Ur Rahman, Arkan A. Ghaib, Talal Ali Al-Hawary, and Arif Mehmood. On the properties and illustrative examples of soft superhypergraphs and rough superhypergraphs. *Prospects for Applied Mathematics and Data Analysis*, 5(1):12–31, 2025.

[24] Takaaki Fujita. Unifying grain boundary networks and crystal graphs: A hypergraph and superhypergraph perspective in material sciences. *Asian Journal of Advanced Research and Reports*, 19(5):344–379, 2025.

[25] Adel Al-Odhari. Neutrosophic power-set and neutrosophic hyper-structure of neutrosophic set of three types. *Annals of Pure and Applied Mathematics*, 31(2):125–146, 2025.

[26] Takaaki Fujita. Toward a unified framework for knot theory, hyperknot theory, and superhyperknot theory via superhyperstructures. *Neutrosophic Knowledge*, 6:55–71, 2025.

[27] Florentin Smarandache. The cardinal of the m-powerset of a set of n elements used in the superhyperstructures and neutrosophic superhyperstructures. *Systems Assessment and Engineering Management*, 2:19–22, 2024.

[28] Ajoy Kanti Das, Rajat Das, Suman Das, Bijoy Krishna Debnath, Carlos Granados, Bimal Shil, and Rakhal Das. A comprehensive study of neutrosophic superhyper bci-semigroups and their algebraic significance. *Transactions on Fuzzy Sets and Systems*, 8(2):80, 2025.

[29] Florentin Smarandache. Superhyperstructure & neutrosophic superhyperstructure, 2024. Accessed: 2024-12-01.

[30] Florentin Smarandache. *SuperHyperFunction, SuperHyperStructure, Neutrosophic SuperHyperFunction and Neutrosophic SuperHyperStructure: Current understanding and future directions*. Infinite Study, 2023.

[31] Alain Bretto. Hypergraph theory. *An introduction. Mathematical Engineering. Cham: Springer*, 1, 2013.

[32] Min Huang, Fenghua Li, et al. Modeling cross-cultural competence in vocational education internationalization using neutrosophic superhyperfunctions and big data driven cultural clusters. *Neutrosophic Sets and Systems*, 88:362–373, 2025.

[33] Yujuan Xie. A neutrosophic superhyper number framework for accurate statistical evaluation of financial performance in high-tech enterprises. *Neutrosophic Sets and Systems*, 88:906–918, 2025.

[34] Takaaki Fujita. Knowledge superhypergraphs, multimodal superhypergraphs, lattice superhypergraphs, and hyperbolic superhypergraphs: Concepts and applications. *Journal of Operational and Strategic Analytics*, 2025.

[35] Takaaki Fujita. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*. Biblio Publishing, 2025.

[36] Lotfi A Zadeh. Fuzzy sets. *Information and control*, 8(3):338–353, 1965.

[37] Lotfi A Zadeh. Fuzzy logic, neural networks, and soft computing. In *Fuzzy sets, fuzzy logic, and fuzzy systems: selected papers by Lotfi A Zadeh*, pages 775–782. World Scientific, 1996.

[38] TM Nishad, Talal Ali Al-Hawary, and B Mohamed Harif. General fuzzy graphs. *Ratio Mathematica*, 47, 2023.

[39] Azriel Rosenfeld. Fuzzy graphs. In *Fuzzy sets and their applications to cognitive and decision processes*, pages 77–95. Elsevier, 1975.

[40] Takaaki Fujita and Florentin Smarandache. A review of the hierarchy of plithogenic, neutrosophic, and fuzzy graphs: Survey and applications. In *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond (Second Volume)*. Biblio Publishing, 2024.

[41] Talal Al-Hawary. Complete fuzzy graphs. *International Journal of Mathematical Combinatorics*, 4:26, 2011.

[42] Mahdi Farshi and Bijan Davvaz. Generalized fuzzy hypergraphs and hypergroupoids. *Filomat*, 30(9):2375–2387, 2016.

[43] Sovan Samanta and Madhumangal Pal. Bipolar fuzzy hypergraphs. *International Journal of Fuzzy Logic Systems*, 2(1):17–28, 2012.

[44] Muhammad Akram and Anam Luqman. Fuzzy hypergraphs and related extensions. In *Studies in Fuzziness and Soft Computing*, 2020.

[45] John N Mordeson and Premchand S Nair. *Fuzzy graphs and fuzzy hypergraphs*, volume 46. Physica, 2012.

[46] Mohammed Alqahtani. Intuitionistic fuzzy quasi-supergraph integration for social network decision making. *International Journal of Analysis and Applications*, 23:137–137, 2025.

[47] Takaaki Fujita. Extensions of multidirected graphs: Fuzzy, neutrosophic, plithogenic, rough, soft, hypergraph, and superhypergraph variants. *International Journal of Topology*, 2(3):11, 2025.

[48] Takaaki Fujita and Florentin Smarandache. Directed n-superhypergraphs incorporating bipolar fuzzy information: A multi-tier framework for modeling bipolar uncertainty in complex networks. *Neutrosophic Sets and Systems*, 88:164–183, 2025.

[49] Takaaki Fujita. A review of fuzzy and neutrosophic offsets: Connections to some set concepts and normalization function. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 74, 2024.

[50] Florentin Smarandache, Miguel A. Quiroz-Martínez, Jesús Estupiñán, Noel Batista Hernández, and Maikel Yelandi Leyva Vazquez. Application of neutrosophic offsets for digital image processing. 2020.

[51] Florentin Smarandache. *Practical Applications of the Independent Neutrosophic Components and of the Neutrosophic Offset Components*. Infinite Study, 2021.

[52] Florentin Smarandache. Interval-valued neutrosophic oversets, neutrosophic undersets, and neutrosophic offsets. *Collected Papers. Volume IX: On Neutrosophic Theory and Its Applications in Algebra*, page 117, 2022.

[53] Florentin Smarandache. Degrees of membership¿ 1 and¡ 0 of the elements with respect to a neutrosophic offset. *Neutrosophic Sets and Systems*, 12:3–8, 2016.

[54] Jayanta Ghosh and Tapas Kumar Samanta. Hyperfuzzy sets and hyperfuzzy group. *Int. J. Adv. Sci. Technol*, 41:27–37, 2012.

[55] Yong Lin Liu, Hee Sik Kim, and J. Neggers. Hyperfuzzy subsets and subgroupoids. *J. Intell. Fuzzy Syst.*, 33:1553–1562, 2017.

[56] Z Nazari and B Mosapour. The entropy of hyperfuzzy sets. *Journal of Dynamical Systems and Geometric Theories*, 16(2):173–185, 2018.

[57] M MAHARIN. An over view on hyper fuzzy subgroups. *Scholar: National School of Leadership*, 9(1.2), 2020.

[58] Seok-Zun Song, Seon Jeong Kim, and Young Bae Jun. Hyperfuzzy ideals in bck/bci-algebras. *Mathematics*, 5(4):81, 2017.

[59] Young Bae Jun, Seok-Zun Song, and Seon Jeong Kim. Distances between hyper structures and length fuzzy ideals of bck/bci-algebras based on hyper structures. *Journal of Intelligent & Fuzzy Systems*, 35(2):2257–2268, 2018.

[60] Said Broumi, Mohamed Talea, Assia Bakali, and Florentin Smarandache. Single valued neutrosophic graphs. *Journal of New theory*, (10):86–101, 2016.

[61] Florentin Smarandache and Maissam Jdid. An overview of neutrosophic and plithogenic theories and applications. 2023.

[62] Said Broumi, Mohamed Talea, Assia Bakali, and Florentin Smarandache. Interval valued neutrosophic graphs. *Critical Review, XII*, 2016:5–33, 2016.

[63] Muhammad Akram, Danish Saleem, and Talal Al-Hawary. Spherical fuzzy graphs with application to decision-making. *Mathematical and Computational Applications*, 25(1):8, 2020.

[64] Fatma Kutlu Gündoğdu and Cengiz Kahraman. Spherical fuzzy sets and spherical fuzzy topsis method. *Journal of intelligent & fuzzy systems*, 36(1):337–352, 2019.

[65] Vicenç Torra and Yasuo Narukawa. On hesitant fuzzy sets and decision. In *2009 IEEE international conference on fuzzy systems*, pages 1378–1382. IEEE, 2009.

[66] Vicenç Torra. Hesitant fuzzy sets. *International journal of intelligent systems*, 25(6):529–539, 2010.

[67] Muhammad Akram, Arooj Adeel, and José Carlos R Alcantud. Hesitant fuzzy n-soft sets: A new model with applications in decision-making. *Journal of Intelligent & Fuzzy Systems*, 36(6):6113–6127, 2019.

[68] Zdzis law Pawlak. Rough sets. *International journal of computer & information sciences*, 11:341–356, 1982.

[69] Zdzis law Pawlak and Andrzej Skowron. Rudiments of rough sets. *Information sciences*, 177(1):3–27, 2007.

[70] Said Broumi, Florentin Smarandache, and Mamoni Dhar. Rough neutrosophic sets. *Infinite Study*, 32:493–502, 2014.

[71] Dmitriy Molodtsov. Soft set theory-first results. *Computers & mathematics with applications*, 37(4-5):19–31, 1999.

[72] Florentin Smarandache. Extension of soft set to hypersoft set, and then to plithogenic hypersoft set. *Neutrosophic sets and systems*, 22(1):168–170, 2018.

[73] Pradip Kumar Maji, Ranjit Biswas, and A Ranjan Roy. Soft set theory. *Computers & mathematics with applications*, 45(4-5):555–562, 2003.

[74] R Radha, A Stanis Arul Mary, and Florentin Smarandache. Quadripartitioned neutrosophic pythagorean soft set. *International Journal of Neutrosophic Science (IJNS) Volume 14, 2021*, page 11, 2021.

[75] Satham Hussain, Jahir Hussain, Isnaini Rosyida, and Said Broumi. Quadripartitioned neutrosophic soft graphs. In *Handbook of Research on Advances and Applications of Fuzzy Sets and Logic*, pages 771–795. IGI Global, 2022.

[76] Florentin Smarandache. *Plithogenic set, an extension of crisp, fuzzy, intuitionistic fuzzy, and neutrosophic sets-revisited*. Infinite study, 2018.

[77] Muhammad Azeem, Humera Rashid, Muhammad Kamran Jamil, Selma Gütmen, and Erfan Babaee Tirkolaee. Plithogenic fuzzy graph: A study of fundamental properties and potential applications. *Journal of Dynamics and Games*, pages 0–0, 2024.

[78] WB Vasantha Kandasamy, K Ilanthenral, and Florentin Smarandache. *Plithogenic Graphs*. Infinite Study, 2020.

[79] Prem Kumar Singh. *Intuitionistic Plithogenic Graph*. Infinite Study, 2022.

[80] Fazeelat Sultana, Muhammad Gulistan, Mumtaz Ali, Naveed Yaqoob, Muhammad Khan, Tabasam Rashid, and Tauseef Ahmed. A study of plithogenic graphs: applications in spreading coronavirus disease (covid-19) globally. *Journal of ambient intelligence and humanized computing*, 14(10):13139–13159, 2023.