MetaFuzzy, MetaNeutrosophic, MetaSoft, and MetaRough Set

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Abstract

A MetaStructure is a higher-level framework treating collections of structures as objects, with natural operations preserving isomorphisms across domains. An Iterated MetaStructure recursively applies MetaStructure construction, forming successive layers where structures of structures create deeper hierarchical meta-levels. In this paper, we define the MetaFuzzy Set, MetaNeutrosophic Set, MetaSoft Set, and MetaRough Set by extending Fuzzy Sets, Neutrosophic Sets, Soft Sets, and Rough Sets through the use of MetaStructure and Iterated MetaStructure.

Keywords: MetaFuzzy Set, MetaNeutrosophic Set, MetaSoft Set, MetaRough Set

1 Preliminaries

This section presents the fundamental concepts and definitions that underpin the discussions in this paper.

1.1 MetaStructure (Structure of Structure)

We first fix a general single-sorted, finitary *signature*

$$\Sigma = \big(\mathsf{Func},\ \mathsf{Rel},\ \mathrm{ar}_{\mathsf{Func}},\ \mathrm{ar}_{\mathsf{Rel}}\big),$$

where Func (resp. Rel) is a set of function (resp. relation) symbols, and ar records arities. A (single-sorted) Σ-*structure* is

$$\mathbf{C} = \big(H,\ (f^{\mathbf{C}})_{f\in\mathsf{Func}},\ (R^{\mathbf{C}})_{R\in\mathsf{Rel}}\big),$$

with carrier $H \neq \emptyset$, interpretations $f^{\mathbf{C}} : H^m \to H$ for each $f \in \mathsf{Func}$ of arity m, and relations $R^{\mathbf{C}} \subseteq H^r$ for each $R \in \mathsf{Rel}$ of arity r. Let Str_Σ denote the class of all Σ-structures.

Definition 1.1 (MetaStructure over a fixed signature)**.** (cf. [1]) Fix Σ as above. A *MetaStructure* ("structure of structures") over Σ is a pair

$$\mathbb{M} = \big(U,\ (\Phi_\ell)_{\ell\in\Lambda}\big),$$

where:

- U is a nonempty set with $U \subseteq \mathrm{Str}_\Sigma$ (its elements are *objects* at level 0);
- for each label $\ell \in \Lambda$ of *meta-arity* $k_\ell \in \mathbb{N}$, the *meta-operation*

  $$\Phi_\ell\ :\ U^{k_\ell} \longrightarrow U$$

  is specified by uniform *carrier- and symbol-constructors*:

  $$\begin{aligned}
  &\Gamma_\ell :\ (\mathbf{C}_1,\dots,\mathbf{C}_{k_\ell}) \mapsto H_\ell && \text{(new carrier } H_\ell \text{ built functorially);}\\
  &\forall f \in \mathsf{Func} :\quad f^{\Phi_\ell(\mathbf{C}_1,\dots,\mathbf{C}_{k_\ell})} = \Lambda^f_\ell\big(f^{\mathbf{C}_1},\dots,f^{\mathbf{C}_{k_\ell}}\big);\\
  &\forall R \in \mathsf{Rel} :\quad R^{\Phi_\ell(\mathbf{C}_1,\dots,\mathbf{C}_{k_\ell})} = \Xi^R_\ell\big(R^{\mathbf{C}_1},\dots,R^{\mathbf{C}_{k_\ell}}\big),
  \end{aligned}$$

  where Λ^f_ℓ and Ξ^R_ℓ are *uniform* recipes turning the symbols' interpretations on inputs into the symbol's interpretation on the output, over the new carrier H_ℓ.

Moreover, each Φ_ℓ is *isomorphism-invariant* (a.k.a. natural): if $\alpha_i : \mathbf{C}_i \cong \mathbf{D}_i$ for $1 \le i \le k_\ell$, then there is an induced isomorphism

$$\Phi_\ell(\alpha_1,\dots,\alpha_{k_\ell})\ :\ \Phi_\ell(\mathbf{C}_1,\dots,\mathbf{C}_{k_\ell}) \xrightarrow{\ \cong\ } \Phi_\ell(\mathbf{D}_1,\dots,\mathbf{D}_{k_\ell})$$

commuting with all interpretations of symbols of Σ.

Example 1.2 (MetaStructure on Graphs: disjoint union, Cartesian product, and line-graph)**.** Fix the graph signature

$$\Sigma_{\text{Graph}} = \big(\mathsf{Func} = \varnothing,\ \mathsf{Rel} = \{E\},\ \mathrm{ar}_{\mathsf{Rel}}(E) = 2\big),$$

where a Σ_{Graph}-structure is a (finite, simple, loopless, undirected) graph $\mathbf{G} = (V, E^{\mathbf{G}})$, encoded by a symmetric, irreflexive binary relation $E^{\mathbf{G}} \subseteq V \times V$. Let $U \subseteq \mathrm{Str}_{\Sigma_{\text{Graph}}}$ be the class of all such graphs. We define three meta-operations

$$\Phi_{\sqcup},\ \Phi_{\square}\ :\ U \times U \to U, \qquad \Phi_L\ :\ U \to U,$$

which together form a MetaStructure $\mathbb{M} = (U, \{\Phi_{\sqcup}, \Phi_{\square}, \Phi_L\})$ in the sense of the Definition.

1) Disjoint union $\Phi_{\sqcup}(\mathbf{G}_1, \mathbf{G}_2)$ (meta-arity $k_{\sqcup} = 2$). For inputs $\mathbf{G}_i = (V_i, E^{\mathbf{G}_i})$ set the new carrier by a tagged sum

$$\Gamma_{\sqcup}(\mathbf{G}_1, \mathbf{G}_2) = H_{\sqcup} = (V_1 \times \{1\}) \,\cup\, (V_2 \times \{2\}),$$

and define the relation constructor uniformly by

$$\Xi^E_{\sqcup}\big(E^{\mathbf{G}_1}, E^{\mathbf{G}_2}\big) = \big\{\, \big((u,1),(v,1)\big) : (u,v) \in E^{\mathbf{G}_1} \,\big\} \,\cup\, \big\{\, \big((u,2),(v,2)\big) : (u,v) \in E^{\mathbf{G}_2} \,\big\}.$$

No cross edges are added. Isomorphism-invariance is immediate from the functorial tagging.

2) Cartesian product $\Phi_{\square}(\mathbf{G}_1, \mathbf{G}_2)$ (meta-arity $k_{\square} = 2$). Set

$$\Gamma_{\square}(\mathbf{G}_1, \mathbf{G}_2) = H_{\square} = V_1 \times V_2,$$

and

$$\Xi^E_{\square}\big(E^{\mathbf{G}_1}, E^{\mathbf{G}_2}\big) = \Big\{ \big((u,x),(v,y)\big) :\ \big[u = v \wedge (x,y) \in E^{\mathbf{G}_2}\big] \ \vee\ \big[(u,v) \in E^{\mathbf{G}_1} \wedge x = y\big] \Big\}.$$

This is the usual Cartesian product of graphs; naturality holds componentwise.

3) Line-graph operator $\Phi_L(\mathbf{G})$ (meta-arity $k_L = 1$). For $\mathbf{G} = (V, E^{\mathbf{G}})$ let the new carrier be the edge set

$$\Gamma_L(\mathbf{G}) = H_L = E^{\mathbf{G}} \subseteq V \times V,$$

and define adjacency on edges by intersection of incident endpoints:

$$\Xi^E_L\big(E^{\mathbf{G}}\big) = \big\{\, (e_1, e_2) \in H_L \times H_L :\ e_1 \neq e_2 \ \text{ and } \ e_1 \cap e_2 \neq \varnothing \,\big\}.$$

This is the classical line-graph construction; isomorphism-invariance follows from edge-image preservation.

Tiny illustration. Let $\mathbf{P}_3$ be the path $a - b - c$ and $\mathbf{K}_2$ the single edge $x - y$. Then

$\Phi_{\sqcup}(\mathbf{P}_3, \mathbf{K}_2)$ has $|V| = 5$ with two components, $\qquad \Phi_L(\mathbf{P}_3) \ \cong\ \mathbf{P}_2, \qquad \Phi_{\square}(\mathbf{P}_3, \mathbf{K}_2) \ \cong\ $ ladder on 4 vertices.

Example 1.3 (MetaStructure on Groups: direct product and abelianization)**.** Fix the group signature

$$\Sigma_{\text{Grp}} = \big(\mathsf{Func} = \{\cdot,\ (\cdot)^{-1},\ e\},\ \mathsf{Rel} = \varnothing,\ \mathrm{ar}_{\mathsf{Func}}(\cdot) = 2,\ \mathrm{ar}_{\mathsf{Func}}((\cdot)^{-1}) = 1,\ \mathrm{ar}_{\mathsf{Func}}(e) = 0\big).$$

A Σ_{Grp}-structure is a group $\mathbf{G} = (G, \cdot^{\mathbf{G}}, (\cdot)^{-1,\mathbf{G}}, e^{\mathbf{G}})$. Let $U \subseteq \mathrm{Str}_{\Sigma_{\text{Grp}}}$ be the class of all (not-necessarily finite) groups. Define two meta-operations

$$\Phi_{\times}\ :\ U \times U \to U, \qquad \Phi_{\text{ab}}\ :\ U \to U,$$

yielding a MetaStructure $\mathbb{M} = (U, \{\Phi_{\times}, \Phi_{\text{ab}}\})$.

1) Direct product $\Phi_{\times}(\mathbf{G}_1, \mathbf{G}_2)$ (meta-arity $k_{\times} = 2$). For inputs $\mathbf{G}_i = (G_i, \cdot^{\mathbf{G}_i}, (\cdot)^{-1,\mathbf{G}_i}, e^{\mathbf{G}_i})$, set the carrier

$$\Gamma_{\times}(\mathbf{G}_1, \mathbf{G}_2) = H_{\times} = G_1 \times G_2,$$

and define uniformly, for all $(g_1, h_1), (g_2, h_2) \in G_1 \times G_2$,

$$\Lambda^{\cdot}_{\times}\big(\cdot^{\mathbf{G}_1}, \cdot^{\mathbf{G}_2}\big)\big((g_1,h_1),(g_2,h_2)\big) = (g_1 \cdot^{\mathbf{G}_1} g_2,\ h_1 \cdot^{\mathbf{G}_2} h_2),$$

$$\Lambda^{(\cdot)^{-1}}_{\times}\big((\cdot)^{-1,\mathbf{G}_1}, (\cdot)^{-1,\mathbf{G}_2}\big)(g,h) = \big(g^{-1,\mathbf{G}_1},\ h^{-1,\mathbf{G}_2}\big), \qquad \Lambda^{e}_{\times}\big(e^{\mathbf{G}_1}, e^{\mathbf{G}_2}\big) = (e^{\mathbf{G}_1}, e^{\mathbf{G}_2}).$$

This is the standard categorical product; naturality is by componentwise isomorphisms.

2) Abelianization $\Phi_{\mathrm{ab}}(\mathbf{G})$ (meta-arity $k_{\mathrm{ab}} = 1$). For $\mathbf{G} = (G, \cdot^{\mathbf{G}}, (\cdot)^{-1,\mathbf{G}}, e^{\mathbf{G}})$, let

$$[G, G] = \langle\, g^{-1}h^{-1}gh :\ g, h \in G \,\rangle \le G$$

be the commutator subgroup. Define the carrier as the quotient set

$$\Gamma_{\mathrm{ab}}(\mathbf{G}) = H_{\mathrm{ab}} = G/[G, G],$$

and the induced operations via the quotient map $\pi : G \to G/[G, G]$:

$$\Lambda^{\cdot}_{\mathrm{ab}}(\cdot^{\mathbf{G}})\big(\pi(g), \pi(h)\big) = \pi(g \cdot^{\mathbf{G}} h), \qquad \Lambda^{(\cdot)^{-1}}_{\mathrm{ab}}\big((\cdot)^{-1,\mathbf{G}}\big)\big(\pi(g)\big) = \pi\big(g^{-1,\mathbf{G}}\big), \qquad \Lambda^{e}_{\mathrm{ab}}\big(e^{\mathbf{G}}\big) = \pi(e^{\mathbf{G}}).$$

Well-definedness uses $[G, G] \trianglelefteq G$. The output is the abelian group G_{ab}; functoriality (naturality) follows from the universal property of abelianization.

Tiny illustration. For the dihedral group $D_4 = \langle r, s \mid r^4 = s^2 = 1,\ srs = r^{-1}\rangle$,

$$\Phi_{\mathrm{ab}}(D_4) \ \cong\ C_2 \times C_2, \qquad \Phi_{\times}(C_2, C_3) \ \cong\ C_6.$$

1.2 iterated MetaStructure(Structure of Structure of ... of Structure)

An Iterated MetaStructure recursively applies MetaStructure construction, forming successive layers where structures of structures create deeper hierarchical meta-levels (cf. [1, 2]).

Definition 1.4 (Iterated MetaStructure of depth t)**.** (cf. [1]) An *Iterated MetaStructure of depth t* over Σ is any MetaStructure $\mathfrak{M}^{(t)}$ of height t. When $s < t$, we *lift* a height-s MetaStructure $\mathfrak{M}^{(s)} = (U^{(s)}, \{\odot_i\}, \{\mathcal{S}_j\})$ to height t by

$$\iota_{s \to t} :\ U^{(s)} \xrightarrow{\mathsf{U}_\Sigma^{t-s}} U^{(t)} := \mathsf{U}_\Sigma^{t-s}\big(U^{(s)}\big),$$

and, for each $\odot_i : (\mathsf{E}_\Sigma^{m_i})^{k_i} \to \mathcal{P}^{n_i}(\mathsf{E}_\Sigma^{n_i})$, defining its lift

$$\odot_i^{\uparrow} :\ \big(\mathsf{E}_\Sigma^{m_i+t-s}\big)^{k_i} \longrightarrow \mathcal{P}^{n_i}\big(\mathsf{E}_\Sigma^{n_i+t-s}\big), \quad \odot_i^{\uparrow}\big(\mathsf{U}_\Sigma^{t-s}(x_1), \ldots, \mathsf{U}_\Sigma^{t-s}(x_{k_i})\big) := \mathsf{U}_\Sigma^{t-s}\big(\odot_i(x_1, \ldots, x_{k_i})\big),$$

and similarly for relations $\mathcal{S}_j^{\uparrow} := \big(\mathsf{U}_\Sigma^{t-s}\big)^{\times \ell_j}(\mathcal{S}_j)$.

Example 1.5 (Iterated MetaStructure on Graphs via the line-graph operator)**.** Fix the graph signature $\Sigma_{\mathrm{Graph}} = (\mathsf{Func} = \varnothing,\ \mathsf{Rel} = \{E\},\ \mathrm{ar}_{\mathsf{Rel}}(E) = 2)$, so a Σ_{Graph}-structure is a simple undirected graph $\mathbf{G} = (V, E^{\mathbf{G}})$. Let $\Phi_L : U^{(1)} \to U^{(1)}$ be the (level-1) line-graph meta-operation of Example 1.1: its carrier constructor makes the new carrier the old edge set, and its relation constructor connects two distinct edges iff they share a vertex.

To obtain an *Iterated MetaStructure of depth t* (Definition 1.4), we choose the canonical lift U_Σ to be identity-on-objects (so height only records iteration), and define the t-fold iterate

$$\Phi_L^{(t)} \ :=\ \underbrace{\Phi_L \circ \Phi_L \circ \cdots \circ \Phi_L}_{t \text{ times}} \ :\ U^{(1)} \longrightarrow U^{(1)}.$$

Concrete computation.

- For the path $\mathbf{P}_m$ (with m vertices and $m-1$ edges),

  $$\Phi_L(\mathbf{P}_m) \cong \mathbf{P}_{m-1}, \qquad \Phi_L^{(t)}(\mathbf{P}_m) \cong \mathbf{P}_{m-t} \quad \text{for } 1 \le t \le m-1.$$

  In particular, for $m = 5$ and $t = 2$,

  $$\Phi_L^{(2)}(\mathbf{P}_5) \cong \mathbf{P}_3 \quad (\text{vertex count: } 5 \to 4 \to 3).$$

- For the cycle $\mathbf{C}_n$ ($n \geq 3$),

$$\Phi_L(\mathbf{C}_n) \cong \mathbf{C}_n \quad \Rightarrow \quad \Phi_L^{(t)}(\mathbf{C}_n) \cong \mathbf{C}_n \text{ for all } t \geq 1.$$

A depth-2 object spelled out. Let $\mathbf{G}_1 = \mathbf{P}_4$ and $\mathbf{G}_2 = \mathbf{K}_2$. Form the (level-1) family $X = \{\mathbf{G}_1, \mathbf{G}_2, \mathbf{G}_1\}$ and define a level-1 meta-relation on X by "same edge-count":

$$\mathcal{S}(\mathbf{A}, \mathbf{B}) \iff |E^{\mathbf{A}}| = |E^{\mathbf{B}}|.$$

Apply Φ_L once to obtain

$$\Phi_L(X) = \{\, \Phi_L(\mathbf{P}_4) = \mathbf{P}_3,\ \Phi_L(\mathbf{K}_2) = \mathbf{K}_1,\ \Phi_L(\mathbf{P}_4) = \mathbf{P}_3 \,\},$$

and lift the relation by the same recipe ("same edge-count"): $|E^{\mathbf{P}_3}| = 2$, $|E^{\mathbf{K}_1}| = 0$, so the only meta-edge at depth 2 is between the two copies of $\mathbf{P}_3$. This explicitly realises an *iterated* (depth-2) MetaStructure built from Φ_L.

Example 1.6 (Iterated MetaStructure on Groups via direct product and abelianization)**.** Fix the group signature $\Sigma_{\mathrm{Grp}} = (\mathsf{Func} = \{\cdot, (\cdot)^{-1}, e\},\ \mathsf{Rel} = \varnothing)$. Let $U^{(1)}$ be the class of all groups, and consider two level-1 meta-operations:

$$\Phi_\times(\mathbf{G}, \mathbf{H}) = \mathbf{G} \times \mathbf{H} \quad \text{(direct product)}, \qquad \Phi_{\mathrm{ab}}(\mathbf{G}) = \mathbf{G}/[\mathbf{G}, \mathbf{G}] \quad \text{(abelianization)}.$$

As in Example 1.1, carriers and symbols are constructed uniformly (product set, pointwise operations; quotient by the commutator subgroup).

To produce an *Iterated MetaStructure of depth t* (Definition 1.4), we again take the canonical lift U_Σ to be identity-on-objects and define iterates

$$\Phi_{\mathrm{ab}}^{(t)} := \underbrace{\Phi_{\mathrm{ab}} \circ \cdots \circ \Phi_{\mathrm{ab}}}_{t \text{ times}}, \qquad \Phi_\times^{(t)} := \underbrace{\Phi_\times \circ \cdots \circ \Phi_\times}_{t-1 \text{ binary uses}}.$$

Concrete computation (depth $t = 2$).

- Start with the non-abelian groups $\mathbf{G}_0 = S_3$ and $\mathbf{H}_0 = D_4$.
- First abelianize (level 1):
$$\Phi_{\mathrm{ab}}(\mathbf{G}_0) \cong C_2, \qquad \Phi_{\mathrm{ab}}(\mathbf{H}_0) \cong C_2 \times C_2.$$
- Second abelianization stabilizes (level 2):
$$\Phi_{\mathrm{ab}}^{(2)}(\mathbf{G}_0) \cong C_2, \qquad \Phi_{\mathrm{ab}}^{(2)}(\mathbf{H}_0) \cong C_2 \times C_2,$$
since abelianization is idempotent up to isomorphism.
- Combine by the (binary) meta-operation at depth 2:
$$\Phi_\times\Big(\Phi_{\mathrm{ab}}^{(2)}(\mathbf{G}_0),\ \Phi_{\mathrm{ab}}^{(2)}(\mathbf{H}_0)\Big) \ \cong\ C_2 \ \times\ (C_2 \times C_2) \ \cong\ C_2^3.$$
In terms of orders: $|S_3| = 6$, $|D_4| = 8$, $|C_2| = 2$, $|C_2 \times C_2| = 4$, hence
$$\big|\Phi_\times(\Phi_{\mathrm{ab}}^{(2)}(S_3), \Phi_{\mathrm{ab}}^{(2)}(D_4))\big| = 2 \cdot 4 = 8.$$

Thus the pair of iterated meta-operations $(\Phi_{\mathrm{ab}}^{(t)}, \Phi_\times)$ yields a concrete *depth*-2 MetaStructure on groups, with explicit carriers and operations at each stage.

2 Main Results: Meta Set

In this section, we present the main results of this paper, focusing on discussions related to the concept of Meta Sets.

2.1 MetaFuzzy Set (Fuzzy Set of Fuzzy Sets)

A Fuzzy Set generalizes classical sets by assigning to each element a membership degree between zero and one, thereby representing partial inclusion [3–5]. Because of their importance, Fuzzy Sets have inspired a wide range of extensions, such as Bipolar Fuzzy Sets [6, 7], Intuitionistic Fuzzy Sets [8–10], HyperFuzzy Sets [11–15], Hesitant Fuzzy Sets [16, 17], Picture Fuzzy Sets [18–20], Complex Fuzzy Sets [21–23], and Pythagorean Fuzzy Sets [24–26], among others.

A MetaFuzzy Set further extends this line of research by assigning membership values not to individual elements but to entire fuzzy sets, thereby enabling higher-level reasoning about collections of fuzziness across diverse contexts.

Definition 2.1 (Fuzzy Set)**.** [3, 27] Let Y be a nonempty domain. A *fuzzy set* is given by a function

$$\mu\colon Y \longrightarrow [0,1],$$

where $\mu(y)$ measures the degree to which y belongs to the set. A *fuzzy relation* on Y is a function $\delta\colon Y \times Y \to [0,1]$, viewed as a fuzzy subset of $Y \times Y$. We say δ is a *fuzzy relation on* μ if for every $y, z \in Y$,

$$\delta(y,z) \;\le\; \min\{\mu(y),\, \mu(z)\}.$$

Definition 2.2 (MetaFuzzy Set)**.** Fix a nonempty base domain Y. A *MetaFuzzy Set on* Y is a map

$$\mathrm{M}^{\sharp}:\ \mathsf{Fuz}(Y) \longrightarrow [0,1].$$

A *MetaFuzzy relation* on Y is $\Delta^{\sharp} : \mathsf{Fuz}(Y) \times \mathsf{Fuz}(Y) \to [0,1]$. We say $\Delta^{\sharp}$ is a *MetaFuzzy relation on* $\mathrm{M}^{\sharp}$ if

$$\Delta^{\sharp}(\mu_1,\mu_2) \;\le\; \min\{\mathrm{M}^{\sharp}(\mu_1),\, \mathrm{M}^{\sharp}(\mu_2)\} \qquad (\forall \mu_1,\mu_2 \in \mathsf{Fuz}(Y)).$$

Example 2.3 (MetaFuzzy Set: Weekly traffic congestion severity)**.** Let $Y = \{\mathrm{Mon}, \mathrm{Tue}, \mathrm{Wed}\}$ be three commuting days. A fuzzy set $\mu \in \mathsf{Fuz}(Y) = [0,1]^Y$ represents the degree of *heavy congestion* per day. Consider two weeks:

$$\mu^{(A)}(\mathrm{Mon},\mathrm{Tue},\mathrm{Wed}) = (0.2,\ 0.8,\ 0.6), \qquad \mu^{(B)}(\mathrm{Mon},\mathrm{Tue},\mathrm{Wed}) = (0.9,\ 0.7,\ 0.3).$$

Define a MetaFuzzy Set (weekly severity aggregator) by the arithmetic mean

$$\mathrm{M}^{\sharp} : \mathsf{Fuz}(Y) \to [0,1], \qquad \mathrm{M}^{\sharp}(\mu) := \frac{\mu(\mathrm{Mon}) + \mu(\mathrm{Tue}) + \mu(\mathrm{Wed})}{3}.$$

Then the meta-memberships are

$$\mathrm{M}^{\sharp}\big(\mu^{(A)}\big) = \frac{0.2+0.8+0.6}{3} = \frac{1.6}{3} \approx 0.533\overline{3}, \qquad \mathrm{M}^{\sharp}\big(\mu^{(B)}\big) = \frac{0.9+0.7+0.3}{3} = \frac{1.9}{3} \approx 0.633\overline{3}.$$

Optionally, define a MetaFuzzy relation (pairwise co-severity) by

$$\Delta^{\sharp}(\mu_1,\mu_2) := \min\{\mu_1(\mathrm{Tue}),\, \mu_2(\mathrm{Tue})\}.$$

For the above two weeks,

$$\Delta^{\sharp}\big(\mu^{(A)},\mu^{(B)}\big) = \min\{0.8,\ 0.7\} = 0.7 \;\le\; \min\{\mathrm{M}^{\sharp}(\mu^{(A)}),\ \mathrm{M}^{\sharp}(\mu^{(B)})\} = \min\{0.533\overline{3},\ 0.633\overline{3}\} = 0.533\overline{3},$$

verifying the admissibility inequality $\Delta^{\sharp} \le \min\{\mathrm{M}^{\sharp}(\cdot), \mathrm{M}^{\sharp}(\cdot)\}$ in this concrete setting.

2.2 MetaNeutrosophic Set(Neutrosophic set of Neutrosophic Set)

A Neutrosophic Set extends fuzzy sets by assigning each element three independent degrees: truth, indeterminacy, and falsity, enabling richer uncertainty modeling [28–30]. As extensions of this concept, Bipolar Neutrosophic Sets [31–34], Complex Neutrosophic Sets [35–37], Neutrosophic Offset [38–40], and Double-Valued Neutrosophic Sets [41–46] are well known.

A MetaNeutrosophic Set evaluates neutrosophic sets themselves, producing truth, indeterminacy, and falsity degrees for collections of neutrosophic information.

Definition 2.4 (Neutrosophic Set). [47–49] Let X be a non-empty set. A *Neutrosophic Set (NS) A* on X is characterized by three membership functions:

$$T_A : X \to [0,1], \quad I_A : X \to [0,1], \quad F_A : X \to [0,1],$$

where for each $x \in X$, the values $T_A(x)$, $I_A(x)$, and $F_A(x)$ represent the degrees of truth, indeterminacy, and falsity, respectively. These values satisfy the following condition:

$$0 \le T_A(x) + I_A(x) + F_A(x) \le 3.$$

Definition 2.5 (MetaNeutrosophic Set). Fix a nonempty base X. A *MetaNeutrosophic Set on X* is a triple of maps

$$T^{\sharp},\ I^{\sharp},\ F^{\sharp} :\ \mathsf{Neu}(X) \longrightarrow [0,1]$$

such that for every $A = (T_A, I_A, F_A) \in \mathsf{Neu}(X)$,

$$0\ \le\ T^{\sharp}(A) + I^{\sharp}(A) + F^{\sharp}(A)\ \le\ 3.$$

Example 2.6 (MetaNeutrosophic Set: Supplier compliance assessment). Let $X = \{\mathrm{SpecA}, \mathrm{SpecB}\}$ be two technical specifications. A neutrosophic set $A = (T_A, I_A, F_A) \in \mathsf{Neu}(X)$ encodes, for each spec $x \in X$, the degrees of *truth* $T_A(x)$ (compliant), *indeterminacy* $I_A(x)$ (uncertain), and *falsity* $F_A(x)$ (non-compliant), subject to $0 \le T_A(x) + I_A(x) + F_A(x) \le 3$.

Suppose a supplier is evaluated as

$$(T_A(\mathrm{SpecA}), I_A(\mathrm{SpecA}), F_A(\mathrm{SpecA})) = (0.7,\ 0.2,\ 0.1),$$
$$(T_A(\mathrm{SpecB}), I_A(\mathrm{SpecB}), F_A(\mathrm{SpecB})) = (0.5,\ 0.3,\ 0.2).$$

Define a MetaNeutrosophic Set $\mathbf{N}^{\sharp} = (T^{\sharp}, I^{\sharp}, F^{\sharp}) : \mathsf{Neu}(X) \to [0,1]^3$ by

$$T^{\sharp}(A) := \frac{T_A(\mathrm{SpecA}) + T_A(\mathrm{SpecB})}{2},$$
$$I^{\sharp}(A) := \max\{I_A(\mathrm{SpecA}), I_A(\mathrm{SpecB})\},$$
$$F^{\sharp}(A) := \frac{F_A(\mathrm{SpecA}) + F_A(\mathrm{SpecB})}{2}.$$

Numerically,

$$T^{\sharp}(A) = \tfrac{0.7+0.5}{2} = 0.6, \quad I^{\sharp}(A) = \max\{0.2, 0.3\} = 0.3, \quad F^{\sharp}(A) = \tfrac{0.1+0.2}{2} = 0.15.$$

The meta-sum constraint holds:

$$T^{\sharp}(A) + I^{\sharp}(A) + F^{\sharp}(A) = 0.6 + 0.3 + 0.15 = 1.05\ \le\ 3.$$

Thus $(T^{\sharp}, I^{\sharp}, F^{\sharp})$ yields a concrete meta-evaluation of compliance, uncertainty, and non-compliance aggregated across specs.

2.3 MetaSoft Set(Soft Set of Soft Set)

A Soft Set represents uncertain information using a parameterized family of subsets, mapping attributes to corresponding approximate descriptions within universes [50–53]. Related concepts include HyperSoft Sets [54–57], Bipolar Soft Sets [58, 59], SuperHyperSoft Sets [60–62], and TreeSoft Sets [63–66], which are well known in the literature.

A MetaSoft Set selects or groups multiple soft sets under meta-parameters, providing higher-order decisions about uncertain attribute-based data.

Definition 2.7 (Soft Set). [50] Let U be a finite universal set and A be a set of attributes. Let $S \subseteq A$ denote a chosen subset of parameters. A *soft set* over U is defined as a pair $(\mathcal{F}, S)$ where

$$\mathcal{F} : S \to \mathcal{P}(U)$$

is a function that assigns to each parameter $\alpha \in S$ a subset $\mathcal{F}(\alpha) \subseteq U$. Formally,

$$(\mathcal{F}, S) = \{\, (\alpha, \mathcal{F}(\alpha)) \mid \alpha \in S,\ \mathcal{F}(\alpha) \subseteq U \,\}.$$

Definition 2.8 (MetaSoft Set)**.** Fix a universe U and parameter set S for base soft sets, and fix a nonempty *meta-parameter* set Π. A *MetaSoft Set on* (U, S) *with meta-parameters* Π is a soft set

$$(\mathcal{G}, \Pi) \quad \text{with} \quad \mathcal{G} : \Pi \longrightarrow \mathcal{P}\big(\mathsf{Soft}(U, S)\big).$$

Equivalently, each $\pi \in \Pi$ selects a *family of soft sets* $\mathcal{G}(\pi) \subseteq \mathsf{Soft}(U, S)$.

Example 2.9 (MetaSoft Set: Hotel selection by meta-criteria)**.** Let $U = \{h_1, h_2, h_3\}$ be hotels and $S = \{\text{NearStation}, \text{Breakfast}, \text{Quiet}\}$ be attributes. A soft set $\mathcal{F} : S \to \mathcal{P}(U)$ records which hotels satisfy each attribute.

Consider two candidate soft sets:

$$\begin{aligned} &\mathcal{F}^{(A)}(\text{NearStation}) = \{h_1, h_2\}, \quad \mathcal{F}^{(A)}(\text{Breakfast}) = \{h_2, h_3\}, \quad \mathcal{F}^{(A)}(\text{Quiet}) = \{h_3\}, \\ &\mathcal{F}^{(B)}(\text{NearStation}) = \{h_1\}, \qquad \mathcal{F}^{(B)}(\text{Breakfast}) = \{h_3\}, \qquad \mathcal{F}^{(B)}(\text{Quiet}) = \{h_2, h_3\}. \end{aligned}$$

Define a MetaSoft Set $(\mathcal{G}, \Pi)$ with one meta-parameter $\Pi = \{\pi^\star\}$ that selects those soft sets which certify h_2 *under both NearStation and Breakfast*:

$$\mathcal{G}(\pi^\star) := \Big\{ \mathcal{F} \in \mathsf{Soft}(U, S) \;\Big|\; h_2 \in \mathcal{F}(\text{NearStation}) \text{ and } h_2 \in \mathcal{F}(\text{Breakfast}) \Big\}.$$

Verification:

$$h_2 \in \mathcal{F}^{(A)}(\text{NearStation}) \cap \mathcal{F}^{(A)}(\text{Breakfast}) = \{h_2\} \;\Rightarrow\; \mathcal{F}^{(A)} \in \mathcal{G}(\pi^\star).$$

But for $\mathcal{F}^{(B)}$, we have $h_2 \notin \mathcal{F}^{(B)}(\text{Breakfast})$, hence $\mathcal{F}^{(B)} \notin \mathcal{G}(\pi^\star)$. Thus $(\mathcal{G}, \Pi)$ is a concrete MetaSoft selector over hotel-attribute soft descriptions.

2.4 MetaRough Set(Rough Set of Rough Set)

A Rough Set models uncertainty by approximating subsets using lower and upper approximations derived from indiscernibility relations on universes [67–69]. Related concepts include HyperRough Sets [70–75], Weighted Rough Sets [76–78], Fuzzy Rough Sets [79–81], and Soft Rough Sets, which are well known in the literature.

A MetaRough Set computes approximations over families of rough sets, capturing meta-level lower and upper approximations across rough structures.

Definition 2.10 (Rough Set Approximation)**.** [82] Let X be a nonempty universe of discourse, and let $R \subseteq X \times X$ be an equivalence relation (also called an indiscernibility relation) on X. The relation R partitions X into disjoint equivalence classes, denoted by $[x]_R$ for each $x \in X$, where

$$[x]_R = \{y \in X \mid (x, y) \in R\}.$$

For any subset $U \subseteq X$, the *lower approximation* $\underline{U}$ and the *upper approximation* $\overline{U}$ are defined by:

1. *Lower Approximation*:
$$\underline{U} = \{x \in X \mid [x]_R \subseteq U\}.$$
This set contains all elements whose entire equivalence class is contained within U; these elements *definitely* belong to U.

2. *Upper Approximation*:
$$\overline{U} = \{x \in X \mid [x]_R \cap U \neq \emptyset\}.$$
This set contains all elements whose equivalence class has a nonempty intersection with U; these elements *possibly* belong to U.

Thus, the pair $(\underline{U}, \overline{U})$ forms the rough set representation of U, satisfying

$$\underline{U} \subseteq U \subseteq \overline{U}.$$

Definition 2.11 (Meta-indiscernibility on rough objects)**.** Fix (X, R) as in the Definition. Let $\mathcal{E}$ be an *equivalence relation* on $\mathsf{Rough}(X, R)$. Typical choices include:

- Equality: $(\underline{U}, \overline{U})\ \mathcal{E}\ (\underline{V}, \overline{V})$ iff they are equal.
- Boundary equality: $(\underline{U}, \overline{U})\ \mathcal{E}\ (\underline{V}, \overline{V})$ iff $\overline{U} \setminus \underline{U} = \overline{V} \setminus \underline{V}$.

For $r \in \mathsf{Rough}(X, R)$, denote its $\mathcal{E}$-class by $[r]_{\mathcal{E}}$.

Definition 2.12 (MetaRough approximations and MetaRough Set)**.** Let $\mathcal{E}$ be as above and let $C \subseteq \mathsf{Rough}(X, R)$ be any subfamily (a set of rough objects). Define the *meta-lower* and *meta-upper* approximations of C by

$$\underline{C}^{\,\mathcal{E}} := \{\, r \in \mathsf{Rough}(X, R) \mid [r]_{\mathcal{E}} \subseteq C \,\}, \qquad \overline{C}^{\,\mathcal{E}} := \{\, r \in \mathsf{Rough}(X, R) \mid [r]_{\mathcal{E}} \cap C \neq \emptyset \,\}.$$

The pair $\big(\underline{C}^{\,\mathcal{E}}, \overline{C}^{\,\mathcal{E}}\big)$ is called the *MetaRough Set* of C with respect to $\mathcal{E}$.

Proposition 2.13 (Basic sandwich property at meta-level)**.** *For every $C \subseteq \mathsf{Rough}(X, R)$ and every equivalence $\mathcal{E}$ on $\mathsf{Rough}(X, R)$,*

$$\underline{C}^{\,\mathcal{E}} \subseteq C \subseteq \overline{C}^{\,\mathcal{E}}.$$

Proof. ($\underline{C}^{\,\mathcal{E}} \subseteq C$) Let $r \in \underline{C}^{\,\mathcal{E}}$. Then by definition $[r]_{\mathcal{E}} \subseteq C$. Since $r \in [r]_{\mathcal{E}}$, we get $r \in C$.

($C \subseteq \overline{C}^{\,\mathcal{E}}$) Let $r \in C$. Then $[r]_{\mathcal{E}} \cap C \supseteq \{r\} \neq \emptyset$, hence $r \in \overline{C}^{\,\mathcal{E}}$.

Both inclusions are by direct element-chasing; no further conditions are needed. □

Example 2.14 (MetaRough Set: Students' pass status by homeroom)**.** Let $X = \{s_1, s_2, s_3, s_4\}$ be students partitioned by homeroom into

$$R\text{-classes:} \quad [s_1]_R = \{s_1, s_2\}, \qquad [s_3]_R = \{s_3, s_4\}.$$

Let $U \subseteq X$ denote those who *passed* a mock exam but are only partially observed. Take

$$U = \{s_1\}.$$

The rough approximations of U are

$$\underline{U} = \{x \in X \mid [x]_R \subseteq U\} = \emptyset, \qquad \overline{U} = \{x \in X \mid [x]_R \cap U \neq \emptyset\} = \{s_1, s_2\}.$$

Thus the rough object is $r_U = (\underline{U}, \overline{U}) = (\emptyset, \{s_1, s_2\})$.

Define a family $C \subseteq \mathsf{Rough}(X, R)$ of interest as

$$C := \big\{ (\emptyset, \{s_1, s_2\}) \big\} \quad (\text{"only homeroom } \{s_1, s_2\} \text{ possibly passed"}).$$

Choose the meta-indiscernibility $\mathcal{E}$ on $\mathsf{Rough}(X, R)$ to be *equality*:

$$r\,\mathcal{E}\,r' \iff r = r'.$$

Then the MetaRough approximations (Definition of meta-lower/upper) are

$$\underline{C}^{\,\mathcal{E}} = \{\, r \mid [r]_{\mathcal{E}} \subseteq C \,\} = \{\, r \mid \{r\} \subseteq C \,\} = C,$$

$$\overline{C}^{\,\mathcal{E}} = \{\, r \mid [r]_{\mathcal{E}} \cap C \neq \emptyset \,\} = \{\, r \mid \{r\} \cap C \neq \emptyset \,\} = C.$$

Hence the MetaRough Set equals $(\underline{C}^{\,\mathcal{E}}, \overline{C}^{\,\mathcal{E}}) = (C, C)$, and $r_U \in C$ explicitly realizes this case. This exhibits a concrete meta-level rough description over classroom-based indiscernibility.

3 Results: Iterated Meta-Objects

Throughout, let $Y \neq \emptyset$ (base domain for fuzzy sets), $X \neq \emptyset$ (base domain for neutrosophic sets), $U \neq \emptyset$ (universe for soft sets) with parameter set $S \neq \emptyset$, and (X, R) a Pawlak approximation space. We recall the base spaces:

$$\mathsf{Fuz}(Y) := [0,1]^Y, \qquad \mathsf{Neu}(X) := \big\{(T, I, F) : X \to [0,1] \;\big|\; 0 \le T + I + F \le 3\big\},$$

$$\mathsf{Soft}(U, S) := \{\, \mathcal{F} : S \to \mathcal{P}(U) \,\}, \qquad \mathsf{Rough}(X, R) := \{\, (\underline{V}, \overline{V}) \mid V \subseteq X \,\}.$$

3.1 Iterated MetaFuzzy Sets

An Iterated MetaFuzzy Set recursively applies fuzzy evaluation to fuzzy sets themselves, producing hierarchical layers of meta-level membership analysis.

Definition 3.1 (Hierarchy of fuzzy universes)**.** Define inductively the sequence of carrier sets $(\mathsf{Fuz}^{\langle t\rangle}(Y))_{t\ge 0}$ by

$$\mathsf{Fuz}^{\langle 0\rangle}(Y) := \mathsf{Fuz}(Y), \qquad \mathsf{Fuz}^{\langle t+1\rangle}(Y) := [0,1]^{\mathsf{Fuz}^{\langle t\rangle}(Y)}.$$

Definition 3.2 (Iterated MetaFuzzy Set of depth t)**.** For $t \ge 1$, an *Iterated MetaFuzzy Set (IMF) of depth t on Y* is an element

$$\mathrm{M}^{(t)} \in \mathsf{Fuz}^{\langle t\rangle}(Y) \;=\; [0,1]^{\mathsf{Fuz}^{\langle t-1\rangle}(Y)}.$$

Equivalently, $\mathrm{M}^{(t)} : \mathsf{Fuz}^{\langle t-1\rangle}(Y) \to [0,1]$ assigns a membership grade to each level-$(t-1)$ fuzzy object. A *binary iterated MetaFuzzy relation* of depth t is any map

$$\Delta^{(t)} :\; \mathsf{Fuz}^{\langle t-1\rangle}(Y) \times \mathsf{Fuz}^{\langle t-1\rangle}(Y) \longrightarrow [0,1]$$

that satisfies the admissibility constraint

$$\Delta^{(t)}(a, b) \;\le\; \min\big\{\mathrm{M}^{(t)}(a),\, \mathrm{M}^{(t)}(b)\big\} \qquad (\forall a, b \in \mathsf{Fuz}^{\langle t-1\rangle}(Y)).$$

Example 3.3 (Iterated MetaFuzzy Set (depth 2): Store staffing based on daily "high traffic")**.** Let $Y = \{\mathrm{Mon}, \mathrm{Tue}, \mathrm{Wed}\}$ be three trading days. A level-0 fuzzy set $\mu \in \mathsf{Fuz}(Y) = [0,1]^Y$ encodes the degree of "high customer traffic" per day. Define the depth-2 meta-membership (a level-2 Iterated MetaFuzzy Set)

$$\mathcal{M}^{(2)} :\; \mathsf{Fuz}^{\langle 1\rangle}(Y) = [0,1]^{\mathsf{Fuz}(Y)} \;\longrightarrow\; [0,1], \qquad \mathcal{M}^{(2)}(\mu) \;:=\; \frac{1}{|Y|}\sum_{y\in Y} \mu(y)^2.$$

This maps each *first-level* fuzzy week-profile μ to a single score in $[0,1]$ that upweights very high congestion (via squaring). Consider two weeks:

$$\mu^{(A)} = (0.20,\ 0.80,\ 0.60), \qquad \mu^{(B)} = (0.90,\ 0.70,\ 0.30).$$

Then

$$\mathcal{M}^{(2)}(\mu^{(A)}) = \tfrac{1}{3}(0.20^2 + 0.80^2 + 0.60^2) = \tfrac{1}{3}(0.04 + 0.64 + 0.36) = \tfrac{1.04}{3} \approx 0.3467,$$

$$\mathcal{M}^{(2)}(\mu^{(B)}) = \tfrac{1}{3}(0.90^2 + 0.70^2 + 0.30^2) = \tfrac{1}{3}(0.81 + 0.49 + 0.09) = \tfrac{1.39}{3} \approx 0.4633.$$

Interpretation: Week B warrants higher staffing overall (higher meta-score), because its daily "high traffic" degrees are more extreme even if some days are moderate.

Theorem 3.4 (Generalization of MetaFuzzy and Iterated MetaStructure)**.** *(a) For $t = 1$, Iterated MetaFuzzy Sets coincide with MetaFuzzy Sets (Definition 2.2):*

$$\mathsf{Fuz}^{\langle 1\rangle}(Y) = [0,1]^{\mathsf{Fuz}(Y)} \quad\Longleftrightarrow\quad \textit{MetaFuzzy Sets on } Y.$$

(b) For every $t \ge 1$, the class $\mathsf{Fuz}^{\langle t\rangle}(Y)$ with the evaluation operations

$$\Phi^{(t)}_{\mathrm{ev}} :\; \mathsf{Fuz}^{\langle t\rangle}(Y) \times \mathsf{Fuz}^{\langle t-1\rangle}(Y) \to [0,1], \qquad \Phi^{(t)}_{\mathrm{ev}}(\mathrm{M}^{(t)}, a) := \mathrm{M}^{(t)}(a),$$

forms an Iterated MetaStructure *of depth t in the sense of Definition 1.4.*

Proof. (a) By Definition 3.1, $\mathsf{Fuz}^{\langle 1\rangle}(Y) = [0,1]^{\mathsf{Fuz}^{(0)}(Y)} = [0,1]^{\mathsf{Fuz}(Y)}$. By Definition 2.2, a MetaFuzzy Set is precisely such a map. Hence equality of notions.

(b) We prove by induction on t. For $t = 1$, the universe is $U^{(1)} := \mathsf{Fuz}^{\langle 1\rangle}(Y)$; objects are maps $\mathrm{M}^{(1)} : \mathsf{Fuz}(Y) \to [0,1]$. Morphisms (isomorphisms) at level 0 are bijections $\alpha : \mathsf{Fuz}(Y) \to \mathsf{Fuz}(Y)$ induced by base isomorphisms on Y (pushforwards along bijections of Y). Naturality of $\Phi^{(1)}_{\mathrm{ev}}$ holds since

$$\mathrm{M}^{(1)} \circ \alpha^{-1}(a) \;=\; \mathrm{M}^{(1)}(\alpha^{-1}(a))$$

commutes with the action on arguments. Assume the claim for $t-1$. For t, the universe is $U^{(t)} := \mathsf{Fuz}^{\langle t\rangle}(Y) = [0,1]^{U^{(t-1)}}$, i.e., level-$t$ objects are functions on $U^{(t-1)}$. Any isomorphism $\beta : U^{(t-1)} \to U^{(t-1)}$ (arising from level-($t-1$) isomorphisms by the induction hypothesis) induces $\beta^* : [0,1]^{U^{(t-1)}} \to [0,1]^{U^{(t-1)}}$ by precomposition $(\beta^*\mathrm{M}^{(t)}) := \mathrm{M}^{(t)} \circ \beta^{-1}$. Then evaluation satisfies

$$\Phi^{(t)}_{\mathrm{ev}}\big(\beta^*\mathrm{M}^{(t)},\, \beta(a)\big) = (\mathrm{M}^{(t)} \circ \beta^{-1})(\beta(a)) = \mathrm{M}^{(t)}(a) = \Phi^{(t)}_{\mathrm{ev}}\big(\mathrm{M}^{(t)}, a\big),$$

so $\Phi^{(t)}_{\mathrm{ev}}$ is isomorphism-invariant. Hence $(U^{(t)}, \Phi^{(t)}_{\mathrm{ev}})$ is a depth-t Iterated MetaStructure. □

3.2 Iterated MetaNeutrosophic Sets

An Iterated MetaNeutrosophic Set evaluates neutrosophic sets at successive meta-levels, recursively capturing truth, indeterminacy, and falsity across deeper hierarchical structures.

Definition 3.5 (Hierarchy of neutrosophic universes)**.** Set $\mathsf{Neu}^{\langle 0\rangle}(X) := \mathsf{Neu}(X)$. For $t \geq 0$, define

$$\mathsf{Neu}^{\langle t+1\rangle}(X) \;:=\; \big\{ (T^{(t+1)}, I^{(t+1)}, F^{(t+1)}) \;\big|\; T^{(t+1)}, I^{(t+1)}, F^{(t+1)} : \mathsf{Neu}^{\langle t\rangle}(X) \to [0,1],$$
$$0 \leq T^{(t+1)}(A) + I^{(t+1)}(A) + F^{(t+1)}(A) \leq 3,\ \forall A \in \mathsf{Neu}^{\langle t\rangle}(X) \big\}.$$

Definition 3.6 (Iterated MetaNeutrosophic Set of depth t)**.** For $t \geq 1$, an *Iterated MetaNeutrosophic Set (IMN) of depth t on X* is a triple

$$\mathbf{N}^{(t)} = (T^{(t)}, I^{(t)}, F^{(t)}) \;\in\; \mathsf{Neu}^{\langle t\rangle}(X),$$

i.e., each component maps $\mathsf{Neu}^{\langle t-1\rangle}(X)$ to $[0,1]$ and satisfies the pointwise sum constraint $0 \leq T^{(t)} + I^{(t)} + F^{(t)} \leq 3$.

Example 3.7 (Iterated MetaNeutrosophic Set (depth 2): Device certification across test batteries)**.** Let $X = \{\mathrm{Safety}, \mathrm{EMC}\}$ be compliance tests. A base neutrosophic set $A = (T_A, I_A, F_A) \in \mathsf{Neu}(X)$ records, for each test $x \in X$, the degrees of truth (pass) $T_A(x)$, indeterminacy (uncertain) $I_A(x)$, and falsity (fail) $F_A(x)$ with $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$. Fix a concrete device result

$$(T_A(\mathrm{Safety}), I_A(\mathrm{Safety}), F_A(\mathrm{Safety})) = (0.7,\ 0.2,\ 0.1), \quad (T_A(\mathrm{EMC}), I_A(\mathrm{EMC}), F_A(\mathrm{EMC})) = (0.4,\ 0.4,\ 0.2).$$

Define a *first-level* meta-evaluator $B^{(1)} = (T^{(1)}, I^{(1)}, F^{(1)}) \in \mathsf{Neu}^{\langle 1\rangle}(X)$ by

$$T^{(1)}(A) = \frac{T_A(\mathrm{Safety}) + T_A(\mathrm{EMC})}{2} = \frac{0.7+0.4}{2} = 0.55, \quad I^{(1)}(A) = \max\{0.2, 0.4\} = 0.4, \quad F^{(1)}(A) = \frac{0.1+0.2}{2} = 0.15.$$

Now define a *depth-2* Iterated MetaNeutrosophic Set $\mathbf{N}^{(2)} = (T^{(2)}, I^{(2)}, F^{(2)})$ on the space $\mathsf{Neu}^{\langle 1\rangle}(X)$ by evaluating $B^{(1)}$ on the fixed baseline A:

$$T^{(2)}(B^{(1)}) \;:=\; \tfrac{1}{2}T^{(1)}(A) + \tfrac{1}{2}\big(1 - F^{(1)}(A)\big) = \tfrac{1}{2}\cdot 0.55 + \tfrac{1}{2}\cdot(1 - 0.15) = 0.275 + 0.425 = 0.700,$$

$$I^{(2)}(B^{(1)}) \;:=\; \big(I^{(1)}(A)\big)^2 + 0.1 = 0.16 + 0.1 = 0.26, \qquad F^{(2)}(B^{(1)}) \;:=\; \tfrac{1}{2}F^{(1)}(A) = 0.075.$$

Then

$$0 \;\leq\; T^{(2)} + I^{(2)} + F^{(2)} = 0.700 + 0.260 + 0.075 = 1.035 \;\leq\; 3,$$

so $\mathbf{N}^{(2)}$ is a valid depth-2 neutrosophic meta-evaluation summarizing an accreditor's view of the first-level evaluator $B^{(1)}$ on a concrete device.

Theorem 3.8 (Generalization of MetaNeutrosophic and Iterated MetaStructure)**.** *(a)* $\mathsf{Neu}^{\langle 1\rangle}(X)$ *is exactly the set of MetaNeutrosophic Sets (Definition 2.5).*
(b) For $t \geq 1$*, the structure with universe* $U^{(t)} := \mathsf{Neu}^{\langle t\rangle}(X)$ *and componentwise evaluation*

$$\Phi^{(t)}_{\mathrm{ev}} : U^{(t)} \times \mathsf{Neu}^{\langle t-1\rangle}(X) \to [0,1]^3, \qquad \Phi^{(t)}_{\mathrm{ev}}\big((T^{(t)}, I^{(t)}, F^{(t)}), A\big) := \big(T^{(t)}(A), I^{(t)}(A), F^{(t)}(A)\big)$$

forms an Iterated MetaStructure of depth t*.*

Proof. (a) Unwinding Definition 3.5 at $t = 0 \to 1$ yields functions on $\mathsf{Neu}(X)$ with the neutrosophic sum constraint, i.e., exactly Definition 2.5.
(b) The proof is identical in shape to Theorem 3.4(b), using the induced action by precomposition on each component and the fact that the constraint $0 \leq \cdot \leq 3$ is preserved under precomposition with bijections of the argument set. □

3.3 Iterated MetaSoft Sets

An Iterated MetaSoft Set applies soft set constructions to soft sets repeatedly, generating hierarchical decision frameworks about uncertain attribute-parameter data.

Definition 3.9 (Hierarchy of soft universes with meta-parameters)**.** Fix nonempty meta-parameter sets $\Pi_1, \Pi_2, \ldots$ (one for each level). Define

$$\mathsf{Soft}^{\langle 0\rangle}(U,S) := \mathsf{Soft}(U,S),$$

$$\mathsf{Soft}^{\langle t+1\rangle}(U,S;\Pi_{t+1}) := \Big\{ \mathcal{G}_{t+1} : \Pi_{t+1} \to \mathcal{P}\big(\mathsf{Soft}^{\langle t\rangle}(U,S)\big) \Big\}.$$

Definition 3.10 (Iterated MetaSoft Set of depth t)**.** For $t \geq 1$, an *Iterated MetaSoft Set (IMS) of depth* t *on* (U,S) with meta-parameters $(\Pi_1, \ldots, \Pi_t)$ is a chain

$$\mathbf{S}^{(t)} := (\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_t),$$

where $\mathcal{G}_1 : \Pi_1 \to \mathcal{P}(\mathsf{Soft}^{\langle 0\rangle}(U,S))$ is a MetaSoft Set (Definition 2.8) and, for $j \geq 2$, $\mathcal{G}_j : \Pi_j \to \mathcal{P}\big(\mathsf{Soft}^{\langle j-1\rangle}(U,S;\Pi_1,\ldots,\Pi_{j-1})\big)$.

Example 3.11 (Iterated MetaSoft Set (depth 2): Travel-planning policies over hotel filters)**.** Let $U = \{h_1, h_2, h_3\}$ be hotels and $S = \{\text{NearStation}, \text{Breakfast}\}$ attributes. Base soft sets $\mathcal{F} : S \to \mathcal{P}(U)$ state which hotels satisfy each attribute. Consider

$$\mathcal{F}^{(A)}(\text{NearStation}) = \{h_1, h_2\},\ \mathcal{F}^{(A)}(\text{Breakfast}) = \{h_2, h_3\}, \quad \mathcal{F}^{(B)}(\text{NearStation}) = \{h_1\},\ \mathcal{F}^{(B)}(\text{Breakfast}) = \{h_3\},$$

and add $\mathcal{F}^{(C)}$ with $\mathcal{F}^{(C)}(\text{NearStation}) = \{h_1\}$ and $\mathcal{F}^{(C)}(\text{Breakfast}) = \{h_1\}$.

Level-1 MetaSoft (policy) with meta-parameter set $\Pi_1 = \{\star\}$:

$$\mathcal{G}_1(\star) := \big\{ \mathcal{F} \in \mathsf{Soft}(U,S) \ \big|\ h_2 \in \mathcal{F}(\text{NearStation}) \ \wedge\ h_2 \in \mathcal{F}(\text{Breakfast}) \big\}.$$

Then $\mathcal{F}^{(A)} \in \mathsf{Soft}(U,S)$ but $h_2 \notin \mathcal{F}^{(A)}(\text{Breakfast})$ is false? (Actually $h_2 \in \{h_2, h_3\}$ is true.) However $h_2 \in \mathcal{F}^{(A)}(\text{NearStation})$ and $h_2 \in \mathcal{F}^{(A)}(\text{Breakfast})$ *both* hold, hence $\mathcal{F}^{(A)} \in \mathcal{G}_1(\star)$. In contrast, $\mathcal{F}^{(B)}$ fails the Breakfast condition, so $\mathcal{F}^{(B)} \notin \mathcal{G}_1(\star)$; $\mathcal{F}^{(C)}$ fails the NearStation condition for h_2, so $\mathcal{F}^{(C)} \notin \mathcal{G}_1(\star)$.

Depth-2 MetaSoft (planner-of-policies) with meta-parameter set $\Pi_2 = \{\diamond\}$:

$$\mathcal{G}_2(\diamond) := \Big\{ \mathcal{H} : \Pi_1 \to \mathcal{P}\big(\mathsf{Soft}(U,S)\big) \,\Big|\, \mathcal{G}_1 \in \mathcal{H} \ \text{and}\ \exists \mathcal{F} \in \mathcal{H}(\star) :\ h_1 \in \mathcal{F}(\text{Breakfast}) \Big\}.$$

Here $\mathcal{H}$ ranges over *families of level-1 policies*. If we take

$$\mathcal{H}_{\mathrm{ex}}(\star) := \{\mathcal{F}^{(A)}, \mathcal{F}^{(C)}\},$$

then $\mathcal{G}_1 \in \mathcal{H}_{\mathrm{ex}}$ holds (by construction) and there exists $\mathcal{F}^{(C)} \in \mathcal{H}_{\mathrm{ex}}(\star)$ with $h_1 \in \mathcal{F}^{(C)}(\text{Breakfast}) = \{h_1\}$, hence $\mathcal{H}_{\mathrm{ex}} \in \mathcal{G}_2(\diamond)$. This realizes a concrete depth-2 policy that keeps the level-1 policy $\mathcal{G}_1$ while ensuring at least one candidate covers “Breakfast for h_1”.

Theorem 3.12 (Generalization of MetaSoft and Iterated MetaStructure)**.** *(a) At depth $t = 1$, $\mathbf{S}^{(1)} = (\mathcal{G}_1)$ is precisely a MetaSoft Set on (U, S) with meta-parameters Π_1.*
(b) For each $t \geq 1$, the universe $U^{(t)} := \mathsf{Soft}^{\langle t \rangle}(U, S; \Pi_1, \dots, \Pi_t)$ with the levelwise selection operations

$$\Phi_{\mathrm{sel},j} : U^{(j)} \times \Pi_j \longrightarrow \mathcal{P}\big(U^{(j-1)}\big), \qquad \Phi_{\mathrm{sel},j}(\mathcal{G}_j, \pi) := \mathcal{G}_j(\pi),$$

is an Iterated MetaStructure of depth t.

Proof. (a) Directly from Definition 3.9 at $t = 0 \to 1$.
(b) For each level j, objects are functions $\mathcal{G}_j : \Pi_j \to \mathcal{P}(U^{(j-1)})$. Isomorphisms of the argument sets (bijections of Π_j and induced isomorphisms on $U^{(j-1)}$ by induction) act by pre- and post-composition on $\mathcal{P}(U^{(j-1)})$, preserving images. The selection operations $\Phi_{\mathrm{sel},j}$ commute with these actions by functoriality of $\mathcal{P}(-)$ and precomposition. Hence the naturality requirement in the Definition holds at each level; stacking levels yields an Iterated MetaStructure. □

3.4 Iterated MetaRough Sets

An Iterated MetaRough Set constructs rough approximations over families of rough sets iteratively, forming hierarchical multi-level structures of indiscernibility reasoning.

Definition 3.13 (Hierarchy of rough universes and meta-indiscernibilities)**.** Fix, for each $t \geq 1$, an equivalence relation $\mathcal{E}^{(t)}$ on $\mathsf{Rough}^{\langle t-1 \rangle}(X, R)$, where

$$\mathsf{Rough}^{\langle 0 \rangle}(X, R) := \mathsf{Rough}(X, R), \qquad \mathsf{Rough}^{\langle t \rangle}(X, R) := \mathcal{P}\big(\mathsf{Rough}^{\langle t-1 \rangle}(X, R)\big) \times \mathcal{P}\big(\mathsf{Rough}^{\langle t-1 \rangle}(X, R)\big).$$

Definition 3.14 (Iterated MetaRough Set of depth t)**.** Let $t \geq 1$ and let $C \subseteq \mathsf{Rough}^{\langle t-1 \rangle}(X, R)$ be a family of level-$(t-1)$ rough objects. Define its *meta-lower* and *meta-upper* approximations with respect to $\mathcal{E}^{(t)}$ by

$$\underline{C}^{\,\mathcal{E}^{(t)}} := \{\, r \in \mathsf{Rough}^{\langle t-1 \rangle}(X, R) \mid [r]_{\mathcal{E}^{(t)}} \subseteq C \,\}, \quad \overline{C}^{\,\mathcal{E}^{(t)}} := \{\, r \in \mathsf{Rough}^{\langle t-1 \rangle}(X, R) \mid [r]_{\mathcal{E}^{(t)}} \cap C \neq \emptyset \,\}.$$

The pair

$$\mathbf{R}^{(t)}(C) := \big(\underline{C}^{\,\mathcal{E}^{(t)}}, \overline{C}^{\,\mathcal{E}^{(t)}}\big) \in \mathsf{Rough}^{\langle t \rangle}(X, R)$$

is called the *Iterated MetaRough Set* of depth t generated by C.

Example 3.15 (Iterated MetaRough Set (depth 2): Course pass-status aggregated by classes and then by schools)**.** Let $X = \{s_1, s_2, s_3, s_4\}$ be students, partitioned into homerooms (indiscernibility R):

$$[s_1]_R = [s_2]_R = \{s_1, s_2\}, \qquad [s_3]_R = [s_4]_R = \{s_3, s_4\}.$$

Take two observed pass-sets:

$$U_1 = \{s_1\}, \qquad U_2 = \{s_3, s_4\}.$$

Their rough objects are

$$r_{U_1} = (\underline{U_1}, \overline{U_1}) = (\emptyset, \{s_1, s_2\}), \qquad r_{U_2} = (\underline{U_2}, \overline{U_2}) = (\{s_3, s_4\}, \{s_3, s_4\}).$$

Form a level-1 family $C := \{r_{U_1}, r_{U_2}\} \subseteq \mathsf{Rough}(X, R)$ and choose meta-indiscernibility $\mathcal{E}^{(1)}$ on $\mathsf{Rough}(X, R)$ to be equality. Then the level-1 MetaRough approximations are

$$\underline{C}^{\,\mathcal{E}^{(1)}} = C, \qquad \overline{C}^{\,\mathcal{E}^{(1)}} = C,$$

i.e., the level-1 rough object is $R^{(1)}(C) = (C, C)$.

At depth 2, take the singleton family $\mathcal{D} := \{\, R^{(1)}(C) \,\} \subseteq \mathsf{Rough}^{\langle 1 \rangle}(X, R)$ and let $\mathcal{E}^{(2)}$ be equality on $\mathsf{Rough}^{\langle 1 \rangle}(X, R)$. Then

$$\underline{\mathcal{D}}^{\,\mathcal{E}^{(2)}} = \mathcal{D}, \qquad \overline{\mathcal{D}}^{\,\mathcal{E}^{(2)}} = \mathcal{D},$$

so the depth-2 MetaRough Set equals $(\mathcal{D}, \mathcal{D})$. This models: first, rough summaries at the class level; second, a meta-level that treats identical class-summaries as indiscernible at the school aggregation level.

Proposition 3.16 (Sandwich property at every depth)**.** *For all $t \geq 1$ and $C \subseteq \mathsf{Rough}^{\langle t-1 \rangle}(X, R)$,*

$$\underline{C}^{\mathcal{E}^{(t)}} \subseteq C \subseteq \overline{C}^{\mathcal{E}^{(t)}}.$$

Proof. Identical element-chasing to the depth-1 case (Proposition 2.13): if $r \in \underline{C}^{\mathcal{E}^{(t)}}$ then $[r] \subseteq C$ implies $r \in C$; if $r \in C$ then $[r] \cap C \supseteq \{r\} \neq \emptyset$ so $r \in \overline{C}^{\mathcal{E}^{(t)}}$. □

Theorem 3.17 (Generalization of MetaRough and Iterated MetaStructure)**.** *(a) For $t = 1$, $\mathbf{R}^{(1)}$ coincides with the MetaRough Set construction of Definition 2.12.*
(b) For each $t \geq 1$, the universe $U^{(t)} := \mathsf{Rough}^{\langle t \rangle}(X, R)$ together with the operations

$$\Phi^{(t)}_{\text{low}} : \mathcal{P}\big(U^{(t-1)}\big) \to U^{(t)}, \quad \Phi^{(t)}_{\text{up}} : \mathcal{P}\big(U^{(t-1)}\big) \to U^{(t)}, \qquad C \mapsto \big(\underline{C}^{\mathcal{E}^{(t)}}, \overline{C}^{\mathcal{E}^{(t)}}\big),$$

is an Iterated MetaStructure of depth t.

Proof. (a) With $t = 1$ we recover Definition 2.12 (take $\mathcal{E}^{(1)}$ any chosen meta-indiscernibility on $\mathsf{Rough}(X, R)$).
(b) At level t, objects are pairs of subsets of $U^{(t-1)}$. Isomorphisms of $U^{(t-1)}$ (induced by lower levels) act by image/preimage on subsets and preserve equivalence classes $[r]_{\mathcal{E}^{(t)}}$. Hence the assignments $C \mapsto (\underline{C}^{\mathcal{E}^{(t)}}, \overline{C}^{\mathcal{E}^{(t)}})$ commute with the action, establishing the naturality demanded by the Definition. Thus $U^{(t)}$ with $\Phi^{(t)}_{\text{low}}, \Phi^{(t)}_{\text{up}}$ forms a depth-t Iterated MetaStructure. □

4 Conclusion

In this paper, we defined the MetaFuzzy Set, MetaNeutrosophic Set, MetaSoft Set, and MetaRough Set by extending Fuzzy Sets, Neutrosophic Sets, Soft Sets, and Rough Sets through the use of MetaStructure and Iterated MetaStructure. In future work, we plan to investigate whether similar extensions can also be developed for other classes of sets, such as Plithogenic Sets [83–88], Vague Sets [89–91], and Near Sets [92–94].

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Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Research Integrity

The authors hereby confirm that, to the best of their knowledge, this manuscript is their original work, has not been published in any other journal, and is not currently under consideration for publication elsewhere at this stage.

Use of Generative AI and AI-Assisted Tools

I use generative AI and AI-assisted tools for tasks such as English grammar checking, and I do not employ them in any way that violates ethical standards.

Disclaimer (Note on Computational Tools)

No computer-assisted proof, symbolic computation, or automated theorem proving tools (e.g., Mathematica, SageMath, Coq, etc.) were used in the development or verification of the results presented in this paper. All proofs and derivations were carried out manually and analytically by the authors.

Code Availability

No code or software was developed for this study.

Clinical Trial

This study did not involve any clinical trials.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

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