Selected Concepts of MetaStructure: Cube, HyperCube, Matrix, Decision-Making, Neural Networks, Geometry, and Functions

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Abstract

A *MetaStructure* is a higher-level framework that regards collections of structures as single objects, endowed with natural operations that preserve isomorphisms across domains. An *Iterated MetaStructure* extends this idea recursively, producing successive layers in which structures of structures give rise to deeper hierarchical meta-levels. In this work, we develop extensions of concepts such as Cube, HyperCube, Matrix, Decision-Making, Neural Networks, Geometry, and Functions within the settings of MetaStructure and Iterated MetaStructure. We further illustrate these extensions through simple yet concrete examples, highlighting both their mathematical generality and intuitive applicability.

Keywords: MetaStructure, Iterated MetaStructure, Cube, HyperCube, Matrix, Decision-Making, Neural Network, Geometry, Function

Contents in this paper

The remainder of this paper is organized as follows. This paper introduces several examples of MetaStructures. Here, the term "Structure" refers to any kind of structure, whether it originates from mathematical theory or from real-world systems.

1 Preliminaries

This section presents the fundamental concepts and definitions that underpin the discussions in this paper. Throughout this paper, all structures and sets are assumed to be finite.

1.1 Classical Structure

In this paper, the term *Structure* refers broadly to a mathematical system, not restricted to a single area, but encompassing domains such as Set Theory, Logic, Probability, Statistics, Algebra, and Geometry.

Definition 1.1 (Classical Structure)**.** (cf. [1, 2]) A *Classical Structure* C is a mathematical object arising from a traditional field—for example Set Theory, Logic, Probability, Statistics, Algebra, Geometry, Graph Theory, Automata Theory, or Game Theory. Formally, it may be represented as a pair

$$C = \big(H,\ \{\#^{(m)}\}_{m\in\mathcal{I}}\big),$$

where:

- H is a nonempty set, often called the *carrier* or *universe*.
- For each $m \in \mathcal{I} \subseteq \mathbb{Z}_{>0}$, there exists an m-ary operation

$$\#^{(m)} : H^m \longrightarrow H,$$

subject to appropriate *axioms* (such as associativity, commutativity, or identity laws), which vary according to the chosen type of structure.

The collection $\{\#^{(m)} : m \in \mathcal{I}\}$ determines the *type* of C. Representative examples include:

- A *Set* $(S, \varnothing)$, consisting solely of a carrier with distinguished elements or relations, but without operations [3,4].
- A *Logic* structure $(L, \wedge, \vee, \neg)$, where $\wedge, \vee$ are binary connectives and $\neg$ a unary connective, satisfying logical axioms [5,6].
- A *Probability* model $(\Omega, \mathcal{F}, P)$, where $P : \mathcal{F} \to [0,1]$ is a probability measure on a sigma-algebra $\mathcal{F} \subseteq \mathcal{P}(\Omega)$ [7–9].
- A *Statistical* model $(X, \mathcal{A}, \theta)$, where θ maps data X into parameters of interest [10,11].
- *Algebraic* structures [12] such as:
  - A *Group* $(G, *)$, with $* : G \times G \to G$ satisfying associativity, identity, and inverses [13,14].
  - A *Ring* $(R, +, \times)$, with two binary operations fulfilling ring axioms [15,16].
  - A *Vector Space* $(V, +, \cdot)$ over a field $\mathbb{F}$, with scalar multiplication $\cdot : \mathbb{F} \times V \to V$ [17,18].
- A *Geometric* structure (X, dist), where $\mathrm{dist} : X \times X \to \mathbb{R}$ satisfies the axioms of a metric [19,20].
- A *Graph* (V, E), where $E \subseteq \{\{u, v\} \mid u, v \in V\}$ for undirected graphs, or $E \subseteq V \times V$ for directed graphs, with adjacency and incidence relations [21–23].
- An *Automaton* $(Q, \Sigma, \delta, q_0, F)$, where Q is a set of states, Σ an input alphabet, $\delta : Q \times \Sigma \to Q$ the transition function, $q_0 \in Q$ the start state, and $F \subseteq Q$ the accepting states [24–26].
- A *Game* $(N, \{A_i\}, \{u_i\})$, where N is the set of players, A_i each player's action set, and $u_i : \prod_{j \in N} A_j \to \mathbb{R}$ the payoff function for player i [27–29].

Related concepts include the *HyperStructure* [30–35] and the *SuperHyperStructure* [36–40], which have also been extensively investigated in recent studies.

1.2 MetaStructure (Structure of Structures)

We begin by fixing a general single-sorted, finitary *signature*

$$\Sigma = \big(\mathsf{Func}, \mathsf{Rel}, \mathrm{ar}_{\mathsf{Func}}, \mathrm{ar}_{\mathsf{Rel}}\big),$$

where Func (resp. Rel) is a set of function (resp. relation) symbols, and ar assigns their arities. A (single-sorted) Σ-*structure* is given by

$$\mathbf{C} = \big(H, (f^{\mathbf{C}})_{f \in \mathsf{Func}}, (R^{\mathbf{C}})_{R \in \mathsf{Rel}}\big),$$

with nonempty carrier H, operations $f^{\mathbf{C}} : H^m \to H$ for each $f \in \mathsf{Func}$ of arity m, and relations $R^{\mathbf{C}} \subseteq H^r$ for each $R \in \mathsf{Rel}$ of arity r. We denote by Str_Σ the class of all such Σ-structures.

Definition 1.2 (MetaStructure over a fixed signature)**.** (cf. [41]) Fix Σ as above. A *MetaStructure* ("structure of structures") over Σ is a pair

$$\mathbb{M} = \big(U, (\Phi_\ell)_{\ell \in \Lambda}\big),$$

where:

- U is a nonempty subset of Str_Σ (its members are the *level 0 objects*);
- for each label $\ell \in \Lambda$ of *meta-arity* $k_\ell \in \mathbb{N}$, the *meta-operation*

$$\Phi_\ell \;:\; U^{k_\ell} \longrightarrow U$$

is specified uniformly by *carrier- and symbol-constructors*:

$$\begin{aligned}
&\Gamma_\ell:\ (\mathbf{C}_1,\dots,\mathbf{C}_{k_\ell}) \mapsto H_\ell, && \text{(a new carrier } H_\ell \text{ defined functorially);}\\
&\forall f\in\mathsf{Func}:\quad f^{\Phi_\ell(\mathbf{C}_1,\dots,\mathbf{C}_{k_\ell})} \;=\; \Lambda_\ell^f\big(f^{\mathbf{C}_1},\dots,f^{\mathbf{C}_{k_\ell}}\big);\\
&\forall R\in\mathsf{Rel}:\quad R^{\Phi_\ell(\mathbf{C}_1,\dots,\mathbf{C}_{k_\ell})} \;=\; \Xi_\ell^R\big(R^{\mathbf{C}_1},\dots,R^{\mathbf{C}_{k_\ell}}\big),
\end{aligned}$$

where Λ_ℓ^f and Ξ_ℓ^R are *uniform recipes* that transform the input interpretations of each symbol into the output interpretation over H_ℓ.

In addition, each Φ_ℓ is required to be *isomorphism-invariant* (natural): if $\alpha_i : \mathbf{C}_i \cong \mathbf{D}_i$ for $1 \le i \le k_\ell$, then there is an induced isomorphism

$$\Phi_\ell(\alpha_1,\dots,\alpha_{k_\ell}) \;:\; \Phi_\ell(\mathbf{C}_1,\dots,\mathbf{C}_{k_\ell}) \xrightarrow{\;\cong\;} \Phi_\ell(\mathbf{D}_1,\dots,\mathbf{D}_{k_\ell}),$$

compatible with all interpretations of the symbols of Σ.

An example of a Metastructure is the *MetaGraph* [42–47], which has been widely studied in the literature.

1.3 Iterated MetaStructure (Structure of Structures of $\cdots$ of Structures)

An Iterated MetaStructure is obtained by recursively applying the MetaStructure construction, producing successive layers in which structures of structures form deeper hierarchical meta-levels (cf. [41, 42]).

Definition 1.3 (Iterated MetaStructure of depth t)**.** (cf. [41]) An *Iterated MetaStructure of depth t* over Σ is a MetaStructure $\mathfrak{M}^{(t)}$ of height t. For $s < t$, we *lift* a height-s MetaStructure $\mathfrak{M}^{(s)} = (U^{(s)}, \{\odot_i\}, \{\mathcal{S}_j\})$ to height t by

$$\iota_{s\to t} \;:\; U^{(s)} \xrightarrow{\;\mathsf{U}_\Sigma^{t-s}\;} U^{(t)} := \mathsf{U}_\Sigma^{t-s}\big(U^{(s)}\big),$$

and, for each $\odot_i : (\mathsf{E}_\Sigma^{m_i})^{k_i} \to \mathcal{P}^{n_i}(\mathsf{E}_\Sigma^{n_i})$, defining its lift by

$$\odot_i^{\uparrow} \;:\; \big(\mathsf{E}_\Sigma^{m_i+t-s}\big)^{k_i} \longrightarrow \mathcal{P}^{n_i}\big(\mathsf{E}_\Sigma^{n_i+t-s}\big),$$

$$\odot_i^{\uparrow}\big(\mathsf{U}_\Sigma^{t-s}(x_1),\dots,\mathsf{U}_\Sigma^{t-s}(x_{k_i})\big) := \mathsf{U}_\Sigma^{t-s}\big(\odot_i(x_1,\dots,x_{k_i})\big),$$

and similarly for relations, with $\mathcal{S}_j^{\uparrow} := \big(\mathsf{U}_\Sigma^{t-s}\big)^{\times \ell_j}(\mathcal{S}_j)$.

2 MetaCube (Cube of Cube)

Cube is a three-dimensional geometric solid with six equal square faces, twelve equal edges, and eight symmetrically arranged vertices [48–52]. MetaCube generalizes cubes by combining or transforming entire cube structures through operations like Cartesian products, subdivisions, or dual representations. Iterated MetaCube recursively applies meta-operations to cubes, creating cube-of-cubes structures that represent multi-level geometric or combinatorial hierarchies.

Definition 2.1 (Cube)**.** A (unit) cube in $\mathbb{R}^3$ is the convex set

$$[0,1]^3 \;=\; \big\{\, x = (x_1,x_2,x_3) \in \mathbb{R}^3 \;\big|\; 0 \le x_i \le 1 \ (i = 1,2,3) \,\big\},$$

equivalently the convex hull $\mathrm{conv}\big(\{0,1\}^3\big)$. Its vertex set is $\{0,1\}^3$, it has $2^3 = 8$ vertices and $3 \cdot 2^{3-1} = 12$ edges.

Fix the *cube signature*

$$\Sigma_{\text{cub}} = \big(\ \mathsf{Func} = \varnothing,\ \mathsf{Rel} = \{\, C,\ V,\ E\,\},\ \mathrm{ar}(C) = \mathrm{ar}(V) = 1,\ \mathrm{ar}(E) = 2\ \big).$$

A Σ_{cub}-structure is a triple

$$\mathbf{Q}_n \ = \ \big(H_n,\ C_n,\ V_n,\ E_n\big),$$

where H_n is a nonempty carrier set, $C_n \subseteq H_n$ (the *cube body*), $V_n \subseteq H_n$ (the *vertex set*), and $E_n \subseteq H_n \times H_n$ (the *edge relation*). We take as our universe

$$U_{\text{cub}} \ = \ \big\{\, \mathbf{Q}_n \ \big|\ n \in \mathbb{N},\ H_n = \mathbb{R}^n,\ C_n = [0,1]^n,\ V_n = \{0,1\}^n,\ E_n = \{(u,v) \in V_n^2 \mid \|u-v\|_1 = 1\}\, \big\} \ \subseteq\ \mathrm{Str}_{\Sigma_{\text{cub}}}.$$

In particular, $\mathbf{Q}_3 = (\mathbb{R}^3, [0,1]^3, \{0,1\}^3, \{(u,v) \in \{0,1\}^3 : \|u-v\|_1 = 1\})$ is the usual 3-cube.

Definition 2.2 (MetaCube)**.** A *MetaCube* is a MetaStructure (Definition 1.2) over Σ_{cub}

$$\mathbb{M}_{\text{cub}} = \big(U_{\text{cub}},\ (\Phi_\ell)_{\ell \in \Lambda}\big),$$

whose meta-operations are generated by the following uniform constructors (each specified by carrier- and symbol-constructors as in Definition 1.2):

(P) Cartesian product (dimension-addition)

$$\Phi_\otimes :\ U_{\text{cub}} \times U_{\text{cub}} \longrightarrow U_{\text{cub}},$$
$$\Phi_\otimes(\mathbf{Q}_m, \mathbf{Q}_n) := \mathbf{Q}_{m+n}$$

with carrier-constructor $\Gamma_\otimes(H_m, H_n) = H_m \times H_n \cong \mathbb{R}^{m+n}$ and symbol-constructors

$$C^{\Phi_\otimes} = C_m \times C_n, \qquad V^{\Phi_\otimes} = V_m \times V_n, \qquad E^{\Phi_\otimes} = \big(E_m \times \Delta_{V_n}\big)\ \cup\ \big(\Delta_{V_m} \times E_n\big),$$

where $\Delta_X = \{(x,x) \mid x \in X\}$.

(S) Scaling by a positive factor $s > 0$

$$\Phi_{\text{scale}(s)}(\mathbf{Q}_n) := \big(\mathbb{R}^n,\ s \cdot C_n,\ s \cdot V_n,\ \{(su, sv) \mid (u,v) \in E_n\}\big).$$

(T) Translation by a vector $c \in \mathbb{R}^n$

$$\Phi_{\text{trans}(c)}(\mathbf{Q}_n) := \big(\mathbb{R}^n,\ C_n + c,\ V_n + c,\ \{(u+c, v+c) \mid (u,v) \in E_n\}\big).$$

(F) Coordinate projection to a face For any index set $I \subseteq \{1,\dots,n\}$ with $|I| = k$ and any assignment $a \in \{0,1\}^{\{1,\dots,n\}\setminus I}$, let $\pi_{I,a} : \mathbb{R}^n \to \mathbb{R}^k$ be the projection onto the I-coordinates with the complementary coordinates fixed to a. Then

$$\Phi_{\text{face}(I,a)}(\mathbf{Q}_n) := \big(\mathbb{R}^k,\ \pi_{I,a}(C_n),\ \pi_{I,a}(V_n),\ \{(\pi_{I,a}(u), \pi_{I,a}(v)) \mid (u,v) \in E_n\}\big) = \mathbf{Q}_k.$$

Example 2.3 (MetaCube: Product-Variant Grid (E-commerce))**.** Consider a laptop SKU space with three binary attributes:

$$\mathsf{CPU} \in \{\text{i5}, \text{i7}\} =: X_1, \quad \mathsf{RAM} \in \{16, 32\}\ (\text{GB}) =: X_2, \quad \mathsf{Storage} \in \{512, 1024\}\ (\text{GB}) =: X_3.$$

Encode each choice by a bit $x_i \in \{0,1\}$ and identify a configuration with $x = (x_1, x_2, x_3) \in \{0,1\}^3$. The configuration space is the vertex set of the cube $\mathbf{Q}_3$ (a MetaCube object), hence it has

$$|V| = 2^3 = 8 \quad \text{variants, and} \quad |E| = 3 \cdot 2^{3-1} = 12 \text{ adjacencies (flip one attribute).}$$

Adding a fourth binary attribute $\mathsf{Color} \in \{\text{Black}, \text{Silver}\} =: X_4$ via the MetaCube product $\Phi_\otimes$ yields $\mathbf{Q}_4$ with

$$|V| = 2^4 = 16, \qquad |E| = 4 \cdot 2^{4-1} = 32.$$

Fixing $\mathsf{RAM} = 32$ (a face operator Φ_{face}) selects a square (2D face) with

$$|V| = 2^2 = 4, \qquad |E| = 2 \cdot 2^{2-1} = 4,$$

i.e., the four variants differing only in CPU and Storage.

Theorem 2.4 (MetaCube is a MetaStructure and generalizes cubes)**.** *The pair* $\mathbb{M}_{\text{cub}} = \big(U_{\text{cub}}, (\Phi_\ell)\big)$ *of Definition 2.2 is a MetaStructure in the sense of Definition 1.2. Moreover, every classical cube* $[0,1]^3 \subset \mathbb{R}^3$ *(more generally* $[0,1]^n$*) appears as an object* $\mathbf{Q}_3 \in U_{\text{cub}}$ *(resp.* $\mathbf{Q}_n \in U_{\text{cub}}$*), hence MetaCube* generalizes *cubes.*

Proof. We verify the two requirements of Definition 1.2.

(1) Uniform constructors. For each meta-operation listed, the carrier-constructor Γ_ℓ and the symbol-constructors $(\Lambda_\ell^{\cdot}, \Xi_\ell^{\cdot})$ are defined uniformly:

(P) The carrier $H_m \times H_n$ is canonically a copy of $\mathbb{R}^{m+n}$, hence in U_{cub}. By definition $C_m \times C_n = [0,1]^m \times [0,1]^n = [0,1]^{m+n}$, $V_m \times V_n = \{0,1\}^{m+n}$, and $E^{\Phi_\otimes}$ is exactly the edge set of the Cartesian product of the m- and n-cubes, which equals the edge set of the $(m+n)$-cube. Thus $\Phi_\otimes(\mathbf{Q}_m, \mathbf{Q}_n) = \mathbf{Q}_{m+n} \in U_{\text{cub}}$.

(S) For $s > 0$, $s \cdot C_n = [0,s]^n$ is an axis-aligned cube affinely isomorphic to $[0,1]^n$, with vertex set $s \cdot V_n$ and edge relation transported coordinatewise; hence the result lies in U_{cub} up to the fixed choice of representing representatives (by the obvious affine isomorphism we identify it with $\mathbf{Q}_n$).

(T) Translation preserves the combinatorial cube structure, merely relocating it; the interpretation of C, V, E is transported by the translation. Again this is (canonically) isomorphic to $\mathbf{Q}_n$.

(F) Fixing coordinates to 0 or 1 and projecting the remainder yields a k-cube in $\mathbb{R}^k$: $\pi_{I,a}([0,1]^n) = [0,1]^k$, $\pi_{I,a}(\{0,1\}^n) = \{0,1\}^k$, and adjacency is preserved under the projection restricted to the face. Therefore $\Phi_{\text{face}(I,a)}(\mathbf{Q}_n) = \mathbf{Q}_k$.

(2) Naturality (isomorphism-invariance). Let $\alpha_i : \mathbf{Q}_{n_i} \xrightarrow{\cong} \mathbf{Q}'_{n_i}$ be isomorphisms of Σ_{cub}-structures (i.e. bijections $\alpha_i : H_{n_i} \to H'_{n_i}$ preserving C, V, E). For (P), define $\alpha_\otimes := \alpha_1 \times \alpha_2 : H_{n_1} \times H_{n_2} \to H'_{n_1} \times H'_{n_2}$. Then

$$\alpha_\otimes(C_{n_1} \times C_{n_2}) = (\alpha_1 C_{n_1}) \times (\alpha_2 C_{n_2}) = C'_{n_1} \times C'_{n_2},$$

$$\alpha_\otimes(V_{n_1} \times V_{n_2}) = (\alpha_1 V_{n_1}) \times (\alpha_2 V_{n_2}) = V'_{n_1} \times V'_{n_2},$$

and for edges, using Δ_X functoriality,

$$\alpha_\otimes\big(E_{n_1} \times \Delta_{V_{n_2}}\big) = (\alpha_1 E_{n_1}) \times \Delta_{\alpha_2 V_{n_2}} = E'_{n_1} \times \Delta_{V'_{n_2}},$$

and similarly for the second summand, hence $\alpha_\otimes$ is an isomorphism

$$\Phi_\otimes(\alpha_1, \alpha_2) : \ \Phi_\otimes(\mathbf{Q}_{n_1}, \mathbf{Q}_{n_2}) \xrightarrow{\cong} \Phi_\otimes(\mathbf{Q}'_{n_1}, \mathbf{Q}'_{n_2}).$$

For (S), define $\alpha_{\text{scale}(s)}(x) = s \cdot \alpha_1(s^{-1}x)$; for (T), $\alpha_{\text{trans}(c)}(x) = \alpha_1(x - c) + c'$ with c' the translation in the codomain; for (F), $\alpha_{\text{face}(I,a)} = \pi'_{I,a} \circ \alpha_1 \circ \iota_{I,a}$ (restriction/section to the face), all of which preserve C, V, E by construction. Thus each Φ_ℓ is natural.

Finally, since $\mathbf{Q}_n \in U_{\text{cub}}$ for every n, classical cubes are (isomorphic to) objects of $\mathbb{M}_{\text{cub}}$. □

Proposition 2.5 (Vertex/edge counts under product)**.** *Let Q_m and Q_n denote the m- and n-dimensional hypercubes (i.e.* $\mathbf{Q}_m, \mathbf{Q}_n$*). Then*

$$|V(Q_m \otimes Q_n)| = 2^{m+n}, \qquad |E(Q_m \otimes Q_n)| = (m+n)2^{m+n-1}.$$

Proof. Vertices: $|V(Q_m \otimes Q_n)| = |V_m \times V_n| = |V_m| \cdot |V_n| = 2^m \cdot 2^n = 2^{m+n}$.

Edges:

$$\begin{aligned} |E(Q_m \otimes Q_n)| &= \big|E_m \times \Delta_{V_n}\big| + \big|\Delta_{V_m} \times E_n\big| \\ &= |E_m| \cdot |V_n| + |V_m| \cdot |E_n| \\ &= \big(m2^{m-1}\big) \cdot 2^n + 2^m \cdot \big(n2^{n-1}\big) \\ &= m\,2^{m+n-1} + n\,2^{m+n-1} = (m+n)\,2^{m+n-1}. \end{aligned}$$

□

Definition 2.6 (Iterated MetaCube of depth t)**.** For $t \in \mathbb{N}$, an *Iterated MetaCube of depth t* is an Iterated MetaStructure (Definition 1.3) over Σ_{cub},

$$\mathfrak{M}^{(t)}_{\text{cub}} = \big(U^{(t)}_{\text{cub}},\ (\odot^{(t)}_{\ell})_{\ell\in\Lambda}\big),$$

obtained by applying the lifting functor $\mathsf{U}_{\Sigma_{\text{cub}}}$ of Definition 1.3 to $\mathbb{M}_{\text{cub}}$ repeatedly t times. Concretely, for $s < t$ and any meta-operation Φ_ℓ of $\mathbb{M}_{\text{cub}}$ with meta-arity k_ℓ, its lift

$$\Phi^{\uparrow}_{\ell} :\ \big(U^{(t)}_{\text{cub}}\big)^{k_\ell} \longrightarrow U^{(t)}_{\text{cub}}$$

is defined on representatives by

$$\Phi^{\uparrow}_{\ell}\big(\mathsf{U}^{t-s}_{\Sigma_{\text{cub}}}(X_1),\dots,\mathsf{U}^{t-s}_{\Sigma_{\text{cub}}}(X_{k_\ell})\big) \ :=\ \mathsf{U}^{t-s}_{\Sigma_{\text{cub}}}\big(\,\Phi_\ell(X_1,\dots,X_{k_\ell})\,\big),$$

and similarly for the relations, as stipulated in Definition 1.3. We call $\odot^{(t)}_{\otimes}$, $\odot^{(t)}_{\text{scale}(s)}$, $\odot^{(t)}_{\text{trans}(c)}$, $\odot^{(t)}_{\text{face}(I,a)}$ the *lifted* product, scaling, translation, and face operators.

Example 2.7 (Iterated MetaCube: Week-by-Day Testing Grid (Depth $t = 1$))**.** For each day $d \in \{\text{Mon},\dots,\text{Sun}\}$ define a 3-cube of test conditions

$$\mathsf{Browser} \in \{\text{Chrome},\text{Firefox}\},\quad \mathsf{OS} \in \{\text{Windows},\text{macOS}\},\quad \mathsf{Network} \in \{\text{WiFi},4\text{G}\},$$

so the daily grid is $\mathbf{Q}^{(d)}_3$ with

$$|V(\mathbf{Q}^{(d)}_3)| = 2^3 = 8,\qquad |E(\mathbf{Q}^{(d)}_3)| = 3\cdot 2^{3-1} = 12.$$

Treat the seven daily cubes $\{\mathbf{Q}^{(d)}_3\}^{7}_{d=1}$ as objects of U_{cub} and form the depth-1 Iterated MetaCube by applying the lifted product $\odot^{(1)}_{\otimes}$ across days. The full weekly plan ("cube-of-cubes") contains

$$7\times 2^3 = 56 \text{ test vertices (configurations).}$$

If a new binary dimension $\mathsf{Region} \in \{\text{JP},\text{US}\}$ is added *uniformly* via the lifted product, the total becomes $2\times 56 = 112$ configurations. A lifted face fixing $\mathsf{Browser} = \text{Chrome}$ across all days reduces each $\mathbf{Q}^{(d)}_3$ to a square with $2^2 = 4$ tests, i.e., $7\times 4 = 28$ tests in the week.

Theorem 2.8 (Iterated MetaCube is an Iterated MetaStructure and generalizes MetaCube)**.** *For every $t \in \mathbb{N}$, $\mathfrak{M}^{(t)}_{\text{cub}}$ of Definition 2.6 is an Iterated MetaStructure in the sense of Definition 1.3. Moreover, for $s < t$ there is a canonical embedding*

$$\iota_{s\to t} :\ \mathfrak{M}^{(s)}_{\text{cub}} \hookrightarrow \mathfrak{M}^{(t)}_{\text{cub}},\qquad X \longmapsto \mathsf{U}^{t-s}_{\Sigma_{\text{cub}}}(X),$$

which is operation-preserving:

$$\Phi^{\uparrow}_{\ell}\big(\iota_{s\to t}(X_1),\dots,\iota_{s\to t}(X_{k_\ell})\big) = \iota_{s\to t}\big(\Phi_\ell(X_1,\dots,X_{k_\ell})\big).$$

In particular, $\mathfrak{M}^{(0)}_{\text{cub}} = \mathbb{M}_{\text{cub}}$ embeds into $\mathfrak{M}^{(t)}_{\text{cub}}$, so Iterated MetaCube generalizes *MetaCube.*

Proof. By Definition 1.3, the construction of $U^{(t)}_{\text{cub}}$ and the lifted operations is functorial in t; hence each $\odot^{(t)}_{\ell}$ is defined via uniform carrier- and symbol-constructors by post-composing the base constructors with $\mathsf{U}^{t-s}_{\Sigma_{\text{cub}}}$. Naturality follows because if $\alpha_i : X_i \to Y_i$ are isomorphisms at level s, then

$$\mathsf{U}^{t-s}_{\Sigma_{\text{cub}}}(\alpha_i) :\ \mathsf{U}^{t-s}_{\Sigma_{\text{cub}}}(X_i) \xrightarrow{\ \cong\ } \mathsf{U}^{t-s}_{\Sigma_{\text{cub}}}(Y_i)$$

are isomorphisms at level t, and

$$\Phi^{\uparrow}_{\ell}\big(\mathsf{U}^{t-s}\alpha_1,\dots,\mathsf{U}^{t-s}\alpha_{k_\ell}\big) = \mathsf{U}^{t-s}\big(\Phi_\ell(\alpha_1,\dots,\alpha_{k_\ell})\big),$$

which is an isomorphism because Φ_ℓ is natural at level s (Theorem 2.4) and U^{t-s} preserves isomorphisms by construction.

The embedding $\iota_{s\to t}$ is exactly the unit of the lifting (Definition 1.3); the operation-preservation identity is the defining axiom of the lift:

$$\Phi^{\uparrow}_{\ell}\big(\mathsf{U}^{t-s}X_1,\dots,\mathsf{U}^{t-s}X_{k_\ell}\big) = \mathsf{U}^{t-s}\big(\Phi_\ell(X_1,\dots,X_{k_\ell})\big).$$

Taking $s = 0$ gives the particular embedding of $\mathbb{M}_{\text{cub}}$ into $\mathfrak{M}^{(t)}_{\text{cub}}$. □

3 MetaHyperCube (Hypercube of Hypercube)

HyperCube is a geometric structure generalizing cubes to any dimension, consisting of vertices, edges, and faces arranged symmetrically [48–52]. MetaHyperCube generalizes hypercubes by combining or transforming entire hypercube structures, enabling higher-level operations like Cartesian products or dimensional dualities. Iterated MetaHyperCube applies meta-operations recursively, producing layered hypercube-of-hypercubes structures that capture complex multi-level geometric and combinatorial relationships.

Definition 3.1 (Hypercube)**.** [53–56] An n-dimensional (unit) hypercube in $\mathbb{R}^n$ is

$$[0,1]^n \;=\; \{\, x = (x_1,\dots,x_n) \in \mathbb{R}^n \;\big|\; 0 \le x_i \le 1 \ (i = 1,\dots,n) \,\},$$

equivalently $\mathrm{conv}\big(\{0,1\}^n\big)$. It has 2^n vertices and $n \cdot 2^{n-1}$ edges.

Fix the *hypercube signature*

$$\Sigma_{\mathrm{hc}} = \Big(\mathsf{Func} = \varnothing, \quad \mathsf{Rel} = \{\, C,\ V,\ E \,\}, \quad \mathrm{ar}(C) = \mathrm{ar}(V) = 1,\ \mathrm{ar}(E) = 2 \,\Big).$$

A Σ_{hc}-structure is a triple

$$\mathbf{H}_n \;=\; \big(H_n,\ C_n,\ V_n,\ E_n\big),$$

where H_n is a nonempty carrier set, $C_n \subseteq H_n$ (the *body*), $V_n \subseteq H_n$ (the *vertex set*), and $E_n \subseteq H_n \times H_n$ (the *edge relation*).

We take as our universe the class of standard hypercubes in all dimensions

$$U_{\mathrm{hc}} \;=\; \Big\{ \mathbf{H}_n \;\Big|\; n \in \mathbb{N},\ H_n = \mathbb{R}^n,\ C_n = [0,1]^n,\ V_n = \{0,1\}^n,\ E_n = \{(u,v) \in V_n^2 :\ \|u - v\|_1 = 1\} \Big\} \;\subseteq\; \mathrm{Str}_{\Sigma_{\mathrm{hc}}}.$$

Thus $\mathbf{H}_n$ is the usual n-dimensional hypercube; in particular, $\mathbf{H}_3$ is the classical cube.

Definition 3.2 (MetaHyperCube)**.** A *MetaHyperCube* is a MetaStructure (Definition 1.2) over Σ_{hc}

$$\mathbb{M}_{\mathrm{hc}} = \big(U_{\mathrm{hc}},\ (\Phi_\ell)_{\ell \in \Lambda}\big),$$

whose meta-operations are generated by the following uniform constructors (each specified by carrier- and symbol-constructors as in Definition 1.2):

(P) Cartesian product (dimension-addition)

$$\Phi_{\otimes} :\ U_{\mathrm{hc}} \times U_{\mathrm{hc}} \longrightarrow U_{\mathrm{hc}},$$
$$\Phi_{\otimes}(\mathbf{H}_m, \mathbf{H}_n) := \mathbf{H}_{m+n}$$

with carrier-constructor $\Gamma_{\otimes}(H_m, H_n) = H_m \times H_n \cong \mathbb{R}^{m+n}$ and symbol-constructors

$$C^{\Phi_\otimes} = C_m \times C_n = [0,1]^{m+n}, \qquad V^{\Phi_\otimes} = V_m \times V_n = \{0,1\}^{m+n},$$

$$E^{\Phi_\otimes} = \big(E_m \times \Delta_{V_n}\big) \ \cup\ \big(\Delta_{V_m} \times E_n\big), \qquad \Delta_X := \{(x,x) \mid x \in X\}.$$

(S) Positive scaling by $s > 0$

$$\Phi_{\mathrm{scale}(s)}(\mathbf{H}_n) := \big(\mathbb{R}^n,\ s \cdot C_n,\ s \cdot V_n,\ \{(su, sv) \mid (u,v) \in E_n\}\big).$$

(T) Translation by $c \in \mathbb{R}^n$

$$\Phi_{\mathrm{trans}(c)}(\mathbf{H}_n) := \big(\mathbb{R}^n,\ C_n + c,\ V_n + c,\ \{(u + c, v + c) \mid (u,v) \in E_n\}\big).$$

(W) Signed coordinate permutations (hyperoctahedral symmetries) For any signed permutation matrix $A \in \mathrm{GL}_n(\mathbb{R})$ with exactly one nonzero entry ± 1 in each row/column,

$$\Phi_A(\mathbf{H}_n) := \big(\mathbb{R}^n,\ AC_n,\ AV_n,\ \{(Au, Av) \mid (u,v) \in E_n\}\big).$$

(F) Coordinate face projection For $I \subseteq \{1, \dots, n\}$ with $|I| = k$ and $a \in \{0,1\}^{\{1,\dots,n\}\setminus I}$, let $\pi_{I,a} : \mathbb{R}^n \to \mathbb{R}^k$ be the projection to I-coordinates with the complementary coordinates fixed to a. Then

$$\Phi_{\mathrm{face}(I,a)}(\mathbf{H}_n) := \big(\mathbb{R}^k,\ \pi_{I,a}(C_n),\ \pi_{I,a}(V_n),\ \{(\pi_{I,a}(u), \pi_{I,a}(v)) \mid (u,v) \in E_n\}\big) = \mathbf{H}_k.$$

Example 3.3 (MetaHyperCube: Feature-Flag Rollout (software release))**.** Consider $n = 5$ binary feature flags (A/B UI, NewSearch, GPU, PaymentsV2, Logging) $\in \{0,1\}^5$. The deployment space is the hypercube $\mathbf{H}_5$ with

$$|V| = 2^5 = 32 \text{ configurations}, \qquad |E| = 5 \cdot 2^{5-1} = 5 \cdot 16 = 80 \text{ single-flag flips}.$$

Adding a new binary flag DarkMode via the product $\Phi_\otimes$ yields $\mathbf{H}_5 \otimes \mathbf{H}_1 \cong \mathbf{H}_6$ with

$$|V| = 2^6 = 64.$$

Fixing two flags, e.g. GPU $= 0$ and PaymentsV2 $= 1$, is a face operation Φ_{face} that reduces to a 3-cube with

$$|V| = 2^3 = 8, \qquad |E| = 3 \cdot 2^2 = 12.$$

Theorem 3.4 (MetaHyperCube is a MetaStructure and generalizes hypercubes)**.** *The pair* $\mathbb{M}_{\mathrm{hc}} = \big(U_{\mathrm{hc}}, (\Phi_\ell)\big)$ *of Definition 3.2 is a MetaStructure in the sense of Definition 1.2. Moreover, every classical hypercube* $[0,1]^n \subset \mathbb{R}^n$ *appears as an object* $\mathbf{H}_n \in U_{\mathrm{hc}}$*, hence MetaHyperCube* generalizes *hypercubes.*

Proof. We verify the two clauses of Definition 1.2.

(1) Uniform constructors and closure. For each meta-operation, the carrier-constructor Γ_ℓ and the symbol-constructors $(\Lambda_\ell^{\cdot}, \Xi_\ell^{\cdot})$ are given as follows, and their outputs lie in U_{hc}:

(P) $\Gamma_\otimes(H_m, H_n) = H_m \times H_n \cong \mathbb{R}^{m+n}$; then $C_m \times C_n = [0,1]^m \times [0,1]^n = [0,1]^{m+n}$, $V_m \times V_n = \{0,1\}^{m+n}$, and

$$E^{\Phi_\otimes} = \big(E_m \times \Delta_{V_n}\big) \cup \big(\Delta_{V_m} \times E_n\big)$$

is the edge set of the Cartesian product graph $Q_m \square Q_n$, which equals the edge set of Q_{m+n}. Hence $\Phi_\otimes(\mathbf{H}_m, \mathbf{H}_n) = \mathbf{H}_{m+n} \in U_{\mathrm{hc}}$.

(S) For $s > 0$, $s \cdot C_n = [0,s]^n$ is affinely isomorphic to $[0,1]^n$; the induced vertex and edge sets are transported coordinatewise, so the result is (canonically) isomorphic to $\mathbf{H}_n$, thus in U_{hc}.

(T) Translation preserves the combinatorial structure: $C_n + c$ is a translate of $[0,1]^n$, with $V_n + c$ and edges transported. This is isomorphic to $\mathbf{H}_n$; hence in U_{hc}.

(W) Signed permutations A map $[0,1]^n$ to another axis-aligned cube of the same size, and map $\{0,1\}^n$ bijectively to itself; adjacency (ℓ^1-distance 1) is preserved by such A. Hence $\Phi_A(\mathbf{H}_n) \cong \mathbf{H}_n \in U_{\mathrm{hc}}$.

(F) Fixing coordinates to $0/1$ and projecting to the remaining k coordinates yields a k-cube: $\pi_{I,a}([0,1]^n) = [0,1]^k$, $\pi_{I,a}(\{0,1\}^n) = \{0,1\}^k$, and edges project to edges. Thus $\Phi_{\mathrm{face}(I,a)}(\mathbf{H}_n) = \mathbf{H}_k \in U_{\mathrm{hc}}$.

(2) Naturality (isomorphism invariance). Let $\alpha_i : \mathbf{H}_{n_i} \xrightarrow{\cong} \mathbf{H}'_{n_i}$ be isomorphisms of Σ_{hc}-structures (bijections preserving C, V, E). We construct induced isomorphisms for each meta-operation:

(P) Set $\alpha_\otimes := \alpha_1 \times \alpha_2 : H_{n_1} \times H_{n_2} \to H'_{n_1} \times H'_{n_2}$. Then

$$\alpha_\otimes(C_{n_1} \times C_{n_2}) = (\alpha_1 C_{n_1}) \times (\alpha_2 C_{n_2}) = C'_{n_1} \times C'_{n_2},$$

$$\alpha_\otimes(V_{n_1}\times V_{n_2}) = (\alpha_1 V_{n_1})\times(\alpha_2 V_{n_2}) = V'_{n_1}\times V'_{n_2},$$

and using functoriality of Δ,

$$\alpha_\otimes\big(E_{n_1}\times\Delta_{V_{n_2}}\big) = (\alpha_1 E_{n_1})\times\Delta_{\alpha_2 V_{n_2}} = E'_{n_1}\times\Delta_{V'_{n_2}},$$

similarly for the second summand. Hence

$$\Phi_\otimes(\alpha_1,\alpha_2):\ \Phi_\otimes(\mathbf{H}_{n_1},\mathbf{H}_{n_2}) \xrightarrow{\cong} \Phi_\otimes(\mathbf{H}'_{n_1},\mathbf{H}'_{n_2}).$$

(S) Define $\alpha_{\mathrm{scale}(s)}(x) = s\cdot\alpha_1(s^{-1}x)$; this preserves $s\cdot C_{n_1}$, $s\cdot V_{n_1}$, and transports edges accordingly.

(T) Define $\alpha_{\mathrm{trans}(c)}(x) = \alpha_1(x-c)+c'$, where c' is the translation in the codomain; this preserves translated C, V, E.

(W) Put $\alpha_A(x) = A\alpha_1(A^{-1}x)$; since A preserves the cube structure combinatorially, α_A is an isomorphism $\Phi_A(\mathbf{H}_{n_1})\to\Phi_A(\mathbf{H}'_{n_1})$.

(F) Let $\alpha_{\mathrm{face}(I,a)} := \pi'_{I,a}\circ\alpha_1\circ\iota_{I,a}$, where $\iota_{I,a}$ is the section of the face embedding; this preserves $[0,1]^k$, its vertex set, and induced edges.

Thus each Φ_ℓ is natural. Since $\mathbf{H}_n\in U_{\mathrm{hc}}$ for all n, classical hypercubes are objects of $\mathbb{M}_{\mathrm{hc}}$. □

Proposition 3.5 (Vertex/edge counts under product)**.** *Let Q_m and Q_n denote the m- and n-dimensional hypercubes (i.e. $\mathbf{H}_m,\mathbf{H}_n$). Then*

$$|V(Q_m\otimes Q_n)| = 2^{m+n}, \qquad |E(Q_m\otimes Q_n)| = (m+n)\,2^{m+n-1}.$$

Proof. Vertices: $|V_m\times V_n| = 2^m\cdot 2^n = 2^{m+n}$. Edges:

$$\begin{aligned}|E_m\times\Delta_{V_n}| + |\Delta_{V_m}\times E_n| &= |E_m|\cdot|V_n| + |V_m|\cdot|E_n|\\ &= \big(m2^{m-1}\big)\,2^n + 2^m\,\big(n2^{n-1}\big) = (m+n)\,2^{m+n-1}.\end{aligned}$$

□

Definition 3.6 (Iterated MetaHyperCube of depth t)**.** For $t\in\mathbb{N}$, an *Iterated MetaHyperCube of depth t* is an Iterated MetaStructure (Definition 1.3) over Σ_{hc},

$$\mathfrak{H}^{(t)}_{\mathrm{hc}} = \big(U^{(t)}_{\mathrm{hc}},\ (\odot^{(t)}_\ell)_{\ell\in\Lambda}\big),$$

obtained by applying the lifting functor $\mathsf{U}_{\Sigma_{\mathrm{hc}}}$ of Definition 1.3 to $\mathbb{M}_{\mathrm{hc}}$ repeatedly t times. Concretely, for $s<t$ and any base meta-operation Φ_ℓ with meta-arity k_ℓ, its lift

$$\Phi^{\uparrow}_\ell:\ \big(U^{(t)}_{\mathrm{hc}}\big)^{k_\ell} \longrightarrow U^{(t)}_{\mathrm{hc}}$$

is defined on representatives by

$$\Phi^{\uparrow}_\ell\big(\mathsf{U}^{t-s}_{\Sigma_{\mathrm{hc}}}(X_1),\dots,\mathsf{U}^{t-s}_{\Sigma_{\mathrm{hc}}}(X_{k_\ell})\big)\ :=\ \mathsf{U}^{t-s}_{\Sigma_{\mathrm{hc}}}\big(\Phi_\ell(X_1,\dots,X_{k_\ell})\big),$$

and similarly for the relations, exactly as in Definition 1.3. We denote by $\odot^{(t)}_\otimes$, $\odot^{(t)}_{\mathrm{scale}(s)}$, $\odot^{(t)}_{\mathrm{trans}(c)}$, $\odot^{(t)}_A$, and $\odot^{(t)}_{\mathrm{face}(I,a)}$ the lifted product, scaling, translation, signed-permutation, and face operators, respectively.

Example 3.7 (Iterated MetaHyperCube (depth $t=1$): Weekly Schedule of Daily Flags)**.** Each day $d\in\{1,\dots,7\}$ uses the same 5-flag hypercube $\mathbf{H}^{(d)}_5$ (as above), so a weekly schedule is a 7-tuple of daily configurations. In the iterated system the lifted product $\odot^{(1)}_\otimes$ forms the "hypercube of hypercubes," whose vertex set size multiplies:

$$\big|V\big(\odot^{(1)}_\otimes(\mathbf{H}^{(1)}_5,\dots,\mathbf{H}^{(7)}_5)\big)\big| = (2^5)^7 = 2^{35} = 34{,}359{,}738{,}368.$$

Applying a lifted face that fixes, say, Logging $=1$ every day reduces each $\mathbf{H}^{(d)}_5$ to $\mathbf{H}^{(d)}_4$, hence

$$(2^4)^7 = 2^{28} = 268{,}435{,}456$$

weekly schedules remain.

Theorem 3.8 (Iterated MetaHyperCube is an Iterated MetaStructure and generalizes MetaHyperCube)**.** *For every $t \in \mathbb{N}$, $\mathfrak{H}_{\mathrm{hc}}^{(t)}$ of Definition 3.6 is an Iterated MetaStructure in the sense of Definition 1.3. Moreover, for $s < t$ there is a canonical embedding*

$$\iota_{s\to t}: \mathfrak{H}_{\mathrm{hc}}^{(s)} \hookrightarrow \mathfrak{H}_{\mathrm{hc}}^{(t)}, \qquad X \longmapsto \mathsf{U}_{\Sigma_{\mathrm{hc}}}^{t-s}(X),$$

which is operation-preserving:

$$\Phi_\ell^{\uparrow}\big(\iota_{s\to t}(X_1), \dots, \iota_{s\to t}(X_{k_\ell})\big) = \iota_{s\to t}\big(\Phi_\ell(X_1, \dots, X_{k_\ell})\big).$$

In particular, $\mathfrak{H}_{\mathrm{hc}}^{(0)} = \mathbb{M}_{\mathrm{hc}}$ embeds into $\mathfrak{H}_{\mathrm{hc}}^{(t)}$, so Iterated MetaHyperCube generalizes *MetaHyperCube.*

Proof. By Definition 1.3, $U_{\mathrm{hc}}^{(t)}$ and each $\odot_\ell^{(t)}$ are obtained by functorially post-composing the base constructors $(\Gamma_\ell, \Lambda_\ell^{\cdot}, \Xi_\ell^{\cdot})$ with $\mathsf{U}_{\Sigma_{\mathrm{hc}}}^{t-s}$, hence they are uniform.

Naturality follows from the preservation of isomorphisms by $\mathsf{U}_{\Sigma_{\mathrm{hc}}}^{t-s}$: if $\alpha_i : X_i \to Y_i$ are level-s isomorphisms, then

$$\mathsf{U}_{\Sigma_{\mathrm{hc}}}^{t-s}(\alpha_i) : \mathsf{U}_{\Sigma_{\mathrm{hc}}}^{t-s}(X_i) \xrightarrow{\cong} \mathsf{U}_{\Sigma_{\mathrm{hc}}}^{t-s}(Y_i)$$

are level-t isomorphisms, and

$$\Phi_\ell^{\uparrow}\big(\mathsf{U}^{t-s}\alpha_1, \dots, \mathsf{U}^{t-s}\alpha_{k_\ell}\big) = \mathsf{U}^{t-s}\big(\Phi_\ell(\alpha_1, \dots, \alpha_{k_\ell})\big),$$

which is an isomorphism since Φ_ℓ is natural at level s (Theorem 3.4).

The embedding $\iota_{s\to t}$ is the unit of the lifting (Definition 1.3) and satisfies

$$\Phi_\ell^{\uparrow}\big(\mathsf{U}^{t-s}X_1, \dots, \mathsf{U}^{t-s}X_{k_\ell}\big) = \mathsf{U}^{t-s}\big(\Phi_\ell(X_1, \dots, X_{k_\ell})\big)$$

by definition, proving operation preservation. Taking $s = 0$ yields the canonical embedding of $\mathbb{M}_{\mathrm{hc}}$ into $\mathfrak{H}_{\mathrm{hc}}^{(t)}$. □

4 MetaVector Space (Vector Space of Vector Space)

Vector Space is a set with vector addition and scalar multiplication, modeling linear structures fundamental in mathematics, physics, and engineering [57–61]. MetaVector Space generalizes vector spaces by combining entire spaces through direct sums, tensor products, or dualization into higher-level linear frameworks. Iterated MetaVector Space applies meta-operations repeatedly, building recursive layers of vector-space-of-vector-spaces, enabling abstraction for complex multi-level linear systems.

Definition 4.1 (Vector Space)**.** (cf. [62–64]) A *vector space* over a field $\mathbb{F}$ is a pair $(V, +, \cdot)$ where V is a nonempty set, $+$ is a binary operation $V \times V \to V$, and $\cdot$ is a scalar multiplication $\mathbb{F} \times V \to V$, such that $(V, +)$ is an abelian group and the following axioms hold for all $u, v \in V$ and $\alpha, \beta \in \mathbb{F}$:

$$\alpha(u + v) = \alpha u + \alpha v, \qquad (\alpha + \beta)u = \alpha u + \beta u, \qquad (\alpha\beta)u = \alpha(\beta u), \qquad 1u = u.$$

Fix a field $\mathbb{F}$. Consider the (single-sorted, finitary) *vector-space signature*

$$\Sigma_{\mathrm{vs}} = \Big(\ \mathsf{Func},\ \mathsf{Rel} = \varnothing,\ \mathrm{ar}\ \Big),$$

where the function symbols are

$$\mathsf{Func} = \big\{ + : 2,\ 0 : 0,\ - : 1,\ m_\alpha : 1\ (\alpha \in \mathbb{F}) \big\}.$$

A Σ_{vs}-structure is a tuple

$$\mathbf{V} = \big(H;\ +^{\mathbf{V}},\ 0^{\mathbf{V}},\ (-)^{\mathbf{V}},\ (m_\alpha^{\mathbf{V}})_{\alpha\in\mathbb{F}}\big),$$

with carrier $H \neq \varnothing$ and interpretations of the symbols as operations on H. We let $U_{\text{vs}} \subseteq \text{Str}_{\Sigma_{\text{vs}}}$ be the class of all *genuine* $\mathbb{F}$*-vector spaces* encoded in this way, i.e.,

$$(H, +, 0, -, (m_\alpha)_{\alpha\in\mathbb{F}})$$

satisfies the vector-space axioms: for all $u, v, w \in H$ and $\alpha, \beta \in \mathbb{F}$,

$$\begin{aligned}
&u + v = v + u, \quad (u + v) + w = u + (v + w), \quad u + 0 = u, \quad u + (-u) = 0,\\
&m_\alpha(u + v) = m_\alpha(u) + m_\alpha(v), \quad m_{\alpha+\beta}(u) = m_\alpha(u) + m_\beta(u),\\
&m_{\alpha\beta}(u) = m_\alpha\big(m_\beta(u)\big), \quad m_1(u) = u.
\end{aligned}$$

Definition 4.2 (MetaVector Space)**.** A *MetaVector Space* is a MetaStructure (Definition 1.2) over Σ_{vs}

$$\mathbb{M}_{\text{vs}} = \big(U_{\text{vs}},\ (\Phi_\ell)_{\ell\in\Lambda}\big),$$

whose meta-operations are given uniformly by the following standard linear constructions:

(DS) Finite direct sum. For $\mathbf{V} = (V, \dots),\ \mathbf{W} = (W, \dots) \in U_{\text{vs}}$ set

$$\Phi_\oplus(\mathbf{V}, \mathbf{W}) := \big(V \oplus W,\ +_\oplus,\ 0_\oplus,\ -_\oplus,\ (m^\oplus_\alpha)_{\alpha\in\mathbb{F}}\big),$$

with carrier-constructor $\Gamma_\oplus(V, W) = V \times W$, and symbol-constructors

$$(u_1, w_1) +_\oplus (u_2, w_2) = (u_1 + u_2,\ w_1 + w_2), \quad 0_\oplus = (0, 0), \quad -(u, w) = (-u, -w), \quad m^\oplus_\alpha(u, w) = (m_\alpha u,\ m_\alpha w).$$

(HOM) Linear hom-space.

$$\Phi_{\text{Hom}}(\mathbf{V}, \mathbf{W}) := \big(\text{Hom}_{\mathbb{F}}(V, W),\ +,\ 0,\ -,\ (m_\alpha)_{\alpha\in\mathbb{F}}\big),$$

with carrier-constructor $\Gamma_{\text{Hom}}(V, W) = \{\, T : V \to W \mid T\ \mathbb{F}\text{-linear} \,\}$ and

$$(T_1 + T_2)(v) = T_1(v) + T_2(v), \quad 0(v) = 0, \quad (-T)(v) = -(T(v)), \quad (m_\alpha T)(v) = m_\alpha\big(T(v)\big).$$

(TENS) Tensor product.

$$\Phi_\otimes(\mathbf{V}, \mathbf{W}) := \big(V \otimes_{\mathbb{F}} W,\ +,\ 0,\ -,\ (m_\alpha)_{\alpha\in\mathbb{F}}\big),$$

with carrier-constructor $\Gamma_\otimes(V, W) = \mathcal{F}(V \times W)/\mathcal{R}_{\text{bilin}}$, the free $\mathbb{F}$-vector space on $V \times W$ quotiented by the bilinear relations

$$\begin{aligned}
&(v_1 + v_2, w) - (v_1, w) - (v_2, w), \quad (v, w_1 + w_2) - (v, w_1) - (v, w_2),\\
&(\alpha v, w) - \alpha\,(v, w), \quad (v, \alpha w) - \alpha\,(v, w),
\end{aligned}$$

so that the class of (v, w) is written $v \otimes w$, with operations induced from the ambient free space.

(DUAL) Algebraic dual.

$$\Phi_*(\mathbf{V}) := \Phi_{\text{Hom}}(\mathbf{V}, \mathbf{F}), \quad \text{where} \quad \mathbf{F} = (\mathbb{F}, +, 0, -, (m_\alpha)_{\alpha\in\mathbb{F}})$$

is the one-dimensional $\mathbb{F}$-vector space.

Example 4.3 (MetaVector Space: Multimodal Biometric Fusion)**.** Face embeddings live in $V_{\text{face}} \cong \mathbb{R}^{128}$ and voice embeddings in $V_{\text{voice}} \cong \mathbb{R}^{64}$. Fusing features by direct sum gives

$$V_{\text{fuse}} = V_{\text{face}} \oplus V_{\text{voice}} \cong \mathbb{R}^{128+64} = \mathbb{R}^{192}.$$

Pairwise cross-modal interactions via tensor product use

$$V_{\text{face}} \otimes V_{\text{voice}} \quad \Rightarrow \quad \dim(V_{\text{face}} \otimes V_{\text{voice}}) = 128 \times 64 = 8192.$$

A linear verification score is a functional $w \in V^*_{\text{fuse}}$ with $s(x) = \langle w, x\rangle$; a bilinear cross-modal score uses $T \in \text{Hom}(V_{\text{face}} \otimes V_{\text{voice}}, \mathbb{F})$ (dimension 8192).

Theorem 4.4 (MetaVector Space is a MetaStructure and generalizes vector spaces)**.** $\mathbb{M}_{\mathrm{vs}} = \big(U_{\mathrm{vs}}, (\Phi_\ell)\big)$ *in Definition 4.2 is a MetaStructure in the sense of Definition 1.2. Moreover, every classical* $\mathbb{F}$*-vector space* $(V, +, 0, -, (m_\alpha))$ *appears as an object of* U_{vs}*, hence MetaVector Space* generalizes *vector spaces.*

Proof. We verify the two MetaStructure clauses.

(1) Uniform constructors and closure. For each meta-operation, the carrier-constructor Γ_ℓ and the symbol-constructors $(+^{\Phi_\ell}, 0^{\Phi_\ell}, (-)^{\Phi_\ell}, (m_\alpha^{\Phi_\ell}))$ are specified above and do not depend on the particular representations of inputs, only on their structures.

<u>(DS)</u> On $V \oplus W = V \times W$, check the axioms (we write $m_\alpha u = \alpha u$ for clarity). For all $(u_i, w_i) \in V \oplus W$, $\alpha, \beta \in \mathbb{F}$,

$$\begin{aligned}
&\text{commutativity: } (u_1, w_1) +_\oplus (u_2, w_2) = (u_1 + u_2, w_1 + w_2) = (u_2 + u_1, w_2 + w_1)\\
&\qquad\qquad\qquad = (u_2, w_2) +_\oplus (u_1, w_1);\\
&\text{associativity: } ((u_1, w_1) +_\oplus (u_2, w_2)) +_\oplus (u_3, w_3)\\
&\qquad = (u_1 + u_2, w_1 + w_2) +_\oplus (u_3, w_3) = (u_1 + u_2 + u_3, w_1 + w_2 + w_3)\\
&\qquad = (u_1, w_1) +_\oplus (u_2 + u_3, w_2 + w_3) = (u_1, w_1) +_\oplus \big((u_2, w_2) +_\oplus (u_3, w_3)\big);\\
&\text{neutral/ inverse: } (u, w) +_\oplus (0, 0) = (u, w), \quad (u, w) +_\oplus (-u, -w) = (0, 0);\\
&\text{distributivities: } \alpha\big((u_1, w_1) +_\oplus (u_2, w_2)\big) = \alpha(u_1 + u_2,\ w_1 + w_2) = (\alpha u_1 + \alpha u_2,\ \alpha w_1 + \alpha w_2)\\
&\qquad = (\alpha u_1, \alpha w_1) +_\oplus (\alpha u_2, \alpha w_2) = \alpha(u_1, w_1) +_\oplus \alpha(u_2, w_2);\\
&(\alpha + \beta)(u, w) = \big((\alpha + \beta)u,\ (\alpha + \beta)w\big) = (\alpha u + \beta u,\ \alpha w + \beta w)\\
&\qquad = \alpha(u, w) +_\oplus \beta(u, w); \qquad \alpha(\beta(u, w)) = (\alpha\beta)(u, w), \quad 1 \cdot (u, w) = (u, w).
\end{aligned}$$

Thus $\Phi_\oplus(\mathbf{V}, \mathbf{W}) \in U_{\mathrm{vs}}$.

<u>(HOM)</u> On $\mathrm{Hom}_{\mathbb{F}}(V, W)$ the operations are pointwise; for $T_i \in \mathrm{Hom}(V, W)$ and $\alpha, \beta \in \mathbb{F}$,

$$\begin{aligned}
&(T_1 + T_2)(v) = (T_1(v) + T_2(v)) = (T_2 + T_1)(v),\\
&((T_1 + T_2) + T_3)(v) = T_1(v) + T_2(v) + T_3(v) = (T_1 + (T_2 + T_3))(v),\\
&0(v) = 0,\ (-T)(v) = -(T(v)),\\
&\alpha(T_1 + T_2)(v) = \alpha(T_1(v) + T_2(v)) = \alpha T_1(v) + \alpha T_2(v) = (\alpha T_1 + \alpha T_2)(v),\\
&(\alpha + \beta)T(v) = (\alpha + \beta)T(v) = \alpha T(v) + \beta T(v) = (\alpha T + \beta T)(v),\\
&\alpha(\beta T)(v) = \alpha\big(\beta T(v)\big) = (\alpha\beta)T(v), \quad 1 \cdot T(v) = T(v),
\end{aligned}$$

so the axioms hold; hence $\Phi_{\mathrm{Hom}}(\mathbf{V}, \mathbf{W}) \in U_{\mathrm{vs}}$.

<u>(TENS)</u> The tensor product carrier is a quotient of a vector space by a linear subspace $\mathcal{R}_{\mathrm{bilin}}$, so it is a vector space; addition, zero, negation, and scalar multiplication are induced from the ambient free space and thus satisfy the axioms. Concretely, if $x = \sum_i \alpha_i(v_i, w_i)$ and $y = \sum_j \beta_j(v'_j, w'_j)$ in the free space, then

$$[x] + [y] = \Big[\sum_i \alpha_i(v_i, w_i) + \sum_j \beta_j(v'_j, w'_j)\Big], \qquad \alpha \cdot [x] = \Big[\sum_i (\alpha\alpha_i)(v_i, w_i)\Big],$$

which are well-defined on equivalence classes because $\mathcal{R}_{\mathrm{bilin}}$ is a linear subspace. Hence $\Phi_\otimes(\mathbf{V}, \mathbf{W}) \in U_{\mathrm{vs}}$.

<u>(DUAL)</u> This is the special case $\mathrm{Hom}_{\mathbb{F}}(V, \mathbb{F})$ of (HOM), thus a vector space.

(2) Naturality (isomorphism invariance). Let $\alpha : \mathbf{V} \xrightarrow{\cong} \mathbf{V}'$ and $\beta : \mathbf{W} \xrightarrow{\cong} \mathbf{W}'$ be Σ_{vs}-isomorphisms (i.e. linear isomorphisms). Then:

<u>(DS)</u> Define $\alpha \oplus \beta : V \oplus W \to V' \oplus W'$ by $(\alpha \oplus \beta)(u, w) = (\alpha u,\ \beta w)$. It preserves all operations componentwise, hence is a Σ_{vs}-isomorphism

$$\Phi_\oplus(\alpha, \beta) :\ \Phi_\oplus(\mathbf{V}, \mathbf{W}) \xrightarrow{\cong} \Phi_\oplus(\mathbf{V}', \mathbf{W}').$$

(HOM) Define $\beta \circ (-) \circ \alpha^{-1} : \mathrm{Hom}(V, W) \to \mathrm{Hom}(V', W')$ by

$$\Phi_{\mathrm{Hom}}(\alpha, \beta)(T) = \beta \circ T \circ \alpha^{-1}.$$

This is linear (composition is bilinear in each slot), bijective, and preserves the pointwise operations.

(TENS) Define $\alpha \otimes \beta : V \otimes W \to V' \otimes W'$ uniquely by

$$(\alpha \otimes \beta)(v \otimes w) = \alpha(v) \otimes \beta(w),$$

which is well-defined because it respects the bilinear relations:

$$\begin{aligned}
&(\alpha \otimes \beta)\big((v_1 + v_2) \otimes w\big) = \alpha(v_1 + v_2) \otimes \beta(w) = \alpha(v_1) \otimes \beta(w) + \alpha(v_2) \otimes \beta(w),\\
&(\alpha \otimes \beta)\big(v \otimes (w_1 + w_2)\big) = \alpha(v) \otimes \beta(w_1) + \alpha(v) \otimes \beta(w_2),\\
&(\alpha \otimes \beta)\big((\gamma v) \otimes w\big) = \alpha(\gamma v) \otimes \beta(w) = \gamma\, \alpha(v) \otimes \beta(w),\\
&(\alpha \otimes \beta)\big(v \otimes (\gamma w)\big) = \alpha(v) \otimes \beta(\gamma w) = \gamma\, \alpha(v) \otimes \beta(w),
\end{aligned}$$

and it is linear and bijective (inverse is $\alpha^{-1} \otimes \beta^{-1}$). Thus it is a Σ_{vs}-isomorphism.

(DUAL) $\alpha^* : V'^* \to V^*$ given by $\alpha^*(f) = f \circ \alpha$ is a linear isomorphism; equivalently, $\Phi_*(\alpha) = (\alpha^{-1})^* : V^* \to V'^*$ is linear and bijective.

Therefore each Φ_ℓ is natural, proving that $\mathbb{M}_{\mathrm{vs}}$ is a MetaStructure. The final claim is immediate from the definition of U_{vs}. □

Proposition 4.5 (Dimensions under ⊕ and ⊗)**.** *If* $\dim V = m$ *and* $\dim W = n$ *(finite), then* $\dim(V \oplus W) = m + n$ *and* $\dim(V \otimes_{\mathbb{F}} W) = mn$.

Proof. For ⊕, concatenate bases. For ⊗, $\{e_i \otimes f_j\}_{i \le m,\ j \le n}$ is a basis. □

Definition 4.6 (Iterated MetaVector Space of depth t)**.** For $t \in \mathbb{N}$, an *Iterated MetaVector Space of depth t* is an Iterated MetaStructure (Definition 1.3) over Σ_{vs},

$$\mathfrak{M}^{(t)}_{\mathrm{vs}} = \big(U^{(t)}_{\mathrm{vs}},\ (\odot^{(t)}_\ell)_{\ell \in \Lambda}\big),$$

obtained by repeatedly applying the lifting $\mathsf{U}_{\Sigma_{\mathrm{vs}}}$ to $\mathbb{M}_{\mathrm{vs}}$. Concretely, for $s < t$ and any base meta-operation Φ_ℓ of arity k_ℓ,

$$\Phi^{\uparrow}_\ell :\ \big(U^{(t)}_{\mathrm{vs}}\big)^{k_\ell} \longrightarrow U^{(t)}_{\mathrm{vs}}$$

is defined on representatives by

$$\Phi^{\uparrow}_\ell\big(\mathsf{U}^{t-s}_{\Sigma_{\mathrm{vs}}}(X_1), \ldots, \mathsf{U}^{t-s}_{\Sigma_{\mathrm{vs}}}(X_{k_\ell})\big) := \mathsf{U}^{t-s}_{\Sigma_{\mathrm{vs}}}\big(\Phi_\ell(X_1, \ldots, X_{k_\ell})\big),$$

and similarly for all function symbols, as stipulated in Definition 1.3. We denote by $\odot^{(t)}_{\oplus}$, $\odot^{(t)}_{\mathrm{Hom}}$, $\odot^{(t)}_{\otimes}$, and $\odot^{(t)}_{*}$ the lifted operations corresponding to (DS), (HOM), (TENS), and (DUAL).

Example 4.7 (Iterated MetaVector Space (depth $t = 1$): Clinic-by-Day Scorers)**.** Let $V = \mathbb{R}^{192}$ be the fused feature space above. For $K = 10$ clinics and $D = 7$ days, consider the lifted direct sum of daily linear scorers

$$\bigoplus_{c=1}^{K} \bigoplus_{d=1}^{D} \mathrm{Hom}(V, \mathbb{F}),$$

whose total dimension is

$$K \cdot D \cdot \dim \mathrm{Hom}(V, \mathbb{F}) = 10 \times 7 \times 192 = 13{,}440.$$

If a 3-dimensional context $C \cong \mathbb{R}^3$ (e.g. workload, weather, promotion level) is incorporated uniformly via the lifted tensor, the scorer space becomes $\bigoplus_{c,d} \mathrm{Hom}(V \otimes C, \mathbb{F})$ with total dimension

$$K \cdot D \cdot \dim(V \otimes C) = 10 \times 7 \times (192 \times 3) = 40{,}320.$$

Theorem 4.8 (Iterated MetaVector Space is an Iterated MetaStructure and generalizes MetaVector Space)**.** *For every $t \in \mathbb{N}$, $\mathfrak{M}^{(t)}_{\mathrm{vs}}$ of Definition 4.6 is an Iterated MetaStructure in the sense of Definition 1.3. Moreover, for $s < t$ the map*

$$\iota_{s\to t} : \mathfrak{M}^{(s)}_{\mathrm{vs}} \hookrightarrow \mathfrak{M}^{(t)}_{\mathrm{vs}}, \qquad X \longmapsto \mathsf{U}^{t-s}_{\Sigma_{\mathrm{vs}}}(X),$$

is an embedding that is operation-preserving:

$$\Phi^{\uparrow}_{\ell}\big(\iota_{s\to t}(X_1), \ldots, \iota_{s\to t}(X_{k_\ell})\big) = \iota_{s\to t}\big(\Phi_\ell(X_1, \ldots, X_{k_\ell})\big).$$

In particular, $\mathfrak{M}^{(0)}_{\mathrm{vs}} = \mathbb{M}_{\mathrm{vs}}$ embeds into $\mathfrak{M}^{(t)}_{\mathrm{vs}}$, so Iterated MetaVector Space generalizes *MetaVector Space.*

Proof. By Definition 1.3, each lifted operation $\Phi^{\uparrow}_{\ell}$ is obtained by post-composing the base constructors $(\Gamma_\ell, +^{\Phi_\ell}, 0^{\Phi_\ell}, (-)^{\Phi_\ell}, (m^{\Phi_\ell}_{\alpha}))$ with $\mathsf{U}^{t-s}_{\Sigma_{\mathrm{vs}}}$; thus the resulting constructors are uniform.

For naturality, let $\alpha_i : X_i \xrightarrow{\cong} Y_i$ be level-s isomorphisms in $U^{(s)}_{\mathrm{vs}}$. Then

$$\mathsf{U}^{t-s}_{\Sigma_{\mathrm{vs}}}(\alpha_i) : \mathsf{U}^{t-s}_{\Sigma_{\mathrm{vs}}}(X_i) \xrightarrow{\cong} \mathsf{U}^{t-s}_{\Sigma_{\mathrm{vs}}}(Y_i)$$

are level-t isomorphisms. Using the defining clause of the lift,

$$\Phi^{\uparrow}_{\ell}\big(\mathsf{U}^{t-s}\alpha_1, \ldots, \mathsf{U}^{t-s}\alpha_{k_\ell}\big) = \mathsf{U}^{t-s}\big(\Phi_\ell(\alpha_1, \ldots, \alpha_{k_\ell})\big).$$

Since each base Φ_ℓ is natural by Theorem 4.4, the right-hand side is an isomorphism; hence $\Phi^{\uparrow}_{\ell}$ is natural.

Finally, the embedding $\iota_{s\to t}$ is precisely the unit of the lifting functor and satisfies

$$\Phi^{\uparrow}_{\ell}\big(\mathsf{U}^{t-s}X_1, \ldots, \mathsf{U}^{t-s}X_{k_\ell}\big) = \mathsf{U}^{t-s}\big(\Phi_\ell(X_1, \ldots, X_{k_\ell})\big)$$

by definition, which is the operation-preservation identity. Taking $s = 0$ gives the canonical embedding of $\mathbb{M}_{\mathrm{vs}}$ into $\mathfrak{M}^{(t)}_{\mathrm{vs}}$. □

5 MetaMatrix (Matrix of Matrix)

Matrix is a rectangular array of numbers representing linear transformations, enabling operations like addition, multiplication, and transposition [65–67]. MetaMatrix generalizes matrices by combining entire matrix structures through block sums, Kronecker products, or multiplication reversal into higher-level algebraic systems. Iterated MetaMatrix repeatedly applies MetaMatrix operations, creating layered matrix-of-matrix structures that enable recursive composition and abstraction of algebraic systems.

Definition 5.1 (Matrix)**.** (cf. [68, 69]) A *matrix* over a field $\mathbb{F}$ is a rectangular array

$$A = (a_{ij}) \quad \text{with } a_{ij} \in \mathbb{F}, \quad 1 \le i \le m,\ 1 \le j \le n,$$

of size $m \times n$, where m is the number of rows and n the number of columns. Matrices can be added and multiplied (when sizes match) and represent linear transformations.

Fix a field $\mathbb{F}$. Consider the (single-sorted, finitary) *matrix-algebra signature*

$$\Sigma_{\mathrm{mat}} = \Big(\ \mathsf{Func},\ \mathsf{Rel} = \varnothing,\ \mathrm{ar}\Big),$$

with function symbols

$$\mathsf{Func} = \big\{ + : 2,\ 0 : 0,\ - : 1,\ m_\alpha : 1\ (\alpha \in \mathbb{F}),\ \cdot : 2,\ T : 1 \big\}.$$

A Σ_{mat}-structure is a tuple

$$\mathbf{M} \ = \ \big(H;\ +^{\mathbf{M}},\ 0^{\mathbf{M}},\ (-)^{\mathbf{M}},\ (m^{\mathbf{M}}_\alpha)_{\alpha\in\mathbb{F}},\ \cdot^{\mathbf{M}},\ T^{\mathbf{M}}\big),$$

with carrier $H \neq \varnothing$ and the obvious interpretations of the symbols as total operations on H.

For each $n \in \mathbb{N}_{\ge 1}$, let $M_n(\mathbb{F})$ denote the $\mathbb{F}$-algebra of $n \times n$ matrices over $\mathbb{F}$. We define the universe

$$U_{\mathrm{mat}} = \Big\{ \mathbf{M}_n \ \Big|\ \mathbf{M}_n = \big(M_n(\mathbb{F});\ +,\ 0,\ -,\ (m_\alpha)_{\alpha\in\mathbb{F}},\ \cdot,\ T\big),\ n \ge 1 \Big\} \ \subseteq\ \mathrm{Str}_{\Sigma_{\mathrm{mat}}},$$

where $+$, 0, $-$, (m_α) are the usual vector-space operations, $\cdot$ is matrix multiplication, and T is the transpose. Each $\mathbf{M}_n$ satisfies the ring/linear axioms and T is $\mathbb{F}$-linear with $(AB)^T = B^T A^T$.

Definition 5.2 (MetaMatrix)**.** A *MetaMatrix* is a MetaStructure (Definition 1.2) over Σ_{mat}

$$\mathbb{M}_{\mathrm{mat}} = \big(U_{\mathrm{mat}},\ (\Phi_\ell)_{\ell\in\Lambda}\big),$$

whose meta-operations are specified uniformly as follows (each with carrier- and symbol-constructors as in Definition 1.2):

(DS) Block direct sum (size addition). For $\mathbf{M}_m, \mathbf{M}_n \in U_{\mathrm{mat}}$,

$$\Phi_{\oplus}(\mathbf{M}_m, \mathbf{M}_n) := \mathbf{M}_{m+n},$$

with carrier-constructor $\Gamma_{\oplus}(M_m, M_n) = M_{m+n}$ and, writing $\mathrm{diag}(A, B)$ for the block diagonal, the symbol-constructors act by transporting the standard operations through

$$\mathrm{diag}(A,B) + \mathrm{diag}(C,D) = \mathrm{diag}(A+C,\ B+D), \quad \mathrm{diag}(A,B)\cdot\mathrm{diag}(C,D) = \mathrm{diag}(AC,\ BD),$$

$$T\big(\mathrm{diag}(A,B)\big) = \mathrm{diag}(A^T, B^T), \qquad m_\alpha\big(\mathrm{diag}(A,B)\big) = \mathrm{diag}(\alpha A,\ \alpha B).$$

(TEN) Kronecker (size multiplication).

$$\Phi_{\otimes}(\mathbf{M}_m, \mathbf{M}_n) := \mathbf{M}_{mn},$$

with carrier-constructor $\Gamma_{\otimes}(M_m, M_n) = M_{mn}$ and symbol-constructors induced via the canonical $\mathbb{F}$-algebra isomorphism $\delta_{m,n} : M_m(\mathbb{F}) \otimes_{\mathbb{F}} M_n(\mathbb{F}) \xrightarrow{\cong} M_{mn}(\mathbb{F})$ sending elementary tensors to the Kronecker product,

$$\delta_{m,n}(A\otimes B) = A \otimes_{\mathrm{K}} B,$$

so that

$$(A\otimes_{\mathrm{K}} B)\cdot(C\otimes_{\mathrm{K}} D) = (AC)\otimes_{\mathrm{K}}(BD), \qquad (A\otimes_{\mathrm{K}} B)^T = A^T\otimes_{\mathrm{K}} B^T,$$

and $+, 0, -, m_\alpha$ are the standard ones on M_{mn}.

(OP) Opposite algebra (multiplication reversal). For $\mathbf{M}_n \in U_{\mathrm{mat}}$,

$$\Phi_{\mathrm{op}}(\mathbf{M}_n) := \big(M_n(\mathbb{F});\ +,\ 0,\ -,\ (m_\alpha),\ \cdot^{\mathrm{op}},\ T\big), \qquad A\cdot^{\mathrm{op}} B := B\cdot A.$$

Example 5.3 (MetaMatrix: 2D DCT in JPEG Blocks)**.** (cf. [70, 71]) Let $C_8 \in \mathbb{R}^{8\times 8}$ be the 1D DCT matrix. For an 8×8 image block vectorized as $x \in \mathbb{R}^{64}$, the separable 2D transform is the Kronecker product

$$K\ :=\ C_8\otimes C_8\ \in\ \mathbb{R}^{64\times 64}, \qquad y\ =\ Kx.$$

Here (TEN) gives size multiplication $8\times 8 \mapsto 64$, and transpose behaves as $(C_8\otimes C_8)^T = C_8^T \otimes C_8^T$. For RGB processing, three independent channels use the block direct sum (DS)

$$K_{\mathrm{RGB}}\ :=\ K\ \oplus\ K\ \oplus\ K\ =\ \mathrm{diag}(K,K,K)\ \in\ \mathbb{R}^{192\times 192}.$$

Theorem 5.4 (MetaMatrix is a MetaStructure and generalizes matrices)**.** *$\mathbb{M}_{\mathrm{mat}} = \big(U_{\mathrm{mat}}, (\Phi_\ell)\big)$ of Definition 5.2 is a MetaStructure in the sense of Definition 1.2. Moreover, for every $n \geq 1$ the classical matrix algebra $\mathbf{M}_n$ is an object of U_{mat}, hence MetaMatrix* generalizes *matrices.*

Proof. We verify the two MetaStructure clauses.

(1) Uniform constructors and closure. Each meta-operation comes with a carrier-constructor Γ_ℓ and uniform symbol-constructors:

<u>(DS)</u> The carrier M_{m+n} is fixed functorially by dimensions. The displayed formulas show that $+, 0, -, m_\alpha, T$ act blockwise and that multiplication is block-diagonal:

$$\mathrm{diag}(A,B)\cdot\mathrm{diag}(C,D) = \mathrm{diag}(AC, BD),$$

which is associative and distributive since matrix multiplication is, and T remains linear with $(XY)^T = Y^T X^T$ verified blockwise. Thus $\Phi_\oplus(\mathbf{M}_m, \mathbf{M}_n) = \mathbf{M}_{m+n} \in U_{\mathrm{mat}}$.

(TEN) The canonical algebra isomorphism $\delta_{m,n}$ intertwines the algebra structure on $M_m \otimes M_n$ with the standard one on M_{mn}. Using the Kronecker identities

$$(A \otimes_{\mathrm{K}} B)(C \otimes_{\mathrm{K}} D) = (AC) \otimes_{\mathrm{K}} (BD), \qquad (A \otimes_{\mathrm{K}} B)^T = A^T \otimes_{\mathrm{K}} B^T,$$

associativity, distributivity and linearity transport to M_{mn}. Hence $\Phi_\otimes(\mathbf{M}_m, \mathbf{M}_n) = \mathbf{M}_{mn} \in U_{\mathrm{mat}}$.

(OP) Reversing multiplication defines the opposite algebra; $+, 0, -, m_\alpha, T$ are unchanged. Associativity holds since $(A \cdot^{\mathrm{op}} B) \cdot^{\mathrm{op}} C = CBA = A \cdot^{\mathrm{op}} (B \cdot^{\mathrm{op}} C)$. Distributivities are immediate. Thus $\Phi_{\mathrm{op}}(\mathbf{M}_n) \in U_{\mathrm{mat}}$.

(2) Naturality (isomorphism invariance). Let $\alpha : \mathbf{M}_m \xrightarrow{\cong} \mathbf{M}'_m$ and $\beta : \mathbf{M}_n \xrightarrow{\cong} \mathbf{M}'_n$ be Σ_{mat}-isomorphisms (in particular, $\mathbb{F}$-algebra isomorphisms preserving T).

For (DS), define $\alpha \oplus \beta : M_{m+n} \to M'_{m+n}$ by

$$(\alpha \oplus \beta)\big(\mathrm{diag}(A, B)\big) := \mathrm{diag}\big(\alpha(A),\ \beta(B)\big),$$

which is a Σ_{mat}-isomorphism because addition, $0, -, m_\alpha, T$ are preserved componentwise and

$$(\alpha \oplus \beta)\big(\mathrm{diag}(A, B) \cdot \mathrm{diag}(C, D)\big) = \mathrm{diag}\big(\alpha(AC),\ \beta(BD)\big) = \mathrm{diag}\big(\alpha(A)\alpha(C),\ \beta(B)\beta(D)\big)$$

$$= \mathrm{diag}\big(\alpha(A), \beta(B)\big) \cdot \mathrm{diag}\big(\alpha(C), \beta(D)\big) = (\alpha \oplus \beta)\big(\mathrm{diag}(A, B)\big) \cdot (\alpha \oplus \beta)\big(\mathrm{diag}(C, D)\big).$$

For (TEN), the map $\alpha \otimes \beta : M_{mn} \to M'_{mn}$ is defined uniquely by

$$(\alpha \otimes \beta)\big(A \otimes_{\mathrm{K}} B\big) := \alpha(A) \otimes_{\mathrm{K}} \beta(B),$$

and extended linearly to all of M_{mn} (this is well-defined because $\delta_{m,n}$ is an algebra isomorphism). Then

$$(\alpha \otimes \beta)\big((A \otimes_{\mathrm{K}} B)(C \otimes_{\mathrm{K}} D)\big) = (\alpha \otimes \beta)\big((AC) \otimes_{\mathrm{K}} (BD)\big) = \alpha(AC) \otimes_{\mathrm{K}} \beta(BD)$$

$$= (\alpha(A)\alpha(C)) \otimes_{\mathrm{K}} (\beta(B)\beta(D)) = \big(\alpha(A) \otimes_{\mathrm{K}} \beta(B)\big) \cdot \big(\alpha(C) \otimes_{\mathrm{K}} \beta(D)\big),$$

showing multiplicativity; linearity and compatibility with T are immediate from $(A \otimes_{\mathrm{K}} B)^T = A^T \otimes_{\mathrm{K}} B^T$ and preservation of T by α, β.

For (OP), the same underlying map $\alpha : M_n \to M'_n$ is an isomorphism $\Phi_{\mathrm{op}}(\mathbf{M}_n) \to \Phi_{\mathrm{op}}(\mathbf{M}'_n)$ because

$$\alpha\big(A \cdot^{\mathrm{op}} B\big) = \alpha(BA) = \alpha(B)\alpha(A) = \alpha(A) \cdot_{\mathrm{op}} \alpha(B),$$

and $+, 0, -, m_\alpha, T$ are preserved.

Therefore each Φ_ℓ is natural, completing the MetaStructure verification. The generalization claim is tautological from the definition of U_{mat}. □

Definition 5.5 (Iterated MetaMatrix of depth t)**.** For $t \in \mathbb{N}$, an *Iterated MetaMatrix of depth t* is an Iterated MetaStructure (Definition 1.3) over Σ_{mat},

$$\mathfrak{M}^{(t)}_{\mathrm{mat}} = \big(U^{(t)}_{\mathrm{mat}},\ (\odot^{(t)}_\ell)_{\ell \in \Lambda}\big),$$

obtained by applying the lifting functor $\mathsf{U}_{\Sigma_{\mathrm{mat}}}$ from Definition 1.3 to $\mathbb{M}_{\mathrm{mat}}$ repeatedly t times. Concretely, for $s < t$ and any base meta-operation Φ_ℓ of arity k_ℓ,

$$\Phi^{\uparrow}_\ell :\ \big(U^{(t)}_{\mathrm{mat}}\big)^{k_\ell} \longrightarrow U^{(t)}_{\mathrm{mat}}$$

is defined on representatives by

$$\Phi^{\uparrow}_\ell\big(\mathsf{U}^{t-s}_{\Sigma_{\mathrm{mat}}}(X_1), \ldots, \mathsf{U}^{t-s}_{\Sigma_{\mathrm{mat}}}(X_{k_\ell})\big) := \mathsf{U}^{t-s}_{\Sigma_{\mathrm{mat}}}\big(\Phi_\ell(X_1, \ldots, X_{k_\ell})\big),$$

and similarly for all symbol interpretations. We write $\odot^{(t)}_\oplus, \odot^{(t)}_\otimes, \odot^{(t)}_{\mathrm{op}}$ for the lifts of (DS), (TEN), (OP).

Example 5.6 (Iterated MetaMatrix (depth $t = 1$): Spatio-Temporal RGB Transform)**.** (cf. [72, 73]) Keep $K = C_8 \otimes C_8 \in \mathbb{R}^{64\times 64}$ per block and let $C_{10} \in \mathbb{R}^{10\times 10}$ be a 1D temporal DCT on a 10-frame GOP. The lifted tensor builds a spatio-temporal transform per channel:

$$K_{\mathrm{st}} \;:=\; K \otimes C_{10} \;\in\; \mathbb{R}^{(64\cdot 10)\times(64\cdot 10)} \;=\; \mathbb{R}^{640\times 640}.$$

Applying the lifted direct sum across RGB yields

$$K_{\mathrm{video}} \;:=\; K_{\mathrm{st}} \,\oplus\, K_{\mathrm{st}} \,\oplus\, K_{\mathrm{st}} \;=\; \mathrm{diag}(K_{\mathrm{st}}, K_{\mathrm{st}}, K_{\mathrm{st}}) \;\in\; \mathbb{R}^{(3\cdot 640)\times(3\cdot 640)} \;=\; \mathbb{R}^{1920\times 1920}.$$

Thus, iteration combines (TEN) for space–time and (DS) for color, with explicit size growth $64 \to 640 \to 1920$.

Theorem 5.7 (Iterated MetaMatrix is an Iterated MetaStructure and generalizes MetaMatrix)**.** *For every $t \in \mathbb{N}$, $\mathfrak{M}^{(t)}_{\mathrm{mat}}$ of Definition 5.5 is an Iterated MetaStructure in the sense of Definition 1.3. Moreover, for $s < t$ the map*

$$\iota_{s\to t} \,:\, \mathfrak{M}^{(s)}_{\mathrm{mat}} \hookrightarrow \mathfrak{M}^{(t)}_{\mathrm{mat}}, \qquad X \;\longmapsto\; \mathsf{U}^{t-s}_{\Sigma_{\mathrm{mat}}}(X),$$

is an embedding that is operation-preserving:

$$\Phi^{\uparrow}_{\ell}\big(\iota_{s\to t}(X_1), \ldots, \iota_{s\to t}(X_{k_\ell})\big) = \iota_{s\to t}\big(\Phi_\ell(X_1, \ldots, X_{k_\ell})\big).$$

In particular, $\mathfrak{M}^{(0)}_{\mathrm{mat}} = \mathbb{M}_{\mathrm{mat}}$ embeds into $\mathfrak{M}^{(t)}_{\mathrm{mat}}$, so Iterated MetaMatrix generalizes *MetaMatrix.*

Proof. By Definition 1.3, the lifted constructors are obtained by post-composing the base constructors $(\Gamma_\ell,\ +^{\Phi_\ell},\ 0^{\Phi_\ell},\ (-)^{\Phi_\ell},\ (m^{\Phi_\ell}_\alpha),\ \cdot^{\Phi_\ell},\ T^{\Phi_\ell})$ with $\mathsf{U}^{t-s}_{\Sigma_{\mathrm{mat}}}$; hence they are uniform.

For naturality, if $\alpha_i : X_i \xrightarrow{\cong} Y_i$ are level-s isomorphisms, then

$$\mathsf{U}^{t-s}_{\Sigma_{\mathrm{mat}}}(\alpha_i) :\ \mathsf{U}^{t-s}_{\Sigma_{\mathrm{mat}}}(X_i) \xrightarrow{\cong} \mathsf{U}^{t-s}_{\Sigma_{\mathrm{mat}}}(Y_i)$$

are level-t isomorphisms, and by the defining equation of the lift,

$$\Phi^{\uparrow}_{\ell}\big(\mathsf{U}^{t-s}\alpha_1, \ldots, \mathsf{U}^{t-s}\alpha_{k_\ell}\big) = \mathsf{U}^{t-s}\big(\Phi_\ell(\alpha_1, \ldots, \alpha_{k_\ell})\big),$$

which is an isomorphism because each base Φ_ℓ is natural (Theorem 5.4).

Operation preservation of $\iota_{s\to t}$ is precisely the lift identity above. Taking $s = 0$ gives the canonical embedding $\mathbb{M}_{\mathrm{mat}} \hookrightarrow \mathfrak{M}^{(t)}_{\mathrm{mat}}$. □

6 Meta-Decision-Making (Decision-Making of Decision-Making)

Decision-Making is the process of selecting the best feasible option from alternatives based on preferences, objectives, or criteria [74–78]. Meta-Decision-Making concerns choosing or structuring how decisions are made, such as combining, filtering, or reframing multiple decision-making processes. Iterated Meta-Decision-Making recursively applies meta-operations to decision processes themselves, enabling layered strategies and adaptive frameworks for complex, multi-level decision contexts.

Definition 6.1 (Decision-Making)**.** (cf. [79–81]) *Decision-making* is the process of selecting one option (or a set of options) from a collection of feasible alternatives, based on given criteria, preferences, or objectives, in order to achieve a desired outcome.

Fix a (single-sorted, finitary) signature for decision problems

$$\Sigma_{\mathrm{dec}} = \Big(\,\mathsf{Func} = \varnothing,\ \mathsf{Rel} = \{\,\mathsf{Feas},\ \succeq\,\},\ \mathrm{ar}(\mathsf{Feas}) = 1,\ \mathrm{ar}(\succeq) = 2\,\Big).$$

A Σ_{dec}-structure is a pair

$$\mathbf{D} \;=\; \big(H,\ \mathsf{Feas}^{\mathbf{D}},\ \succeq^{\mathbf{D}}\big),$$

where $H \neq \varnothing$ is the *set of alternatives*, $\mathsf{Feas}^{\mathbf{D}} \subseteq H$ is the *feasible set*, and $\succeq^{\mathbf{D}} \subseteq H \times H$ is a *preorder* (reflexive and transitive) interpreting the (weak) *preference* or *at-least-as-good-as* relation.

Definition 6.2 (Decision-Making instance and choice correspondence)**.** A *decision-making instance* is any Σ_{dec}-structure $\mathbf{D} = (H, \mathsf{Feas}, \succeq)$ with $\succeq$ a preorder. Its (set-valued) *choice correspondence* is the set of $\succeq$-maximal feasible alternatives:

$$\text{Choice}(\mathbf{D}) \;:=\; \text{Max}_{\succeq}(\mathsf{Feas}) \;=\; \big\{x \in \mathsf{Feas} \;\big|\; \forall y \in \mathsf{Feas} : \; y \succeq x \;\Rightarrow\; x \succeq y\big\}.$$

Let $U_{\text{dec}} \subseteq \text{Str}_{\Sigma_{\text{dec}}}$ be the class of all such instances.

Definition 6.3 (MetaDecision-Making)**.** A *MetaDecision-Making system* is a MetaStructure (Definition 1.2) over Σ_{dec} of the form

$$\mathbb{M}_{\text{dec}} = \big(U_{\text{dec}},\; (\Phi_\ell)_{\ell\in\Lambda}\big),$$

where the following meta-operations are given uniformly by carrier- and relation-constructors:

(PROD) Product (independent joint decision). For $\mathbf{D}_1 = (H_1, \mathsf{Feas}_1, \succeq_1)$ and $\mathbf{D}_2 = (H_2, \mathsf{Feas}_2, \succeq_2)$ put

$$\Phi_\otimes(\mathbf{D}_1, \mathbf{D}_2) := \big(H_1 \times H_2,\; \mathsf{Feas}_1 \times \mathsf{Feas}_2,\; \succeq_\times\big),$$

with componentwise preorder

$$(x_1, x_2) \;\succeq_\times\; (y_1, y_2) \quad\Longleftrightarrow\quad x_1 \succeq_1 y_1 \;\text{ and }\; x_2 \succeq_2 y_2.$$

(FIL) Feasible-set filtering by a parameter $S \subseteq H$. For $\mathbf{D} = (H, \mathsf{Feas}, \succeq)$ and any subset $S \subseteq H$,

$$\Phi_{\text{filter}(S)}(\mathbf{D}) := \big(S,\; \mathsf{Feas} \cap S,\; \succeq\restriction_{S\times S}\big).$$

(PUSH) Pushforward along a bijection $f : H \to K$ (renaming alternatives).

$$\Phi_{\text{push}(f)}(\mathbf{D}) := \big(K,\; f[\mathsf{Feas}],\; \succeq^f\big), \quad \text{where} \quad u \succeq^f v \iff f^{-1}(u) \succeq f^{-1}(v).$$

Example 6.4 (MetaDecision-Making: Joint Trip Booking via Product + Filter)**. Flights.** Alternatives $F = \{A, B, C\}$. Prices (kJPY): $p(A) = 48,\ p(B) = 52,\ p(C) = 44$. Durations (h): $d(A) = 1.5,\ d(B) = 1.0,\ d(C) = 2.0$. Budget filter $S = \{x \in F \mid p(x) \le 50\} = \{A, C\}$ (FIL). Preference by utility $u_F(x) = -\frac{p(x)}{10} - d(x)$ ($x \succeq_F y \iff u_F(x) \ge u_F(y)$):

$$u_F(A) = -4.8 - 1.5 = -6.3, \quad u_F(C) = -4.4 - 2.0 = -6.4 \Rightarrow \text{Choice}(\mathbf{D}_F) = \{A\}.$$

Hotels. Alternatives $H = \{H_1, H_2\}$. Ratings $r(H_1) = 4.3,\ r(H_2) = 4.0$, prices (kJPY) $q(H_1) = 11,\ q(H_2) = 9$. Utility $u_H(h) = r(h) - 0.1\,q(h)$:

$$u_H(H_1) = 4.3 - 1.1 = 3.2, \quad u_H(H_2) = 4.0 - 0.9 = 3.1 \Rightarrow \text{Choice}(\mathbf{D}_H) = \{H_1\}.$$

Product (PROD).

$$\text{Choice}\big(\Phi_\otimes(\mathbf{D}_F, \mathbf{D}_H)\big) = \text{Choice}(\mathbf{D}_F) \times \text{Choice}(\mathbf{D}_H) = \{(A, H_1)\}.$$

Thus the MetaDecision selects the package (Flight A, Hotel H_1).

Theorem 6.5 (MetaDecision-Making is a MetaStructure and generalizes decision-making)**.** *$\mathbb{M}_{\text{dec}}$ of Definition 6.3 is a MetaStructure in the sense of Definition 1.2. Moreover, every (classical) decision-making instance* $\mathbf{D} = (H, \mathsf{Feas}, \succeq)$ *is an object of U_{dec}; hence MetaDecision-Making* generalizes *decision-making.*

Proof. We verify the two MetaStructure clauses.

(1) Uniform constructors and closure. For (PROD), $H_1 \times H_2 \neq \varnothing$ when both carriers are nonempty; $\mathsf{Feas}_1 \times \mathsf{Feas}_2 \subseteq H_1 \times H_2$. The relation $\succeq_\times$ is a preorder: reflexivity holds since $x_i \succeq_i x_i$, and transitivity follows from transitivity of $\succeq_1, \succeq_2$:

$$(x_1, x_2) \succeq_\times (y_1, y_2) \;\wedge\; (y_1, y_2) \succeq_\times (z_1, z_2) \;\Rightarrow\; x_i \succeq_i z_i \;(i = 1, 2) \;\Rightarrow\; (x_1, x_2) \succeq_\times (z_1, z_2).$$

Thus $\Phi_\otimes(\mathbf{D}_1, \mathbf{D}_2) \in U_{\text{dec}}$.

For (FIL), the carrier is S; $\mathsf{Feas} \cap S \subseteq S$ and the restricted relation $\succeq\restriction_{S\times S}$ is again a preorder. Hence $\Phi_{\text{filter}(S)}(\mathbf{D}) \in U_{\text{dec}}$.

For (PUSH), K is nonempty; $f[\mathsf{Feas}] \subseteq K$; and $\succeq^f$ is a preorder because

$$u \succeq^f u \iff f^{-1}(u) \succeq f^{-1}(u), \qquad u \succeq^f v \wedge v \succeq^f w \Rightarrow f^{-1}(u) \succeq f^{-1}(v) \succeq f^{-1}(w) \Rightarrow u \succeq^f w.$$

Hence $\Phi_{\text{push}(f)}(\mathbf{D}) \in U_{\text{dec}}$.

(2) Naturality (isomorphism invariance). An isomorphism $\alpha : \mathbf{D} \to \mathbf{D}'$ of Σ_{dec}-structures is a bijection $\alpha : H \to H'$ with $\alpha[\mathsf{Feas}] = \mathsf{Feas}'$ and $(x \succeq y) \Rightarrow (\alpha x \succeq' \alpha y)$, with the converse via α^{-1}. For (PROD), let $\alpha_i : \mathbf{D}_i \to \mathbf{D}_i'$ be isomorphisms and put $\alpha_\times := \alpha_1 \times \alpha_2 : H_1 \times H_2 \to H_1' \times H_2'$. Then

$$\alpha_\times[\mathsf{Feas}_1 \times \mathsf{Feas}_2] = \alpha_1[\mathsf{Feas}_1] \times \alpha_2[\mathsf{Feas}_2] = \mathsf{Feas}_1' \times \mathsf{Feas}_2',$$

and

$$\begin{aligned} (x_1, x_2) \succeq_\times (y_1, y_2) &\iff x_1 \succeq_1 y_1 \ \wedge \ x_2 \succeq_2 y_2 \\ &\iff \alpha_1 x_1 \succeq_1' \alpha_1 y_1 \ \wedge \ \alpha_2 x_2 \succeq_2' \alpha_2 y_2 \\ &\iff \alpha_\times(x_1, x_2) \succeq_\times' \alpha_\times(y_1, y_2). \end{aligned}$$

Thus $\Phi_\otimes(\alpha_1, \alpha_2)$ is an isomorphism.

For (FIL), let $\alpha : \mathbf{D} \to \mathbf{D}'$ and $S \subseteq H$, $S' := \alpha[S] \subseteq H'$. Then $\alpha : S \to S'$ is a bijection with $\alpha[\mathsf{Feas} \cap S] = \mathsf{Feas}' \cap S'$ and it preserves the restricted relation, so $\Phi_{\text{filter}(S)}(\alpha)$ is an isomorphism.

For (PUSH), if $f : H \to K$ and $f' : H' \to K'$ with $\beta : K \to K'$ satisfying $f' = \beta \circ \alpha \circ f^{-1}$ on $f[H]$, then β intertwines the pushforwards (a routine diagram chase). Hence each Φ_ℓ is natural. The generalization claim is tautological since every decision instance lies in U_{dec}. □

Proposition 6.6 (Choice behavior under META operations)**.** *Let* $\mathbf{D}_i = (H_i, \mathsf{Feas}_i, \succeq_i)$*,* $i = 1, 2$*.*

(i) Product law:

$$\text{Choice}\big(\Phi_\otimes(\mathbf{D}_1, \mathbf{D}_2)\big) \;=\; \text{Choice}(\mathbf{D}_1) \;\times\; \text{Choice}(\mathbf{D}_2).$$

(ii) Filtering law: for any $S \subseteq H$,

$$\text{Choice}\big(\Phi_{\text{filter}(S)}(\mathbf{D})\big) \;=\; \text{Choice}(\mathbf{D}) \;\cap\; S.$$

(iii) Pushforward law: for any bijection $f : H \to K$,

$$\text{Choice}\big(\Phi_{\text{push}(f)}(\mathbf{D})\big) \;=\; f\big[\text{Choice}(\mathbf{D})\big].$$

Proof. (i) "$\subseteq$": If (x_1, x_2) is $\succeq_\times$-maximal in $\mathsf{Feas}_1 \times \mathsf{Feas}_2$ and, say, $x_1 \notin \text{Max}_{\succeq_1}(\mathsf{Feas}_1)$, then $\exists y_1 \in \mathsf{Feas}_1$ with $y_1 \succeq_1 x_1$ and $\neg(x_1 \succeq_1 y_1)$. Then $(y_1, x_2) \in \mathsf{Feas}_1 \times \mathsf{Feas}_2$ and $(y_1, x_2) \succeq_\times (x_1, x_2)$ but not conversely, contradicting maximality. Hence $x_i \in \text{Max}_{\succeq_i}(\mathsf{Feas}_i)$ for $i = 1, 2$.

"$\supseteq$": If $x_i \in \text{Max}_{\succeq_i}(\mathsf{Feas}_i)$ and $(y_1, y_2) \in \mathsf{Feas}_1 \times \mathsf{Feas}_2$ with $(y_1, y_2) \succeq_\times (x_1, x_2)$, then $y_i \succeq_i x_i$ for $i = 1, 2$, so by maximality also $x_i \succeq_i y_i$. Hence $(x_1, x_2) \succeq_\times (y_1, y_2)$, proving maximality of (x_1, x_2).

(ii) If $x \in \text{Choice}(\Phi_{\text{filter}(S)}(\mathbf{D}))$, then $x \in \mathsf{Feas} \cap S$ and maximal within $\mathsf{Feas} \cap S$, hence $x \in \text{Choice}(\mathbf{D}) \cap S$. Conversely, if $x \in \text{Choice}(\mathbf{D}) \cap S$ and $y \in \mathsf{Feas} \cap S$ with $y \succeq x$, then by maximality in Feas we also have $x \succeq y$, so x is maximal in $\mathsf{Feas} \cap S$.

(iii) By definition of $\succeq^f$ and $f[\mathsf{Feas}]$, $u \in \text{Choice}(\Phi_{\text{push}(f)}(\mathbf{D}))$ iff $u \in f[\mathsf{Feas}]$ and for all $v \in f[\mathsf{Feas}]$, $v \succeq^f u \Rightarrow u \succeq^f v$, i.e., for all $y \in \mathsf{Feas}$, $y \succeq f^{-1}(u) \Rightarrow f^{-1}(u) \succeq y$, which is equivalent to $f^{-1}(u) \in \text{Choice}(\mathbf{D})$. Hence $u \in f[\text{Choice}(\mathbf{D})]$. □

Definition 6.7 (Iterated MetaDecision-Making of depth t)**.** For $t \in \mathbb{N}$, an *Iterated MetaDecision-Making system of depth t* is an Iterated MetaStructure (Definition 1.3) over Σ_{dec},

$$\mathfrak{M}^{(t)}_{\text{dec}} = \big(U^{(t)}_{\text{dec}},\ (\odot^{(t)}_{\ell})_{\ell \in \Lambda}\big),$$

obtained by applying the lifting $\mathsf{U}_{\Sigma_{\text{dec}}}$ to $\mathbb{M}_{\text{dec}}$ repeatedly t times. Concretely, for $s < t$ and any base meta-operation Φ_ℓ of meta-arity k_ℓ,

$$\Phi^{\uparrow}_{\ell} :\ \big(U^{(t)}_{\text{dec}}\big)^{k_\ell} \longrightarrow U^{(t)}_{\text{dec}}$$

is defined on representatives by

$$\Phi^{\uparrow}_{\ell}\big(\mathsf{U}^{t-s}_{\Sigma_{\text{dec}}}(X_1), \dots, \mathsf{U}^{t-s}_{\Sigma_{\text{dec}}}(X_{k_\ell})\big) \ :=\ \mathsf{U}^{t-s}_{\Sigma_{\text{dec}}}\big(\,\Phi_\ell(X_1, \dots, X_{k_\ell})\,\big),$$

and similarly for all relations, as in Definition 1.3. We denote by $\odot^{(t)}_{\otimes}$, $\odot^{(t)}_{\text{filter}(S)}$, and $\odot^{(t)}_{\text{push}(f)}$ the lifts of (PROD), (FIL), and (PUSH), respectively.

Example 6.8 (Iterated MetaDecision-Making (depth $t = 1$): Choose a Rule, then Choose a Hotel)**. Meta-level (rules).** Rules $R = \{r_1, r_2\}$ with validation scores $v(r_1) = 0.82$, $v(r_2) = 0.85$. Preference $r \succeq_v r' \iff v(r) \geq v(r') \Rightarrow \text{Choice}(\mathbf{R}) = \{r_2\}$.

Base-level (hotels under a rule). Let $H' = \{H_1, H_2, H_3\}$ with ratings $(4.3, 4.1, 4.0)$, prices (kJPY) $(11, 9, 8)$, breakfast flags $(1, 0, 1)$. Define scores

$$s_{r_1}(h) = r(h) - 0.10\, q(h), \qquad s_{r_2}(h) = r(h) - 0.05\, q(h) + 0.5\, \mathbf{1}_{\{\text{breakfast}\}},$$

and $h \succeq^{(r)} h' \iff s_r(h) \geq s_r(h')$. Under the chosen rule r_2:

$$\begin{aligned} s_{r_2}(H_1) &= 4.3 - 0.05 \cdot 11 + 0.5 = 4.25, \\ s_{r_2}(H_2) &= 4.1 - 0.05 \cdot 9 + 0 = 3.65, \qquad \Rightarrow \quad \text{Choice}(\mathbf{D}^{(r_2)}_{H'}) = \{H_1\}. \\ s_{r_2}(H_3) &= 4.0 - 0.05 \cdot 8 + 0.5 = 4.10, \end{aligned}$$

Lifted product outcome.

$$\text{Choice}\big(\odot^{(1)}_{\otimes}(\mathbf{R}, \mathbf{D}_{H'})\big) = \text{Choice}(\mathbf{R}) \times \text{Choice}(\mathbf{D}^{(r_2)}_{H'}) = \{(r_2, H_1)\}.$$

Thus the iterated meta-level first selects the *decision rule* r_2, then the hotel H_1 under that rule.

Theorem 6.9 (Iterated MetaDecision-Making is an Iterated MetaStructure and generalizes MetaDecision–Making)**.** *For every $t \in \mathbb{N}$, $\mathfrak{M}^{(t)}_{\text{dec}}$ of Definition 6.7 is an Iterated MetaStructure in the sense of Definition 1.3. Moreover, for $s < t$ there is a canonical embedding*

$$\iota_{s \to t} :\ \mathfrak{M}^{(s)}_{\text{dec}} \hookrightarrow \mathfrak{M}^{(t)}_{\text{dec}}, \qquad X \ \longmapsto\ \mathsf{U}^{t-s}_{\Sigma_{\text{dec}}}(X),$$

which preserves all meta-operations:

$$\Phi^{\uparrow}_{\ell}\big(\iota_{s\to t}(X_1), \dots, \iota_{s\to t}(X_{k_\ell})\big) = \iota_{s\to t}\big(\Phi_\ell(X_1, \dots, X_{k_\ell})\big).$$

In particular, $\mathfrak{M}^{(0)}_{\text{dec}} = \mathbb{M}_{\text{dec}}$ embeds into $\mathfrak{M}^{(t)}_{\text{dec}}$, so Iterated MetaDecision-Making generalizes *MetaDecision-Making.*

Proof. By Definition 1.3, each lifted operation $\Phi^{\uparrow}_{\ell}$ is defined by post-composing the base constructors with $\mathsf{U}^{t-s}_{\Sigma_{\text{dec}}}$, hence the constructors are uniform at every level. If $\alpha_i : X_i \to Y_i$ are isomorphisms at level s, then

$$\mathsf{U}^{t-s}_{\Sigma_{\text{dec}}}(\alpha_i) :\ \mathsf{U}^{t-s}_{\Sigma_{\text{dec}}}(X_i) \xrightarrow{\ \cong\ } \mathsf{U}^{t-s}_{\Sigma_{\text{dec}}}(Y_i)$$

are isomorphisms at level t, and by the defining lift identity

$$\Phi^{\uparrow}_{\ell}\big(\mathsf{U}^{t-s}\alpha_1, \dots, \mathsf{U}^{t-s}\alpha_{k_\ell}\big) = \mathsf{U}^{t-s}\big(\Phi_\ell(\alpha_1, \dots, \alpha_{k_\ell})\big),$$

the left-hand side is an isomorphism since each base Φ_ℓ is natural by Theorem 6.5. The operation-preservation for $\iota_{s\to t}$ is exactly the same identity with X_i in place of α_i. Taking $s = 0$ gives the desired embedding of $\mathbb{M}_{\text{dec}}$ into every $\mathfrak{M}^{(t)}_{\text{dec}}$. □

7 Meta-Function (Function of Function)

A Meta-Function is a higher-level structure that manipulates functions themselves, combining, composing, or transforming them into new functions. An Iterated Meta-Function extends this by repeatedly applying such meta-operations to functions-of-functions, enabling layered and recursive transformations.

Definition 7.1 (Function)**.** [37, 82] A *function* from a set X to a set Y is a mapping

$$f : X \to Y$$

such that for every $x \in X$ there exists a unique $y \in Y$ with $f(x) = y$. The set X is called the *domain*, and Y the *codomain* of f.

Fix the (single-sorted, finitary) *function signature*

$$\Sigma_{\text{fun}} = \Big(\mathsf{Func} = \varnothing, \quad \mathsf{Rel} = \{ \mathsf{Dom}, \mathsf{Cod}, \mathsf{Graph} \}, \quad \mathrm{ar}(\mathsf{Dom}) = \mathrm{ar}(\mathsf{Cod}) = 1, \ \mathrm{ar}(\mathsf{Graph}) = 2 \Big).$$

A Σ_{fun}-structure is a triple

$$\mathbf{F} = (H, \ \mathsf{Dom}^{\mathbf{F}}, \ \mathsf{Cod}^{\mathbf{F}}, \ \mathsf{Graph}^{\mathbf{F}}),$$

where $H \neq \varnothing$ is the carrier set, $\mathsf{Dom}^{\mathbf{F}}, \mathsf{Cod}^{\mathbf{F}} \subseteq H$ (the intended domain/codomain), and $\mathsf{Graph}^{\mathbf{F}} \subseteq H \times H$. We write $x \in \mathsf{Dom}$ (resp. $y \in \mathsf{Cod}$) for $x \in \mathsf{Dom}^{\mathbf{F}}$ (resp. $y \in \mathsf{Cod}^{\mathbf{F}}$), and $(x, y) \in \mathsf{Graph}$ for $(x, y) \in \mathsf{Graph}^{\mathbf{F}}$.

Definition 7.2 (Function as a structure)**.** Let $U_{\text{fun}} \subseteq \mathrm{Str}_{\Sigma_{\text{fun}}}$ be the class of all Σ_{fun}-structures $\mathbf{F} = (H, \mathsf{Dom}, \mathsf{Cod}, \mathsf{Graph})$ such that:

(F1) *Typing:* $\mathsf{Graph} \subseteq \mathsf{Dom} \times \mathsf{Cod}$.

(F2) *Totality:* $\forall x \in \mathsf{Dom} \ \exists y \in \mathsf{Cod} : \ (x, y) \in \mathsf{Graph}$.

(F3) *Functionality:* $\forall x \in \mathsf{Dom} \ \forall y, z \in \mathsf{Cod} : \ [(x, y) \in \mathsf{Graph} \ \wedge \ (x, z) \in \mathsf{Graph}] \Rightarrow y = z$.

Thus each object of U_{fun} encodes an ordinary function $f : \mathsf{Dom} \to \mathsf{Cod}$ via its graph.

Remark 7.3 (Classical function inside U_{fun})**.** Given any usual function $f : X \to Y$, put $H := X \sqcup Y := \{0\} \times X \ \cup \ \{1\} \times Y$,

$$\mathsf{Dom} := \{0\} \times X, \qquad \mathsf{Cod} := \{1\} \times Y, \qquad \mathsf{Graph} := \big\{ ((0, x), (1, f(x))) \ \big| \ x \in X \big\}.$$

Then $\mathbf{F}_f := (H, \mathsf{Dom}, \mathsf{Cod}, \mathsf{Graph}) \in U_{\text{fun}}$.

Definition 7.4 (MetaFunction)**.** A *MetaFunction* is a MetaStructure (Definition 1.2) over Σ_{fun}

$$\mathbb{M}_{\text{fun}} = \big(U_{\text{fun}}, \ (\Phi_\ell)_{\ell \in \Lambda}\big),$$

whose meta-operations are specified uniformly as follows (each by carrier- and relation-constructors, as in Definition 1.2):

(PROD) Product of functions. For $\mathbf{F}_1 = (H_1, \mathsf{Dom}_1, \mathsf{Cod}_1, \mathsf{Graph}_1)$ and $\mathbf{F}_2 = (H_2, \mathsf{Dom}_2, \mathsf{Cod}_2, \mathsf{Graph}_2)$ define

$$\Phi_\times(\mathbf{F}_1, \mathbf{F}_2) := (H_\times, \mathsf{Dom}_\times, \mathsf{Cod}_\times, \mathsf{Graph}_\times),$$

on the disjoint tagged carrier

$$H_\times := \underbrace{\{0\} \times \mathsf{Dom}_1 \times \mathsf{Dom}_2}_{\text{tagged domain}} \ \cup \ \underbrace{\{1\} \times \mathsf{Cod}_1 \times \mathsf{Cod}_2}_{\text{tagged codomain}},$$

with

$$\mathsf{Dom}_\times := \{0\} \times \mathsf{Dom}_1 \times \mathsf{Dom}_2, \qquad \mathsf{Cod}_\times := \{1\} \times \mathsf{Cod}_1 \times \mathsf{Cod}_2,$$

$$\mathsf{Graph}_\times := \Big\{ ((0, x_1, x_2), (1, y_1, y_2)) \ \Big| \ (x_1, y_1) \in \mathsf{Graph}_1 \ \wedge \ (x_2, y_2) \in \mathsf{Graph}_2 \Big\}.$$

(COPROD) Coproduct (disjoint union) of functions. Define

$$\Phi_{\sqcup}(\mathbf{F}_1, \mathbf{F}_2) := (H_{\sqcup}, \mathsf{Dom}_{\sqcup}, \mathsf{Cod}_{\sqcup}, \mathsf{Graph}_{\sqcup}),$$

with

$$\begin{aligned} H_{\sqcup} &:= \{0\} \times \mathsf{Dom}_1 \ \cup \ \{1\} \times \mathsf{Dom}_2 \ \cup \ \{2\} \times \mathsf{Cod}_1 \ \cup \ \{3\} \times \mathsf{Cod}_2, \\ \mathsf{Dom}_{\sqcup} &:= \{0\} \times \mathsf{Dom}_1 \ \cup \ \{1\} \times \mathsf{Dom}_2, \\ \mathsf{Cod}_{\sqcup} &:= \{2\} \times \mathsf{Cod}_1 \ \cup \ \{3\} \times \mathsf{Cod}_2, \\ \mathsf{Graph}_{\sqcup} &:= \Big\{\big((0,x),(2,y)\big) \mid (x,y) \in \mathsf{Graph}_1\Big\} \ \cup \ \Big\{\big((1,u),(3,v)\big) \mid (u,v) \in \mathsf{Graph}_2\Big\}. \end{aligned}$$

(COMP) Composition along a gluing bijection. Given $\mathbf{F}_1 = (H_1, \mathsf{Dom}_1, \mathsf{Cod}_1, \mathsf{Graph}_1), \mathbf{F}_2 = (H_2, \mathsf{Dom}_2, \mathsf{Cod}_2, \mathsf{Graph}_2)$ and a bijection $\psi : \mathsf{Cod}_1 \xrightarrow{\cong} \mathsf{Dom}_2$, put

$$\Phi_{\circ,\psi}(\mathbf{F}_1, \mathbf{F}_2) := (H_\circ, \mathsf{Dom}_\circ, \mathsf{Cod}_\circ, \mathsf{Graph}_\circ),$$

with

$$H_\circ := \{0\} \times \mathsf{Dom}_1 \ \cup \ \{1\} \times \mathsf{Cod}_2, \qquad \mathsf{Dom}_\circ := \{0\} \times \mathsf{Dom}_1, \quad \mathsf{Cod}_\circ := \{1\} \times \mathsf{Cod}_2,$$

$$\mathsf{Graph}_\circ := \Big\{\big((0,x),(1,z)\big) \ \Big| \ \exists y \in \mathsf{Cod}_1 : \ (x,y) \in \mathsf{Graph}_1 \ \wedge \ \big(\psi(y), z\big) \in \mathsf{Graph}_2\Big\}.$$

(RES) Domain restriction by $S \subseteq \mathsf{Dom}$. For $\mathbf{F} = (H, \mathsf{Dom}, \mathsf{Cod}, \mathsf{Graph})$ and $S \subseteq \mathsf{Dom}$,

$$\Phi_{\mathrm{res}(S)}(\mathbf{F}) := \big(\{0\} \times S \ \cup \ \{1\} \times \mathsf{Cod}, \ \{0\} \times S, \ \{1\} \times \mathsf{Cod}, \ \mathsf{Graph}'\big),$$

$$\mathsf{Graph}' := \Big\{\big((0,x),(1,y)\big) \ \Big| \ x \in S, \ (x,y) \in \mathsf{Graph}\Big\}.$$

(PUSH) Relabeling (pushforward) along a bijection respecting colors. If $f : H \xrightarrow{\cong} K$ is a bijection with $f[\mathsf{Dom}] = \mathsf{Dom}'$ and $f[\mathsf{Cod}] = \mathsf{Cod}'$, define

$$\Phi_{\mathrm{push}(f)}(\mathbf{F}) := \big(K, \ \mathsf{Dom}', \ \mathsf{Cod}', \ \mathsf{Graph}'\big), \qquad \mathsf{Graph}' := \big\{(f(x), f(y)) \mid (x,y) \in \mathsf{Graph}\big\}.$$

Example 7.5 (MetaFunction: Checkout charges via product of functions)**.** Let

$$f_{\mathrm{ship}} : (w,d) \mapsto 300 + 50\lceil w \rceil + 0.5\,d \ \ (\mathrm{JPY}), \qquad f_{\mathrm{tax}} : (s, \mathrm{reg}) \mapsto r(\mathrm{reg})\, s \ \ (\mathrm{JPY}),$$

where w=kg, d=km, s=subtotal, and $r(\mathrm{Tokyo}) = 0.10$. The MetaFunction product $\Phi_\times$ yields

$$(f_{\mathrm{ship}} \times f_{\mathrm{tax}})\big((w,d),(s,\mathrm{reg})\big) = \big(f_{\mathrm{ship}}(w,d), \ f_{\mathrm{tax}}(s,\mathrm{reg})\big).$$

For a concrete cart: $(w, d, s, \mathrm{reg}) = (2.0, 100, 12{,}000, \mathrm{Tokyo})$,

$$f_{\mathrm{ship}} = 300 + 50 \cdot 2 + 0.5 \cdot 100 = 450 \ \mathrm{JPY}, \qquad f_{\mathrm{tax}} = 0.10 \cdot 12{,}000 = 1{,}200 \ \mathrm{JPY},$$

so the product returns (450, 1,200), i.e. (shipping fee, tax).

Theorem 7.6 (MetaFunction is a MetaStructure and generalizes functions)**.** $\mathbb{M}_{\mathrm{fun}} = \big(U_{\mathrm{fun}}, (\Phi_\ell)\big)$ *of Definition 7.4 is a MetaStructure in the sense of Definition 1.2. Moreover, every classical function $f : X \to Y$ embeds as an object of U_{fun} (Remark 7.3); hence MetaFunction* generalizes *functions.*

Proof. We verify the two MetaStructure clauses.

(1) Uniform constructors and closure. For each meta-operation we check (F1)–(F3).

(PROD). Typing: by construction $\mathsf{Graph}_\times \subseteq \mathsf{Dom}_\times \times \mathsf{Cod}_\times$. Totality: fix $(x_1, x_2) \in \mathsf{Dom}_1 \times \mathsf{Dom}_2$. By (F2) for $\mathbf{F}_i$, there exist $y_i \in \mathsf{Cod}_i$ with $(x_i, y_i) \in \mathsf{Graph}_i$. Then $\big((0, x_1, x_2), (1, y_1, y_2)\big) \in \mathsf{Graph}_\times$. Functionality: assume

$$\big((0,x_1,x_2),(1,y_1,y_2)\big), \ \big((0,x_1,x_2),(1,y_1',y_2')\big) \in \mathsf{Graph}_\times.$$

Then $(x_i, y_i), (x_i, y_i') \in \mathsf{Graph}_i$, so $y_i = y_i'$ by (F3) in $\mathbf{F}_i$ $(i = 1, 2)$.

(COPROD). Typing is immediate. Totality: for $(0, x) \in \mathsf{Dom}_\sqcup$ choose y with $(x, y) \in \mathsf{Graph}_1$; then $\big((0, x), (2, y)\big) \in \mathsf{Graph}_\sqcup$. Similarly for $(1, u)$. Functionality: if $\big((0, x), (2, y)\big)$ and $\big((0, x), (2, y')\big)$ lie in $\mathsf{Graph}_\sqcup$, then $(x, y), (x, y') \in \mathsf{Graph}_1$ so $y = y'$, and likewise on the second summand.

(COMP). Typing is clear. Totality: for $x \in \mathsf{Dom}_1$ pick $y \in \mathsf{Cod}_1$ with $(x, y) \in \mathsf{Graph}_1$ ((F2)); then pick $z \in \mathsf{Cod}_2$ with $\big(\psi(y), z\big) \in \mathsf{Graph}_2$; hence $\big((0, x), (1, z)\big) \in \mathsf{Graph}_\circ$. Functionality: suppose $\big((0, x), (1, z)\big), \big((0, x), (1, z')\big) \in \mathsf{Graph}_\circ$. Then there exist $y, y' \in \mathsf{Cod}_1$ with $(x, y), (x, y') \in \mathsf{Graph}_1$ and $(\psi(y), z), (\psi(y'), z') \in \mathsf{Graph}_2$. By (F3) in $\mathbf{F}_1$, $y = y'$, hence $(\psi(y), z), (\psi(y), z') \in \mathsf{Graph}_2$ and by (F3) in $\mathbf{F}_2$, $z = z'$.

(RES). Typing and functionality are inherited; totality holds because for each $x \in S \subseteq \mathsf{Dom}$ there exists y with $(x, y) \in \mathsf{Graph}$, hence $\big((0, x), (1, y)\big) \in \mathsf{Graph}'$.

(PUSH). Typing: if $(x, y) \in \mathsf{Graph}$, then $(f(x), f(y)) \in \mathsf{Graph}' \subseteq \mathsf{Dom}' \times \mathsf{Cod}'$. Totality: given $u = f(x) \in \mathsf{Dom}'$, choose y with $(x, y) \in \mathsf{Graph}$ and set $v := f(y) \in \mathsf{Cod}'$ so $(u, v) \in \mathsf{Graph}'$. Functionality: if $(u, v), (u, v') \in \mathsf{Graph}'$, write $u = f(x)$, $v = f(y)$, $v' = f(y')$ with $(x, y), (x, y') \in \mathsf{Graph}$; then $y = y'$ by (F3), hence $v = v'$.

(2) Naturality (isomorphism invariance). An isomorphism $\alpha : \mathbf{F} \to \mathbf{F}'$ of Σ_{fun}-structures is a bijection $\alpha : H \to H'$ with $\alpha[\mathsf{Dom}] = \mathsf{Dom}'$, $\alpha[\mathsf{Cod}] = \mathsf{Cod}'$, and

$$(x, y) \in \mathsf{Graph} \iff (\alpha x, \alpha y) \in \mathsf{Graph}'.$$

Given isomorphisms $\alpha_i : \mathbf{F}_i \to \mathbf{F}_i'$, define:

$$\alpha_\times(t, x_1, x_2) := (t, \alpha_1 x_1, \alpha_2 x_2), \quad \alpha_\sqcup(t, \cdot) := (t, \alpha_i(\cdot)) \text{ on each tagged part,}$$

$$\alpha_\circ(0, x) := (0, \alpha_1 x), \quad \alpha_\circ(1, z) := (1, \alpha_2 z),$$

and for (RES) the restriction of α to the corresponding tagged carrier; for (PUSH) take the given f and the identity on K. Each map preserves the colored domain/codomain and carries the defining pairs of the graphs to those of the target (componentwise for (PROD)/(COPROD), by the defining existence in (COMP), and tautologically for (RES)/(PUSH)). Hence each Φ_ℓ is natural. The last claim follows from Remark 7.3. □

Proposition 7.7 (Behavior of classical operations)**.** *Let $f_i : X_i \to Y_i$ be functions and encode them as $\mathbf{F}_{f_i} \in U_{\text{fun}}$. Then*

$$\Phi_\times(\mathbf{F}_{f_1}, \mathbf{F}_{f_2}) \cong \mathbf{F}_{f_1 \times f_2}, \qquad \Phi_\sqcup(\mathbf{F}_{f_1}, \mathbf{F}_{f_2}) \cong \mathbf{F}_{f_1 \sqcup f_2},$$

and for any bijection $\psi : Y_1 \xrightarrow{\cong} X_2$,

$$\Phi_{\circ,\psi}(\mathbf{F}_{f_1}, \mathbf{F}_{f_2}) \cong \mathbf{F}_{f_2 \circ \psi \circ f_1}.$$

Proof. Immediate from the tagged-carrier constructions and the defining equations of the graphs. □

Definition 7.8 (Iterated MetaFunction of depth t)**.** For $t \in \mathbb{N}$, an *Iterated MetaFunction of depth t* is an Iterated MetaStructure (Definition 1.3) over Σ_{fun},

$$\mathfrak{M}^{(t)}_{\text{fun}} = \big(U^{(t)}_{\text{fun}}, \ (\odot^{(t)}_\ell)_{\ell \in \Lambda}\big),$$

obtained by applying the lifting $\mathsf{U}_{\Sigma_{\text{fun}}}$ to $\mathbb{M}_{\text{fun}}$ repeatedly t times. Concretely, for $s < t$ and any base meta-operation Φ_ℓ (of meta-arity k_ℓ),

$$\Phi^\uparrow_\ell : \ \big(U^{(t)}_{\text{fun}}\big)^{k_\ell} \longrightarrow U^{(t)}_{\text{fun}}$$

is defined on representatives by

$$\Phi^\uparrow_\ell\big(\mathsf{U}^{t-s}_{\Sigma_{\text{fun}}}(X_1), \ldots, \mathsf{U}^{t-s}_{\Sigma_{\text{fun}}}(X_{k_\ell})\big) := \mathsf{U}^{t-s}_{\Sigma_{\text{fun}}}\big(\Phi_\ell(X_1, \ldots, X_{k_\ell})\big),$$

and similarly for all relations. We denote the lifts of (PROD), (COPROD), (COMP), (RES), (PUSH) by $\odot^{(t)}_\times$, $\odot^{(t)}_\sqcup$, $\odot^{(t)}_{\circ,\psi}$, $\odot^{(t)}_{\text{res}(S)}$, $\odot^{(t)}_{\text{push}(f)}$, respectively.

Example 7.9 (Iterated MetaFunction (depth $t = 1$): Promotions as operators on price functions)**.** Define a *function-of-function* (discount operator) that maps any price function $g : X \to \mathbb{R}_{\geq 0}$ to a new function

$$\mathcal{G}_{\text{promo},\delta}(g) : x \longmapsto (1-\delta)\, g(x) \quad (0 < \delta < 1).$$

Applying it to the shipping function above with a 10% weekend promotion,

$$\tilde{f}_{\text{ship}} := \mathcal{G}_{\text{promo},0.10}\big(f_{\text{ship}}\big), \qquad \tilde{f}_{\text{ship}}(2.0, 100) = 0.9 \cdot 450 = 405 \text{ JPY}.$$

Thus the iterated meta-level modifies the *function itself* (not just inputs/outputs) and then evaluates it.

Theorem 7.10 (Iterated MetaFunction is an Iterated MetaStructure and generalizes MetaFunction)**.** *For every $t \in \mathbb{N}$, $\mathfrak{M}^{(t)}_{\text{fun}}$ of Definition 7.8 is an Iterated MetaStructure in the sense of Definition 1.3. Moreover, for $s < t$ the map*

$$\iota_{s\to t} : \mathfrak{M}^{(s)}_{\text{fun}} \hookrightarrow \mathfrak{M}^{(t)}_{\text{fun}}, \qquad X \longmapsto \mathsf{U}^{t-s}_{\Sigma_{\text{fun}}}(X),$$

is an embedding that preserves all meta-operations:

$$\Phi^{\uparrow}_{\ell}\big(\iota_{s\to t}(X_1), \dots, \iota_{s\to t}(X_{k_\ell})\big) = \iota_{s\to t}\big(\Phi_\ell(X_1, \dots, X_{k_\ell})\big).$$

In particular, $\mathfrak{M}^{(0)}_{\text{fun}} = \mathbb{M}_{\text{fun}}$ embeds into $\mathfrak{M}^{(t)}_{\text{fun}}$, so Iterated MetaFunction generalizes *MetaFunction.*

Proof. By Definition 1.3, each lifted constructor is obtained by post-composing the base constructors with $\mathsf{U}^{t-s}_{\Sigma_{\text{fun}}}$, hence uniform at all levels. If $\alpha_i : X_i \xrightarrow{\cong} Y_i$ are level-s isomorphisms, then

$$\mathsf{U}^{t-s}_{\Sigma_{\text{fun}}}(\alpha_i) : \mathsf{U}^{t-s}_{\Sigma_{\text{fun}}}(X_i) \xrightarrow{\cong} \mathsf{U}^{t-s}_{\Sigma_{\text{fun}}}(Y_i)$$

are level-t isomorphisms, and

$$\Phi^{\uparrow}_{\ell}\big(\mathsf{U}^{t-s}\alpha_1, \dots, \mathsf{U}^{t-s}\alpha_{k_\ell}\big) = \mathsf{U}^{t-s}\big(\Phi_\ell(\alpha_1, \dots, \alpha_{k_\ell})\big)$$

is an isomorphism because each base Φ_ℓ is natural by Theorem 7.6. The operation-preservation for $\iota_{s\to t}$ is the same identity with X_i in place of α_i. Taking $s = 0$ yields the claimed embedding. □

8 Meta-Probability (Probability of Probability)

Probability provides a mathematical framework for modeling uncertainty, assigning numerical likelihoods to events within a sample space under axioms [7, 83–90]. Meta-Probability regards probability spaces as objects, enabling uniform meta-operations that transform or combine them, thereby generalizing classical probability theory systematically. Iterated Meta-Probability recursively applies Meta-Probability, constructing hierarchical levels where probability of probabilities emerges, extending uncertainty modeling into higher abstract layers.

Definition 8.1 (Classical Probability)**.** (cf. [91–93]) A *Classical Probability* space is a triple $(\Omega, \mathcal{F}, P)$ where Ω is the sample space, $\mathcal{F}$ is a sigma-algebra of events, and $P : \mathcal{F} \to [0, 1]$ is a probability measure satisfying Kolmogorov's axioms.

Fix the following single-sorted, finitary signature for (finite) probability spaces:

$$\Sigma_{\text{prob}} = \Big(\ \mathsf{Func} = \varnothing, \quad \mathsf{Rel} = \{\,\Omega,\ \mathsf{Ev},\ \mathsf{I},\ \mathsf{Inc},\ W\,\}, \quad \text{ar}(\Omega) = \text{ar}(\mathsf{Ev}) = \text{ar}(\mathsf{I}) = 1,\ \text{ar}(\mathsf{Inc}) = 2,\ \text{ar}(W) = 2\ \Big).$$

A Σ_{prob}-structure is a tuple

$$\mathbf{P} = \big(H;\ \Omega^{\mathbf{P}},\ \mathsf{Ev}^{\mathbf{P}},\ \mathsf{I}^{\mathbf{P}},\ \mathsf{Inc}^{\mathbf{P}},\ W^{\mathbf{P}}\big),$$

where $H \neq \varnothing$ is the carrier, $\Omega^{\mathbf{P}}$ (outcomes), $\mathsf{Ev}^{\mathbf{P}}$ (events), $\mathsf{I}^{\mathbf{P}}$ (probability values) are unary "colors", $\mathsf{Inc}^{\mathbf{P}} \subseteq \Omega^{\mathbf{P}} \times \mathsf{Ev}^{\mathbf{P}}$ encodes membership $x \in E$, and $W^{\mathbf{P}} \subseteq \Omega^{\mathbf{P}} \times \mathsf{I}^{\mathbf{P}}$ encodes point-masses $w : \Omega \to [0, 1]$.

Definition 8.2 (Finite Probability encoded in Σ_{prob})**.** Let $U_{\text{prob}} \subseteq \text{Str}_{\Sigma_{\text{prob}}}$ be the class of all $\mathbf{P} = (H; \Omega, \mathsf{Ev}, \mathsf{I}, \mathsf{Inc}, W)$ such that:

(P1) **Typing/finite:** Ω is finite and nonempty; Ev is in bijection with $\mathcal{P}(\Omega)$ via Inc; i.e., for each $E \in \mathsf{Ev}$ the fiber $\{x \in \Omega : \mathsf{Inc}(x, E)\}$ is a subset of Ω, and every subset arises from a unique E.

(P2) **Values:** I is (identified with) $[0,1]$; more precisely, each $\mathbf{P}$ comes with a fixed identification $\iota_{\mathbf{P}} : \mathsf{I} \xrightarrow{\cong} [0,1]$ (not a symbol of the signature) used below.

(P3) **Point weights:** W is the graph of a total functional relation $w : \Omega \to \mathsf{I}$ (so for each $x \in \Omega$ there is a unique $p \in \mathsf{I}$ with $W(x, p)$), with $\sum_{x\in\Omega} \iota_{\mathbf{P}}(w(x)) = 1$.

(P4) **Event probabilities (derived):** For each $E \in \mathsf{Ev}$, define

$$P(E) \ := \sum_{x\in\Omega:\ \mathsf{Inc}(x,E)} \iota_{\mathbf{P}}\big(w(x)\big) \ \in \ [0,1].$$

Then $P : \mathsf{Ev} \to [0,1]$ satisfies the (finite) Kolmogorov axioms.

In particular, each $\mathbf{P} \in U_{\mathrm{prob}}$ deterministically encodes a finite probability space $(\Omega, \mathcal{P}(\Omega), P)$ via w.

Remark 8.3 (Embedding classical finite probability)**.** Given a classical finite probability space $(\Omega, \mathcal{P}(\Omega), P)$, define

$$H := \{0\} \times \Omega \ \cup \ \{1\} \times \mathcal{P}(\Omega) \ \cup \ \{2\} \times [0,1],$$
$$\Omega := \{0\} \times \Omega, \qquad \mathsf{Ev} := \{1\} \times \mathcal{P}(\Omega), \qquad \mathsf{I} := \{2\} \times [0,1],$$
$$\mathsf{Inc}\big((0,x),(1,E)\big) \iff x \in E, \qquad W\big((0,x),(2,p)\big) \iff p = P(\{x\}).$$

With $\iota_{\mathbf{P}}(2, p) = p$, we obtain $\mathbf{P} \in U_{\mathrm{prob}}$.

Definition 8.4 (Meta-Probability)**.** A *Meta-Probability* is a MetaStructure (Definition 1.2) over Σ_{prob}

$$\mathbb{M}_{\mathrm{prob}} = \big(U_{\mathrm{prob}},\ (\Phi_\ell)_{\ell\in\Lambda}\big),$$

whose meta-operations are given uniformly by the following standard probabilistic constructions (each specified by carrier- and relation-constructors):

(PROD) Independent product. For $\mathbf{P}_i = (H_i; \Omega_i, \mathsf{Ev}_i, \mathsf{I}_i, \mathsf{Inc}_i, W_i) \in U_{\mathrm{prob}}$ $(i = 1, 2)$ define

$$\Phi_\otimes(\mathbf{P}_1, \mathbf{P}_2) := (H_\times; \Omega_\times, \mathsf{Ev}_\times, \mathsf{I}_\times, \mathsf{Inc}_\times, W_\times),$$

on the tagged carrier

$$H_\times := \underbrace{\{0\} \times \Omega_1 \times \Omega_2}_{\text{outcomes}} \ \cup \ \underbrace{\{1\} \times \mathcal{P}(\Omega_1 \times \Omega_2)}_{\text{events}} \ \cup \ \underbrace{\{2\} \times [0,1]}_{\text{probabilities}},$$

with

$$\Omega_\times := \{0\} \times \Omega_1 \times \Omega_2, \quad \mathsf{Ev}_\times := \{1\} \times \mathcal{P}(\Omega_1 \times \Omega_2), \quad \mathsf{I}_\times := \{2\} \times [0,1],$$
$$\mathsf{Inc}_\times\big((0,x_1,x_2),(1,E)\big) \iff (x_1,x_2) \in E,$$

and point-weights

$$W_\times\big((0,x_1,x_2),(2,p)\big) \iff p = \iota_{\mathbf{P}_1}\big(w_1(x_1)\big) \cdot \iota_{\mathbf{P}_2}\big(w_2(x_2)\big),$$

where W_i encodes $w_i : \Omega_i \to \mathsf{I}_i$ and $\iota_{\mathbf{P}_\times}(2, p) = p$.

(MIX) Convex mixture (same outcome set). If $\mathbf{P}_1, \mathbf{P}_2 \in U_{\mathrm{prob}}$ have the *same* outcome set Ω (up to isomorphism) and $\lambda \in (0,1)$, set

$$\Phi_{\mathrm{mix}(\lambda)}(\mathbf{P}_1, \mathbf{P}_2) := (H; \Omega, \mathsf{Ev}, \mathsf{I}, \mathsf{Inc}, W),$$

on the tagged carrier $H := \{0\} \times \Omega \ \cup \ \{1\} \times \mathcal{P}(\Omega) \ \cup \ \{2\} \times [0,1]$ with the obvious $\Omega, \mathsf{Ev}, \mathsf{I}, \mathsf{Inc}$, and

$$W\big((0,x),(2,p)\big) \iff p = \lambda\, \iota_{\mathbf{P}_1}(w_1(x)) + (1-\lambda)\, \iota_{\mathbf{P}_2}(w_2(x)).$$

(RES) Conditioning on an event $A \subseteq \Omega$ with $P(A) > 0$. For $\mathbf{P} \in U_{\mathrm{prob}}$ and $A \subseteq \Omega$ with $P(A) := \sum_{x\in A} \iota_{\mathbf{P}}(w(x)) > 0$, define

$$\Phi_{\mathrm{cond}(A)}(\mathbf{P}) := (H'; \Omega', \mathsf{Ev}', \mathsf{I}', \mathsf{Inc}', W'),$$

with $H' = \{0\}\times A \,\cup\, \{1\}\times\mathcal{P}(A) \,\cup\, \{2\}\times[0,1]$, the obvious induced colors $\Omega', \mathsf{Ev}', \mathsf{I}', \mathsf{Inc}'$, and point-weights

$$W'\big((0,x),(2,p)\big) \iff p = \frac{\iota_{\mathbf{P}}(w(x))}{P(A)} \quad (x \in A).$$

(PUSH) Pushforward along a bijection $f : \Omega \to K$ (relabeling outcomes). For $\mathbf{P} = (H; \Omega, \mathsf{Ev}, \mathsf{I}, \mathsf{Inc}, W)$ and any bijection $f : \Omega \to K$ set

$$\Phi_{\mathrm{push}(f)}(\mathbf{P}) := (\widehat{H}; \widehat{\Omega}, \widehat{\mathsf{Ev}}, \widehat{\mathsf{I}}, \widehat{\mathsf{Inc}}, \widehat{W}),$$

on $\widehat{H} = \{0\} \times K \,\cup\, \{1\} \times \mathcal{P}(K) \,\cup\, \{2\} \times [0,1]$, with

$$\widehat{\Omega} = \{0\} \times K, \quad \widehat{\mathsf{Ev}} = \{1\} \times \mathcal{P}(K), \quad \widehat{\mathsf{I}} = \{2\} \times [0,1], \quad \widehat{\mathsf{Inc}}\big((0,k),(1,B)\big) \iff k \in B,$$

$$\widehat{W}\big((0,k),(2,p)\big) \iff p = \iota_{\mathbf{P}}\big(w(f^{-1}(k))\big).$$

Example 8.5 (Meta–Probability: Sensor–fusion for home safety)**. Context.** A smart home uses two independent detectors: a smoke sensor ($\Omega_1 = \{\mathrm{ok}, \mathrm{alarm}\}$) and a heat sensor ($\Omega_2 = \{\mathrm{ok}, \mathrm{alarm}\}$). Each device has a calibrated point–mass distribution $w_i : \Omega_i \to [0,1]$.

Independent product. Fuse the devices into a single probabilistic object via the Meta–Probability product

$$\mathbf{P}_{\mathrm{fused}} \;:=\; \Phi_{\otimes}\big(\mathbf{P}_{\mathrm{smoke}}, \mathbf{P}_{\mathrm{heat}}\big),$$

so that for any pair $(x_1, x_2) \in \Omega_1 \times \Omega_2$,

$$\mathbb{P}_{\mathrm{fused}}\big((x_1,x_2)\big) \;=\; w_1(x_1)\, w_2(x_2).$$

Conditioning on an alert. Let $A := \{(\mathrm{alarm}, \mathrm{ok}), (\mathrm{ok}, \mathrm{alarm}), (\mathrm{alarm}, \mathrm{alarm})\}$. If $\mathbb{P}_{\mathrm{fused}}(A) > 0$, form

$$\mathbf{P}_{\mathrm{alert}} \;:=\; \Phi_{\mathrm{cond}(A)}\big(\mathbf{P}_{\mathrm{fused}}\big),$$

which reweights outcomes within A to yield the posterior distribution given that at least one device fired. This directly supports operations such as "probability that *both* devices agree" versus "exactly one device alarmed," useful for triaging responses.

Label management. If the vendor firmware renames states (e.g. ok $\mapsto$ normal), apply a bijective relabeling $\Phi_{\mathrm{push}}(f)$; naturality ensures the probabilities are preserved.

Example 8.6 (Meta–Probability: A/B email campaign as a convex mixture)**. Context.** A marketer splits traffic between two subject lines, A and B, over the same outcome set $\Omega = \{\text{no click}, \text{click}\}$. Historical click distributions are $w_A, w_B : \Omega \to [0,1]$. A traffic allocation $\lambda \in (0,1)$ sends a λ-fraction to A and $1-\lambda$ to B.

Mixture model. The overall campaign distribution is the Meta–Probability mixture

$$\mathbf{P}_{\mathrm{campaign}} \;:=\; \Phi_{\mathrm{mix}(\lambda)}\big(\mathbf{P}_A, \mathbf{P}_B\big), \qquad \mathbb{P}_{\mathrm{campaign}}(x) = \lambda\, w_A(x) + (1-\lambda)\, w_B(x).$$

Condition on opens. Let $O \subseteq \Omega_{\mathrm{rich}}$ be the event "email opened" in a finer outcome alphabet (e.g. $\{\text{no open}, \text{open\&no click}, \text{open\&click}\}$). Condition via $\Phi_{\mathrm{cond}}(O)$ to compute post–open click–through rates. Adjusting λ explores trade–offs between exploration and exploitation while remaining within one Meta–Probability object. Relabeling UI codes (e.g. ESP vendor changes) is handled by $\Phi_{\mathrm{push}}(f)$.

Theorem 8.7 (Meta-Probability is a MetaStructure and generalizes probability)**.** $\mathbb{M}_{\mathrm{prob}} = \big(U_{\mathrm{prob}}, (\Phi_\ell)\big)$ *in Definition 8.4 is a MetaStructure (Definition 1.2). Moreover, every classical finite probability space* $(\Omega, \mathcal{P}(\Omega), P)$ *appears (via Remark 8.3) as an object of* U_{prob}*; hence Meta-Probability* generalizes *probability.*

Proof. Uniform constructors and closure. (PROD): $\Omega_\times$ is finite and nonempty; $\mathsf{Ev}_\times \cong \mathcal{P}(\Omega_\times)$ via $\mathsf{Inc}_\times$. $W_\times$ is total and functional by definition, with

$$\sum_{(x_1,x_2)\in\Omega_1\times\Omega_2} \iota_{\mathbf{P}_\times}\big(w_\times(x_1,x_2)\big) = \Big(\sum_{x_1} \iota_{\mathbf{P}_1}(w_1(x_1))\Big)\Big(\sum_{x_2} \iota_{\mathbf{P}_2}(w_2(x_2))\Big) = 1,$$

so (P3)–(P4) hold.

(MIX): For each $x \in \Omega$, $w(x) := \lambda w_1(x) + (1-\lambda)w_2(x) \in [0,1]$, and $\sum_x w(x) = \lambda \sum_x w_1(x) + (1-\lambda)\sum_x w_2(x) = 1$, so (P3) holds; (P1),(P2),(P4) are immediate.

(RES): A is finite and nonempty since $P(A) > 0$; the renormalized weights $w'(x) = w(x)/P(A)$ are nonnegative and sum to 1, hence (P3) holds on A; the induced Ev' and Inc' still satisfy (P1).

(PUSH): Transporting labels by f preserves finiteness and (P1); $\widehat{w}(k) := w(f^{-1}(k))$ is total, functional, and $\sum_k \widehat{w}(k) = \sum_x w(x) = 1$, so (P3) holds.

Naturality (isomorphism invariance). An isomorphism of Σ_{prob}-structures is a bijection $\alpha : H \to H'$ preserving the colored parts $\Omega, \mathsf{Ev}, \mathsf{I}$ and the relations Inc, W. Given isomorphisms $\alpha_i : \mathbf{P}_i \to \mathbf{P}'_i$, define the induced maps on the tagged carriers componentwise (Cartesian product on outcomes for (PROD), identity on tags and underlying bijection on labels for (MIX), restriction for (RES), and the given relabeling for (PUSH)). Each induced map preserves $\Omega, \mathsf{Ev}, \mathsf{I}$ and carries the defining pairs of Inc, W to the corresponding pairs in the target construction, hence it is an isomorphism of the outputs. Therefore, each Φ_ℓ is natural. The generalization claim follows from Remark 8.3. □

Definition 8.8 (Iterated Meta-Probability of depth t)**.** For $t \in \mathbb{N}$, an *Iterated Meta-Probability of depth t* is an Iterated MetaStructure (Definition 1.3) over Σ_{prob},

$$\mathfrak{M}^{(t)}_{\text{prob}} = \big(U^{(t)}_{\text{prob}},\ (\odot^{(t)}_\ell)_{\ell\in\Lambda}\big),$$

obtained by applying the lifting functor $\mathsf{U}_{\Sigma_{\text{prob}}}$ to $\mathbb{M}_{\text{prob}}$ repeatedly t times. Concretely, if $s < t$ and Φ_ℓ has meta-arity k_ℓ, its lift

$$\Phi^{\uparrow}_\ell :\ \big(U^{(t)}_{\text{prob}}\big)^{k_\ell} \longrightarrow U^{(t)}_{\text{prob}}$$

is defined on representatives by

$$\Phi^{\uparrow}_\ell\big(\mathsf{U}^{t-s}_{\Sigma_{\text{prob}}}(X_1),\dots,\mathsf{U}^{t-s}_{\Sigma_{\text{prob}}}(X_{k_\ell})\big) := \mathsf{U}^{t-s}_{\Sigma_{\text{prob}}}\big(\Phi_\ell(X_1,\dots,X_{k_\ell})\big),$$

and similarly for all relations. We write $\odot^{(t)}_\otimes, \odot^{(t)}_{\text{mix}(\lambda)}, \odot^{(t)}_{\text{cond}(A)}, \odot^{(t)}_{\text{push}(f)}$ for the lifts of (PROD), (MIX), (RES), (PUSH).

Example 8.9 (Iterated Meta–Probability: Multi–day precipitation planning with regime ensembles)**. Context.** A city operations team plans for T consecutive days. For each day t, a calibrated discrete precipitation model over $\Omega_t = \{\text{none}, \text{light}, \text{moderate}, \text{heavy}\}$ is given by $\mathbf{P}_t \in U_{\text{prob}}$. Additionally, two climate regimes (e.g. El Niño vs. La Niña) provide alternative daily distributions $\mathbf{P}^{(\text{E})}_t, \mathbf{P}^{(\text{L})}_t$ with regime prior $\pi \in (0,1)$.

Iterated construction. At the base level, the joint weather across all T days is the lifted product

$$\mathbf{P}_{1:T} := \odot^{(T)}_\otimes\big(\mathbf{P}_1, \mathbf{P}_2, \dots, \mathbf{P}_T\big),$$

a probability measure on $\Omega_1 \times \cdots \times \Omega_T$. To incorporate regime uncertainty *at the next meta–level*, first build the two day–wise products

$$\mathbf{P}^{(\text{E})}_{1:T} = \odot^{(T)}_\otimes\big(\mathbf{P}^{(\text{E})}_1, \dots, \mathbf{P}^{(\text{E})}_T\big), \qquad \mathbf{P}^{(\text{L})}_{1:T} = \odot^{(T)}_\otimes\big(\mathbf{P}^{(\text{L})}_1, \dots, \mathbf{P}^{(\text{L})}_T\big),$$

then form the regime–level mixture (lifted)

$$\mathbf{P}^{\text{ens}}_{1:T} := \odot^{(T)}_{\text{mix}(\pi)}\big(\mathbf{P}^{(\text{E})}_{1:T}, \mathbf{P}^{(\text{L})}_{1:T}\big).$$

This yields an ensemble distribution over full T-day sequences that first composes independent daily uncertainties and then mixes entire scenarios by regime.

Operational queries.

- *Run–of–days thresholds:* let A be the event "at least two heavy days." Compute $\mathbb{P}^{\mathrm{ens}}_{1:T}(A)$ or condition on A via $\odot^{(T)}_{\mathrm{cond}(A)}$ for contingency planning.
- *Calendar relabeling:* if dates are renumbered, a bijection f on $\Omega_1 \times \cdots \times \Omega_T$ implements consistent outcome renaming with $\odot^{(T)}_{\mathrm{push}}(f)$, preserving probabilities by naturality.

Thus Iterated Meta–Probability provides a clean, hierarchical way to combine *within–day* independence and *across–scenario* model averaging in one uniform framework.

Theorem 8.10 (Iterated Meta-Probability is an Iterated MetaStructure and generalizes Meta-Probability)**.** *For every $t \in \mathbb{N}$, $\mathfrak{M}^{(t)}_{\mathrm{prob}}$ of Definition 8.8 is an Iterated MetaStructure in the sense of Definition 1.3. Moreover, for $s < t$ the embedding*

$$\iota_{s\to t} : \mathfrak{M}^{(s)}_{\mathrm{prob}} \hookrightarrow \mathfrak{M}^{(t)}_{\mathrm{prob}}, \qquad X \longmapsto \mathsf{U}^{t-s}_{\Sigma_{\mathrm{prob}}}(X),$$

preserves all meta-operations:

$$\Phi^{\uparrow}_{\ell}\big(\iota_{s\to t}(X_1), \dots, \iota_{s\to t}(X_{k_\ell})\big) = \iota_{s\to t}\big(\Phi_\ell(X_1, \dots, X_{k_\ell})\big).$$

In particular, $\mathfrak{M}^{(0)}_{\mathrm{prob}} = \mathbb{M}_{\mathrm{prob}}$ embeds into $\mathfrak{M}^{(t)}_{\mathrm{prob}}$, so Iterated Meta-Probability generalizes *Meta-Probability.*

Proof. By Definition 1.3, each lifted constructor is obtained by post-composing the base constructors with $\mathsf{U}^{t-s}_{\Sigma_{\mathrm{prob}}}$; hence uniformity holds at every level. If $\alpha_i : X_i \xrightarrow{\cong} Y_i$ are level-s isomorphisms, then

$$\mathsf{U}^{t-s}_{\Sigma_{\mathrm{prob}}}(\alpha_i) : \mathsf{U}^{t-s}_{\Sigma_{\mathrm{prob}}}(X_i) \xrightarrow{\cong} \mathsf{U}^{t-s}_{\Sigma_{\mathrm{prob}}}(Y_i)$$

are level-t isomorphisms, and

$$\Phi^{\uparrow}_{\ell}\big(\mathsf{U}^{t-s}\alpha_1, \dots, \mathsf{U}^{t-s}\alpha_{k_\ell}\big) = \mathsf{U}^{t-s}\big(\Phi_\ell(\alpha_1, \dots, \alpha_{k_\ell})\big)$$

is an isomorphism because each base Φ_ℓ is natural (Theorem 8.7). The operation-preservation of $\iota_{s\to t}$ is exactly the same identity with X_i in place of α_i. Taking $s = 0$ yields the desired embedding of $\mathbb{M}_{\mathrm{prob}}$ into every $\mathfrak{M}^{(t)}_{\mathrm{prob}}$. □

9 Meta-Geometry (Geometry of Geometry)

Geometry studies points, lines, shapes, and spatial relations, focusing on properties like distance, congruence, and transformations in structured spaces [94–98, 98]. Meta-Geometry treats geometrical systems as objects, applying higher-level operations to combine, transform, or analyze them uniformly across diverse geometric frameworks. Iterated Meta-Geometry recursively lifts Meta-Geometry, creating hierarchical levels where structures of geometries interact, enabling increasingly abstract geometric generalizations.

Definition 9.1 (Classical Geometry)**.** (cf. [99–101]) *Classical Geometry* refers to the Euclidean framework in which points, lines, and planes are introduced axiomatically, and geometric figures are studied through congruence, similarity, and distance relations.

Fix the single-sorted, finitary signature

$$\Sigma_{\mathrm{geo}} = \Big(\ \mathsf{Func} = \varnothing, \quad \mathsf{Rel} = \{\mathsf{Points},\ \mathsf{Vals},\ \mathsf{Dist}\}, \quad \mathrm{ar}(\mathsf{Points}) = \mathrm{ar}(\mathsf{Vals}) = 1,\ \mathrm{ar}(\mathsf{Dist}) = 3\ \Big).$$

A Σ_{geo}-structure is a triple

$$\mathbf{G} = \big(H;\ \mathsf{Points}^{\mathbf{G}},\ \mathsf{Vals}^{\mathbf{G}},\ \mathsf{Dist}^{\mathbf{G}}\big),$$

where $H \neq \varnothing$ is the carrier, $\mathsf{Points}^{\mathbf{G}}$ is the set of points, $\mathsf{Vals}^{\mathbf{G}}$ is the set of nonnegative distance-values, and $\mathsf{Dist}^{\mathbf{G}} \subseteq \mathsf{Points}^{\mathbf{G}} \times \mathsf{Points}^{\mathbf{G}} \times \mathsf{Vals}^{\mathbf{G}}$ is the (typed) graph of the distance map.

Definition 9.2 (Finite metric geometries encoded in Σ_{geo})**.** Let $U_{\mathrm{geo}} \subseteq \mathrm{Str}_{\Sigma_{\mathrm{geo}}}$ be the class of all $\mathbf{G} = (H; \mathsf{Points}, \mathsf{Vals}, \mathsf{Dist})$ satisfying:

(G1) **Typing/finite:** Points is finite and nonempty.

(G2) **Value identification:** There is a fixed identification $\iota_{\mathbf{G}} : \mathsf{Vals} \xrightarrow{\cong} [0,\infty)$ (used only to state axioms).

(G3) **Total functional distance:** Dist is the graph of a total functional map $d : \mathsf{Points} \times \mathsf{Points} \to \mathsf{Vals}$: for every $(x, y) \in \mathsf{Points}^2$ there is a unique $r \in \mathsf{Vals}$ with $\mathsf{Dist}(x, y, r)$.

(G4) **Metric axioms:** Writing $d_{\mathbb{R}}(x, y) := \iota_{\mathbf{G}}\big(d(x, y)\big) \in [0,\infty)$, for all $x, y, z \in \mathsf{Points}$:

$$\begin{aligned}
&d_{\mathbb{R}}(x, y) \geq 0, \qquad d_{\mathbb{R}}(x, y) = 0 \iff x = y,\\
&d_{\mathbb{R}}(x, y) = d_{\mathbb{R}}(y, x),\\
&d_{\mathbb{R}}(x, z) \leq d_{\mathbb{R}}(x, y) + d_{\mathbb{R}}(y, z).
\end{aligned}$$

Thus each $\mathbf{G} \in U_{\text{geo}}$ canonically encodes a finite metric space $(\mathsf{Points}, d_{\mathbb{R}})$.

Remark 9.3 (Embedding finite Euclidean configurations)**.** Given a finite set $X \subset \mathbb{R}^n$ with Euclidean distance $d_2(x, y) = \|x - y\|_2$, set

$$H := \{0\} \times X \;\cup\; \{1\} \times [0,\infty), \qquad \mathsf{Points} := \{0\} \times X, \quad \mathsf{Vals} := \{1\} \times [0,\infty),$$

and

$$\mathsf{Dist}\big((0, x), (0, y), (1, r)\big) \iff r = d_2(x, y).$$

With $\iota_{\mathbf{G}}(1, r) = r$ this yields $\mathbf{G} \in U_{\text{geo}}$.

Definition 9.4 (Meta-Geometry)**.** A *Meta-Geometry* is a MetaStructure (Definition 1.2) over Σ_{geo},

$$\mathbb{M}_{\text{geo}} = \big(U_{\text{geo}},\ (\Phi_\ell)_{\ell\in\Lambda}\big),$$

whose meta-operations are specified uniformly by the following standard geometric constructions (each via carrier- and relation-constructors):

(PROD) ℓ_2 product of metric spaces. For $\mathbf{G}_i = (H_i; \mathsf{Points}_i, \mathsf{Vals}_i, \mathsf{Dist}_i) \in U_{\text{geo}}$ $(i = 1, 2)$ define

$$\Phi_\otimes(\mathbf{G}_1, \mathbf{G}_2) := (H_\times; \mathsf{Points}_\times, \mathsf{Vals}_\times, \mathsf{Dist}_\times)$$

on the tagged carrier $H_\times := \{0\} \times (\mathsf{Points}_1 \times \mathsf{Points}_2) \;\cup\; \{1\} \times [0,\infty)$ with

$$\mathsf{Points}_\times := \{0\} \times (\mathsf{Points}_1 \times \mathsf{Points}_2), \qquad \mathsf{Vals}_\times := \{1\} \times [0,\infty),$$

and

$$\mathsf{Dist}_\times\Big((0, (x_1, x_2)), (0, (y_1, y_2)), (1, r)\Big) \iff r = \sqrt{\iota_{\mathbf{G}_1} d_1(x_1, y_1)^2 + \iota_{\mathbf{G}_2} d_2(x_2, y_2)^2}\,.$$

(SCALE) Positive scaling $s > 0$. For $\mathbf{G} \in U_{\text{geo}}$ and $s > 0$ let

$$\Phi_{\text{scale}(s)}(\mathbf{G}) := (H'; \mathsf{Points}', \mathsf{Vals}', \mathsf{Dist}')$$

on $H' := \{0\} \times \mathsf{Points} \;\cup\; \{1\} \times [0,\infty)$ with $\mathsf{Points}' := \{0\} \times \mathsf{Points}$, $\mathsf{Vals}' := \{1\} \times [0,\infty)$, and

$$\mathsf{Dist}'\big((0, x), (0, y), (1, r)\big) \iff r = s \cdot \iota_{\mathbf{G}}\big(d(x, y)\big).$$

(RES) Metric subspace restriction by $S \subseteq \mathsf{Points}$.

$$\Phi_{\text{res}(S)}(\mathbf{G}) := (H_S; \mathsf{Points}_S, \mathsf{Vals}_S, \mathsf{Dist}_S),$$

with $H_S := \{0\} \times S \;\cup\; \{1\} \times [0,\infty)$, $\mathsf{Points}_S := \{0\} \times S$, $\mathsf{Vals}_S := \{1\} \times [0,\infty)$, and

$$\mathsf{Dist}_S\big((0, x), (0, y), (1, r)\big) \iff r = \iota_{\mathbf{G}}\big(d(x, y)\big) \quad (x, y \in S).$$

(PUSH) Relabeling by a bijection $f : \mathsf{Points} \xrightarrow{\cong} K$.

$$\Phi_{\text{push}(f)}(\mathbf{G}) := (\widehat{H}; \widehat{\mathsf{Points}}, \widehat{\mathsf{Vals}}, \widehat{\mathsf{Dist}}),$$

with $\widehat{H} := \{0\} \times K \;\cup\; \{1\} \times [0,\infty)$, $\widehat{\mathsf{Points}} := \{0\} \times K$, $\widehat{\mathsf{Vals}} := \{1\} \times [0,\infty)$, and

$$\widehat{\mathsf{Dist}}\big((0, f(x)), (0, f(y)), (1, r)\big) \iff r = \iota_{\mathbf{G}}\big(d(x, y)\big).$$

Example 9.5 (Meta–Geometry: Multi–floor office routing with elevators)**. Context.** A company wants a single distance model that accounts for (i) walking on a floor and (ii) vertical travel by elevator between floors.

Floor geometries. For each floor $k \in \{1, \dots, L\}$, let $X_k \subset \mathbb{R}^2$ be the finite set of key locations (desks, meeting rooms, printers). Encode the Euclidean distances on X_k as $\mathbf{G}_k \in U_{\text{geo}}$ (Definition 9.2, Remark 9.3).

Vertical (floor) geometry. Let $F := \{1, \dots, L\}$ with discrete metric $d_F(i, j) = |i - j|$. Encode F as $\mathbf{G}_F \in U_{\text{geo}}$. Choose a scale $s > 0$ (seconds per floor) and form $\Phi_{\text{scale}(s)}(\mathbf{G}_F)$.

Product construction. Define the point set $\{(x, k) \mid x \in X_k,\ k \in F\}$. The Meta–Geometry product

$$\mathbf{G}_{\text{office}} \ := \ \Phi_{\otimes}\Big(\bigsqcup_k \mathbf{G}_k \ , \ \Phi_{\text{scale}(s)}(\mathbf{G}_F) \Big)$$

implements the ℓ_2–combination of horizontal and vertical components: for $(x, k), (y, \ell)$,

$$d_{\text{office}}\big((x, k), (y, \ell)\big) \ = \ \sqrt{\|x - y\|_2^2 \ + \ \big(s\,|k - \ell|\big)^2}\ .$$

(Here the disjoint tagging of floors is represented in the carrier; the metric itself is the uniform ℓ_2 product from Definition 9.4.)

Operations in practice.

- *Policy change in elevator speed:* update s via $\Phi_{\text{scale}(s')}$.
- *Visitor-only zones:* restrict to a subset $S \subseteq \{(x, k)\}$ by $\Phi_{\text{res}(S)}$.
- *Room renumbering:* relabel points by a bijection $f : \{(x, k)\} \xrightarrow{\cong} K$ with $\Phi_{\text{push}(f)}$.

Thus a single Meta–Geometry object cleanly captures realistic travel costs across the building.

Example 9.6 (Meta–Geometry: Store layout combining shelf position and product category)**. Context.** A retailer wants a proximity metric reflecting both physical shelf location and semantic similarity of product categories.

Spatial and semantic parts. Let $S \subset \mathbb{R}^2$ be a finite set of shelf coordinates; encode it as $\mathbf{G}_{\text{pos}} \in U_{\text{geo}}$ with Euclidean distance. Let C be a finite set of categories with the discrete metric $d_C(c, c') = \mathbf{1}_{c \neq c'}$; encode it as $\mathbf{G}_{\text{cat}}$. Pick a weight $\lambda > 0$ to tune the category effect and form $\Phi_{\text{scale}(\lambda)}(\mathbf{G}_{\text{cat}})$.

Product construction. The Meta–Geometry product

$$\mathbf{G}_{\text{store}} \ := \ \Phi_{\otimes}\Big(\mathbf{G}_{\text{pos}} \ , \ \Phi_{\text{scale}(\lambda)}(\mathbf{G}_{\text{cat}}) \Big)$$

assigns to $(s, c), (s', c')$ the distance

$$d_{\text{store}}\big((s, c), (s', c')\big) \ = \ \sqrt{\|s - s'\|_2^2 \ + \ \big(\lambda\, \mathbf{1}_{c \neq c'}\big)^2}\ .$$

Use. This metric favors recommendations that are near in space and, when desired, in the same category (via small λ), or emphasizes category coherence (via large λ). Seasonal planogram changes are handled by a relabeling $\Phi_{\text{push}}(f)$, and promotion endcaps can be isolated with $\Phi_{\text{res}(S)}$.

Theorem 9.7 (Meta-Geometry is a MetaStructure and generalizes geometry)**.** $\mathbb{M}_{\text{geo}} = \big(U_{\text{geo}}, (\Phi_\ell)\big)$ *of Definition 9.4 is a MetaStructure (Definition 1.2). Moreover, every finite Euclidean configuration* (X, d_2) *embeds as an object of* U_{geo} *(Remark 9.3); hence Meta-Geometry* generalizes *geometry.*

Proof. Uniform constructors and closure. (PROD) The carrier $\mathsf{Points}_\times$ is finite and nonempty; $\mathsf{Dist}_\times$ is total and functional by definition. Put

$$D((x_1, x_2), (y_1, y_2)) := \sqrt{d_1(x_1, y_1)^2 + d_2(x_2, y_2)^2},$$

where d_i are the real-valued distances induced by $\mathbf{G}_i$. Nonnegativity and symmetry are immediate. Identity of indiscernibles holds since $D = 0$ implies $d_i(x_i, y_i) = 0$ so $x_i = y_i$. For the triangle inequality, write $a = d_1(x_1, z_1)$, $b = d_2(x_2, z_2)$ and $c = d_1(x_1, y_1)$, $d = d_1(y_1, z_1)$, $e = d_2(x_2, y_2)$, $f = d_2(y_2, z_2)$. Then $a \le c + d$ and $b \le e + f$ by the triangle inequalities in the factors. Using the Minkowski inequality in $\mathbb{R}^2$,

$$\sqrt{a^2 + b^2} \;\le\; \sqrt{c^2 + e^2} \;+\; \sqrt{d^2 + f^2},$$

which is exactly $D((x_1, x_2), (z_1, z_2)) \le D((x_1, x_2), (y_1, y_2)) + D((y_1, y_2), (z_1, z_2))$. Hence $\Phi_\otimes(\mathbf{G}_1, \mathbf{G}_2) \in U_{\text{geo}}$.

(SCALE) If d is a metric then sd is a metric for $s > 0$; axioms are preserved verbatim, so $\Phi_{\text{scale}(s)}(\mathbf{G}) \in U_{\text{geo}}$.

(RES) A restriction of a metric to a subset is a metric; finiteness is preserved. Thus $\Phi_{\text{res}(S)}(\mathbf{G}) \in U_{\text{geo}}$.

(PUSH) Relabeling by a bijection preserves all relations by transport of structure; hence the result lies in U_{geo}.

Naturality (isomorphism invariance). An isomorphism $\alpha : \mathbf{G} \to \mathbf{G}'$ is a bijection of carriers preserving the colored parts $\mathsf{Points}, \mathsf{Vals}$ and the ternary relation Dist. Given isomorphisms $\alpha_i : \mathbf{G}_i \to \mathbf{G}_i'$, define the induced maps on outputs componentwise: Cartesian product on point tags for (PROD), identity on values and the obvious transport on (SCALE), restriction of α for (RES), and the given relabeling for (PUSH). In each case the defining triples of Dist are carried to those of the target, so the induced maps are isomorphisms. Finally, Remark 9.3 embeds every finite Euclidean configuration into U_{geo}. □

Definition 9.8 (Iterated Meta-Geometry of depth t)**.** For $t \in \mathbb{N}$, an *Iterated Meta-Geometry of depth t* is an Iterated MetaStructure (Definition 1.3) over Σ_{geo},

$$\mathfrak{M}^{(t)}_{\text{geo}} = \big(U^{(t)}_{\text{geo}},\ (\odot^{(t)}_\ell)_{\ell \in \Lambda}\big),$$

obtained by repeatedly applying the lifting functor $\mathsf{U}_{\Sigma_{\text{geo}}}$ to $\mathbb{M}_{\text{geo}}$. Concretely, if $s < t$ and Φ_ℓ has meta-arity k_ℓ, its lift

$$\Phi^{\uparrow}_\ell :\ \big(U^{(t)}_{\text{geo}}\big)^{k_\ell} \longrightarrow U^{(t)}_{\text{geo}}$$

is defined on representatives by

$$\Phi^{\uparrow}_\ell\big(\mathsf{U}^{t-s}_{\Sigma_{\text{geo}}}(X_1), \ldots, \mathsf{U}^{t-s}_{\Sigma_{\text{geo}}}(X_{k_\ell})\big) := \ \mathsf{U}^{t-s}_{\Sigma_{\text{geo}}}\big(\,\Phi_\ell(X_1, \ldots, X_{k_\ell})\,\big),$$

and similarly for all relations. We write $\odot^{(t)}_\otimes$, $\odot^{(t)}_{\text{scale}(s)}$, $\odot^{(t)}_{\text{res}(S)}$, $\odot^{(t)}_{\text{push}(f)}$ for the lifts of (PROD), (SCALE), (RES), (PUSH).

Example 9.9 (Iterated Meta–Geometry: Multi–day itinerary planning)**. Context.** A travel planner scores entire itineraries over T days, where each day has its own set of points of interest and an associated daily distance.

Daily geometries. For each day $t \in \{1, \ldots, T\}$, let X_t be a finite set of candidate POIs (including the hotel) and encode the street-network or walking metric on X_t as $\mathbf{G}_t \in U_{\text{geo}}$. Optionally scale day-specific effort by $w_t > 0$ to reflect fatigue or time budget: $\widetilde{\mathbf{G}}_t := \Phi_{\text{scale}(w_t)}(\mathbf{G}_t)$.

Iterated product (lifted). The Iterated MetaStructure (Definition 1.3) applies the uniform product constructor layer by layer to obtain a geometry on $\prod_{t=1}^{T} X_t$ (tuples of daily choices):

$$\mathfrak{G}^{(T)} \ := \ \odot^{(T)}_\otimes\big(\widetilde{\mathbf{G}}_1,\ \widetilde{\mathbf{G}}_2,\ \ldots,\ \widetilde{\mathbf{G}}_T\big).$$

For two itineraries $\mathbf{x} = (x_1, \ldots, x_T)$ and $\mathbf{y} = (y_1, \ldots, y_T)$, the induced distance is the ℓ_2 aggregation of daily costs:

$$d_{\text{itinerary}}(\mathbf{x}, \mathbf{y}) \ = \ \sqrt{\sum_{t=1}^{T} \big(w_t\, d_t(x_t, y_t)\big)^2}\ .$$

Operations in practice.

- *Exclude closed venues on day t:* apply a restriction $\Phi_{\mathrm{res}(S_t)}(\mathbf{G}_t)$ before lifting.
- *Rename POIs due to data refresh:* apply $\Phi_{\mathrm{push}}(f_t)$ at the day level; naturality guarantees the lifted product updates coherently.
- *Reweight fatigue or weather:* modify w_t via $\Phi_{\mathrm{scale}(w_t)}$.

Thus the Iterated Meta–Geometry yields a principled metric on *whole* itineraries, built uniformly from daily geometries and stable under natural relabelings and filters.

Theorem 9.10 (Iterated Meta-Geometry is an Iterated MetaStructure and generalizes Meta-Geometry)**.** *For every $t \in \mathbb{N}$, $\mathfrak{M}^{(t)}_{\mathrm{geo}}$ of Definition 9.8 is an Iterated MetaStructure in the sense of Definition 1.3. Moreover, for $s < t$ the embedding*

$$\iota_{s\to t} : \mathfrak{M}^{(s)}_{\mathrm{geo}} \hookrightarrow \mathfrak{M}^{(t)}_{\mathrm{geo}}, \qquad X \longmapsto \mathsf{U}^{t-s}_{\Sigma_{\mathrm{geo}}}(X),$$

preserves all meta-operations:

$$\Phi^{\uparrow}_{\ell}\big(\iota_{s\to t}(X_1), \ldots, \iota_{s\to t}(X_{k_\ell})\big) = \iota_{s\to t}\big(\Phi_\ell(X_1, \ldots, X_{k_\ell})\big).$$

In particular, $\mathfrak{M}^{(0)}_{\mathrm{geo}} = \mathbb{M}_{\mathrm{geo}}$ embeds into $\mathfrak{M}^{(t)}_{\mathrm{geo}}$, so Iterated Meta-Geometry generalizes *Meta-Geometry.*

Proof. By Definition 1.3, each lifted constructor is obtained by post-composing the base constructors with $\mathsf{U}^{t-s}_{\Sigma_{\mathrm{geo}}}$, so uniformity holds at every level. If $\alpha_i : X_i \xrightarrow{\cong} Y_i$ are level-s isomorphisms, then

$$\mathsf{U}^{t-s}_{\Sigma_{\mathrm{geo}}}(\alpha_i) : \mathsf{U}^{t-s}_{\Sigma_{\mathrm{geo}}}(X_i) \xrightarrow{\cong} \mathsf{U}^{t-s}_{\Sigma_{\mathrm{geo}}}(Y_i)$$

are level-t isomorphisms, and

$$\Phi^{\uparrow}_{\ell}\big(\mathsf{U}^{t-s}\alpha_1, \ldots, \mathsf{U}^{t-s}\alpha_{k_\ell}\big) = \mathsf{U}^{t-s}\big(\Phi_\ell(\alpha_1, \ldots, \alpha_{k_\ell})\big)$$

is an isomorphism because each base Φ_ℓ is natural by Theorem 9.7. Operation preservation of $\iota_{s\to t}$ is the same identity with X_i in place of α_i. Taking $s = 0$ yields the claimed embedding of $\mathbb{M}_{\mathrm{geo}}$ into every $\mathfrak{M}^{(t)}_{\mathrm{geo}}$. □

10 Meta-Functorial Structure (Functorial Structure of Functorial Structure)

A Functorial Structure assigns to every object in a category a set, with morphisms inducing consistent pushforwards between these sets [1]. A Meta-Functorial Structure treats functorial structures themselves as objects, applying uniform meta-operations to combine or transform them while preserving naturality. An Iterated Meta-Functorial Structure repeatedly lifts Meta-Functorial Structures, forming hierarchical layers where structures of structures enable deeper categorical generalizations.

Definition 10.1 (Functorial Set)**.** [1] Let C be a category and

$$F : C \longrightarrow \mathbf{Set}$$

be a (covariant) endofunctor. For any object $X \in \mathrm{Ob}(C)$, an *F-set over X* is an element

$$s \in F(X).$$

We denote the collection of all F-sets over X simply by $F(X)$. A morphism $f : X \to Y$ in C induces a *pushforward*

$$F(f) : F(X) \longrightarrow F(Y), \qquad s \mapsto F(f)(s).$$

Example 10.2 (Power-set functor on **Set**)**.** Let $C = \mathbf{Set}$ and define $F : \mathbf{Set} \to \mathbf{Set}$ by $F(X) = \mathcal{P}(X)$ (the set of all subsets of X). For a function $f : X \to Y$, set

$$F(f) : \mathcal{P}(X) \to \mathcal{P}(Y), \qquad F(f)(S) := f[S] = \{\, f(x) \mid x \in S \,\}.$$

Then $F(\mathrm{id}_X) = \mathrm{id}_{\mathcal{P}(X)}$ and $F(g \circ f) = F(g) \circ F(f)$ (since $(g \circ f)[S] = g[f[S]]$). An F-set over X is simply a subset $S \subseteq X$.

Example 10.3 (Finite-list (free monoid) functor on **Set**)**.** Let C = **Set** and define $F(X) = X^*$, the set of all finite lists (over X), including the empty list [ ]. For a function $f : X \to Y$, let

$$F(f) : X^* \to Y^*, \qquad F(f)([x_1, \dots, x_n]) := [f(x_1), \dots, f(x_n)].$$

Clearly $F(\mathrm{id}_X) = \mathrm{id}_{X^*}$ and $F(g \circ f) = F(g) \circ F(f)$ by entrywise application. An F-set over X is a specific finite list of elements of X.

Definition 10.4 (Functorial Structure)**.** [1] Let C be a category. A *Functorial Structure* on C is simply a covariant functor

$$F : C \longrightarrow \mathbf{Set}.$$

For each object $X \in \mathrm{Ob}(C)$, an element

$$s \in F(X)$$

is called an *F-structure on X*. Every morphism $f\colon X \to Y$ in C induces a *pushforward*

$$F(f)\colon F(X) \longrightarrow F(Y), \qquad s \longmapsto F(f)(s),$$

and the usual functoriality conditions $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ and $F(g \circ f) = F(g) \circ F(f)$ hold.

Example 10.5 (Probability measures as a functor **Meas** → **Set**)**.** Let C = **Meas** (measurable spaces and measurable maps). Define $F(X, \Sigma_X) = \mathsf{Prob}(X)$, the set of all probability measures on (X, Σ_X). Given a measurable map $f : (X, \Sigma_X) \to (Y, \Sigma_Y)$, define the pushforward

$$F(f) : \mathsf{Prob}(X) \to \mathsf{Prob}(Y), \qquad (F(f)(\mu))(A) := \mu\big(f^{-1}(A)\big) \quad (A \in \Sigma_Y).$$

Then $F(\mathrm{id}) = \mathrm{id}$ and $F(g \circ f) = F(g) \circ F(f)$ by standard properties of pushforwards. An F-structure on X is a probability measure μ on X.

Example 10.6 (Path components as a functor **Top** → **Set**)**.** Let C = **Top** (topological spaces and continuous maps). Define $F(X) = \pi_0(X)$, the set of path-connected components of X. For a continuous map $f : X \to Y$, put

$$F(f) : \pi_0(X) \to \pi_0(Y), \qquad F(f)([x]) := [\, f(x)\,],$$

where $[x]$ denotes the path-component of x. This is well-defined, $F(\mathrm{id}) = \mathrm{id}$, and $F(g \circ f) = F(g) \circ F(f)$. An F-structure on X is a chosen path component of X.

Fix once and for all a small category

$$C = (\mathrm{Ob}, \mathrm{Mor}, \mathrm{dom}, \mathrm{cod}, \mathrm{id}, \circ),$$

and a concrete set-theoretic presentation of C. We work with a single-sorted, finitary signature

$$\Sigma_{\mathrm{ffs}} = \Big(\mathsf{Func} = \varnothing,\ \mathsf{Rel} = \{\mathrm{Obj}, \mathrm{Mor}, \mathrm{El}, \mathrm{Dom}, \mathrm{Cod}, \mathrm{Id}, \mathrm{Comp}, \mathrm{Base}, \mathrm{Act}\},\ \mathrm{ar}(\cdot) \Big),$$

whose relation symbols have the following intended arities and meanings:

$$\begin{aligned}
\mathrm{Obj}, \mathrm{Mor}, \mathrm{El} &\subseteq H \quad \text{(unary tags for objects, morphisms, elements);}\\
\mathrm{Dom}, \mathrm{Cod} &\subseteq H \times H \quad \text{(domain/codomain of a morphism);}\\
\mathrm{Id} &\subseteq H \times H \quad \text{(identity morphism at an object);}\\
\mathrm{Comp} &\subseteq H \times H \times H \quad \text{(composition of morphisms);}\\
\mathrm{Base} &\subseteq H \times H \quad \text{(membership of an element in a fiber over an object);}\\
\mathrm{Act} &\subseteq H \times H \times H \quad \text{(action of a morphism on elements).}
\end{aligned}$$

Definition 10.7 (Functorial-structure universe)**.** Let $U_{\mathrm{ffs}} \subseteq \mathrm{Str}_{\Sigma_{\mathrm{ffs}}}$ be the class of all Σ_{ffs}-structures

$$\mathbf{F} = \big(H;\ \mathrm{Obj}, \mathrm{Mor}, \mathrm{El}, \mathrm{Dom}, \mathrm{Cod}, \mathrm{Id}, \mathrm{Comp}, \mathrm{Base}, \mathrm{Act}\big)$$

satisfying the following axioms:

(F1) *Rigid categorical skeleton.* The substructure determined by Obj, Mor, Dom, Cod, Id, Comp is (literally) equal to the fixed presentation of C; in particular, Obj = Ob, Mor = Mor, and $\mathrm{Dom}(f,x)$ (resp. $\mathrm{Cod}(f,y)$) holds iff $\mathrm{dom}(f)=x$ (resp. $\mathrm{cod}(f)=y$); $\mathrm{Id}(x,f)$ holds iff $f=\mathrm{id}_x$; and $\mathrm{Comp}(f,g,h)$ holds iff $h=g\circ f$ in C.

(F2) *Typed elements.* $\mathrm{El}\subseteq H$ is disjoint from $\mathrm{Ob}\cup\mathrm{Mor}$. The relation $\mathrm{Base}(e,x)$ implies $e\in\mathrm{El}$ and $x\in\mathrm{Obj}$, and every $e\in\mathrm{El}$ has a unique base object $x\in\mathrm{Obj}$ with $\mathrm{Base}(e,x)$.

(F3) *Action is total and single-valued along morphisms.*
If $\mathrm{Base}(e,x)$ and $\mathrm{Dom}(f,x)$, then there exists a unique $e'\in\mathrm{El}$ and $y\in\mathrm{Obj}$ with $\mathrm{Cod}(f,y)$, $\mathrm{Base}(e',y)$, and $\mathrm{Act}(e,f,e')$. Moreover, if $\mathrm{Act}(e,f,e_1)$ and $\mathrm{Act}(e,f,e_2)$ then $e_1=e_2$.

(F4) *Functoriality (identities and composition).*
If $\mathrm{Base}(e,x)$ and $\mathrm{Id}(x,f)$ then $\mathrm{Act}(e,f,e)$.
If $\mathrm{Act}(e,f,e_1)$, $\mathrm{Act}(e_1,g,e_2)$ and $\mathrm{Comp}(f,g,h)$, then $\mathrm{Act}(e,h,e_2)$.

Remark 10.8 (Classical functor inside U_{ffs})**.** Given any functor $F:C\to\mathbf{Set}$, define

$$H\ :=\ \mathrm{Ob}\ \cup\ \mathrm{Mor}\ \cup\ \bigsqcup_{x\in\mathrm{Ob}}\big(\{x\}\times F(x)\big),$$

declare Obj = Ob, Mor = Mor, $\mathrm{El}=\bigsqcup_x\{x\}\times F(x)$ and

$$\mathrm{Base}\big((x,s),x\big)\ \text{holds},\qquad \mathrm{Act}\big((x,s),f,(y,t)\big)\iff \mathrm{dom}(f)=x,\ \mathrm{cod}(f)=y,\ t=F(f)(s).$$

Then $\mathbf{F}_F\in U_{\mathrm{ffs}}$ and the axioms (F1)–(F4) are straightforward consequences of the functor laws.

Definition 10.9 (Meta-Functorial Structure)**.** A *Meta-Functorial Structure* is a MetaStructure over Σ_{ffs},

$$\mathbb{M}_{\mathrm{ffs}}=\big(U_{\mathrm{ffs}},\ (\Phi_\ell)_{\ell\in\Lambda}\big),$$

whose meta-operations are given uniformly as follows (the categorical skeleton Ob, Mor, Dom, Cod, Id, Comp is kept literally fixed by every operation):

(PROD) Pointwise product. For $\mathbf{F}_1,\mathbf{F}_2\in U_{\mathrm{ffs}}$ put

$$\Phi_\times(\mathbf{F}_1,\mathbf{F}_2)=(H;\ \dots,\ \mathrm{El}_\times,\ \mathrm{Base}_\times,\ \mathrm{Act}_\times)$$

with the same categorical part as in (F1) and

$$\mathrm{El}_\times\ :=\ \Big\{(e_1,e_2)\in\mathrm{El}_1\times\mathrm{El}_2\ \Big|\ \exists x\in\mathrm{Obj}:\ \mathrm{Base}_1(e_1,x)\wedge\mathrm{Base}_2(e_2,x)\Big\},$$

$$\mathrm{Base}_\times\big((e_1,e_2),x\big)\iff\mathrm{Base}_1(e_1,x)\ \wedge\ \mathrm{Base}_2(e_2,x),$$
$$\mathrm{Act}_\times\big((e_1,e_2),f,(e_1',e_2')\big)\iff\mathrm{Act}_1(e_1,f,e_1')\ \wedge\ \mathrm{Act}_2(e_2,f,e_2').$$

(COPROD) Pointwise disjoint union. Define $\Phi_\sqcup(\mathbf{F}_1,\mathbf{F}_2)$ by

$$\mathrm{El}_\sqcup:=\{0\}\times\mathrm{El}_1\ \cup\ \{1\}\times\mathrm{El}_2,\quad \mathrm{Base}_\sqcup((i,e),x)\iff\begin{cases}\mathrm{Base}_1(e,x), & i=0,\\ \mathrm{Base}_2(e,x), & i=1,\end{cases}$$

$$\mathrm{Act}_\sqcup\big((i,e),f,(i,e')\big)\iff\begin{cases}\mathrm{Act}_1(e,f,e'), & i=0,\\ \mathrm{Act}_2(e,f,e'), & i=1,\end{cases}$$

and $\mathrm{Act}_\sqcup((0,\cdot),f,(1,\cdot))$ and $\mathrm{Act}_\sqcup((1,\cdot),f,(0,\cdot))$ never hold.

(RES) Fiberwise restriction by a predicate $S\subseteq\mathrm{El}$. If $S\subseteq\mathrm{El}$ is such that

$$\forall e\in S\,\forall f\in\mathrm{Mor}\,\forall e'\ \big(\mathrm{Act}(e,f,e')\Rightarrow e'\in S\big),$$

put $\Phi_{\mathrm{res}(S)}(\mathbf{F})$ by restricting El, Base, Act to S.

(PUSH) Relabeling (element renaming). If $g:\mathrm{El}\xrightarrow{\cong}\mathrm{El}'$ is a bijection such that

$$\mathrm{Base}'\big(g(e),x\big)\iff\mathrm{Base}(e,x),\qquad \mathrm{Act}'\big(g(e),f,g(e')\big)\iff\mathrm{Act}(e,f,e'),$$

define $\Phi_{\mathrm{push}(g)}(\mathbf{F})$ by transporting El, Base, Act along g.

Example 10.10 (Meta–Functorial Structure for Regional Catalog and Pricing)**. Setting.** Let C be the category whose objects are sales regions (NA, EU, APAC, . . .) and whose morphisms are logistics/routing maps (e.g., reassignment of a product from region X to region Y). Consider two functors (Definition 10.4):

$$F_{\text{cat}}, F_{\text{price}} : C \longrightarrow \mathbf{Set},$$

where for a region R:

- $F_{\text{cat}}(R)$ is the set of *active SKUs* carried in R; a morphism $f : R \to R'$ pushes SKUs forward via catalog rules (e.g., relabeling, compliance filtering), yielding $F_{\text{cat}}(f)$.
- $F_{\text{price}}(R)$ is the set of *price records* in R (SKU, currency, price); a morphism $f : R \to R'$ pushes records forward by currency conversion and SKU remapping, yielding $F_{\text{price}}(f)$.

Meta-level construction. Encode each functor as an object $\mathbf{F}_{\text{cat}}, \mathbf{F}_{\text{price}} \in U_{\text{ffs}}$ (Definition 10.9). The following uniform meta-operations (Definition 10.9) realize common multi-region workflows:

1. *Pointwise product (PROD).* The Meta–Functorial product

   $$\Phi_{\times}\big(\mathbf{F}_{\text{cat}}, \mathbf{F}_{\text{price}}\big) \in U_{\text{ffs}}$$

   assigns to each region R the set of *paired* data (SKU, price) $\in F_{\text{cat}}(R) \times F_{\text{price}}(R)$, functorially propagated along any route $f : R \to R'$.
2. *Fiberwise restriction (RES).* For a product segment $S \subseteq \text{El}$ (e.g., "eco line"), $\Phi_{\text{res}(S)}$ keeps exactly those pairs whose SKU lies in S, closed under the action along $\text{Mor}(C)$.
3. *Relabeling (PUSH).* When migrating to a new ERP, a bijection g of SKU/record identifiers induces $\Phi_{\text{push}(g)}$ that transports elements without changing the underlying category C.
4. *Pointwise coproduct (COPROD).* To aggregate two marketplaces with disjoint catalogs, $\Phi_{\sqcup}$ forms a tagged union at each region, preserving functorial pushforwards.

Outcome. The resulting object in U_{ffs} is a *Meta–Functorial Structure* that combines, filters, and relabels region-wise data in a way that is automatically consistent with all inter-region mappings (i.e., natural with respect to morphisms in C).

Theorem 10.11 (Meta-Functorial Structure is a MetaStructure and generalizes functors)**.** $\mathbb{M}_{\text{ffs}} = \big(U_{\text{ffs}}, (\Phi_\ell)\big)$ *of Definition 10.9 is a MetaStructure over* Σ_{ffs} *in the sense of Definition 1.2. Moreover, every classical functor* $F : C \to \mathbf{Set}$ *appears (via Remark 10.8) as an object of* U_{ffs}*; hence Meta-Functorial Structure* generalizes *Functorial Structure.*

Proof. Uniform constructors and closure. In each meta-operation the categorical relations Obj, Mor, Dom, Cod, Id, Comp are left unchanged (hence remain equal to the fixed presentation of C). For (PROD), typing of $\text{Base}_{\times}$ is immediate, and $\text{Act}_{\times}$ is total/single-valued because each component action is; identities and composition hold componentwise, so (F3)–(F4) are preserved. For (COPROD), disjoint tags keep fibers separated; totality and single-valuedness hold on each summand, and $\text{Act}_{\sqcup}$ never mixes tags, so functoriality holds. For (RES), the closure condition on S guarantees totality of the restricted action and preserves functoriality. For (PUSH), typing, totality, and functoriality follow by transport along the bijection g. Hence every output lies in U_{ffs}.

Naturality (isomorphism invariance). An isomorphism $\alpha : \mathbf{F} \to \mathbf{F}'$ in $\text{Str}_{\Sigma_{\text{ffs}}}$ must fix the categorical part and restrict to a bijection $\alpha : \text{El} \xrightarrow{\cong} \text{El}'$ with $\text{Base}'(\alpha(e), x) \iff \text{Base}(e, x)$ and $\text{Act}'(\alpha(e), f, \alpha(e')) \iff \text{Act}(e, f, e')$. For (PROD), define $\alpha_{\times}(e_1, e_2) := (\alpha_1(e_1), \alpha_2(e_2))$; this preserves $\text{Base}_{\times}$ and $\text{Act}_{\times}$ componentwise. For (COPROD) use $\alpha_{\sqcup}(i, e) := (i, \alpha_i(e))$. For (RES), restrict α to S; for (PUSH), take the given g. Each yields an induced isomorphism between outputs, proving naturality. The generalization claim is exactly Remark 10.8. □

Definition 10.12 (Iterated Meta-Functorial Structure of depth t)**.** For $t \in \mathbb{N}$, an *Iterated Meta-Functorial Structure of depth t* is an Iterated MetaStructure over Σ_{ffs},

$$\mathfrak{M}^{(t)}_{\text{ffs}} = \left(U^{(t)}_{\text{ffs}},\ (\odot^{(t)}_{\ell})_{\ell\in\Lambda}\right),$$

obtained by applying the lifting functor $\mathsf{U}_{\Sigma_{\text{ffs}}}$ of Definition 1.3 to $\mathbb{M}_{\text{ffs}}$ repeatedly t times. Concretely, for $s < t$ and any base meta-operation Φ_ℓ of meta-arity k_ℓ,

$$\Phi^{\uparrow}_{\ell} :\ \left(U^{(t)}_{\text{ffs}}\right)^{k_\ell} \longrightarrow U^{(t)}_{\text{ffs}}$$

is defined on representatives by

$$\Phi^{\uparrow}_{\ell}\left(\mathsf{U}^{t-s}_{\Sigma_{\text{ffs}}}(X_1), \ldots, \mathsf{U}^{t-s}_{\Sigma_{\text{ffs}}}(X_{k_\ell})\right) \ :=\ \mathsf{U}^{t-s}_{\Sigma_{\text{ffs}}}\big(\,\Phi_\ell(X_1, \ldots, X_{k_\ell})\,\big),$$

and similarly for all relations (in particular, the categorical part remains rigid at every height). We denote by $\odot^{(t)}_{\times}$, $\odot^{(t)}_{\sqcup}$, $\odot^{(t)}_{\text{res}(S)}$, and $\odot^{(t)}_{\text{push}(g)}$ the respective lifts of (PROD), (COPROD), (RES), and (PUSH).

Example 10.13 (Iterated Meta–Functorial Structure for a Multi-Tier Supply Chain)**. Setting.** Fix a base category C whose objects are supply-chain stages $\{\text{Plant}, \text{DC}, \text{Store}\}$ for each company, and whose morphisms encode admissible flows (shipment lanes, consolidation, or splits). For a given company k, consider two functors

$$F^{(k)}_{\text{inv}},\ F^{(k)}_{\text{fcst}} :\ C \to \mathbf{Set},$$

mapping each stage to its *inventory snapshots* and *demand forecasts*, with morphisms pushing data forward via routing and temporal alignment.

Level 1: Meta across signals within a company. Treat $\mathbf{F}^{(k)}_{\text{inv}}, \mathbf{F}^{(k)}_{\text{fcst}} \in U_{\text{ffs}}$. Using the meta-constructors of Definition 10.9:

1. $\Phi_\times\left(\mathbf{F}^{(k)}_{\text{inv}}, \mathbf{F}^{(k)}_{\text{fcst}}\right)$ yields a company-level Meta–Functorial object whose fiber at each stage pairs inventory with the corresponding forecast, functorially consistent along lanes.
2. $\Phi_{\text{res}(S)}$ restricts to a product family (e.g., cold chain) that is closed under lane actions.
3. $\Phi_{\text{push}}(g)$ aligns identifiers after an internal SKU renaming.

The outcome lives in U_{ffs} and represents a *height-1* meta-layer that fuses signals per company.

Level 2: Iterated meta across companies in a consortium. Suppose companies $k = 1, \ldots, K$ cooperate in a logistics consortium and expose their level-1 meta-objects. The Iterated MetaStructure lift (Definition 10.12) applies the same uniform constructors at the *next* layer:

$$\odot^{(2)}_{\times}\Big(\Phi^{\uparrow}_{\times}\big(\mathbf{F}^{(1)}_{\text{inv}}, \mathbf{F}^{(1)}_{\text{fcst}}\big),\ \ldots,\ \Phi^{\uparrow}_{\times}\big(\mathbf{F}^{(K)}_{\text{inv}}, \mathbf{F}^{(K)}_{\text{fcst}}\big)\Big)\ \in\ \mathfrak{M}^{(2)}_{\text{ffs}}.$$

Here the lift $(\,\cdot\,)^{\uparrow}$ keeps the categorical skeleton rigid while operating pointwise on the exposed element fibers, enabling:

- consortium-wide *disjoint unions* of non-overlapping assortments via $\odot^{(2)}_{\sqcup}$,
- *fiber restrictions* enforcing shared cold-chain constraints via $\odot^{(2)}_{\text{res}(S)}$,
- *identifier harmonization* across members via $\odot^{(2)}_{\text{push}}(g)$.

Outcome. The height-2 object in $\mathfrak{M}^{(2)}_{\text{ffs}}$ captures “structures of functorial structures”: it consistently combines per-company meta-fusions into a consortium-level view, with every construction remaining natural with respect to the common base category C and its morphisms.

Theorem 10.14 (Iterated Meta-Functorial Structure is an Iterated MetaStructure and generalizes Meta-Functorial)**.** *For every $t \in \mathbb{N}$, $\mathfrak{M}^{(t)}_{\mathrm{ffs}}$ of Definition 10.12 is an Iterated MetaStructure in the sense of Definition 1.3. Moreover, for $s < t$ the map*

$$\iota_{s\to t} : \mathfrak{M}^{(s)}_{\mathrm{ffs}} \hookrightarrow \mathfrak{M}^{(t)}_{\mathrm{ffs}}, \qquad X \longmapsto \mathsf{U}^{t-s}_{\Sigma_{\mathrm{ffs}}}(X),$$

is an embedding preserving all lifted meta-operations. In particular, $\mathfrak{M}^{(0)}_{\mathrm{ffs}} = \mathbb{M}_{\mathrm{ffs}}$ embeds into $\mathfrak{M}^{(t)}_{\mathrm{ffs}}$; hence Iterated Meta-Functorial Structure generalizes *Meta-Functorial Structure.*

Proof. By construction, each $\Phi^{\uparrow}_{\ell}$ is obtained by post-composing the base constructors with $\mathsf{U}^{t-s}_{\Sigma_{\mathrm{ffs}}}$, hence the lifted constructors are uniform at height t. If $\alpha_i : X_i \xrightarrow{\cong} Y_i$ are level-s isomorphisms, then $\mathsf{U}^{t-s}_{\Sigma_{\mathrm{ffs}}}(\alpha_i)$ are level-t isomorphisms and

$$\Phi^{\uparrow}_{\ell}\big(\mathsf{U}^{t-s}\alpha_1, \ldots, \mathsf{U}^{t-s}\alpha_{k_\ell}\big) = \mathsf{U}^{t-s}\big(\Phi_\ell(\alpha_1, \ldots, \alpha_{k_\ell})\big),$$

which is an isomorphism because each base Φ_ℓ is natural (Theorem 10.11). The operation-preserving property of $\iota_{s\to t}$ is the same identity with the X_i in place of the α_i. Taking $s = 0$ yields the claimed embedding of $\mathbb{M}_{\mathrm{ffs}}$ into every height t. □

11 Meta-GNNs(graph neural networks of graph neural networks)

Graph Neural Networks are models where node features are iteratively updated by aggregating neighbor information, enabling both node-level and graph-level learning [102–104]. Meta-GNNs are higher-level frameworks that treat entire GNNs as objects, applying uniform operations to generalize and integrate multiple architectures. Iterated Meta-GNNs recursively extend Meta-GNNs, forming hierarchical layers where networks of GNNs produce deeper generalized meta-level learning frameworks.

Definition 11.1 (Graph Neural Network)**.** (cf. [105–107]) Let $G = (V, E)$ be a finite simple graph and $h^{(0)} : V \to \mathbb{R}^{d_0}$ initial node features. A T-layer message-passing GNN specifies, for each $t = 0, \ldots, T-1$ and each $m \in \mathbb{N}$, a permutation-invariant *aggregator*

$$\mathrm{Agg}^{(t)}_m : (\mathbb{R}^{d_t})^m \to \mathbb{R}^{d_{t+1}} \quad \text{(i.e., } \mathrm{Agg}^{(t)}_m(x_{\sigma(1)}, \ldots, x_{\sigma(m)}) = \mathrm{Agg}^{(t)}_m(x_1, \ldots, x_m) \text{ for all } \sigma \in S_m)$$

and an *update*

$$\mathrm{Upd}^{(t)} : \mathbb{R}^{d_t} \times \mathbb{R}^{d_{t+1}} \to \mathbb{R}^{d_{t+1}}.$$

The layerwise evolution is

$$m^{(t)}_v = \mathrm{Agg}^{(t)}_{\deg(v)}\big((h^{(t)}_u)_{u\in N(v)}\big), \qquad h^{(t+1)}_v = \mathrm{Upd}^{(t)}\big(h^{(t)}_v,\, m^{(t)}_v\big),$$

yielding embeddings $h^{(T)} : V \to \mathbb{R}^{d_T}$ (optionally pooled by some $\rho_m : (\mathbb{R}^{d_T})^m \to \mathbb{R}^q$).

Fix a depth $T \in \mathbb{N}$. Consider the (single-sorted, finitary) signature

$$\Sigma_{\mathrm{gnn}} = \Big(\, \mathsf{Func},\ \mathsf{Rel} = \varnothing,\ \mathrm{ar} \Big),$$

with function symbols

$$\mathsf{Func} = \big\{\mathrm{Agg}^{(t)}_m : m \ \ (t = 0, \ldots, T-1,\ m \in \mathbb{N}), \quad \mathrm{Upd}^{(t)} : 2 \ \ (t = 0, \ldots, T-1)\big\} \ \cup \ \big\{\rho_m : m \ \ (m \in \mathbb{N})\big\}.$$

A Σ_{gnn}-structure is a tuple

$$\mathbf{G} \ = \ \big(H;\ (\mathrm{Agg}^{(t)}_m)_{t,m},\ (\mathrm{Upd}^{(t)})_t,\ (\rho_m)_m\big),$$

with carrier $H \neq \varnothing$ and each symbol interpreted as a total operation $H^{\mathrm{arity}} \to H$.

Definition 11.2 (Universe of GNN architectures)**.** Let $U_{\mathrm{gnn}} \subseteq \mathrm{Str}_{\Sigma_{\mathrm{gnn}}}$ be the class of all structures $\mathbf{G}$ whose carrier has the form

$$H \ = \ \Big(\bigsqcup_{t=0}^{T} \mathbb{R}^{d_t}\Big) \ \sqcup \ \mathbb{R}^q,$$

and whose function symbols are interpreted as the *typed* maps described above, namely:

- for each t, m, $\mathrm{Agg}_m^{(t)} : (\mathbb{R}^{d_t})^m \to \mathbb{R}^{d_{t+1}}$ is permutation-invariant in its m inputs and extended to H by identity on components not of the required type;
- for each t, $\mathrm{Upd}^{(t)} : \mathbb{R}^{d_t} \times \mathbb{R}^{d_{t+1}} \to \mathbb{R}^{d_{t+1}}$ (extended by identity elsewhere);
- for each m, $\rho_m : (\mathbb{R}^{d_T})^m \to \mathbb{R}^q$ (extended by identity elsewhere).

Given $\mathbf{G} \in U_{\mathrm{gnn}}$, the induced *forward semantics* on any finite graph $G = (V, E)$ and $h^{(0)} : V \to \mathbb{R}^{d_0}$ is the message-passing recursion above, with $\mathrm{Agg}_{\deg(v)}^{(t)}$ and $\mathrm{Upd}^{(t)}$ taken from $\mathbf{G}$, and optional readout $\rho_{|V|}$.

Definition 11.3 (Meta-GNN)**.** A *Meta-GNN* is a MetaStructure (Definition 1.2) over Σ_{gnn},

$$\mathbb{M}_{\mathrm{gnn}} = \big(U_{\mathrm{gnn}},\ (\Phi_\ell)_{\ell \in \Lambda}\big),$$

whose meta-operations are the following uniform constructors, each given by carrier- and symbol-constructors.

(PAR) Parallel/direct-sum of architectures. For $\mathbf{G}_1, \mathbf{G}_2 \in U_{\mathrm{gnn}}$ with depth T and dimensions $(d_t^{(1)})$, $(d_t^{(2)})$, set

$$\Phi_\oplus(\mathbf{G}_1, \mathbf{G}_2) \in U_{\mathrm{gnn}}$$

with carrier $H = \bigsqcup_t \mathbb{R}^{d_t^{(1)}+d_t^{(2)}} \sqcup \mathbb{R}^{q_1+q_2}$ and, blockwise for $x = (x^{(1)}, x^{(2)})$,

$$\mathrm{Agg}_m^{(t)}\big((x_i^{(1)}, x_i^{(2)})_{i=1}^m\big) := \big(\mathrm{Agg}_m^{(t,1)}(x_i^{(1)})_i,\ \mathrm{Agg}_m^{(t,2)}(x_i^{(2)})_i\big),$$

$$\mathrm{Upd}^{(t)}\big((h^{(1)}, h^{(2)}), (m^{(1)}, m^{(2)})\big) := \big(\mathrm{Upd}^{(t,1)}(h^{(1)}, m^{(1)}),\ \mathrm{Upd}^{(t,2)}(h^{(2)}, m^{(2)})\big),$$

$$\rho_m\big((z_i^{(1)}, z_i^{(2)})_i\big) := \big(\rho_m^{(1)}(z_i^{(1)})_i,\ \rho_m^{(2)}(z_i^{(2)})_i\big).$$

(SER) Serial/stacked composition. If $\mathbf{G}_1, \mathbf{G}_2 \in U_{\mathrm{gnn}}$ with depths T_1, T_2 satisfy $d_{T_1}^{(1)} = d_0^{(2)}$, define

$$\Phi_{\mathrm{stack}}(\mathbf{G}_1, \mathbf{G}_2) \in U_{\mathrm{gnn}}$$

by concatenating layers: the depth is $T_1 + T_2$ and $(\mathrm{Agg}_m^{(t)}, \mathrm{Upd}^{(t)})$ are those of $\mathbf{G}_1$ for $t < T_1$ and of $\mathbf{G}_2$ reindexed for $t \geq T_1$ (same for ρ).

(PUSH) Feature-space relabeling (change of basis). For invertible linear maps $A_t : \mathbb{R}^{d_t} \xrightarrow{\cong} \mathbb{R}^{d_t}$ and $B : \mathbb{R}^q \xrightarrow{\cong} \mathbb{R}^q$, define

$$\Phi_{\mathrm{push}}(A_\bullet, B)(\mathbf{G}) \in U_{\mathrm{gnn}}$$

on the same carrier with symbols conjugated by $A_\bullet, B$:

$$\mathrm{Agg}_m^{(t)\,\prime} = A_{t+1} \circ \mathrm{Agg}_m^{(t)} \circ (A_t^{-1})^{\times m}, \qquad \mathrm{Upd}^{(t)\,\prime} = A_{t+1} \circ \mathrm{Upd}^{(t)} \circ (A_t^{-1} \times A_{t+1}^{-1}),$$

$$\rho_m' = B \circ \rho_m \circ (A_T^{-1})^{\times m}.$$

(TRUNC) Depth truncation. For $K \leq T$, let $\Phi_{\leq K}(\mathbf{G})$ be the architecture obtained by keeping layers $0, \ldots, K-1$ and replacing ρ_m by any map $\tilde{\rho}_m : (\mathbb{R}^{d_K})^m \to \mathbb{R}^q$ (e.g. original ρ precomposed with the removed layers).

Example 11.4 (Meta-GNN for Cross-Store Product Recommendation)**. Setting.** Consider a retail company operating two online stores, S_1 and S_2. For each store S_i we build a user–item interaction graph $G_i = (V_i, E_i)$ whose nodes carry feature vectors (e.g., user profiles, item attributes, and interaction statistics). A baseline GNN $\mathbf{G}_i \in U_{\mathrm{gnn}}$ is trained on G_i to produce recommendation embeddings and scores.

Meta-level construction. To transfer knowledge and make *cross-store* recommendations, we combine the two store-specific GNNs by a Meta-GNN:

1. *Parallel/direct sum of architectures.* Use $\Phi_{\oplus}(\mathbf{G}_1, \mathbf{G}_2)$ to form a blockwise model that preserves both representation pipelines while sharing higher-level supervision (e.g., a unified ranking loss).

2. *Feature-space relabeling (change of basis).* Because the hidden dimensions and bases may differ, apply $\Phi_{\text{push}}(A_{\bullet}, B)$ with invertible linear maps A_t and B to align latent spaces across stores before joint training and inference.

3. *Serial stacking.* Build a lightweight "broker" head $\mathbf{G}_{\text{broker}}$ that consumes the aligned store embeddings on a *store-overlap graph H* (nodes: stores, edges weighted by shared customers or catalog similarity). Then stack via
$$\Phi_{\text{stack}}\Big(\Phi_{\oplus}(\mathbf{G}_1, \mathbf{G}_2),\ \mathbf{G}_{\text{broker}}\Big) \in U_{\text{gnn}},$$
so cross-store signals propagate to final scores.

4. *Depth truncation for edge deployment.* For on-device recommendations, export a smaller variant $\Phi_{\le K}(\,\cdot\,)$ that keeps the first K layers and a compact readout $\tilde{\rho}$.

Outcome. The resulting Meta-GNN integrates two *base* GNNs into a single architecture that aligns features, shares information through a broker layer, and yields improved cold-start and long-tail recommendations across S_1 and S_2.

Theorem 11.5 (Meta-GNN is a MetaStructure and generalizes GNNs)**.** $\mathbb{M}_{\text{gnn}} = \big(U_{\text{gnn}}, (\Phi_\ell)\big)$ *of Definition 11.3 is a MetaStructure in the sense of Definition 1.2. Moreover, every classical message-passing GNN (sum/mean/attention-type) appears as an object of* U_{gnn}*, hence Meta-GNNs* generalize *GNNs.*

Proof. Uniform constructors and closure. For (PAR) the carrier is a disjoint union of Euclidean spaces, hence of the required form. Blockwise definitions preserve permutation-invariance of $\text{Agg}_m^{(t)}$, totality and typing; updates and readouts are componentwise, so $\Phi_{\oplus}(\mathbf{G}_1, \mathbf{G}_2) \in U_{\text{gnn}}$. For (SER) concatenation simply reindexes the given families $(\text{Agg}_m^{(t)}, \text{Upd}^{(t)}, \rho_m)$, so the result lies in U_{gnn}. For (PUSH) conjugation by linear isomorphisms preserves permutation-invariance and typing, hence the pushed structure is again in U_{gnn}. For (TRUNC) the remaining symbols are a subsequence of the originals; choosing any $\tilde{\rho}_m$ yields a valid member of U_{gnn}.

Naturality (isomorphism invariance). An isomorphism $\alpha : \mathbf{G} \xrightarrow{\cong} \mathbf{G}'$ in $\text{Str}_{\Sigma_{\text{gnn}}}$ is precisely a family of linear bijections $(A_t)_t$ and B on the tagged carrier that conjugate all symbols as in (PUSH). Given isomorphisms $\alpha_i : \mathbf{G}_i \to \mathbf{G}_i'$ $(i = 1, 2)$, the induced maps
$$\alpha_{\oplus} := (A_{\bullet}^{(1)} \oplus A_{\bullet}^{(2)},\ B^{(1)} \oplus B^{(2)}), \qquad \alpha_{\text{stack}} := (A_{\bullet}^{(1)} \parallel A_{\bullet}^{(2)},\ B^{(2)}),$$
and $\alpha_{\le K}$, α_{push} conjugate the corresponding symbols componentwise, yielding Σ_{gnn}-isomorphisms
$$\Phi_\ell(\alpha_1, \alpha_2) :\ \Phi_\ell(\mathbf{G}_1, \mathbf{G}_2) \xrightarrow{\cong} \Phi_\ell(\mathbf{G}_1', \mathbf{G}_2').$$
Hence each Φ_ℓ is natural.

Generalization. Any standard MPNN is obtained by choosing $\text{Agg}_m^{(t)}(x_1, \dots, x_m) = \sigma\big(W^{(t)} \sum_i x_i\big)$ (sum/mean variants similarly) and $\text{Upd}^{(t)}(h, m) = \sigma\big(U^{(t)} h + V^{(t)} m\big)$, with ρ_m any permutation-invariant pooling; these are permutation-invariant and fit U_{gnn}. Attention-type layers are also included by letting $\text{Agg}_m^{(t)}$ compute $\sum_i \alpha_i(x_1, \dots, x_m)\, W x_i$ with α_i symmetric in the multiset arguments (e.g. via DeepSets), which is again permutation-invariant. Thus classical GNNs embed in U_{gnn}. □

Definition 11.6 (Iterated Meta-GNN of depth t)**.** For $t \in \mathbb{N}$, an *Iterated Meta-GNN of depth t* is an Iterated MetaStructure (Definition 1.3) over Σ_{gnn},
$$\mathfrak{M}_{\text{gnn}}^{(t)} = \big(U_{\text{gnn}}^{(t)},\ (\odot_\ell^{(t)})_{\ell \in \Lambda}\big),$$
obtained by repeatedly applying the lifting functor $\mathsf{U}_{\Sigma_{\text{gnn}}}$ to $\mathbb{M}_{\text{gnn}}$. Concretely, for $s < t$ and any base meta-operation Φ_ℓ of meta-arity k_ℓ,
$$\Phi_\ell^{\uparrow} :\ (U_{\text{gnn}}^{(t)})^{k_\ell} \longrightarrow U_{\text{gnn}}^{(t)}, \quad \Phi_\ell^{\uparrow}\big(\mathsf{U}_{\Sigma_{\text{gnn}}}^{t-s}(X_1), \dots, \mathsf{U}_{\Sigma_{\text{gnn}}}^{t-s}(X_{k_\ell})\big) := \mathsf{U}_{\Sigma_{\text{gnn}}}^{t-s}\big(\Phi_\ell(X_1, \dots, X_{k_\ell})\big),$$
and similarly for all symbol interpretations.

Example 11.7 (Iterated Meta-GNN for Multi-Scale Traffic Control)**. Setting.** A transportation authority manages traffic at three scales: *(i)* intersections within a city, *(ii)* the citywide road network, *(iii)* a regional network of cities. Graphs exist at each scale, with sensors, lanes, or zones providing node/edge features.

Level 0 (base GNNs). For each intersection v we form a micro-graph G_v^{int} (detectors, phases, lanes) and train a GNN $\mathbf{G}_v^{\mathrm{int}} \in U_{\mathrm{gnn}}$ that outputs a control embedding z_v and suggested signal timings.

Level 1 (Meta-GNN over intersections). Construct a city graph G^{city} whose nodes are intersections and edges connect physically adjacent or flow-coupled junctions. Treat the collection $\{\mathbf{G}_v^{\mathrm{int}}\}_v$ as inputs to a Meta-GNN:

1. Align local embeddings with $\Phi_{\mathrm{push}}(A_\bullet, B)$ so z_v share a common latent basis.
2. Aggregate the aligned models in parallel via repeated $\Phi_\oplus$ (folded over intersections) to preserve local specializations while exposing them to city-level coordination.
3. Stack a city controller $\mathbf{G}^{\mathrm{city}}$ with Φ_{stack} to propagate congestion signals and produce citywide control actions (e.g., offsets, green waves).

This yields a height-1 object in $\mathfrak{M}^{(1)}_{\mathrm{gnn}}$.

Level 2 (Iterated Meta-GNN over cities). Now consider several cities $C_1, \ldots, C_M$ connected by highways in a regional graph G^{reg} (nodes: cities, edges weighted by intercity traffic flow). Each city C_j already hosts a level-1 Meta-GNN as above. We lift once more (Definition 11.6) to form a height-2 Iterated Meta-GNN:

$$\Phi^{\uparrow}_{\mathrm{stack}}\Big(\underbrace{\Phi^{\uparrow}_{\oplus}\big(\mathbf{G}^{\mathrm{city}}_{C_1}, \ldots, \mathbf{G}^{\mathrm{city}}_{C_M}\big)}_{\text{parallel composition at meta-level}},\ \mathbf{G}^{\mathrm{reg}} \Big) \in \mathfrak{M}^{(2)}_{\mathrm{gnn}},$$

coordinating ramp metering, incident response, and diversion at the regional scale.

Outcome. The pipeline realizes *Iterated* Meta-GNNs: base GNNs at intersections (level 0), a Meta-GNN coordinating a city (level 1), and a second lift coordinating multiple cities (level 2). Each lift uses the uniform constructors ($\Phi_\oplus$, Φ_{push}, Φ_{stack}, $\Phi_{\le K}$) and remains within the Iterated MetaStructure $\mathfrak{M}^{(t)}_{\mathrm{gnn}}$.

Theorem 11.8 (Iterated Meta-GNN is an Iterated MetaStructure and generalizes Meta-GNN)**.** *For every $t \in \mathbb{N}$, $\mathfrak{M}^{(t)}_{\mathrm{gnn}}$ in Definition 11.6 is an Iterated MetaStructure (Definition 1.3). Moreover, for $s < t$ the embedding*

$$\iota_{s\to t} : \mathfrak{M}^{(s)}_{\mathrm{gnn}} \hookrightarrow \mathfrak{M}^{(t)}_{\mathrm{gnn}}, \qquad X \longmapsto \mathsf{U}^{t-s}_{\Sigma_{\mathrm{gnn}}}(X),$$

preserves all lifted meta-operations, so Iterated Meta-GNNs generalize *Meta-GNNs.*

Proof. By Definition 1.3, each lifted constructor $\Phi^{\uparrow}_{\ell}$ is obtained by post-composing the base constructors of Φ_ℓ with $\mathsf{U}^{t-s}_{\Sigma_{\mathrm{gnn}}}$, hence uniform at level t. If $\alpha_i : X_i \xrightarrow{\cong} Y_i$ are level-s isomorphisms, then $\mathsf{U}^{t-s}_{\Sigma_{\mathrm{gnn}}}(\alpha_i)$ are level-t isomorphisms and

$$\Phi^{\uparrow}_{\ell}\big(\mathsf{U}^{t-s}\alpha_1, \ldots, \mathsf{U}^{t-s}\alpha_{k_\ell}\big) = \mathsf{U}^{t-s}\big(\Phi_\ell(\alpha_1, \ldots, \alpha_{k_\ell})\big),$$

which is an isomorphism since each base Φ_ℓ is natural by Theorem 11.5. The claimed operation-preservation of $\iota_{s\to t}$ is the same identity with X_i in place of α_i. Taking $s = 0$ shows $\mathbb{M}_{\mathrm{gnn}}$ embeds into every $\mathfrak{M}^{(t)}_{\mathrm{gnn}}$. □

12 Conclusion

In this work, we developed extensions of concepts such as Cube, HyperCube, Matrix, Decision-Making, Neural Networks, Geometry, and Functions within the settings of MetaStructure and Iterated MetaStructure. In future work, we aim to explore the relationships between the concepts of MetaStructure and Iterated MetaStructure and other frameworks such as Fuzzy Sets [108–110], Intuitionistic Fuzzy Sets [111–113], Plithogenic Sets [114–117], Neutrosophic Sets [118–120], and Quadripartitioned Neutrosophic Sets [121–123], including possible extensions incorporating these structures.

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Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Research Integrity

The authors hereby confirm that, to the best of their knowledge, this manuscript is their original work, has not been published in any other journal, and is not currently under consideration for publication elsewhere at this stage.

Use of Generative AI and AI-Assisted Tools

I use generative AI and AI-assisted tools for tasks such as English grammar checking, and I do not employ them in any way that violates ethical standards.

Disclaimer (Note on Computational Tools)

No computer-assisted proof, symbolic computation, or automated theorem proving tools (e.g., Mathematica, SageMath, Coq, etc.) were used in the development or verification of the results presented in this paper. All proofs and derivations were carried out manually and analytically by the authors.

Code Availability

No code or software was developed for this study.

Clinical Trial

This study did not involve any clinical trials.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

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