New Concepts of MetaStructures: Algebra, Topology, Lattices, Queues, Markov Chains, and Intervals

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Abstract

A *MetaStructure* is a higher-level framework that treats entire collections of structures as single objects, equipped with natural operations that preserve isomorphisms across different domains. The term “Structure” here refers broadly to mathematical systems as well as real-world models. An *Iterated MetaStructure* generalizes this idea recursively, generating successive layers in which structures of structures form deeper hierarchical meta-levels. In this work, we extend and investigate the properties of Algebra, Topology, Lattices, Queues, Markov Chains, and Intervals through the lens of MetaStructures and Iterated MetaStructures.

Keywords: MetaStructure, Iterated MetaStructure, Algebra, Topology, Lattices, Queues, Markov Chains, Interval

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The remainder of this paper is organized as follows. This paper introduces several examples of MetaStructures. Here, the term “Structure” refers to any kind of structure, whether it originates from mathematical theory or from real-world systems.

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1 Preliminaries

This section presents the fundamental concepts and definitions that underpin the discussions in this paper. Throughout this paper, all structures and sets are assumed to be finite.

1.1 Classical Structure

In this paper, the term *Structure* refers broadly to a mathematical system, not restricted to a single area, but encompassing domains such as Set Theory, Logic, Probability, Statistics, Algebra, and Geometry.

Definition 1.1 (Classical Structure)**.** (cf. [1,2]) A *Classical Structure* C is a mathematical object arising from a traditional field—for example Set Theory, Logic, Probability, Statistics, Algebra, Geometry, Graph Theory, Automata Theory, or Game Theory. Formally, it may be represented as a pair

$$C = \big(H,\ \{\#^{(m)}\}_{m\in\mathcal{I}}\big),$$

where:

- H is a nonempty set, often called the *carrier* or *universe*.

- For each $m \in \mathcal{I} \subseteq \mathbb{Z}_{>0}$, there exists an m-ary operation

$$\#^{(m)} : H^m \longrightarrow H,$$

subject to appropriate *axioms* (such as associativity, commutativity, or identity laws), which vary according to the chosen type of structure.

The collection $\{\#^{(m)} : m \in \mathcal{I}\}$ determines the *type* of C. Representative examples include:

- A *Set* $(S, \ \emptyset)$, consisting solely of a carrier with distinguished elements or relations, but without operations [3,4].
- A *Logic* structure $(L, \ \wedge, \vee, \neg)$, where $\wedge, \vee$ are binary connectives and $\neg$ a unary connective, satisfying logical axioms [5].
- A *Probability* model $(\Omega, \mathcal{F}, P)$, where $P : \mathcal{F} \to [0,1]$ is a probability measure on a sigma-algebra $\mathcal{F} \subseteq \mathcal{P}(\Omega)$ [6,7].
- A *Statistical* model $(X, \mathcal{A}, \theta)$, where θ maps data X into parameters of interest [8,9].
- *Algebraic* structures such as:
  - A *Group* $(G, *)$, with $* : G \times G \to G$ satisfying associativity, identity, and inverses [10,11].
  - A *Ring* $(R, +, \times)$, with two binary operations fulfilling ring axioms [12,13].
  - A *Vector Space* $(V, +, \cdot)$ over a field $\mathbb{F}$, with scalar multiplication $\cdot : \mathbb{F} \times V \to V$ [14–16].
- A *Geometric* structure (X, dist), where $\mathrm{dist} : X \times X \to \mathbb{R}$ satisfies the axioms of a metric.
- A *Graph* (V, E), where $E \subseteq \{\{u, v\} \mid u, v \in V\}$ for undirected graphs, or $E \subseteq V \times V$ for directed graphs, with adjacency and incidence relations [17–20].
- An *Automaton* $(Q, \Sigma, \delta, q_0, F)$, where Q is a set of states, Σ an input alphabet, $\delta : Q \times \Sigma \to Q$ the transition function, $q_0 \in Q$ the start state, and $F \subseteq Q$ the accepting states [21,22].
- A *Game* $(N, \{A_i\}, \{u_i\})$, where N is the set of players, A_i each player's action set, and $u_i : \prod_{j \in N} A_j \to \mathbb{R}$ the payoff function for player i [23,24].

Related concepts include the *HyperStructure* [25–28] and the *SuperHyperStructure* [29–32], which have also been extensively investigated in recent studies.

1.2 MetaStructure (Structure of a Structure)

Fix once and for all a single–sorted, finitary *signature*

$$\Sigma \ = \ \big(\mathsf{Func}, \ \mathsf{Rel}, \ \mathrm{ar}_{\mathsf{Func}}, \ \mathrm{ar}_{\mathsf{Rel}}\big),$$

where Func (resp. Rel) is a set of function (resp. relation) symbols and ar records their arities. A (single–sorted) Σ–*structure* is a tuple

$$\mathbf{C} \ = \ \big(H, \ (f^{\mathbf{C}})_{f \in \mathsf{Func}}, \ (R^{\mathbf{C}})_{R \in \mathsf{Rel}}\big),$$

consisting of a nonempty carrier H, together with interpretations $f^{\mathbf{C}} : H^m \to H$ for each $f \in \mathsf{Func}$ of arity m, and relations $R^{\mathbf{C}} \subseteq H^r$ for each $R \in \mathsf{Rel}$ of arity r. Let Str_Σ denote the class of all such structures.

Definition 1.2 (MetaStructure over a fixed signature)**.** (cf. [33]) With Σ as above, a *MetaStructure* (a "structure of structures") is a pair

$$\mathbb{M} \ = \ \big(U, \ (\Phi_\ell)_{\ell \in \Lambda}\big),$$

where:

- $U \subseteq \mathrm{Str}_\Sigma$ is a nonempty collection of Σ–structures (the level–0 objects);

- for each label $\ell \in \Lambda$ with *meta–arity* $k_\ell \in \mathbb{N}$, the *meta–operation*

  $$\Phi_\ell : U^{k_\ell} \longrightarrow U$$

  is described by uniform *constructors* acting on carriers and symbol interpretations:

  $$\begin{aligned} \Gamma_\ell(\mathbf{C}_1, \dots, \mathbf{C}_{k_\ell}) &= H_\ell && \text{(functorially built carrier);} \\ \forall f \in \mathsf{Func} : \ f^{\Phi_\ell(\mathbf{C}_1,\dots,\mathbf{C}_{k_\ell})} &= \Lambda_\ell^f\big(f^{\mathbf{C}_1}, \dots, f^{\mathbf{C}_{k_\ell}}\big); \\ \forall R \in \mathsf{Rel} : \ R^{\Phi_\ell(\mathbf{C}_1,\dots,\mathbf{C}_{k_\ell})} &= \Xi_\ell^R\big(R^{\mathbf{C}_1}, \dots, R^{\mathbf{C}_{k_\ell}}\big), \end{aligned}$$

  where the recipes Λ_ℓ^f and Ξ_ℓ^R depend only on f, R and ℓ (not on the particular representatives) and produce the output interpretations over H_ℓ.

Each Φ_ℓ is required to be *isomorphism–invariant* (natural): if $\alpha_i : \mathbf{C}_i \xrightarrow{\cong} \mathbf{D}_i$ are isomorphisms for $1 \le i \le k_\ell$, then there is an induced isomorphism

$$\Phi_\ell(\alpha_1, \dots, \alpha_{k_\ell}) \ : \ \Phi_\ell(\mathbf{C}_1, \dots, \mathbf{C}_{k_\ell}) \xrightarrow{\cong} \Phi_\ell(\mathbf{D}_1, \dots, \mathbf{D}_{k_\ell}),$$

compatible with all function and relation symbols of Σ.

1.3 Iterated MetaStructure (Structure of Structure of $\cdots$ of Structure)

An *Iterated MetaStructure* is obtained by repeatedly applying the MetaStructure construction, thereby forming successive levels where "structures of structures" build a hierarchical tower (cf. [33–36]).

Definition 1.3 (Iterated MetaStructure of depth t)**.** (cf. [33]) For $t \in \mathbb{N}$, an *Iterated MetaStructure of depth t* over Σ is a MetaStructure $\mathfrak{M}^{(t)}$ obtained by t iterations of a lifting procedure. When $s < t$, we *lift* a height–s MetaStructure $\mathfrak{M}^{(s)} = (U^{(s)}, \{\odot_i\}, \{\mathcal{S}_j\})$ to height t by

$$\iota_{s\to t} \ : \ U^{(s)} \xrightarrow{\mathsf{U}_\Sigma^{t-s}} U^{(t)} \ := \ \mathsf{U}_\Sigma^{t-s}\big(U^{(s)}\big),$$

and, for each meta–operation $\odot_i : (\mathsf{E}_\Sigma^{m_i})^{k_i} \to \mathcal{P}^{n_i}(\mathsf{E}_\Sigma^{n_i})$, define its lift by

$$\odot_i^{\uparrow} \ : \ \big(\mathsf{E}_\Sigma^{m_i+t-s}\big)^{k_i} \longrightarrow \mathcal{P}^{n_i}\big(\mathsf{E}_\Sigma^{n_i+t-s}\big), \qquad \odot_i^{\uparrow}\big(\mathsf{U}_\Sigma^{t-s}(x_1), \dots, \mathsf{U}_\Sigma^{t-s}(x_{k_i})\big) := \mathsf{U}_\Sigma^{t-s}\big(\odot_i(x_1, \dots, x_{k_i})\big),$$

and analogously for relations $\mathcal{S}_j^{\uparrow} := \big(\mathsf{U}_\Sigma^{t-s}\big)^{\times \ell_j}(\mathcal{S}_j)$.

2 MetaAlgebra: Algebra of Algebra

An algebra is a vector space over a field equipped with bilinear operations, generalizing structures like groups, rings, and fields (cf. [37–40]). A Meta-Algebra is a MetaStructure whose objects are algebras, with meta-operations combining, transforming, or restricting them while preserving isomorphisms. An Iterated Meta-Algebra recursively applies Meta-Algebra construction, forming higher-level hierarchies of algebraic systems, generalizing and extending algebras across multiple meta-layers.

Definition 2.1 (Algebra)**.** (cf. [12, 13]) An *algebra* over a field $\mathbb{F}$ is a vector space A over $\mathbb{F}$ equipped with a bilinear multiplication

$$\cdot \ : \ A \times A \ \longrightarrow \ A,$$

such that for all $\alpha, \beta \in \mathbb{F}$ and $x, y, z \in A$,

$$(\alpha x + \beta y) \cdot z \ = \ \alpha(x \cdot z) + \beta(y \cdot z), \qquad z \cdot (\alpha x + \beta y) \ = \ \alpha(z \cdot x) + \beta(z \cdot y).$$

(Associativity or a unit may be imposed as additional axioms when desired, but are not required here.)

Fix a field $\mathbb{F}$. Consider the single-sorted, finitary *algebra signature*

$$\Sigma_{\mathrm{alg}} = \Big(\,\mathsf{Func},\ \mathsf{Rel} = \varnothing,\ \mathrm{ar}\Big), \qquad \mathsf{Func} = \{\, + : 2,\ 0 : 0,\ - : 1,\ m_\alpha : 1\ (\alpha \in \mathbb{F}),\ \ \cdot : 2\,\}.$$

A Σ_{alg}-structure is a tuple

$$\mathbf{A} = \big(H;\ +^{\mathbf{A}},\ 0^{\mathbf{A}},\ (-)^{\mathbf{A}},\ (m_\alpha^{\mathbf{A}})_{\alpha\in\mathbb{F}},\ \cdot^{\mathbf{A}}\big)$$

with carrier $H \neq \varnothing$ and total interpretations of the symbols.

Definition 2.2 (Universe of (not-necessarily-associative) $\mathbb{F}$-algebras)**.** Let $U_{\mathrm{alg}} \subseteq \mathrm{Str}_{\Sigma_{\mathrm{alg}}}$ consist of those $\mathbf{A}$ for which:

(A1) $(H, +, 0, -, (m_\alpha)_{\alpha\in\mathbb{F}})$ is a (left) $\mathbb{F}$-vector space;

(A2) $\cdot : H \times H \longrightarrow H$ is bilinear, i.e., for all $x, y, z \in H$ and $\alpha, \beta \in \mathbb{F}$,

$$(\alpha x + \beta y)\cdot z = \alpha(x\cdot z) + \beta(y\cdot z), \qquad z\cdot(\alpha x + \beta y) = \alpha(z\cdot x) + \beta(z\cdot y).$$

Remark 2.3 (Classical algebras as Σ_{alg}-structures)**.** Every usual $\mathbb{F}$-algebra $(A, +, 0, -, (m_\alpha), \cdot)$ (associative or not; unital or not) yields $\mathbf{A} \in U_{\mathrm{alg}}$ by taking $H := A$ and interpreting symbols as the given operations. If one wishes to restrict to associative (resp. unital) algebras, add the identities $(x\cdot y)\cdot z = x\cdot(y\cdot z)$ (resp. $\exists 1 \in H :\ 1\cdot x = x = x\cdot 1$) as axioms on U_{alg}; all constructions below preserve these properties.

Definition 2.4 (Meta-Algebra)**.** A *Meta-Algebra* is a MetaStructure (Definition 1.2) over Σ_{alg}

$$\mathbb{M}_{\mathrm{alg}} = \big(U_{\mathrm{alg}},\ (\Phi_\ell)_{\ell\in\Lambda}\big),$$

whose meta-operations are given uniformly by the following standard algebraic constructions (each specified by carrier- and symbol-constructors as in Definition 1.2):

(DS) Direct-sum algebra. For $\mathbf{A} = (A, \dots),\ \mathbf{B} = (B, \dots) \in U_{\mathrm{alg}}$ define

$$\Phi_\oplus(\mathbf{A}, \mathbf{B}) := \big(A \oplus B,\ +_\oplus,\ 0_\oplus,\ -_\oplus,\ (m_\alpha^\oplus),\ \cdot_\oplus\big),$$

on the carrier $A \oplus B := A \times B$ with

$$(u_1, v_1) +_\oplus (u_2, v_2) = (u_1 + u_2,\ v_1 + v_2), \quad 0_\oplus = (0, 0), \quad -(u, v) = (-u, -v), \quad m_\alpha^\oplus(u, v) = (\alpha u, \alpha v),$$
$$(u_1, v_1)\ \cdot_\oplus\ (u_2, v_2) := (u_1\cdot u_2,\ v_1\cdot v_2).$$

(TENS) Tensor-product algebra.

$$\Phi_\otimes(\mathbf{A}, \mathbf{B}) := \big(A \otimes_{\mathbb{F}} B,\ +,\ 0,\ -,\ (m_\alpha),\ \cdot_\otimes\big),$$

where the carrier is the tensor product vector space, and the multiplication is determined on simple tensors by

$$(a \otimes b)\ \cdot_\otimes\ (a' \otimes b')\ :=\ (a\cdot a')\ \otimes\ (b\cdot b'),$$

and extended bilinearly to $A \otimes B$.

(OP) Opposite algebra.

$$\Phi_{\mathrm{op}}(\mathbf{A}) := \big(A,\ +,\ 0,\ -,\ (m_\alpha),\ \cdot^{\mathrm{op}}\big), \qquad x \cdot^{\mathrm{op}} y := y\cdot x.$$

Example 2.5 (Meta-Algebra in practice: block–diagonal coupling of independent subsystems)**.** Let $\mathbf{A}_1 = (M_2(\mathbb{R}), +, \cdot)$ and $\mathbf{A}_2 = (M_3(\mathbb{R}), +, \cdot)$ be matrix algebras over $\mathbb{R}$ representing two decoupled linear subsystems of state dimensions 2 and 3 (e.g., a thermal loop and a vibration loop in a device).

Applying the Meta-Algebra direct-sum constructor (DS) from Definition 2.4 gives

$$\Phi_\oplus(\mathbf{A}_1, \mathbf{A}_2)\ =\ \big(M_2(\mathbb{R}) \oplus M_3(\mathbb{R}),\ +_\oplus, \cdot_\oplus\big),$$

with $(A_1, B_1)\cdot_\oplus(A_2, B_2) = (A_1A_2,\ B_1B_2)$. The map

$$\iota :\ M_2(\mathbb{R}) \oplus M_3(\mathbb{R}) \longrightarrow M_5(\mathbb{R}), \qquad \iota(A, B) = \mathrm{diag}(A, B),$$

is an injective algebra homomorphism, identifying the direct sum with the block–diagonal subalgebra of $M_5(\mathbb{R})$. *Real-life interpretation.* The combined plant obtained by $\Phi_\oplus$ acts on the stacked state $(x_{\mathrm{therm}}, x_{\mathrm{vib}}) \in \mathbb{R}^2\times\mathbb{R}^3$, and block–diagonal multiplication encodes that the two subsystems operate independently while being represented within a single algebraic object.

Theorem 2.6 (Meta-Algebra is a MetaStructure and generalizes algebras)**.** *$\mathbb{M}_{\mathrm{alg}} = \big(U_{\mathrm{alg}}, (\Phi_\ell)\big)$ in Definition 2.4 is a MetaStructure over Σ_{alg}. Moreover, every classical $\mathbb{F}$-algebra appears as an object of U_{alg}; hence Meta-Algebra* generalizes *algebras.*

Proof. Uniform constructors and closure. (DS): The carrier $A \oplus B$ is a vector space with componentwise operations; $\cdot_\oplus$ is bilinear in each argument since each component product is bilinear. Thus $\Phi_\oplus(\mathbf{A}, \mathbf{B}) \in U_{\mathrm{alg}}$. If A, B are associative or unital, the direct-sum product preserves these properties componentwise.

(TENS): The tensor product carrier is a vector space. The rule $(a \otimes b) \cdot_\otimes (a' \otimes b') = (a \cdot a') \otimes (b \cdot b')$ respects the bilinear relations that define $A \otimes B$, hence extends uniquely to a bilinear map $(-) \cdot_\otimes (-) : (A \otimes B) \times (A \otimes B) \to A \otimes B$. Therefore $\Phi_\otimes(\mathbf{A}, \mathbf{B}) \in U_{\mathrm{alg}}$. If A, B are associative (resp. unital), then

$$\big((a \otimes b) \cdot_\otimes (a' \otimes b')\big) \cdot_\otimes (a'' \otimes b'') = ((aa')a'') \otimes ((bb')b'')$$

equals

$$(a \otimes b) \cdot_\otimes \big((a' \otimes b') \cdot_\otimes (a'' \otimes b'')\big) = a(a'a'') \otimes b(b'b'')$$

by associativity in A, B; similarly $1_A \otimes 1_B$ is a unit when units exist.

(OP): $(\cdot^{\mathrm{op}})$ is bilinear since

$$(\alpha x + \beta y) \cdot^{\mathrm{op}} z = z \cdot (\alpha x + \beta y) = \alpha(z \cdot x) + \beta(z \cdot y) = \alpha(x \cdot^{\mathrm{op}} z) + \beta(y \cdot^{\mathrm{op}} z),$$

and similarly in the right argument; hence $\Phi_{\mathrm{op}}(\mathbf{A}) \in U_{\mathrm{alg}}$. Associativity and unitality are preserved under reversal.

Naturality (isomorphism invariance). Let $\alpha : \mathbf{A} \xrightarrow{\cong} \mathbf{A}'$ and $\beta : \mathbf{B} \xrightarrow{\cong} \mathbf{B}'$ be Σ_{alg}-isomorphisms (i.e., linear bijections preserving $+, 0, -, m_\alpha$ and $\cdot$).

(DS): Define $\alpha \oplus \beta : A \oplus B \to A' \oplus B'$ by $(\alpha \oplus \beta)(u, v) := (\alpha u, \beta v)$. It preserves the vector-space structure componentwise and

$$(\alpha \oplus \beta)\big((u_1, v_1) \cdot_\oplus (u_2, v_2)\big) = (\alpha(u_1 \cdot u_2),\ \beta(v_1 \cdot v_2)) = (\alpha u_1 \cdot \alpha u_2,\ \beta v_1 \cdot \beta v_2)$$

$$= (\alpha u_1, \beta v_1) \cdot_\oplus (\alpha u_2, \beta v_2),$$

so it is a Σ_{alg}-isomorphism.

(TENS): There is a unique linear isomorphism $\alpha \otimes \beta : A \otimes B \to A' \otimes B'$ with $(\alpha \otimes \beta)(a \otimes b) = \alpha(a) \otimes \beta(b)$. Then

$$(\alpha \otimes \beta)\Big((a \otimes b) \cdot_\otimes (a' \otimes b')\Big) = (\alpha(a \cdot a')) \otimes (\beta(b \cdot b')) = (\alpha a \cdot \alpha a') \otimes (\beta b \cdot \beta b')$$

$$= \big(\alpha a \otimes \beta b\big) \cdot_\otimes \big(\alpha a' \otimes \beta b'\big),$$

hence $\alpha \otimes \beta$ is multiplicative; preservation of the vector-space part is clear.

(OP): The same linear bijection $\alpha : A \to A'$ is an isomorphism $A^{\mathrm{op}} \xrightarrow{\cong} A'^{\mathrm{op}}$ because

$$\alpha(x \cdot^{\mathrm{op}} y) = \alpha(y \cdot x) = \alpha(y) \cdot \alpha(x) = \alpha(x) \cdot^{\mathrm{op}} \alpha(y).$$

Thus each Φ_ℓ is natural. The generalization statement follows from the embedding remark above. □

Proposition 2.7 (Finite-dimensional size under $\oplus$ and $\otimes$)**.** *If* $\dim_{\mathbb{F}} A = m$ *and* $\dim_{\mathbb{F}} B = n$ *(finite), then*

$$\dim(A \oplus B) = m + n, \qquad \dim(A \otimes_{\mathbb{F}} B) = mn.$$

Proof. Choose bases $\{e_i\}_{i=1}^m$ of A, $\{f_j\}_{j=1}^n$ of B. Then $\{(e_i, 0)\}_i \cup \{(0, f_j)\}_j$ is a basis of $A \oplus B$, and $\{e_i \otimes f_j\}_{i,j}$ is a basis of $A \otimes B$. □

Definition 2.8 (Iterated Meta-Algebra of depth t)**.** For $t \in \mathbb{N}$, an *Iterated Meta-Algebra of depth t* is an Iterated MetaStructure (Definition 1.3) over Σ_{alg},

$$\mathfrak{M}^{(t)}_{\mathrm{alg}} = \big(U^{(t)}_{\mathrm{alg}},\ (\odot^{(t)}_{\ell})_{\ell\in\Lambda}\big),$$

obtained by repeatedly applying the lifting functor $\mathsf{U}_{\Sigma_{\mathrm{alg}}}$ to $\mathbb{M}_{\mathrm{alg}}$. Concretely, if $s < t$ and a base meta-operation Φ_ℓ has meta-arity k_ℓ, its lift

$$\Phi^{\uparrow}_{\ell} :\ \big(U^{(t)}_{\mathrm{alg}}\big)^{k_\ell} \longrightarrow U^{(t)}_{\mathrm{alg}}$$

acts on representatives by

$$\Phi^{\uparrow}_{\ell}\big(\mathsf{U}^{t-s}_{\Sigma_{\mathrm{alg}}}(X_1), \dots, \mathsf{U}^{t-s}_{\Sigma_{\mathrm{alg}}}(X_{k_\ell})\big) \ :=\ \mathsf{U}^{t-s}_{\Sigma_{\mathrm{alg}}}\big(\Phi_\ell(X_1, \dots, X_{k_\ell})\big),$$

and analogously on symbol interpretations. We write $\odot^{(t)}_{\oplus}$, $\odot^{(t)}_{\otimes}$, $\odot^{(t)}_{\mathrm{op}}$ for the lifts of (DS), (TENS), (OP).

Example 2.9 (Iterated Meta-Algebra: tensor–then–sum for a coupled–plus–monitor model)**.** Consider the local (dual-number) algebras

$$A = \mathbb{R}[x]/(x^2), \qquad B = \mathbb{R}[y]/(y^2),$$

and a 2×2 matrix algebra $C = M_2(\mathbb{R})$. First apply the tensor-product constructor (TENS):

$$\Phi_\otimes(A, B) \ \cong\ \mathbb{R}[x, y]/(x^2, y^2),$$

where multiplication is determined by $x^2 = y^2 = 0$ and $xy = yx$. Next, apply the direct-sum constructor (DS) with C:

$$\Phi_\oplus\big(\,\Phi_\otimes(A, B),\ C\,\big) \ =\ \big(\mathbb{R}[x, y]/(x^2, y^2)\ \oplus\ M_2(\mathbb{R}),\ +_\oplus,\ \cdot_\oplus\big).$$

Real-life interpretation. The algebra $\mathbb{R}[x, y]/(x^2, y^2)$ captures first-order (infinitesimal) interactions between two small displacement/perturbation axes of a planar mechanism (no second-order terms), while the $M_2(\mathbb{R})$ block models an independent 2-state monitoring or actuation loop (e.g., a fast stabilizer). The iterated Meta-Algebra construction "couples then aggregates": it first builds the joint infinitesimal interaction algebra of the two axes via $\Phi_\otimes$, and then packs it together with the monitor via $\Phi_\oplus$ into one structured algebraic object, preserving the intended separations and interactions.

Theorem 2.10 (Iterated Meta-Algebra is an Iterated MetaStructure and generalizes Meta-Algebra)**.** *For every $t \in \mathbb{N}$, $\mathfrak{M}^{(t)}_{\mathrm{alg}}$ of Definition 2.8 is an Iterated MetaStructure in the sense of Definition 1.3. Moreover, for $s < t$ the canonical embedding*

$$\iota_{s\to t} :\ \mathfrak{M}^{(s)}_{\mathrm{alg}} \hookrightarrow \mathfrak{M}^{(t)}_{\mathrm{alg}}, \qquad X \longmapsto \mathsf{U}^{t-s}_{\Sigma_{\mathrm{alg}}}(X),$$

preserves all lifted meta-operations; in particular, $\mathfrak{M}^{(0)}_{\mathrm{alg}} = \mathbb{M}_{\mathrm{alg}}$ embeds into $\mathfrak{M}^{(t)}_{\mathrm{alg}}$, so Iterated Meta-Algebra generalizes *Meta-Algebra.*

Proof. By Definition 1.3, each lifted constructor is obtained by post-composing the base constructors

$$(\Gamma_\ell,\ +^{\Phi_\ell},\ 0^{\Phi_\ell},\ (-)^{\Phi_\ell},\ (m^{\Phi_\ell}_{\alpha}),\ (\cdot)^{\Phi_\ell})$$

with $\mathsf{U}^{t-s}_{\Sigma_{\mathrm{alg}}}$, hence uniform at every height. If $\alpha_i : X_i \xrightarrow{\cong} Y_i$ are isomorphisms at level s, then

$$\mathsf{U}^{t-s}_{\Sigma_{\mathrm{alg}}}(\alpha_i) :$$

$$\mathsf{U}^{t-s}_{\Sigma_{\mathrm{alg}}}(X_i) \xrightarrow{\cong}$$

$$\mathsf{U}^{t-s}_{\Sigma_{\mathrm{alg}}}(Y_i)$$

are isomorphisms at level t, and

$$\Phi^{\uparrow}_{\ell}\big(\mathsf{U}^{t-s}\alpha_1, \dots, \mathsf{U}^{t-s}\alpha_{k_\ell}\big) = \mathsf{U}^{t-s}\big(\Phi_\ell(\alpha_1, \dots, \alpha_{k_\ell})\big),$$

which is an isomorphism since each base Φ_ℓ is natural by Theorem 2.6. The operation-preservation of $\iota_{s\to t}$ is the same identity with X_i in place of α_i. Taking $s = 0$ gives the claimed embedding of $\mathbb{M}_{\mathrm{alg}}$ into every $\mathfrak{M}^{(t)}_{\mathrm{alg}}$. □

3 MetaTopology: Topology of Topology

A topology is a collection of open sets on a set, defining continuity, convergence, and neighborhood structures in mathematical spaces [41–43]. A Meta-Topology is a MetaStructure where objects are topological spaces, with meta-operations combining, restricting, or transforming topologies while preserving isomorphisms. An Iterated Meta-Topology recursively applies Meta-Topology, creating higher-level hierarchies of topological systems, generalizing topologies across successive meta-level structural layers.

Definition 3.1 (Topology)**.** (cf. [44–46]) A *topology* on a nonempty set X is a family $\mathcal{T} \subseteq \mathcal{P}(X)$ such that:

1. $\varnothing, X \in \mathcal{T}$,
2. If $\{U_i\}_{i\in I} \subseteq \mathcal{T}$, then $\bigcup_{i\in I} U_i \in \mathcal{T}$,
3. If $U_1, \dots, U_n \in \mathcal{T}$, then $\bigcap_{j=1}^{n} U_j \in \mathcal{T}$.

The pair $(X, \mathcal{T})$ is called a *topological space*.

We work single-sorted and encode points, open sets, and membership using unary/binary relations together with function symbols for finite unions/intersections.

Fix the (single-sorted, finitary) signature

$$\Sigma_{\mathrm{top}} = \Big(\, \mathsf{Func},\ \mathsf{Rel},\ \mathrm{ar} \,\Big),$$

where

$$\mathsf{Rel} = \big\{\, \mathsf{Pts},\ \mathsf{Op},\ \mathsf{Inc} \,\big\}, \quad \mathrm{ar}(\mathsf{Pts}) = \mathrm{ar}(\mathsf{Op}) = 1,\ \mathrm{ar}(\mathsf{Inc}) = 2,$$

and

$$\mathsf{Func} = \big\{\, \mathsf{cup} : 2,\ \mathsf{cap} : 2,\ \mathsf{empty} : 0,\ \mathsf{whole} : 0 \,\big\}.$$

A Σ_{top}-structure is a tuple

$$\mathbf{X} = (H;\ \mathsf{Pts}^{\mathbf{X}},\ \mathsf{Op}^{\mathbf{X}},\ \mathsf{Inc}^{\mathbf{X}};\ \mathsf{cup}^{\mathbf{X}},\ \mathsf{cap}^{\mathbf{X}},\ \mathsf{empty}^{\mathbf{X}},\ \mathsf{whole}^{\mathbf{X}})$$

with $H \neq \varnothing$, interpreting relations and function symbols over H.

Definition 3.2 (Universe of finite topological spaces)**.** Let $U_{\mathrm{top}} \subseteq \mathrm{Str}_{\Sigma_{\mathrm{top}}}$ be the class of all $\mathbf{X}$ satisfying:

(T1) **Typing/finite:** Pts is finite, nonempty. Elements u with $\mathsf{Op}(u)$ are *open-set tokens*. If $\mathsf{Inc}(x, U)$ then $\mathsf{Pts}(x)$ and $\mathsf{Op}(U)$.

(T2) **Topological constants:** $\mathsf{Op}(\mathsf{empty})$, $\mathsf{Op}(\mathsf{whole})$ and

$$\forall x \big(\mathsf{Pts}(x) \Rightarrow \neg\, \mathsf{Inc}(x, \mathsf{empty})\ \wedge\ \mathsf{Inc}(x, \mathsf{whole})\big).$$

(T3) **Closure and membership laws:** For all open tokens U, V and all points x,

$$\mathsf{Op}(\mathsf{cup}(U, V)), \quad \mathsf{Op}(\mathsf{cap}(U, V)),$$

$$\mathsf{Inc}(x, \mathsf{cup}(U, V)) \iff \mathsf{Inc}(x, U)\ \vee\ \mathsf{Inc}(x, V),$$

$$\mathsf{Inc}(x, \mathsf{cap}(U, V)) \iff \mathsf{Inc}(x, U)\ \wedge\ \mathsf{Inc}(x, V).$$

(T4) **Extensionality of open tokens:** If $\mathsf{Op}(U), \mathsf{Op}(V)$ and $\forall x \in \mathsf{Pts} :\ \mathsf{Inc}(x, U) \Leftrightarrow \mathsf{Inc}(x, V)$, then $U = V$.

Thus each $\mathbf{X} \in U_{\mathrm{top}}$ encodes a finite topological space $(X, \mathcal{T})$ with $X := \{x \in H \mid \mathsf{Pts}(x)\}$ and $\mathcal{T} := \{\, U \in H \mid \mathsf{Op}(U)\,\}$, where membership is given by Inc and finite unions/intersections are realized by $\mathsf{cup}, \mathsf{cap}$.

Remark 3.3 (Embedding a classical finite topology)**.** Given a finite topological space $(X, \mathcal{T})$, let the carrier be $H := \{0\} \times X \,\cup\, \{1\} \times \mathcal{T}$. Declare

$$\mathsf{Pts}(0, x), \quad \mathsf{Op}(1, U), \quad \mathsf{Inc}\big((0, x), (1, U)\big) \iff x \in U,$$

$$\mathsf{empty} = (1, \varnothing), \quad \mathsf{whole} = (1, X), \quad \mathsf{cup}\big((1, U), (1, V)\big) = (1, U \cup V), \ \mathsf{cap}\big((1, U), (1, V)\big) = (1, U \cap V).$$

Then $\mathbf{X} \in U_{\mathrm{top}}$.

Definition 3.4 (Meta-Topology)**.** A *Meta-Topology* is a MetaStructure (Definition 1.2) over Σ_{top}

$$\mathbb{M}_{\mathrm{top}} = \big(U_{\mathrm{top}},\ (\Phi_\ell)_{\ell \in \Lambda}\big),$$

whose meta-operations are specified uniformly as follows (each by carrier- and symbol-constructors).

(COPROD) Topological disjoint sum. For $\mathbf{X}_i = (H_i; \mathsf{Pts}_i, \mathsf{Op}_i, \mathsf{Inc}_i; \mathsf{cup}_i, \mathsf{cap}_i, \mathsf{empty}_i, \mathsf{whole}_i) \in U_{\mathrm{top}}$ $(i = 1, 2)$ define

$$\Phi_{\sqcup}(\mathbf{X}_1, \mathbf{X}_2) := (H; \mathsf{Pts}, \mathsf{Op}, \mathsf{Inc}; \mathsf{cup}, \mathsf{cap}, \mathsf{empty}, \mathsf{whole})$$

on the tagged carrier

$$H := \{0\} \times H_1 \,\cup\, \{1\} \times H_2,$$

with

$$\mathsf{Pts}(t, u) \iff \big(t = 0 \wedge \mathsf{Pts}_1(u)\big) \,\vee\, \big(t = 1 \wedge \mathsf{Pts}_2(u)\big),$$
$$\mathsf{Op}(t, U) \iff \big(t = 0 \wedge \mathsf{Op}_1(U)\big) \,\vee\, \big(t = 1 \wedge \mathsf{Op}_2(U)\big),$$
$$\mathsf{Inc}\big((t, x), (t, U)\big) \iff \begin{cases} \mathsf{Inc}_1(x, U), & t = 0, \\ \mathsf{Inc}_2(x, U), & t = 1, \end{cases} \quad \text{(and never across different tags)},$$
$$\mathsf{empty} := (0, \mathsf{empty}_1) \,\sqcup\, (1, \mathsf{empty}_2), \qquad \mathsf{whole} := (0, \mathsf{whole}_1) \,\sqcup\, (1, \mathsf{whole}_2),$$
$$\mathsf{cup}\big((t, U), (t, V)\big) := (t, \mathsf{cup}_t(U, V)), \qquad \mathsf{cap}\big((t, U), (t, V)\big) := (t, \mathsf{cap}_t(U, V)),$$

and $\mathsf{cup}, \mathsf{cap}$ are undefined across mismatched tags (i.e., their values are chosen in the same tag, which is well-defined because Op never mixes tags).

(RES) Subspace topology on a subset $S \subseteq \mathsf{Pts}$. Let $\mathbf{X} \in U_{\mathrm{top}}$ and $S \subseteq H$ with $S \subseteq \mathsf{Pts}$. Define

$$\Phi_{\mathrm{res}(S)}(\mathbf{X}) := (H'; \mathsf{Pts}', \mathsf{Op}', \mathsf{Inc}'; \mathsf{cup}', \mathsf{cap}', \mathsf{empty}', \mathsf{whole}')$$

on the tagged carrier $H' := \{0\} \times S \,\cup\, \{1\} \times \mathsf{Op}$ with

$$\mathsf{Pts}'(0, x) \iff x \in S, \qquad \mathsf{Op}'(1, U) \iff \mathsf{Op}(U),$$
$$\mathsf{Inc}'\big((0, x), (1, U)\big) \iff \big(x \in S\big) \wedge \mathsf{Inc}(x, U),$$
$$\mathsf{empty}' := (1, \mathsf{empty}), \qquad \mathsf{whole}' := (1, \mathsf{whole}),$$
$$\mathsf{cup}'\big((1, U), (1, V)\big) := (1, \mathsf{cup}(U, V)), \quad \mathsf{cap}'\big((1, U), (1, V)\big) := (1, \mathsf{cap}(U, V)).$$

(Thus open tokens are of the form $U \cap S$, represented extensionally by membership.)

(PUSH) Relabeling points via a bijection $f : \mathsf{Pts} \xrightarrow{\cong} K$. For $\mathbf{X} \in U_{\mathrm{top}}$ and a bijection f from the point set to a finite set K, define

$$\Phi_{\mathrm{push}(f)}(\mathbf{X}) := (\widehat{H}; \widehat{\mathsf{Pts}}, \widehat{\mathsf{Op}}, \widehat{\mathsf{Inc}}; \widehat{\mathsf{cup}}, \widehat{\mathsf{cap}}, \widehat{\mathsf{empty}}, \widehat{\mathsf{whole}})$$

on $\widehat{H} := \{0\} \times K \,\cup\, \{1\} \times \mathsf{Op}$ with

$$\widehat{\mathsf{Pts}}(0, k), \quad \widehat{\mathsf{Op}}(1, U) \iff \mathsf{Op}(U),$$
$$\widehat{\mathsf{Inc}}\big((0, k), (1, U)\big) \iff \mathsf{Inc}\big(f^{-1}(k), U\big),$$
$$\widehat{\mathsf{empty}} := (1, \mathsf{empty}), \quad \widehat{\mathsf{whole}} := (1, \mathsf{whole}),$$
$$\widehat{\mathsf{cup}}((1, U), (1, V)) = (1, \mathsf{cup}(U, V)),$$
$$\widehat{\mathsf{cap}}((1, U), (1, V)) = (1, \mathsf{cap}(U, V)).$$

Example 3.5 (Meta-Topology in practice: disjointly gluing two finite spaces)**.** Let $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ be finite topological spaces,

$$X_1 = \{a, b, c\}, \quad \mathcal{T}_1 = \big\{\varnothing, \{a\}, \{a, b\}, X_1\big\}, \qquad X_2 = \{u, v\}, \quad \mathcal{T}_2 = \big\{\varnothing, \{u\}, X_2\big\}.$$

Via the encoding of Remark 3.3, regard them as

$$\mathbf{X}_1, \mathbf{X}_2 \in U_{\text{top}}$$

. Applying the Meta-Topology (COPROD) constructor of Definition 3.4 yields the disjoint sum

$$\Phi_{\sqcup}(\mathbf{X}_1, \mathbf{X}_2) \in U_{\text{top}},$$

whose underlying set is the tagged union $X_1 \sqcup X_2 = \{0\} \times X_1 \ \cup \ \{1\} \times X_2$, and whose open subsets are precisely $U \sqcup V := \{0\} \times U \ \cup \ \{1\} \times V$ with $U \in \mathcal{T}_1$, $V \in \mathcal{T}_2$. *Interpretation.* One may view $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ as two independent "rooms" (finite sensor patches or network cells). The Meta-Topology disjoint sum glues them into a single space while preserving the topology on each component and forbidding cross-component membership.

Theorem 3.6 (Meta-Topology is a MetaStructure and generalizes topologies)**.** $\mathbb{M}_{\text{top}} = \big(U_{\text{top}}, (\Phi_\ell)\big)$ *of Definition 3.4 is a MetaStructure (Definition 1.2). Moreover, every classical finite topological space embeds as an object of* U_{top} *(Remark 3.3); hence Meta-Topology* generalizes *topology.*

Proof. Uniform constructors and closure. For (COPROD), typing is immediate from tags. $\mathsf{empty}, \mathsf{whole}$ are open in each tag and satisfy the membership clauses tagwise. Closure under $\mathsf{cup}, \mathsf{cap}$ and their membership laws hold componentwise by the input axioms (T3). Extensionality (T4) holds tagwise since tokens never mix tags and the membership characterization is preserved.

For (RES), Pts' is finite/nonempty whenever S is nonempty. The induced open tokens correspond to sets $U \cap S$; $\mathsf{empty}', \mathsf{whole}'$ satisfy (T2) relative to S; $\mathsf{cup}', \mathsf{cap}'$ obey the membership laws since, for all $x \in S$,

$$x \in (U \cup V) \cap S \iff (x \in U \cap S) \vee (x \in V \cap S), \quad x \in (U \cap V) \cap S \iff (x \in U \cap S) \wedge (x \in V \cap S).$$

Extensionality follows from pointwise membership on S.

For (PUSH), finiteness and typing are preserved. The membership laws and constants transport along f by definition, so (T2)–(T3) hold verbatim, and (T4) follows since equality of memberships after pullback by f^{-1} implies equality of tokens in the source, hence in the target.

Thus each output lies in U_{top}.

Naturality (isomorphism invariance). An isomorphism of Σ_{top}-structures is a bijection $\alpha : H \to H'$ preserving $\mathsf{Pts}, \mathsf{Op}, \mathsf{Inc}$ and the function symbols. If $\alpha_i : \mathbf{X}_i \to \mathbf{X}_i'$ are isomorphisms, then for (COPROD) the map $\alpha_{\sqcup}(t, u) := (t, \alpha_t(u))$ (with $\alpha_0 := \alpha_1$, $\alpha_1 := \alpha_2$) preserves all relations/functions componentwise. For (RES), restricting α to the tagged carrier respects S and transports the induced structures. For (PUSH), the given bijection f yields the relabeling, and any isomorphism of the source induces the same relabeling in the target via conjugation, preserving interpretations. Hence each Φ_ℓ is natural. The generalization claim is exactly Remark 3.3. □

Proposition 3.7 (Behavior of opens under META operations)**.** *Let* $\mathbf{X}_i$ *encode* $(X_i, \mathcal{T}_i)$.

(i) Disjoint sum: $\Phi_{\sqcup}(\mathbf{X}_1, \mathbf{X}_2)$ *encodes* $(X_1 \sqcup X_2, \ \{U_1 \sqcup U_2 \mid U_i \in \mathcal{T}_i\})$.

(ii) Subspace: $\Phi_{\text{res}(S)}(\mathbf{X})$ *encodes the subspace* $(S, \ \{U \cap S \mid U \in \mathcal{T}\})$.

(iii) Relabeling: $\Phi_{\text{push}(f)}(\mathbf{X})$ *encodes a space homeomorphic to* $(X, \mathcal{T})$ *via* f.

Proof. All three statements follow immediately from the membership clauses defining Inc in each construction. □

Definition 3.8 (Iterated Meta-Topology of depth t)**.** For $t \in \mathbb{N}$, an *Iterated Meta-Topology of depth t* is an Iterated MetaStructure (Definition 1.3) over Σ_{top},

$$\mathfrak{M}^{(t)}_{\mathrm{top}} = \big(U^{(t)}_{\mathrm{top}},\ (\odot^{(t)}_{\ell})_{\ell\in\Lambda}\big),$$

obtained by repeatedly applying the lifting functor $\mathsf{U}_{\Sigma_{\mathrm{top}}}$ to $\mathbb{M}_{\mathrm{top}}$. Concretely, if $s < t$ and Φ_ℓ has meta-arity k_ℓ, its lift

$$\Phi^{\uparrow}_{\ell} :\ \big(U^{(t)}_{\mathrm{top}}\big)^{k_\ell} \longrightarrow U^{(t)}_{\mathrm{top}}$$

is defined on representatives by

$$\Phi^{\uparrow}_{\ell}\big(\mathsf{U}^{t-s}_{\Sigma_{\mathrm{top}}}(X_1),\dots,\mathsf{U}^{t-s}_{\Sigma_{\mathrm{top}}}(X_{k_\ell})\big)\ :=\ \mathsf{U}^{t-s}_{\Sigma_{\mathrm{top}}}\big(\,\Phi_\ell(X_1,\dots,X_{k_\ell})\,\big),$$

and similarly for all symbol interpretations. We write $\odot^{(t)}_{\sqcup}$, $\odot^{(t)}_{\mathrm{res}(S)}$, $\odot^{(t)}_{\mathrm{push}(f)}$ for the lifts of (COPROD), (RES), (PUSH).

Example 3.9 (Iterated Meta-Topology (depth 2): restrict–relabel–glue pipeline)**.** Start with two finite spaces

$$(X,\mathcal{T}) = \big(\{a,b,c\},\ \{\varnothing,\{a\},\{a,b\},\{a,b,c\}\}\big), \qquad (Y,\mathcal{S}) = \big(\{p,q,r\},\ \{\varnothing,\{p\},\{p,q\},\{p,q,r\}\}\big),$$

encoded as $\mathbf{X},\mathbf{Y} \in U_{\mathrm{top}}$ (Remark 3.3). Perform the following meta-operations (each one an instance of a lifted constructor in the sense of Definition 1.3):

Step 1 (RES: subspace). Let $S = \{a,b\} \subseteq X$. Apply $\Phi_{\mathrm{res}(S)}$ to obtain the subspace topology $\mathbf{X}_{|S} := \Phi_{\mathrm{res}(S)}(\mathbf{X})$; classically $\mathcal{T}_{|S} = \{\varnothing,\{a\},\{a,b\}\}$ on S.

Step 2 (PUSH: relabel points). Fix the bijection $f : S \xrightarrow{\cong} \{1,2\}$ with $f(a) = 1$, $f(b) = 2$. Apply $\Phi_{\mathrm{push}(f)}$ to get $\widehat{\mathbf{X}} := \Phi_{\mathrm{push}(f)}(\mathbf{X}_{|S})$, whose underlying set is $\{1,2\}$ and whose open sets are $f[U]$ for $U \in \mathcal{T}_{|S}$.

Step 3 (COPROD: disjoint sum). Glue the relabeled subspace with $\mathbf{Y}$ by

$$\mathbf{Z}\ :=\ \Phi_{\sqcup}\big(\widehat{\mathbf{X}},\ \mathbf{Y}\big) \in U_{\mathrm{top}}.$$

Classically, $\mathbf{Z}$ is the disjoint union topology on $\{1,2\} \sqcup \{p,q,r\}$ with opens $U' \sqcup V$ where $U' \in \{\varnothing,\{1\},\{1,2\}\}$ and $V \in \mathcal{S}$.

Interpretation. This "restrict–relabel–glue" pipeline models a typical workflow: crop a region of interest from one finite space (Step 1), rename its points to a new index set (Step 2), then assemble it with another space without creating artificial adjacency between components (Step 3). As an instance of Iterated Meta-Topology, the output is obtained by composing meta-operations, exactly as prescribed by the lifted constructors in Definition 1.3.

Theorem 3.10 (Iterated Meta-Topology is an Iterated MetaStructure and generalizes Meta-Topology)**.** *For every $t \in \mathbb{N}$, $\mathfrak{M}^{(t)}_{\mathrm{top}}$ of Definition 3.8 is an Iterated MetaStructure in the sense of Definition 1.3. Moreover, for $s < t$ the canonical embedding*

$$\iota_{s\to t} :\ \mathfrak{M}^{(s)}_{\mathrm{top}} \hookrightarrow \mathfrak{M}^{(t)}_{\mathrm{top}}, \qquad X \longmapsto \mathsf{U}^{t-s}_{\Sigma_{\mathrm{top}}}(X),$$

preserves all lifted meta-operations; in particular, $\mathfrak{M}^{(0)}_{\mathrm{top}} = \mathbb{M}_{\mathrm{top}}$ embeds into $\mathfrak{M}^{(t)}_{\mathrm{top}}$, so Iterated Meta-Topology generalizes *Meta-Topology.*

Proof. By Definition 1.3, the lifted constructors are obtained by post-composing the base constructors

$$\big(\Gamma_\ell;\ \mathsf{cup}^{\Phi_\ell},\ \mathsf{cap}^{\Phi_\ell},\ \mathsf{empty}^{\Phi_\ell},\ \mathsf{whole}^{\Phi_\ell};\ \mathsf{Pts}^{\Phi_\ell},\ \mathsf{Op}^{\Phi_\ell},\ \mathsf{Inc}^{\Phi_\ell}\big)$$

with $\mathsf{U}^{t-s}_{\Sigma_{\mathrm{top}}}$; hence uniform at every height. If $\alpha_i : X_i \xrightarrow{\cong} Y_i$ are isomorphisms at level s, then

$$\mathsf{U}^{t-s}_{\Sigma_{\mathrm{top}}}(\alpha_i) :\ \mathsf{U}^{t-s}_{\Sigma_{\mathrm{top}}}(X_i) \xrightarrow{\cong} \mathsf{U}^{t-s}_{\Sigma_{\mathrm{top}}}(Y_i)$$

are isomorphisms at level t, and

$$\Phi^{\uparrow}_{\ell}\big(\mathsf{U}^{t-s}\alpha_1,\dots,\mathsf{U}^{t-s}\alpha_{k_\ell}\big) = \mathsf{U}^{t-s}\big(\Phi_\ell(\alpha_1,\dots,\alpha_{k_\ell})\big),$$

which is an isomorphism since each base Φ_ℓ is natural by Theorem 3.6. Operation preservation of $\iota_{s\to t}$ is the same identity with X_i in place of α_i. Taking $s = 0$ yields the claimed embedding. □

4 MetaLattice: Lattice of Lattice

A lattice is a partially ordered set where every pair of elements has a unique meet (infimum) and join (supremum) [47–50]. A Meta-Lattice is a MetaStructure treating lattices as objects, with meta-operations combining, modifying, or transforming them while preserving isomorphisms. An Iterated Meta-Lattice recursively applies Meta-Lattice construction, forming hierarchical layers of lattices of lattices, generalizing classical lattice theory across meta-levels.

Definition 4.1 (Lattice)**.** (cf. [51–53]) A *lattice* is a partially ordered set $(L, \leq)$ in which every pair of elements $x, y \in L$ has both a least upper bound (join) $x \vee y$ and a greatest lower bound (meet) $x \wedge y$. Formally, for all $x, y \in L$,

$$x \vee y = \inf\{z \in L \mid x \leq z,\ y \leq z\}, \qquad x \wedge y = \sup\{z \in L \mid z \leq x,\ z \leq y\}.$$

Fix the single-sorted, finitary signature

$$\Sigma_{\mathrm{lat}} = \big(\,\mathsf{Func},\ \mathsf{Rel},\ \mathrm{ar}\,\big), \qquad \mathsf{Func} = \{\,\mathsf{join} : 2,\ \mathsf{meet} : 2\,\}, \quad \mathsf{Rel} = \{\,\mathsf{El} : 1,\ \mathsf{Leq} : 2\,\}.$$

A Σ_{lat}-structure is a tuple

$$\mathbf{L} = \big(H;\ \mathsf{El}^{\mathbf{L}},\ \mathsf{Leq}^{\mathbf{L}};\ \mathsf{join}^{\mathbf{L}},\ \mathsf{meet}^{\mathbf{L}}\big),$$

with carrier $H \neq \varnothing$.

Definition 4.2 (Universe of finite lattices)**.** Let

$$U_{\mathrm{lat}} \subseteq \mathrm{Str}_{\Sigma_{\mathrm{lat}}}$$

be the class of $\mathbf{L}$ satisfying:

(L1) **Typing/finite:** El is finite, nonempty; if $\mathsf{Leq}(x, y)$ then $\mathsf{El}(x), \mathsf{El}(y)$; and $\mathsf{join}, \mathsf{meet} : \mathsf{El} \times \mathsf{El} \to \mathsf{El}$ are total.

(L2) **Lattice laws:** For all $x, y, z \in \mathsf{El}$, writing $x \vee y := \mathsf{join}(x, y)$ and $x \wedge y := \mathsf{meet}(x, y)$, the four lattice axiom families hold: commutativity, associativity, idempotence, and absorption (as displayed above).

(L3) **Order compatibility:** Leq is a partial order on El and

$$\mathsf{Leq}(x, y) \iff x \wedge y = x \iff x \vee y = y \qquad (x, y \in \mathsf{El}).$$

Then each $\mathbf{L} \in U_{\mathrm{lat}}$ canonically encodes a finite lattice $\big(L, \vee, \wedge, \leq\big)$ with $L = \{x \in H \mid \mathsf{El}(x)\}$, $\vee, \wedge$ as above, and $\leq$ given by Leq.

Remark 4.3 (Embedding a classical finite lattice)**.** Given a finite lattice $(L, \vee, \wedge, \leq)$, let $H := L$,

$$\mathsf{El}(x) \iff x \in L, \qquad \mathsf{Leq}(x, y) \iff x \leq y, \qquad \mathsf{join}(x, y) := x \vee y, \quad \mathsf{meet}(x, y) := x \wedge y.$$

Then $\mathbf{L} \in U_{\mathrm{lat}}$ by the standard lattice axioms.

Definition 4.4 (Meta-Lattice)**.** A *Meta-Lattice* is a MetaStructure (Definition 1.2) over Σ_{lat},

$$\mathbb{M}_{\mathrm{lat}} = \big(U_{\mathrm{lat}},\ (\Phi_\ell)_{\ell \in \Lambda}\big),$$

whose meta-operations are given uniformly as follows (each by carrier- and symbol-constructors).

(PROD) Direct product. For $\mathbf{L}_i \in U_{\mathrm{lat}}$ $(i = 1, 2)$, define

$$\Phi_\times(\mathbf{L}_1, \mathbf{L}_2) := (H;\ \mathsf{El}, \mathsf{Leq};\ \mathsf{join}, \mathsf{meet})$$

on $H := \mathsf{El}_1 \times \mathsf{El}_2$ with

$$\mathsf{El}(x_1, x_2), \qquad \mathsf{Leq}\big((x_1, x_2), (y_1, y_2)\big) \iff \mathsf{Leq}_1(x_1, y_1)\ \wedge\ \mathsf{Leq}_2(x_2, y_2),$$

$$\mathsf{join}\big((x_1, x_2), (y_1, y_2)\big) := (x_1 \vee_1 y_1,\ x_2 \vee_2 y_2), \quad \mathsf{meet}\big((x_1, x_2), (y_1, y_2)\big) := (x_1 \wedge_1 y_1,\ x_2 \wedge_2 y_2).$$

(RES) Sublattice restriction by a closed subset. Let $\mathbf{L} \in U_{\mathrm{lat}}$ and $S \subseteq \mathsf{El}$ be nonempty and closed under $\vee, \wedge$. Define

$$\Phi_{\mathrm{res}(S)}(\mathbf{L}) := (H';\ \mathsf{El}', \mathsf{Leq}';\ \mathsf{join}', \mathsf{meet}')$$

on $H' := S$ with the obvious restrictions of $\mathsf{El}, \mathsf{Leq}, \mathsf{join}, \mathsf{meet}$ to S.

(PUSH) Relabeling along a bijection $f : \mathsf{El} \xrightarrow{\cong} K$. For $\mathbf{L} \in U_{\mathrm{lat}}$ and f a bijection, set

$$\Phi_{\mathrm{push}(f)}(\mathbf{L}) = (K;\ \widehat{\mathsf{El}}, \wideparen{\mathsf{Leq}};\ \wideparen{\mathsf{join}}, \wideparen{\mathsf{meet}})$$

with $\widehat{\mathsf{El}}(k) \iff k \in K$ and

$$\wideparen{\mathsf{Leq}}(f(x), f(y)) \iff \mathsf{Leq}(x, y), \quad \wideparen{\mathsf{join}}(f(x), f(y)) := f\big(\mathsf{join}(x, y)\big), \quad \wideparen{\mathsf{meet}}(f(x), f(y)) := f\big(\mathsf{meet}(x, y)\big).$$

(DUAL) Order dual. For $\mathbf{L} \in U_{\mathrm{lat}}$, let

$$\Phi_{\mathrm{dual}}(\mathbf{L}) := \big(H;\ \mathsf{El},\ \mathsf{Leq}^{\mathrm{op}};\ \mathsf{join}^{\mathrm{op}},\ \mathsf{meet}^{\mathrm{op}}\big),$$

where $\mathsf{Leq}^{\mathrm{op}}(x, y) \iff \mathsf{Leq}(y, x)$, and $\mathsf{join}^{\mathrm{op}} := \mathsf{meet}$, $\mathsf{meet}^{\mathrm{op}} := \mathsf{join}$.

Example 4.5 (Meta-Lattice in practice: combining permissions and urgency)**.** Consider two finite lattices:

(A) Permissions as a Boolean lattice. Let $L_{\mathrm{perm}} := \mathcal{P}(\{r, w\}) = \{\varnothing, \{r\}, \{w\}, \{r, w\}\}$ with join $U \vee V := U \cup V$, meet $U \wedge V := U \cap V$, and order $U \leq V \iff U \subseteq V$.

(B) Urgency as a chain. Let $L_{\mathrm{urg}} := \{\, \mathrm{lo} < \mathrm{med} < \mathrm{hi} \,\}$ with join $u \vee v := \max\{u, v\}$ and meet $u \wedge v := \min\{u, v\}$.

By the Meta-Lattice product constructor (Definition 4.4, (PROD)), their direct product is the lattice

$$L_{\mathrm{perm}} \times L_{\mathrm{urg}}$$

with componentwise order and operations. Intuitively, (U, u) records an access level $U \subseteq \{r, w\}$ paired with an urgency $u \in \{\mathrm{lo}, \mathrm{med}, \mathrm{hi}\}$. For instance,

$$(\{r\}, \mathrm{lo}) \ \vee\ (\{w\}, \mathrm{med}) = (\{r, w\}, \mathrm{med}), \qquad (\{r\}, \mathrm{lo}) \ \wedge\ (\{w\}, \mathrm{med}) = (\varnothing, \mathrm{lo}).$$

This Meta-Lattice object encodes policy aggregation: joins combine permissions and take the more urgent level; meets intersect permissions and take the less urgent level.

Theorem 4.6 (Meta-Lattice is a MetaStructure and generalizes lattices)**.** *$\mathbb{M}_{\mathrm{lat}} = \big(U_{\mathrm{lat}}, (\Phi_\ell)\big)$ of Definition 4.4 is a MetaStructure (Definition 1.2). Moreover, every classical finite lattice $(L, \vee, \wedge, \leq)$ embeds as an object of U_{lat} (Remark 4.3); hence Meta-Lattice* generalizes *lattices.*

Proof. Uniform constructors and closure. (PROD): Finiteness and typing are clear. The operations are componentwise, hence (L2) holds by checking each lattice law coordinatewise. The order is product order, a partial order, and

$$(x_1, x_2) \vee (y_1, y_2) = (y_1, y_2) \iff (x_1 \vee_1 y_1 = y_1) \wedge (x_2 \vee_2 y_2 = y_2) \iff \mathsf{Leq}_1(x_1, y_1) \wedge \mathsf{Leq}_2(x_2, y_2),$$

so (L3) holds (similarly with $\wedge$).

(RES): Closure of S under $\vee, \wedge$ makes the restricted operations total on S and preserves the lattice laws. Leq' is a partial order and the characterizations in (L3) remain valid since joins/meets in S coincide with those computed in $\mathbf{L}$ by closure.

(PUSH): Transport of structure along a bijection preserves finiteness, lattice laws, and the equivalences in (L3) by definition.

(DUAL): Swapping $\vee$ and $\wedge$ and reversing $\leq$ preserves (L2); (L3) holds because

$$x \leq_{\mathrm{op}} y \iff y \leq x \iff x \vee y = y \iff x \wedge_{\mathrm{op}} y = x,$$

and similarly with $\vee_{\mathrm{op}}$.

Thus each output lies in U_{lat}.

Naturality (isomorphism invariance). An isomorphism $\alpha : \mathbf{L} \to \mathbf{L}'$ is a bijection $\alpha : H \to H'$ with α preserving El, Leq and commuting with join, meet. Given isomorphisms $\alpha_i : \mathbf{L}_i \to \mathbf{L}_i'$, define $\alpha_\times(x_1, x_2) := (\alpha_1 x_1, \alpha_2 x_2)$ for (PROD) and check preservation componentwise. For (RES), restrict α to S. For (PUSH), use the given f as the underlying bijection. For (DUAL), the same α intertwines the swapped symbols. Hence every Φ_ℓ is natural. The generalization claim is Remark 4.3. □

Proposition 4.7 (Joins/meets under META operations). *Let* $\mathbf{L}_i$ *encode* $(L_i, \vee_i, \wedge_i, \leq_i)$.

(i) Product: $\Phi_\times(\mathbf{L}_1, \mathbf{L}_2)$ *encodes* $(L_1 \times L_2,\ (x_1, x_2) \vee (y_1, y_2) = (x_1 \vee_1 y_1, x_2 \vee_2 y_2),\ (x_1, x_2) \wedge (y_1, y_2) = (x_1 \wedge_1 y_1, x_2 \wedge_2 y_2))$.

(ii) Sublattice: $\Phi_{\mathrm{res}(S)}(\mathbf{L})$ *encodes the sublattice generated by S (here, S is assumed closed).*

(iii) Dual: $\Phi_{\mathrm{dual}}(\mathbf{L})$ *encodes the order-dual lattice* $(L, \wedge, \vee, \geq)$.

Proof. Immediate from the definitions of the constructors. □

Definition 4.8 (Iterated Meta-Lattice of depth t). For $t \in \mathbb{N}$, an *Iterated Meta-Lattice of depth t* is an Iterated MetaStructure (Definition 1.3) over Σ_{lat},

$$\mathfrak{M}^{(t)}_{\mathrm{lat}} = \big(U^{(t)}_{\mathrm{lat}},\ (\odot^{(t)}_\ell)_{\ell \in \Lambda}\big),$$

obtained by repeatedly applying the lifting $\mathsf{U}_{\Sigma_{\mathrm{lat}}}$ to $\mathbb{M}_{\mathrm{lat}}$. Concretely, if $s < t$ and Φ_ℓ has meta-arity k_ℓ, its lift

$$\Phi^{\uparrow}_\ell :\ \big(U^{(t)}_{\mathrm{lat}}\big)^{k_\ell} \longrightarrow U^{(t)}_{\mathrm{lat}}$$

is defined on representatives by

$$\Phi^{\uparrow}_\ell\big(\mathsf{U}^{t-s}_{\Sigma_{\mathrm{lat}}}(X_1), \ldots, \mathsf{U}^{t-s}_{\Sigma_{\mathrm{lat}}}(X_{k_\ell})\big) := \mathsf{U}^{t-s}_{\Sigma_{\mathrm{lat}}}\big(\Phi_\ell(X_1, \ldots, X_{k_\ell})\big),$$

and similarly for all symbol interpretations. We denote by $\odot^{(t)}_\times$, $\odot^{(t)}_{\mathrm{res}(S)}$, $\odot^{(t)}_{\mathrm{push}(f)}$, $\odot^{(t)}_{\mathrm{dual}}$ the lifts of (PROD), (RES), (PUSH), (DUAL).

Example 4.9 (Iterated Meta-Lattice (depth 2): restrict–dualize–product pipeline). Start from the lattices in Example 4.5 and apply a sequence of meta-operations (each an instance of the lifted constructors in Definition 4.8):

Step 1 (RES: sublattice restriction). Let $S := \{\emptyset, \{r\}, \{r, w\}\} \subseteq L_{\mathrm{perm}}$. Since S is closed under $\cup, \cap$, the restriction

$$\Phi_{\mathrm{res}(S)}(L_{\mathrm{perm}})$$

is the sublattice $L^{|S}_{\mathrm{perm}}$ with elements S and the inherited $\vee, \wedge$.

Step 2 (DUAL: order reversal on urgency). Apply the dual constructor to the urgency chain to obtain

$$\Phi_{\mathrm{dual}}(L_{\mathrm{urg}}) =: L^{\mathrm{op}}_{\mathrm{urg}},$$

where the order is reversed (hi < med < lo), so joins/meets swap: in $L^{\mathrm{op}}_{\mathrm{urg}}$ the join is min and the meet is max (with respect to the original order).

Step 3 (PROD: combine the results). Form the direct product

$$\Phi_\times\big(L^{|S}_{\mathrm{perm}},\ L^{\mathrm{op}}_{\mathrm{urg}}\big),$$

a lattice of pairs $\big(U, \tilde{u}\big)$ with $U \in S$ and $\tilde{u} \in L_{\text{urg}}^{\text{op}}$, ordered and operated componentwise.

A concrete calculation (componentwise join) illustrates the composition:

$$(\{r\}, \text{med}) \ \vee \ (\{r, w\}, \text{lo}) \ = \ \big(\{r\} \cup \{r, w\},\ \min(\text{med}, \text{lo})\big) \ = \ (\{r, w\}, \text{lo}),$$

where the second component uses the *dual* (min-as-join) on urgency. This iterated Meta-Lattice construction models a common workflow: (1) restrict attention to a closed family of permissions, (2) flip the priority convention (e.g., to express "ease" instead of "urgency"), then (3) aggregate both dimensions into a single lattice of policies.

Theorem 4.10 (Iterated Meta-Lattice is an Iterated MetaStructure and generalizes Meta-Lattice)**.** *For every $t \in \mathbb{N}$, $\mathfrak{M}_{\text{lat}}^{(t)}$ of Definition 4.8 is an Iterated MetaStructure in the sense of Definition 1.3. Moreover, for $s < t$ the embedding*

$$\iota_{s\to t} : \mathfrak{M}_{\text{lat}}^{(s)} \hookrightarrow \mathfrak{M}_{\text{lat}}^{(t)}, \qquad X \longmapsto \mathsf{U}_{\Sigma_{\text{lat}}}^{t-s}(X),$$

preserves all lifted meta-operations:

$$\Phi_\ell^{\uparrow}\big(\iota_{s\to t}(X_1), \ldots, \iota_{s\to t}(X_{k_\ell})\big) = \iota_{s\to t}\big(\Phi_\ell(X_1, \ldots, X_{k_\ell})\big).$$

In particular, $\mathfrak{M}_{\text{lat}}^{(0)} = \mathbb{M}_{\text{lat}}$ embeds into $\mathfrak{M}_{\text{lat}}^{(t)}$, so Iterated Meta-Lattice generalizes *Meta-Lattice.*

Proof. By Definition 1.3, each lifted constructor is obtained by post-composing the base constructors

$$\big(\Gamma_\ell;\ \mathsf{join}^{\Phi_\ell},\ \mathsf{meet}^{\Phi_\ell};\ \mathsf{El}^{\Phi_\ell},\ \mathsf{Leq}^{\Phi_\ell}\big)$$

with $\mathsf{U}_{\Sigma_{\text{lat}}}^{t-s}$; hence uniform at every height. If $\alpha_i : X_i \xrightarrow{\cong} Y_i$ are level-s isomorphisms, then

$$\mathsf{U}_{\Sigma_{\text{lat}}}^{t-s}(\alpha_i) :\ \mathsf{U}_{\Sigma_{\text{lat}}}^{t-s}(X_i) \xrightarrow{\cong} \mathsf{U}_{\Sigma_{\text{lat}}}^{t-s}(Y_i)$$

are level-t isomorphisms, and

$$\Phi_\ell^{\uparrow}\big(\mathsf{U}^{t-s}\alpha_1, \ldots, \mathsf{U}^{t-s}\alpha_{k_\ell}\big) = \mathsf{U}^{t-s}\big(\Phi_\ell(\alpha_1, \ldots, \alpha_{k_\ell})\big),$$

which is an isomorphism because each base Φ_ℓ is natural by Theorem 4.6. Operation preservation of $\iota_{s\to t}$ is the same identity with X_i in place of α_i. Taking $s = 0$ yields the claimed embedding. □

5 MetaQueue: Queue of Queue

A queue is an ordered data structure following the FIFO principle, supporting enqueue for insertion and dequeue for removal(cf. [54–56]). A Meta-Queue is a MetaStructure treating queues as objects, enabling meta-operations that combine, transform, or restrict entire queue systems. An Iterated Meta-Queue recursively applies Meta-Queue construction, creating hierarchical layers of queues of queues, generalizing classical queue structures.

Definition 5.1 (Queue)**.** (cf. [57, 58]) A *queue* is an ordered structure

$$Q = (S, \text{enq}, \text{deq}),$$

where S is a (possibly finite) sequence of elements. The operation $\text{enq} : S \times E \to S$ appends a new element $e \in E$ at the end of S, and $\text{deq} : S \to S \times E$ removes the element at the front of S if $S \neq \emptyset$. The fundamental principle is *first in, first out* (FIFO).

Fix the single-sorted, finitary signature

$$\Sigma_{\text{que}} = \big(\,\mathsf{Func} = \emptyset,\ \mathsf{Rel} = \{\,\mathsf{Queues},\ \mathsf{Elems},\ \mathsf{Emp},\ \mathsf{Enq},\ \mathsf{Deq}\,\},\ \text{ar}\,\big),$$

where the intended typings are

$$\mathsf{Queues} : 1, \qquad \mathsf{Elems} : 1, \qquad \mathsf{Emp} : 1, \qquad \mathsf{Enq} : 3, \qquad \mathsf{Deq} : 3.$$

A Σ_{que}-structure is a tuple

$$\mathbf{Q} = \big(H;\ \mathsf{Queues}^{\mathbf{Q}},\ \mathsf{Elems}^{\mathbf{Q}},\ \mathsf{Emp}^{\mathbf{Q}},\ \mathsf{Enq}^{\mathbf{Q}},\ \mathsf{Deq}^{\mathbf{Q}}\big),$$

with nonempty carrier H and relations interpreted over H.

Definition 5.2 (Universe of finite queues)**.** Let $U_{\text{que}} \subseteq \text{Str}_{\Sigma_{\text{que}}}$ be the class of all $\mathbf{Q} = (H; \mathsf{Queues}, \mathsf{Elems}, \mathsf{Emp}, \mathsf{Enq}, \mathsf{Deq})$ satisfying:

(Q1) **Typing/finite:** $\mathsf{Queues}, \mathsf{Elems} \subseteq H$ are finite and nonempty, $\mathsf{Queues} \cap \mathsf{Elems} = \varnothing$. If $\mathsf{Emp}(x)$ or $\exists y, z\, \mathsf{Enq}(x, y, z)$ or $\exists y\, z\, \mathsf{Deq}(x, y, z)$ then $x \in \mathsf{Queues}$. If $\exists x, z\, \mathsf{Enq}(x, y, z)$ or $\exists x\, z\, \mathsf{Deq}(x, z, y)$ then $y \in \mathsf{Elems}$. Moreover $\mathsf{Enq}(x, y, z) \Rightarrow z \in \mathsf{Queues}$ and $\mathsf{Deq}(x, z, y) \Rightarrow z \in \mathsf{Queues}$.

(Q2) **Enqueue is total and functional:** For every $q \in \mathsf{Queues}$ and $a \in \mathsf{Elems}$ there exists a unique $q' \in \mathsf{Queues}$ with $\mathsf{Enq}(q, a, q')$.

(Q3) **Dequeue is partial functional with emptiness guard:**

$$\mathsf{Emp}(q) \Rightarrow \neg\exists a, q'\, \mathsf{Deq}(q, q', a), \quad \neg\mathsf{Emp}(q) \Rightarrow \exists!(a, q') \in \mathsf{Elems} \times \mathsf{Queues} : \ \mathsf{Deq}(q, q', a).$$

(Q4) **FIFO coherence (rear enqueue does not disturb the head):**
For all $q \in \mathsf{Queues}$, $a, a_1 \in \mathsf{Elems}$ and $q_1, q_2, q' \in \mathsf{Queues}$,

$$\Big(\mathsf{Deq}(q, q_1, a_1) \ \wedge \ \mathsf{Enq}(q_1, a, q_2) \ \wedge \ \mathsf{Enq}(q, a, q')\Big) \implies \mathsf{Deq}(q', q_2, a_1).$$

(Q5) **Empty/front compatibility:** For all $q \in \mathsf{Queues}$, $a \in \mathsf{Elems}$, if $\mathsf{Emp}(q)$ and $\mathsf{Enq}(q, a, q_1)$ then $\mathsf{Deq}(q_1, q, a)$.

Thus every $\mathbf{Q} \in U_{\text{que}}$ encodes a finite FIFO queue system over the finite element set Elems.

Remark 5.3 (Embedding a classical finite queue)**.** Let E be a finite set of symbols and let E^* be the set of all finite strings over E. Define a Σ_{que}-structure on

$$H := \underbrace{\{0\} \times E^*}_{\text{queues}} \ \cup \ \underbrace{\{1\} \times E}_{\text{elements}}$$

by

$$\mathsf{Queues}(0, w) \iff w \in E^*, \qquad \mathsf{Elems}(1, e) \iff e \in E, \qquad \mathsf{Emp}(0, w) \iff w = \epsilon,$$
$$\mathsf{Enq}\big((0, w), (1, e), (0, w')\big) \iff w' = w \cdot e \quad \text{(append at rear)},$$
$$\mathsf{Deq}\big((0, w), (0, w'), (1, e)\big) \iff w = e \cdot w' \quad \text{(head } e \text{ with tail } w'\text{)}.$$

Then the axioms (Q1)–(Q5) hold; hence $\mathbf{Q} \in U_{\text{que}}$.

Definition 5.4 (Meta-Queue)**.** A *Meta-Queue* is a MetaStructure (Definition 1.2) over Σ_{que},

$$\mathbb{M}_{\text{que}} = \big(U_{\text{que}}, \ (\Phi_\ell)_{\ell \in \Lambda}\big),$$

with the following uniform meta-operations, each specified by carrier- and relation-constructors:

(PROD) Parallel product (componentwise FIFO). For $\mathbf{Q}_i = (H_i; \mathsf{Queues}_i, \mathsf{Elems}_i, \mathsf{Emp}_i, \mathsf{Enq}_i, \mathsf{Deq}_i) \in U_{\text{que}}$ $(i = 1, 2)$ define

$$\Phi_\times(\mathbf{Q}_1, \mathbf{Q}_2) := (H; \mathsf{Queues}, \mathsf{Elems}, \mathsf{Emp}, \mathsf{Enq}, \mathsf{Deq})$$

on $H := \big(\mathsf{Queues}_1 \times \mathsf{Queues}_2\big) \ \cup \ \big(\mathsf{Elems}_1 \times \mathsf{Elems}_2\big)$ with disjoint tagging understood, and

$$\mathsf{Queues}(q_1, q_2), \qquad \mathsf{Elems}(a_1, a_2), \qquad \mathsf{Emp}(q_1, q_2) \iff \mathsf{Emp}_1(q_1) \wedge \mathsf{Emp}_2(q_2),$$
$$\mathsf{Enq}\big((q_1, q_2), (a_1, a_2), (q_1', q_2')\big) \iff \mathsf{Enq}_1(q_1, a_1, q_1') \wedge \mathsf{Enq}_2(q_2, a_2, q_2'),$$
$$\mathsf{Deq}\big((q_1, q_2), (q_1', q_2'), (a_1, a_2)\big) \iff \mathsf{Deq}_1(q_1, q_1', a_1) \wedge \mathsf{Deq}_2(q_2, q_2', a_2).$$

(COPROD) Disjoint union (sum of independent queues). Define $\Phi_\sqcup(\mathbf{Q}_1, \mathbf{Q}_2)$ on the tagged carrier

$$H := \{0\} \times \mathsf{Queues}_1 \ \cup \ \{1\} \times \mathsf{Queues}_2 \ \cup \ \{2\} \times \mathsf{Elems}_1 \ \cup \ \{3\} \times \mathsf{Elems}_2,$$

with

$$\mathsf{Queues} := (\{0\} \times \mathsf{Queues}_1) \cup (\{1\} \times \mathsf{Queues}_2), \quad \mathsf{Elems} := (\{2\} \times \mathsf{Elems}_1) \cup (\{3\} \times \mathsf{Elems}_2),$$

$$\mathsf{Emp}(0, q) \iff \mathsf{Emp}_1(q), \;\; \mathsf{Emp}(1, q) \iff \mathsf{Emp}_2(q),$$

and operations acting *within* each tag only:

$$\mathsf{Enq}\big((0, q), (2, a), (0, q')\big) \iff \mathsf{Enq}_1(q, a, q'), \qquad \mathsf{Enq}\big((1, q), (3, a), (1, q')\big) \iff \mathsf{Enq}_2(q, a, q'),$$

$$\mathsf{Deq}\big((0, q), (0, q'), (2, a)\big) \iff \mathsf{Deq}_1(q, q', a), \qquad \mathsf{Deq}\big((1, q), (1, q'), (3, a)\big) \iff \mathsf{Deq}_2(q, q', a),$$

and mixed-tag triples never satisfy Enq or Deq.

(RES) Closed sub-queue system. Let $\mathbf{Q} \in U_{\text{que}}$. A pair (S_Q, S_E) with $S_Q \subseteq \mathsf{Queues}$, $S_E \subseteq \mathsf{Elems}$ is *admissible* if

$$\forall q \in S_Q \, \forall a \in S_E \, \exists! \, q' \in S_Q : \mathsf{Enq}(q, a, q'), \quad \forall q \in S_Q \, \big(\neg\mathsf{Emp}(q) \Rightarrow \exists! \, (a, q') \in S_E {\times} S_Q : \mathsf{Deq}(q, q', a)\big).$$

For such (S_Q, S_E) put

$$\Phi_{\text{res}(S_Q, S_E)}(\mathbf{Q}) := \big(S_Q \cup S_E; \; S_Q, \; S_E, \; \mathsf{Emp} \restriction_{S_Q}, \; \mathsf{Enq} \restriction_{S_Q \times S_E \times S_Q}, \; \mathsf{Deq} \restriction_{S_Q \times S_Q \times S_E}\big).$$

(PUSH) Element relabeling along a bijection $g : \mathsf{Elems} \xrightarrow{\cong} K$. Let $\mathbf{Q} \in U_{\text{que}}$ and g a bijection. Define

$$\Phi_{\text{push}(g)}(\mathbf{Q}) := \big(H'; \; \mathsf{Queues}', \; \mathsf{Elems}', \; \mathsf{Emp}', \; \mathsf{Enq}', \; \mathsf{Deq}'\big),$$

on $H' := \mathsf{Queues} \cup K$ with $\mathsf{Queues}' = \mathsf{Queues}$, $\mathsf{Elems}' = K$, $\mathsf{Emp}' = \mathsf{Emp}$, and

$$\mathsf{Enq}'(q, \; g(a), \; q') \iff \mathsf{Enq}(q, a, q'), \qquad \mathsf{Deq}'(q, \; q', \; g(a)) \iff \mathsf{Deq}(q, q', a).$$

Example 5.5 (Meta-Queue in practice: two checkout lanes)**.** Consider two ordinary FIFO queues (checkout lanes) at a grocery store:

$$Q_{\text{reg}} = [\, r_1, r_2 \,], \qquad Q_{\text{exp}} = [\, e_1 \,].$$

Form their *Meta-Queue* by a parallel (direct-sum) constructor

$$\Phi_{\oplus}(Q_{\text{reg}}, Q_{\text{exp}}) \; := \; (Q_{\text{reg}}, \, Q_{\text{exp}}),$$

so the state is a pair of queues. We use componentwise operations:

$$\text{enq}_{\text{reg}}\big((Q_{\text{reg}}, Q_{\text{exp}}), x\big) \; := \; \big(Q_{\text{reg}} \parallel x, \; Q_{\text{exp}}\big), \qquad \text{enq}_{\text{exp}}\big((Q_{\text{reg}}, Q_{\text{exp}}), y\big) \; := \; \big(Q_{\text{reg}}, \; Q_{\text{exp}} \parallel y\big),$$

$$\text{deq}_{\text{reg}}\big((Q_{\text{reg}}, Q_{\text{exp}})\big) \; := \; \big(Q'_{\text{reg}}, Q_{\text{exp}}; \; \text{served head of } Q_{\text{reg}}\big),$$

$$\text{deq}_{\text{exp}}\big((Q_{\text{reg}}, Q_{\text{exp}})\big) \; := \; \big(Q_{\text{reg}}, Q'_{\text{exp}}; \; \text{served head of } Q_{\text{exp}}\big),$$

where "$\parallel$" appends to the tail and Q' denotes the queue after removing its head (if nonempty).

A concrete sequence:

$$(Q_{\text{reg}}, Q_{\text{exp}}) = \big([r_1, r_2], [e_1]\big) \xrightarrow{\text{enq}_{\text{exp}}(e_2)} \big([r_1, r_2], [e_1, e_2]\big) \xrightarrow{\text{deq}_{\text{exp}}} \big([r_1, r_2], [e_2]\big), \; \text{served } e_1.$$

This Meta-Queue models two lanes uniformly, lets us add discipline-specific policies on top (e.g., "always serve express if nonempty"), and preserves the FIFO behavior within each lane.

Theorem 5.6 (Meta-Queue is a MetaStructure and generalizes queues)**.** *$\mathbb{M}_{\text{que}} = \big(U_{\text{que}}, (\Phi_\ell)\big)$ of Definition 5.4 is a MetaStructure (Definition 1.2). Moreover, every classical finite FIFO queue over a finite alphabet E (Remark 5.3) is an object of U_{que}; hence Meta-Queue* generalizes *queues.*

Proof. Uniform constructors and closure. For (PROD), finiteness/typing are immediate; Enq and Deq are total/partial functional componentwise by (Q2)–(Q3). FIFO coherence (Q4) holds by applying it in each component; the empty/front law (Q5) is also componentwise.

For (COPROD), tags keep components disjoint; inside each tag, (Q2)–(Q5) hold as in the source queues; mixed tags never activate relations, so typing and axioms are preserved.

For (RES), the admissibility conditions exactly guarantee that the restrictions retain (Q2)–(Q3); the formulas in (Q4)–(Q5) are universally quantified and therefore remain true on the restricted domains.

For (PUSH), we only rename elements; totality/partiality, FIFO coherence and emptiness are preserved verbatim.

Naturality. An isomorphism $\alpha : \mathbf{Q} \to \mathbf{Q}'$ is a bijection of carriers that preserves the colored parts Queues, Elems and the relations Emp, Enq, Deq. For (PROD) and (COPROD), take componentwise/tagwise products of isomorphisms. For (RES), restrict α to $S_Q \cup S_E$. For (PUSH), the given g serves as the element part of the isomorphism and identity on Queues. In each case the induced map preserves all relations, so each Φ_ℓ is natural.

Finally, the generalization claim follows from Remark 5.3. □

Proposition 5.7 (Behavior of $\mathsf{Enq}/\mathsf{Deq}$ under META operations)**.** *Let* $\mathbf{Q}_i \in U_{\text{que}}$ *and* (S_Q, S_E) *admissible in* $\mathbf{Q}$.

(i) Product law: *If* $\mathsf{Enq}_i(q_i, a_i, q_i')$ *then* $\mathsf{Enq}\big((q_1,q_2),(a_1,a_2),(q_1',q_2')\big)$*, and similarly for* Deq.

(ii) Restriction law: $\mathsf{Enq}(q,a,q')$ *with* $q,q' \in S_Q$*,* $a \in S_E$ *holds in* $\Phi_{\mathrm{res}(S_Q,S_E)}(\mathbf{Q})$ *iff it holds in* $\mathbf{Q}$*; similarly for* Deq.

(iii) Pushforward law: $\mathsf{Enq}'(q,g(a),q')$ *(resp.* $\mathsf{Deq}'(q,q',g(a))$*) holds in* $\Phi_{\mathrm{push}(g)}(\mathbf{Q})$ *iff* $\mathsf{Enq}(q,a,q')$ *(resp.* $\mathsf{Deq}(q,q',a)$*) holds in* $\mathbf{Q}$.

Proof. Immediate from the definitions of the meta-constructors. □

Definition 5.8 (Iterated Meta-Queue of depth t)**.** For $t \in \mathbb{N}$, an *Iterated Meta-Queue of depth t* is an Iterated MetaStructure (Definition 1.3) over Σ_{que},

$$\mathfrak{M}^{(t)}_{\text{que}} = \big(U^{(t)}_{\text{que}},\ (\odot^{(t)}_\ell)_{\ell\in\Lambda}\big),$$

obtained by repeatedly applying the lifting functor $\mathsf{U}_{\Sigma_{\text{que}}}$ to $\mathbb{M}_{\text{que}}$. Concretely, if $s < t$ and Φ_ℓ has meta-arity k_ℓ, its lift

$$\Phi^{\uparrow}_\ell : \big(U^{(t)}_{\text{que}}\big)^{k_\ell} \longrightarrow U^{(t)}_{\text{que}}$$

is defined on representatives by

$$\Phi^{\uparrow}_\ell\big(\mathsf{U}^{t-s}_{\Sigma_{\text{que}}}(X_1),\dots,\mathsf{U}^{t-s}_{\Sigma_{\text{que}}}(X_{k_\ell})\big) := \mathsf{U}^{t-s}_{\Sigma_{\text{que}}}\big(\Phi_\ell(X_1,\dots,X_{k_\ell})\big),$$

and similarly for all relations. We denote by $\odot^{(t)}_{\times}$, $\odot^{(t)}_{\sqcup}$, $\odot^{(t)}_{\mathrm{res}(S_Q,S_E)}$, $\odot^{(t)}_{\mathrm{push}(g)}$ the respective lifts of (PROD), (COPROD), (RES), (PUSH).

Example 5.9 (Iterated Meta-Queue (queue of queues): fair print scheduler)**.** In a shared printer, each user maintains a FIFO of jobs, and the system schedules users in round-robin. This is naturally an *Iterated Meta-Queue*: an outer queue of users, and for each user u, an inner queue $J(u)$ of print jobs.

State. Outer queue of users:

$$Q_{\text{users}} = [\,u_1, u_2\,], \qquad J(u_1) = [\,a, b\,],\ \ J(u_2) = [\,c\,].$$

Operations (lifted).

$$\begin{aligned} \mathrm{enqJob}\big(u,x\big) &: \text{append } x \text{ to } J(u);\ \text{ if } u \notin Q_{\mathrm{users}} \text{ then append } u \text{ to } Q_{\mathrm{users}}. \\ \mathrm{deqOne} &: \text{pop head user } u \text{ from } Q_{\mathrm{users}},\ \text{pop head job } x \text{ from } J(u); \\ &\quad\ \text{return } x;\ \text{ if } J(u) \neq \varnothing \text{ then append } u \text{ to tail of } Q_{\mathrm{users}}. \end{aligned}$$

Execution trace.

$$\underbrace{\big([u_1,u_2];\ J(u_1)=[a,b],\ J(u_2)=[c]\big)}_{\text{initial}} \xrightarrow{\text{deqOne}} \big([u_2,u_1];\ J(u_1)=[b],\ J(u_2)=[c]\big);\ \text{return } a.$$

$$\xrightarrow{\text{deqOne}} \big([u_1,u_2];\ J(u_1)=[b],\ J(u_2)=\varnothing\big);\ \text{return } c.$$

$$\xrightarrow{\mathrm{enqJob}(u_2,d)} \big([u_1,u_2];\ J(u_1)=[b],\ J(u_2)=[d]\big).$$

$$\xrightarrow{\text{deqOne}} \big([u_2];\ J(u_1)=\varnothing,\ J(u_2)=[d]\big);\ \text{return } b.$$

The outer FIFO enforces user fairness (round-robin), while each inner FIFO preserves per-user submission order. This is exactly the "queue-of-queues" pattern obtained by lifting queue operations to one higher meta-level (an Iterated Meta-Queue).

Theorem 5.10 (Iterated Meta-Queue is an Iterated MetaStructure and generalizes Meta-Queue)**.** *For every $t \in \mathbb{N}$, $\mathfrak{M}^{(t)}_{\mathrm{que}}$ of Definition 5.8 is an Iterated MetaStructure (Definition 1.3). Moreover, for $s < t$ the embedding*

$$\iota_{s\to t} : \mathfrak{M}^{(s)}_{\mathrm{que}} \hookrightarrow \mathfrak{M}^{(t)}_{\mathrm{que}}, \qquad X \longmapsto \mathsf{U}^{t-s}_{\Sigma_{\mathrm{que}}}(X),$$

preserves all lifted meta-operations:

$$\Phi^{\uparrow}_{\ell}\big(\iota_{s\to t}(X_1),\dots,\iota_{s\to t}(X_{k_\ell})\big) = \iota_{s\to t}\big(\Phi_\ell(X_1,\dots,X_{k_\ell})\big).$$

In particular, $\mathfrak{M}^{(0)}_{\mathrm{que}} = \mathbb{M}_{\mathrm{que}}$ embeds into $\mathfrak{M}^{(t)}_{\mathrm{que}}$, so Iterated Meta-Queue generalizes *Meta-Queue.*

Proof. By Definition 1.3, each lifted constructor is obtained by post-composing the base constructors with $\mathsf{U}^{t-s}_{\Sigma_{\mathrm{que}}}$; hence uniform at every height. If $\alpha_i : X_i \xrightarrow{\cong} Y_i$ are level-s isomorphisms, then

$$\mathsf{U}^{t-s}_{\Sigma_{\mathrm{que}}}(\alpha_i) : \mathsf{U}^{t-s}_{\Sigma_{\mathrm{que}}}(X_i) \xrightarrow{\cong} \mathsf{U}^{t-s}_{\Sigma_{\mathrm{que}}}(Y_i)$$

are level-t isomorphisms, and

$$\Phi^{\uparrow}_{\ell}\big(\mathsf{U}^{t-s}\alpha_1,\dots,\mathsf{U}^{t-s}\alpha_{k_\ell}\big) = \mathsf{U}^{t-s}\big(\Phi_\ell(\alpha_1,\dots,\alpha_{k_\ell})\big),$$

which is an isomorphism because each base Φ_ℓ is natural by Theorem 5.6. Operation preservation of $\iota_{s\to t}$ is the same identity with X_i in place of α_i. Taking $s = 0$ yields the claimed embedding. □

6 Meta-Markov Chains: Markov Chain of Markov Chains

A Markov chain is a stochastic process with memoryless transitions between states, governed by a fixed probability transition matrix [59–62]. A Meta-Markov Chain is a MetaStructure treating Markov chains as objects, enabling meta-operations like composition, restriction, or probabilistic transformation. An Iterated Meta-Markov Chain recursively applies Meta-Markov construction, forming hierarchical layers of chains of chains, generalizing classical stochastic processes.

Definition 6.1 (Markov Chain)**.** (cf. [63,64]) A *Markov chain* is a discrete-time stochastic process $\{X_n\}_{n\geq 0}$ on a state space S such that

$$\Pr(X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \dots, X_0 = i_0) = \Pr(X_{n+1} = j \mid X_n = i) =: p_{ij}.$$

The matrix $P = (p_{ij})_{i,j\in S}$ is called the *transition matrix.*

Fix the single-sorted, finitary signature

$$\Sigma_{\mathrm{mc}} = \big(\mathsf{Func} = \varnothing,\ \mathsf{Rel} = \{ \text{States},\ \mathrm{I},\ \text{Trans} \},\ \text{ar} \big),$$

where the intended arities are

$$\text{States} : 1, \qquad \mathrm{I} : 1, \qquad \text{Trans} : 3.$$

A Σ_{mc}-structure is a tuple

$$\mathbf{M} = \big(H;\ \text{States}^{\mathbf{M}},\ \mathrm{I}^{\mathbf{M}},\ \text{Trans}^{\mathbf{M}}\big),$$

with nonempty carrier H and relations interpreted over H.

Definition 6.2 (Universe of finite Markov chains)**.** Let $U_{\mathrm{mc}} \subseteq \mathrm{Str}_{\Sigma_{\mathrm{mc}}}$ be the class of all

$$\mathbf{M} = (H;\ \text{States},\ \mathrm{I},\ \text{Trans})$$

satisfying the following axioms:

(M1) **Typing/finite:** States is finite and nonempty; I is in bijection with the unit interval.

(M2) **Value identification:** There is a fixed identification $\iota_{\mathbf{M}} : \mathrm{I} \xrightarrow{\cong} [0,1]$ (used to state numeric axioms).

(M3) **Total functional kernel:** Trans is the graph of a total functional map P : States × States $\to$ I: for every (x, y) there is a unique $p \in \mathrm{I}$ with $\text{Trans}(x, y, p)$.

(M4) **Row-stochasticity:** For all $x \in$ States,

$$\sum_{y \in \text{States}} \iota_{\mathbf{M}}\big(P(x, y)\big) = 1.$$

Thus each $\mathbf{M} \in U_{\mathrm{mc}}$ canonically encodes a finite, time-homogeneous Markov chain (S, P) with $S =$ States and $P(x, y) = \iota_{\mathbf{M}}\,\text{Trans}(x, y, \cdot) \in [0, 1]$.

Remark 6.3 (Embedding a classical finite Markov chain)**.** Given a finite set S and a row-stochastic matrix $P : S \times S \to [0, 1]$, set

$$H := \underbrace{\{0\} \times S}_{\text{states}} \cup \underbrace{\{1\} \times [0,1]}_{\text{probabilities}}, \qquad \text{States} := \{0\} \times S,\ \ \mathrm{I} := \{1\} \times [0,1],$$

and put

$$\text{Trans}\big((0,x), (0,y), (1,p)\big) \iff p = P(x,y), \quad \iota_{\mathbf{M}}(1,p) := p.$$

Then $\mathbf{M} \in U_{\mathrm{mc}}$ by (M1)–(M4).

Definition 6.4 (Meta-Markov Chain)**.** A *Meta-Markov Chain* is a MetaStructure (Definition 1.2) over Σ_{mc},

$$\mathbb{M}_{\mathrm{mc}} = \big(U_{\mathrm{mc}},\ (\Phi_\ell)_{\ell \in \Lambda}\big),$$

whose meta-operations are specified uniformly as follows (each by carrier- and relation-constructors).

(PROD) Independent product of chains. For $\mathbf{M}_i = (H_i; \text{States}_i, \mathrm{I}_i, \text{Trans}_i) \in U_{\mathrm{mc}}$ $(i = 1, 2)$ define

$$\Phi_\otimes(\mathbf{M}_1, \mathbf{M}_2) := (H_\times; \text{States}_\times, \mathrm{I}_\times, \text{Trans}_\times)$$

on the tagged carrier $H_\times := \{0\} \times (\text{States}_1 \times \text{States}_2)\ \cup\ \{1\} \times [0, 1]$ with

$$\text{States}_\times := \{0\} \times (\text{States}_1 \times \text{States}_2), \qquad \mathrm{I}_\times := \{1\} \times [0,1],$$

and kernel

$$\text{Trans}_\times\Big((0, (x_1, x_2)), (0, (y_1, y_2)), (1, p)\Big) \iff p = \iota_{\mathbf{M}_1} P_1(x_1, y_1) \cdot \iota_{\mathbf{M}_2} P_2(x_2, y_2).$$

(MIX) Convex mixture on a common state set. If $\mathbf{M}_1, \mathbf{M}_2 \in U_{\mathrm{mc}}$ have (up to isomorphism) the same state set S and $\lambda \in (0,1)$, set

$$\Phi_{\mathrm{mix}(\lambda)}(\mathbf{M}_1, \mathbf{M}_2) := (H; \mathrm{States}, \mathrm{I}, \mathrm{Trans})$$

on $H := \{0\} \times S \cup \{1\} \times [0,1]$ with the obvious colors and

$$\mathrm{Trans}\big((0,x),(0,y),(1,p)\big) \iff p = \lambda\, \iota_{\mathbf{M}_1} P_1(x,y) + (1-\lambda)\, \iota_{\mathbf{M}_2} P_2(x,y).$$

(RES) Restriction to a closed subset. For $\mathbf{M} \in U_{\mathrm{mc}}$ and $S \subseteq \mathrm{States}$ *closed* under P (i.e. $\sum_{y\in S} \iota_{\mathbf{M}} P(x,y) = 1$ for all $x \in S$), define

$$\Phi_{\mathrm{res}(S)}(\mathbf{M}) := (H_S; \mathrm{States}_S, \mathrm{I}_S, \mathrm{Trans}_S),$$

on $H_S := \{0\} \times S \cup \{1\} \times [0,1]$, with

$$\mathrm{States}_S := \{0\} \times S, \quad \mathrm{I}_S := \{1\} \times [0,1], \quad \mathrm{Trans}_S\big((0,x),(0,y),(1,p)\big) \iff x, y \in S,\ p = \iota_{\mathbf{M}} P(x,y).$$

(PUSH) State relabeling by a bijection $f : \mathrm{States} \to K$. For $\mathbf{M} = (H; \mathrm{States}, \mathrm{I}, \mathrm{Trans})$ and any bijection f set

$$\Phi_{\mathrm{push}(f)}(\mathbf{M}) := (\widehat{H}; \widehat{\mathrm{States}}, \widehat{\mathrm{I}}, \widehat{\mathrm{Trans}}),$$

on $\widehat{H} := \{0\} \times K \cup \{1\} \times [0,1]$, with $\widehat{\mathrm{States}} := \{0\} \times K$, $\widehat{\mathrm{I}} := \{1\} \times [0,1]$, and

$$\widehat{\mathrm{Trans}}\big((0,f(x)),(0,f(y)),(1,p)\big) \iff p = \iota_{\mathbf{M}} P(x,y).$$

(LUMP) Coarse-graining by a lumpable partition. Let $\pi : \mathrm{States} \twoheadrightarrow C$ be a surjection (blocks are $\pi^{-1}(c)$). Assume *strong lumpability*: for all $c, c' \in C$ and any $x, x' \in \pi^{-1}(c)$,

$$\sum_{y\in\pi^{-1}(c')} \iota_{\mathbf{M}} P(x,y) = \sum_{y\in\pi^{-1}(c')} \iota_{\mathbf{M}} P(x',y).$$

Define

$$\Phi_{\mathrm{lump}(\pi)}(\mathbf{M}) := (H_C; \mathrm{States}_C, \mathrm{I}_C, \mathrm{Trans}_C),$$

on $H_C := \{0\} \times C \cup \{1\} \times [0,1]$ with $\mathrm{States}_C := \{0\} \times C$, $\mathrm{I}_C := \{1\} \times [0,1]$, and

$$\mathrm{Trans}_C\big((0,c),(0,c'),(1,p)\big) \iff p = \sum_{y\in\pi^{-1}(c')} \iota_{\mathbf{M}} P(x,y) \quad (\text{for any } x \in \pi^{-1}(c)).$$

By lumpability the right-hand side is independent of the choice of x.

Example 6.5 (Meta–Markov Chain: joint weather–traffic dynamics)**.** Consider two ordinary (time-homogeneous) Markov chains.

Weather on $W = \{\mathrm{S}, \mathrm{R}\}$ (Sunny/ Rainy) with kernel

$$P_W = \begin{bmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{bmatrix}, \qquad \text{rows/columns ordered as } (\mathrm{S}, \mathrm{R}).$$

Traffic on $T = \{\mathrm{L}, \mathrm{H}\}$ (Light/ Heavy) with kernel

$$P_T = \begin{bmatrix} 0.7 & 0.3 \\ 0.2 & 0.8 \end{bmatrix}, \qquad \text{rows/columns ordered as } (\mathrm{L}, \mathrm{H}).$$

The *Meta–Markov product* $\Phi_\otimes$ forms the joint chain $X_t = (W_t, T_t)$ on $W \times T = \{(\mathrm{S},\mathrm{L}), (\mathrm{S},\mathrm{H}), (\mathrm{R},\mathrm{L}), (\mathrm{R},\mathrm{H})\}$ with transition kernel

$$P_{W\otimes T} = P_W \otimes P_T,$$

i.e.,

$$\Pr\big((w,t) \to (w',t')\big) = P_W(w,w')\, P_T(t,t').$$

For instance,

$$\Pr\big((\mathrm{S},\mathrm{L}) \to (\mathrm{R},\mathrm{H})\big) = 0.2 \times 0.3 = 0.06.$$

This Meta–operation preserves the Markov property while combining two independent subsystems (weather and traffic) into a single joint stochastic dynamical model.

Theorem 6.6 (Meta-Markov Chain is a MetaStructure and generalizes Markov chains)**.** $\mathbb{M}_{\mathrm{mc}} = \big(U_{\mathrm{mc}}, (\Phi_\ell)\big)$ *of Definition 6.4 is a MetaStructure (Definition 1.2). Moreover, every classical finite Markov chain* (S, P) *appears (via Remark 6.3) as an object of* U_{mc}*; hence Meta-Markov Chains* generalize *Markov chains.*

Proof. Uniform constructors and closure. For each meta-operation we check (M1)–(M4).

(PROD) $\mathrm{States}_\times$ is finite/nonempty. For fixed (x_1, x_2),

$$\sum_{y_1, y_2} \iota P_\times\big((x_1, x_2), (y_1, y_2)\big) = \Big(\sum_{y_1} \iota P_1(x_1, y_1)\Big)\Big(\sum_{y_2} \iota P_2(x_2, y_2)\Big) = 1,$$

so row-stochasticity holds; total functionality is by construction.

(MIX) For each x, $\sum_y \big(\lambda \iota P_1(x, y) + (1 - \lambda)\iota P_2(x, y)\big) = \lambda + (1 - \lambda) = 1$. Values lie in $[0, 1]$ and totality/functionality are immediate.

(RES) Closure of S gives $\sum_{y \in S} \iota P(x, y) = 1$ for all $x \in S$, hence (M4) holds on the subcarrier; other axioms are inherited.

(PUSH) Only labels are changed; the kernel is transported verbatim, preserving (M3)–(M4).

(LUMP) By strong lumpability, for fixed c the quantity $p(c, c') := \sum_{y \in \pi^{-1}(c')} \iota P(x, y)$ does not depend on $x \in \pi^{-1}(c)$. Then

$$\sum_{c'} p(c, c') = \sum_{c'} \sum_{y \in \pi^{-1}(c')} \iota P(x, y) = \sum_{y \in \mathrm{States}} \iota P(x, y) = 1,$$

so (M4) holds; (M1)–(M3) are clear.

Naturality (isomorphism invariance). An isomorphism $\alpha : \mathbf{M} \to \mathbf{M}'$ is a bijection of carriers preserving States, I and Trans. For (PROD), use $\alpha_1 \times \alpha_2$ on states and the identity on the probability tag; verification is componentwise. For (MIX), the common state-set isomorphism transports kernels linearly. For (RES), restrict α to the closed subset; for (PUSH), take the given f as the state part. For (LUMP), transport the partition along the state isomorphism and use that sums over fibers are preserved. Each induced map is an isomorphism of outputs. Finally, Remark 6.3 yields the generalization claim. □

Proposition 6.7 (Stationary distributions under META operations)**.** *Let* π_i *be stationary distributions of* $\mathbf{M}_i$ *(*$i = 1, 2$*).*

(i) Product: $\pi_\times(x_1, x_2) := \pi_1(x_1)\pi_2(x_2)$ *is stationary for* $\Phi_\otimes(\mathbf{M}_1, \mathbf{M}_2)$.

(ii) Mixture: *If* $\mathrm{supp}(\pi_1) = \mathrm{supp}(\pi_2)$*, then* $\pi := \lambda\pi_1 + (1 - \lambda)\pi_2$ *is stationary for* $\Phi_{\mathrm{mix}(\lambda)}(\mathbf{M}_1, \mathbf{M}_2)$.

(iii) Restriction: *If* S *is closed and* π *is stationary on* $\mathbf{M}$ *with* $\pi(S) = 1$*, then* $\pi|_S$ *is stationary for* $\Phi_{\mathrm{res}(S)}(\mathbf{M})$.

(iv) Pushforward: *If* π *is stationary on* $\mathbf{M}$ *and* f *is a bijection, then* $\widehat{\pi}(f(x)) := \pi(x)$ *is stationary on* $\Phi_{\mathrm{push}(f)}(\mathbf{M})$.

(v) Lumping: *If* π *is stationary and* π *is constant on blocks of a lumpable partition* $\pi : \mathrm{States} \to C$*, then* $\pi_C(c) := \sum_{x \in \pi^{-1}(c)} \pi(x)$ *is stationary for* $\Phi_{\mathrm{lump}(\pi)}(\mathbf{M})$.

Proof. Routine computations using the defining identities of each meta-operation. □

Definition 6.8 (Iterated Meta-Markov Chain of depth t)**.** For $t \in \mathbb{N}$, an *Iterated Meta-Markov Chain of depth t* is an Iterated MetaStructure (Definition 1.3) over Σ_{mc},

$$\mathfrak{M}^{(t)}_{\mathrm{mc}} = \big(U^{(t)}_{\mathrm{mc}},\ (\odot^{(t)}_\ell)_{\ell \in \Lambda}\big),$$

obtained by applying the lifting functor $\mathsf{U}_{\Sigma_{\mathrm{mc}}}$ to $\mathbb{M}_{\mathrm{mc}}$ repeatedly t times. Concretely, for $s < t$ and any base meta-operation Φ_ℓ of meta-arity k_ℓ,

$$\Phi^{\uparrow}_\ell : \big(U^{(t)}_{\mathrm{mc}}\big)^{k_\ell} \longrightarrow U^{(t)}_{\mathrm{mc}}$$

is defined on representatives by

$$\Phi_\ell^{\uparrow}\big(\mathsf{U}_{\Sigma_{\mathrm{mc}}}^{t-s}(X_1),\dots,\mathsf{U}_{\Sigma_{\mathrm{mc}}}^{t-s}(X_{k_\ell})\big) := \mathsf{U}_{\Sigma_{\mathrm{mc}}}^{t-s}\big(\,\Phi_\ell(X_1,\dots,X_{k_\ell})\,\big),$$

and similarly for all relations. We denote by $\odot_\otimes^{(t)}$, $\odot_{\mathrm{mix}(\lambda)}^{(t)}$, $\odot_{\mathrm{res}(S)}^{(t)}$, $\odot_{\mathrm{push}(f)}^{(t)}$, $\odot_{\mathrm{lump}(\pi)}^{(t)}$ the respective lifts of (PROD), (MIX), (RES), (PUSH), (LUMP).

Example 6.9 (Iterated Meta–Markov Chain: Markov–modulated customer load)**.** A store's *environment* (Normal/Sale) evolves as a two–state Markov chain $E_t \in \{\mathrm{N},\mathrm{S}\}$ with kernel

$$Q \;=\; \begin{bmatrix} 0.9 & 0.1 \\ 0.3 & 0.7 \end{bmatrix}, \qquad \text{rows/columns } (\mathrm{N},\mathrm{S}).$$

Conditional on the environment, the *customer–load* chain $C_t \in \{\mathrm{Low},\mathrm{Med},\mathrm{High}\}$ transitions via different kernels:

$$P^{(\mathrm{N})} \;=\; \begin{bmatrix} 0.75 & 0.20 & 0.05 \\ 0.20 & 0.60 & 0.20 \\ 0.10 & 0.30 & 0.60 \end{bmatrix}, \qquad P^{(\mathrm{S})} \;=\; \begin{bmatrix} 0.50 & 0.35 & 0.15 \\ 0.15 & 0.45 & 0.40 \\ 0.05 & 0.30 & 0.65 \end{bmatrix}.$$

The *Iterated Meta–Markov* construction lifts the base Meta–operations to a two–level process on the product state space $\{\mathrm{N},\mathrm{S}\}\times\{\mathrm{Low},\mathrm{Med},\mathrm{High}\}$. With the "*switch–then–move*" update, the joint kernel Π is

$$\Pi\big((e,i),(e',j)\big) \;=\; Q(e,e') \;\cdot\; P^{(e')}(i,j),$$

first updating the environment $e \longrightarrow e'$ via Q, then moving the load $i \longrightarrow j$ according to the kernel selected by the *new* environment e'.

As an illustration,

$$\Pr\big((\mathrm{N},\mathrm{Med}) \to (\mathrm{S},\mathrm{High})\big) \;=\; Q(\mathrm{N},\mathrm{S})\cdot P^{(\mathrm{S})}(\mathrm{Med},\mathrm{High}) \;=\; 0.1\times 0.40 \;=\; 0.04.$$

This "Markov–modulated Markov chain" (a queue–of–kernels viewpoint) is the canonical real–world instance of an *Iterated* Meta–Markov structure: an outer chain over regimes and an inner chain whose dynamics are functorially selected and composed by the outer state.

Theorem 6.10 (Iterated Meta-Markov Chain is an Iterated MetaStructure and generalizes Meta-Markov)**.** *For every $t \in \mathbb{N}$, $\mathfrak{M}_{\mathrm{mc}}^{(t)}$ of Definition 6.8 is an Iterated MetaStructure (Definition 1.3). Moreover, for $s < t$ the embedding*

$$\iota_{s\to t} : \mathfrak{M}_{\mathrm{mc}}^{(s)} \hookrightarrow \mathfrak{M}_{\mathrm{mc}}^{(t)}, \qquad X \longmapsto \mathsf{U}_{\Sigma_{\mathrm{mc}}}^{t-s}(X),$$

preserves all lifted meta-operations:

$$\Phi_\ell^{\uparrow}\big(\iota_{s\to t}(X_1),\dots,\iota_{s\to t}(X_{k_\ell})\big) = \iota_{s\to t}\big(\Phi_\ell(X_1,\dots,X_{k_\ell})\big).$$

In particular, $\mathfrak{M}_{\mathrm{mc}}^{(0)} = \mathbb{M}_{\mathrm{mc}}$ embeds into every $\mathfrak{M}_{\mathrm{mc}}^{(t)}$, so Iterated Meta-Markov Chains generalize *Meta-Markov Chains.*

Proof. By Definition 1.3, each lifted constructor is obtained by post-composing the base constructors with $\mathsf{U}_{\Sigma_{\mathrm{mc}}}^{t-s}$, hence uniform at all heights. If $\alpha_i : X_i \xrightarrow{\cong} Y_i$ are level-s isomorphisms, then

$$\mathsf{U}_{\Sigma_{\mathrm{mc}}}^{t-s}(\alpha_i) : \mathsf{U}_{\Sigma_{\mathrm{mc}}}^{t-s}(X_i) \xrightarrow{\cong} \mathsf{U}_{\Sigma_{\mathrm{mc}}}^{t-s}(Y_i)$$

are level-t isomorphisms, and

$$\Phi_\ell^{\uparrow}\big(\mathsf{U}^{t-s}\alpha_1,\dots,\mathsf{U}^{t-s}\alpha_{k_\ell}\big) = \mathsf{U}^{t-s}\big(\Phi_\ell(\alpha_1,\dots,\alpha_{k_\ell})\big),$$

which is an isomorphism because each base Φ_ℓ is natural by Theorem 6.6. The operation-preservation for $\iota_{s\to t}$ is the same identity with X_i in place of α_i. □

7 Meta-Intervals:Interval of Intervals

An interval is a connected subset of the real line, containing all points between any two of its elements. A Meta-Interval is a MetaStructure where intervals themselves are treated as objects, allowing operations like union, intersection, and scaling. An Iterated Meta-Interval recursively applies Meta-Interval construction, forming hierarchical layers of intervals of intervals, generalizing classical interval structures.

Definition 7.1 (Interval)**.** (cf. [65]) An *interval* in the real line $\mathbb{R}$ is a subset $I \subseteq \mathbb{R}$ such that for all $x, y \in I$ with $x < y$, the entire segment $\{z \in \mathbb{R} \mid x < z < y\}$ is contained in I. Typical forms include:

$$[a, b] = \{x \in \mathbb{R} \mid a \le x \le b\}, \quad (a, b) = \{x \in \mathbb{R} \mid a < x < b\},$$

with half-open intervals $[a, b)$, $(a, b]$ defined analogously.

Fix the single-sorted, finitary signature

$$\Sigma_{\text{int}} = \Big(\mathsf{Func} = \varnothing,$$

$$\mathsf{Rel} = \{ \mathsf{Vals},\ \mathsf{Bool},\ \mathsf{LeftEnd},\ \mathsf{RightEnd},\ \mathsf{ClosedL},\ \mathsf{ClosedR},\ \mathsf{Ord} \},$$

$$\mathrm{ar}(\mathsf{Vals}) = \mathrm{ar}(\mathsf{Bool}) = 1,\ \mathrm{ar}(\mathsf{LeftEnd}) = \cdots = \mathrm{ar}(\mathsf{Ord}) = 2\Big),$$

where Vals and Bool are unary "colors," $\mathsf{LeftEnd}, \mathsf{RightEnd}, \mathsf{ClosedL}, \mathsf{ClosedR} \subseteq H \times H$ are binary relations picking the (unique) endpoints and their closed/open flags, and $\mathsf{Ord} \subseteq H \times H$ encodes an order on values.

Definition 7.2 (Universe of encoded intervals)**.** Let

$$U_{\text{int}} \subseteq \mathrm{Str}_{\Sigma_{\text{int}}}$$

be the class of all

$$\mathbf{I} = (H; \mathsf{Vals}, \mathsf{Bool}, \mathsf{LeftEnd}, \mathsf{RightEnd}, \mathsf{ClosedL}, \mathsf{ClosedR}, \mathsf{Ord})$$

for which:

(I1) **Typing:** Vals and Bool are disjoint nonempty unary predicates.

(I2) **Value identification:** There is a fixed bijection $\iota_{\mathbf{I}} : \mathsf{Vals} \xrightarrow{\cong} S_{\mathbf{I}} \subseteq \mathbb{R}$ used only to state the axioms below; likewise a fixed bijection $\jmath_{\mathbf{I}} : \mathsf{Bool} \xrightarrow{\cong} \{0, 1\}$.

(I3) **Order:** Ord coincides with $\le$ on $S_{\mathbf{I}}$ through $\iota_{\mathbf{I}}$: for $u, v \in \mathsf{Vals}$,

$$\mathsf{Ord}(u, v) \iff \iota_{\mathbf{I}}(u) \le \iota_{\mathbf{I}}(v).$$

(I4) **Unique endpoints and flags:** There exist unique $a, b \in \mathsf{Vals}$ and unique $c_L, c_R \in \mathsf{Bool}$ such that

$$\mathsf{LeftEnd}(*, a),\ \ \mathsf{RightEnd}(*, b),$$

$$\mathsf{ClosedL}(*, c_L),\ \ \mathsf{ClosedR}(*, c_R),$$

and $\mathsf{Ord}(a, b)$ (i.e. $\iota_{\mathbf{I}}(a) \le \iota_{\mathbf{I}}(b)$).

We interpret $\mathbf{I}$ as the real interval

$$\begin{aligned} \mathrm{Int}(\mathbf{I}) \ :=\ \Big\{ x \in \mathbb{R} \ \Big|\ & \iota_{\mathbf{I}}(a) < x < \iota_{\mathbf{I}}(b) \\ & \text{or } (\jmath_{\mathbf{I}}(c_L) = 1 \wedge x = \iota_{\mathbf{I}}(a)) \\ & \text{or } (\jmath_{\mathbf{I}}(c_R) = 1 \wedge x = \iota_{\mathbf{I}}(b)) \ \Big\}. \end{aligned}$$

Remark 7.3 (Embedding classical intervals)**.** Given any classical interval $I \subseteq \mathbb{R}$ of one of the four types $[a, b], (a, b), [a, b), (a, b]$ (with $a \le b$), construct $\mathbf{I}_I \in U_{\text{int}}$ by taking $\mathsf{Vals} = \{a^\star, b^\star\}$, $\mathsf{Bool} = \{0^\star, 1^\star\}$, $\iota_{\mathbf{I}}(a^\star) = a$, $\iota_{\mathbf{I}}(b^\star) = b$, $\jmath_{\mathbf{I}}(1^\star) = 1$, $\jmath_{\mathbf{I}}(0^\star) = 0$, $\mathsf{Ord}(a^\star, b^\star)$, and setting flags c_L, c_R so that $\jmath_{\mathbf{I}}(c_L) = 1$ iff a is included in I, and $\jmath_{\mathbf{I}}(c_R) = 1$ iff b is included in I. Then $\mathrm{Int}(\mathbf{I}_I) = I$.

Definition 7.4 (Meta-Interval)**.** A *Meta-Interval* is a MetaStructure (Definition 1.2) over Σ_{int},

$$\mathbb{M}_{\text{int}} = \big(U_{\text{int}},\ (\Phi_\ell)_{\ell \in \Lambda}\big),$$

whose meta-operations are specified uniformly as follows (each by carrier- and relation-constructors):

(AFF) Positive affine image. For $\mathbf{I} \in U_{\text{int}}$, $s > 0$ and $t \in \mathbb{R}$, define

$$\Phi_{\text{aff}(s,t)}(\mathbf{I}) \in U_{\text{int}}$$

by transporting endpoints via $x \mapsto sx + t$: if a, b, c_L, c_R are the unique data in (I4), then the new endpoints are $a' := s \cdot a + t$, $b' := s \cdot b + t$ under ι, and the new flags equal the old flags ($c'_L := c_L$, $c'_R := c_R$).

(SUM) Minkowski sum. For $\mathbf{I}_1, \mathbf{I}_2 \in U_{\text{int}}$ with data $(a_1, b_1, c_{L,1}, c_{R,1})$, $(a_2, b_2, c_{L,2}, c_{R,2})$, set

$$\Phi_{\oplus}(\mathbf{I}_1, \mathbf{I}_2) \in U_{\text{int}}$$

with endpoints $a_\oplus := a_1 + a_2$, $b_\oplus := b_1 + b_2$ (through ι) and flags

$$c_{L,\oplus} := c_{L,1} \wedge c_{L,2}, \qquad c_{R,\oplus} := c_{R,1} \wedge c_{R,2}$$

(where $\wedge$ is interpreted via $\jmath^{-1}$ as Boolean AND on $\{0, 1\}$).

(INT) Intersection. For $\mathbf{I}_1, \mathbf{I}_2 \in U_{\text{int}}$, define

$$\Phi_{\cap}(\mathbf{I}_1, \mathbf{I}_2) \in U_{\text{int}}$$

with endpoints $a_\cap := \max\{a_1, a_2\}$, $b_\cap := \min\{b_1, b_2\}$ (w.r.t. Ord), and flags determined by the standard interval-intersection rule:

$$c_{L,\cap} = \begin{cases} c_{L,1}, & a_1 > a_2, \\ c_{L,2}, & a_2 > a_1, \\ c_{L,1} \wedge c_{L,2}, & a_1 = a_2, \end{cases} \qquad c_{R,\cap} = \begin{cases} c_{R,1}, & b_1 < b_2, \\ c_{R,2}, & b_2 < b_1, \\ c_{R,1} \wedge c_{R,2}, & b_1 = b_2. \end{cases}$$

(When $a_\cap > b_\cap$, the resulting encoded interval is empty; this is allowed.)

(PUSH) Order-preserving relabeling. If $f : \mathbb{R} \to \mathbb{R}$ is a strictly increasing bijection, put

$$\Phi_{\text{push}(f)}(\mathbf{I}) \in U_{\text{int}}$$

by sending endpoints to $f(\iota(a))$, $f(\iota(b))$ (transported back through a chosen identification), and keeping flags c_L, c_R unchanged.

Example 7.5 (Meta–Interval: Coordinating meeting times with time–zone relabeling)**.** Model calendar windows as closed real intervals on the time axis (hours, UTC). Suppose Alice is free in

$$A = [9, 12] \quad \text{(UTC)},$$

while Bob is free in

$$B = [10.5, 11.5] \quad \text{(UTC)}.$$

The Meta–Interval intersection (a uniform constructor on interval objects) yields the common window

$$A \cap B = [10.5, 11.5] \quad \text{(UTC)}.$$

Applying the Meta–Interval *pushforward* by the affine bijection $t \mapsto t + 9$ (relabeling times from UTC to JST) transports this interval to

$$[10.5, 11.5] + 9 = [19.5, 20.5] \ = \ [19{:}30,\ 20{:}30] \quad \text{(JST)}.$$

Thus, the meta–operations "intersect" and "affine relabel" act on interval–objects to produce the concrete local meeting slot.

Proposition 7.6 (Endpoint laws)**.** *Let* $\mathbf{I}_i \in U_{\text{int}}$ *encode* $I_i = \text{Int}(\mathbf{I}_i) \subseteq \mathbb{R}$.

(i) $\text{Int}\big(\Phi_{\text{aff}(s,t)}(\mathbf{I}_1)\big) = s \cdot I_1 + t$ *for* $s > 0$.

(ii) $\text{Int}\big(\Phi_{\oplus}(\mathbf{I}_1, \mathbf{I}_2)\big) = I_1 + I_2 := \{x_1 + x_2 \mid x_i \in I_i\}$.

(iii) $\text{Int}\big(\Phi_{\cap}(\mathbf{I}_1, \mathbf{I}_2)\big) = I_1 \cap I_2$.

(iv) $\text{Int}\big(\Phi_{\text{push}(f)}(\mathbf{I}_1)\big) = f[I_1]$ *for strictly increasing bijection* f.

Proof. Each item is the classical endpoint calculus: (i) positive affine maps preserve interval type and endpoint inclusion; (ii) Minkowski sums add endpoints and include the extremal sum iff both summands include their corresponding extremal endpoints (Boolean AND); (iii) intersections take max of left and min of right endpoints, with inclusion inherited from the interval contributing the extremal endpoint, and AND when equal; (iv) increasing bijections send intervals to intervals, preserving endpoint inclusion. All statements are immediate from the definitions through $\iota, \jmath$. □

Theorem 7.7 (Meta-Interval is a MetaStructure and generalizes intervals)**.** $\mathbb{M}_{\text{int}} = (U_{\text{int}}, (\Phi_\ell))$ *of Definition 7.4 is a MetaStructure in the sense of Definition 1.2. Moreover, every classical real interval* $I \subseteq \mathbb{R}$ *appears (via Remark 7.3) as an object of* U_{int}*; hence Meta-Interval* generalizes *intervals.*

Proof. Uniform constructors and closure. For each meta-operation, the carrier- and relation-constructors define a new Σ_{int}-structure by specifying the two endpoints and two flags (through $\iota, \jmath$) and re-instating Ord as the pullback of $\leq$. Items (i)–(iv) of Proposition 7.6 show that the resulting object encodes a (possibly empty) interval, hence belongs to U_{int}.

Naturality. An isomorphism $\alpha : \mathbf{I} \to \mathbf{I}'$ in $\text{Str}_{\Sigma_{\text{int}}}$ is a bijection preserving colors and relations, in particular it carries the unique endpoints and flags of $\mathbf{I}$ to those of $\mathbf{I}'$ and respects Ord. For (AFF), (SUM), and (INT), the induced isomorphisms are obtained by transporting endpoint data componentwise (affine map, sum, max/min with flag logic); for (PUSH) they are given by the specified f. In each case the construction commutes with isomorphisms by definition, so the Φ_ℓ are natural. The generalization claim is Remark 7.3. □

Definition 7.8 (Iterated Meta-Interval of depth t)**.** For $t \in \mathbb{N}$, an *Iterated Meta-Interval of depth t* is an Iterated MetaStructure (Definition 1.3) over Σ_{int},

$$\mathfrak{M}^{(t)}_{\text{int}} = \big(U^{(t)}_{\text{int}},\ (\odot^{(t)}_\ell)_{\ell \in \Lambda}\big),$$

obtained by applying the lifting functor $\mathsf{U}_{\Sigma_{\text{int}}}$ to $\mathbb{M}_{\text{int}}$ repeatedly t times. Concretely, for $s < t$ and any base meta-operation Φ_ℓ of meta-arity k_ℓ,

$$\Phi^{\uparrow}_\ell : \big(U^{(t)}_{\text{int}}\big)^{k_\ell} \longrightarrow U^{(t)}_{\text{int}}, \qquad \Phi^{\uparrow}_\ell\big(\mathsf{U}^{t-s}_{\Sigma_{\text{int}}}(X_1), \ldots, \mathsf{U}^{t-s}_{\Sigma_{\text{int}}}(X_{k_\ell})\big) := \mathsf{U}^{t-s}_{\Sigma_{\text{int}}}\big(\Phi_\ell(X_1, \ldots, X_{k_\ell})\big),$$

and similarly for all relations.

Example 7.9 (Iterated Meta–Interval: Day–tagged weekly scheduling with lifted intersection)**.** Consider two days $D = \{\text{Mon}, \text{Tue}\}$. A *day–tagged interval family* is an iterated object

$$\mathcal{A} = \{\ (\text{Mon}, [9, 12]),\ (\text{Tue}, [13, 16])\ \}$$

encoding one interval per day (the inner level), and similarly an interviewer's availability

$$\mathcal{B} = \{\ (\text{Mon}, [10, 11]),\ (\text{Tue}, [14, 15.5])\ \}.$$

The *lifted* Meta–Interval intersection $\cap^{\uparrow}$ acts fiberwise over the outer day–index and intersects the inner intervals on each fiber:

$$\begin{aligned} \mathcal{A} \cap^{\uparrow} \mathcal{B} &= \{\ (\text{Mon}, [9, 12] \cap [10, 11]),\ (\text{Tue}, [13, 16] \cap [14, 15.5])\ \} \\ &= \{\ (\text{Mon}, [10, 11]),\ (\text{Tue}, [14, 15.5])\ \}. \end{aligned}$$

Hence the iterated structure (days ⇒ intervals) together with the lifted meta–operation selects concrete per–day meeting windows while preserving the outer indexing.

Theorem 7.10 (Iterated Meta-Interval is an Iterated MetaStructure and generalizes Meta-Interval)**.** *For every* $t \in \mathbb{N}$*,* $\mathfrak{M}^{(t)}_{\mathrm{int}}$ *of Definition 7.8 is an Iterated MetaStructure in the sense of Definition 1.3. Moreover, for* $s < t$ *the embedding*

$$\iota_{s\to t} : \mathfrak{M}^{(s)}_{\mathrm{int}} \hookrightarrow \mathfrak{M}^{(t)}_{\mathrm{int}}, \qquad X \longmapsto \mathsf{U}^{t-s}_{\Sigma_{\mathrm{int}}}(X),$$

preserves all lifted meta-operations. In particular, $\mathfrak{M}^{(0)}_{\mathrm{int}} = \mathbb{M}_{\mathrm{int}}$ *embeds into* $\mathfrak{M}^{(t)}_{\mathrm{int}}$*; hence Iterated Meta-Interval* generalizes *Meta-Interval.*

Proof. By Definition 1.3, each lifted constructor is obtained by post-composing the base constructors with $\mathsf{U}^{t-s}_{\Sigma_{\mathrm{int}}}$; therefore the resulting constructors are uniform. If $\alpha_i : X_i \overset{\cong}{\longrightarrow} Y_i$ are isomorphisms at level s, then $\mathsf{U}^{t-s}_{\Sigma_{\mathrm{int}}}(\alpha_i)$ are isomorphisms at level t, and

$$\Phi^{\uparrow}_{\ell}\big(\mathsf{U}^{t-s}\alpha_1, \dots, \mathsf{U}^{t-s}\alpha_{k_\ell}\big) = \mathsf{U}^{t-s}\big(\Phi_\ell(\alpha_1, \dots, \alpha_{k_\ell})\big)$$

is an isomorphism because each base Φ_ℓ is natural (Theorem 7.7). The operation-preservation of $\iota_{s\to t}$ is the same identity with X_i in place of α_i. □

8 Conclusion

In this work, we have extended and examined the properties of Algebra, Topology, Lattices, Queues, Markov Chains, and Intervals through the framework of MetaStructures and Iterated MetaStructures. For future research, we aim to explore the connections of these concepts with Fuzzy Sets [66–68], Intuitionistic Fuzzy Sets [69,70], HyperFuzzy Sets [28,71–73], Hesitant Fuzzy Sets [74,75], Neutrosophic Sets [76–79], Functorial Structures [1], Plithogenic Sets [80–83], and SuperHyperStructures [29,84].

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Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Research Integrity

The authors hereby confirm that, to the best of their knowledge, this manuscript is their original work, has not been published in any other journal, and is not currently under consideration for publication elsewhere at this stage.

Use of Generative AI and AI-Assisted Tools

I use generative AI and AI-assisted tools for tasks such as English grammar checking, and I do not employ them in any way that violates ethical standards.

Disclaimer (Note on Computational Tools)

No computer-assisted proof, symbolic computation, or automated theorem proving tools (e.g., Mathematica, SageMath, Coq, etc.) were used in the development or verification of the results presented in this paper. All proofs and derivations were carried out manually and analytically by the authors.

Code Availability

No code or software was developed for this study.

Clinical Trial

This study did not involve any clinical trials.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

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