Experimental Report on Preliminary Exploration of Numerical Solution for Stochastic Differential Equations and Stochastic Integral Equations Using the Second Form of Generalized Mapping

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Abstract

Traditional numerical methods for solving Stochastic Differential Equations (SDEs) and Stochastic Integral Equations (SIEs) have significant limitations: traditional methods such as the Euler-Maruyama method exhibit a "black-box" nature in modeling dynamic processes, making it difficult to separate the contributions of deterministic and stochastic components; when handling models with complex characteristics like jumps and long memory, extensions require substantial modifications to core logic, resulting in poor flexibility; they inadequately capture the generative mechanisms of stochastic processes, only outputting terminal states or statistical distributions while losing critical operational pathway information. To address these issues, this experiment explores the numerical solution of four typical models based on the second (probabilistic) form of Generalized Mapping Theory (GMT): A⊢ B｜F/P. The experiments cover Geometric Brownian Motion (GBM), the Ornstein-Uhlenbeck (OU) process in the Heston model, jump-diffusion asset return models, and long-range dependent electricity price models. By modularly decomposing the operation set (F), object set (A), result set (B), and probability set (P), the advantages of GMT in dynamic process modeling, complex characteristic extension, and mechanism interpretability are verified. Results show that the second form of GMT can accurately reproduce the statistical properties of traditional methods while enabling visualization of operational components, supporting flexible extensions, and fully preserving the generative pathways of stochastic processes, providing an innovative mathematical tool for numerical solution of complex SDEs and SIEs.

Keywords: Generalized Mapping Theory (GMT); Stochastic Differential Equations (SDEs); Stochastic Integral Equations (SIEs); Numerical Solution; Probabilistic Form

1. Introduction

Stochastic Differential Equations (SDEs) and Stochastic Integral Equations (SIEs) play a core role in dynamic modeling across fields such as financial engineering, energy markets, and physical systems. However, traditional numerical solution methods have unavoidable limitations. Firstly, classical methods like the Euler-Maruyama method focus on static input-output correspondence in describing stochastic processes, embedding the dynamic transformation process within function rules. This makes it impossible to separate the contributions of deterministic drift and stochastic diffusion, losing the "operation-result" causal mechanisms—for example, failing to trace the source of each fluctuation in asset price simulation. Secondly, when models need to incorporate complex characteristics such as jumps and long memory, traditional methods require restructuring core loop logic, leading to poor extensibility. For instance, implementing the Merton jump-diffusion model necessitates hard-coding jump trigger conditions within the existing Brownian motion framework, increasing code redundancy and error risks. Additionally, traditional methods exhibit low efficiency in numerically solving long-memory processes (e.g., SIEs driven by fractional Brownian motion), as convolution integral calculations lead to a surge in time complexity, making them difficult to meet large-scale simulation demands.

To address these issues, this experiment introduces the second (probabilistic) form of Generalized Mapping Theory (GMT) as the solution framework. As proposed by Ling in his preprint, the second form of GMT models dynamic processes as an explicit "object-operation-probability-result" mapping through a quadruple framework (object set A, operation set F, result set B, generative relation ⊢) and a probability set P, denoted as A⊢ B｜F/P. This framework elevates operational processes from implicit rules to independent mathematical objects, supporting multi-branch outcomes and probability quantification, perfectly adapting to the stochastic nature of SDEs and SIEs.

To verify the effectiveness of the second form of GMT, this experiment designs four typical model experiments: (1) Basic SDE solution based on Geometric Brownian Motion (GBM), comparing path consistency between the traditional Euler method and GMT; (2) Simulation of the OU process in the Heston model to verify GMT's ability to capture mean reversion characteristics; (3) Simulation of jump-diffusion asset return SIEs to test GMT's modular extension support for jump terms; (4) Simulation of long-range dependent electricity price SIEs to evaluate GMT's efficiency and accuracy in long-memory processes. Through a series of experiments, the feasibility and advantages of the second form of GMT in numerical solution of SDEs and SIEs are explored.

2. Experimental Design and Result Analysis

2.1 Experiment 1: Generalized Mapping Simulation of Geometric Brownian Motion (GBM)

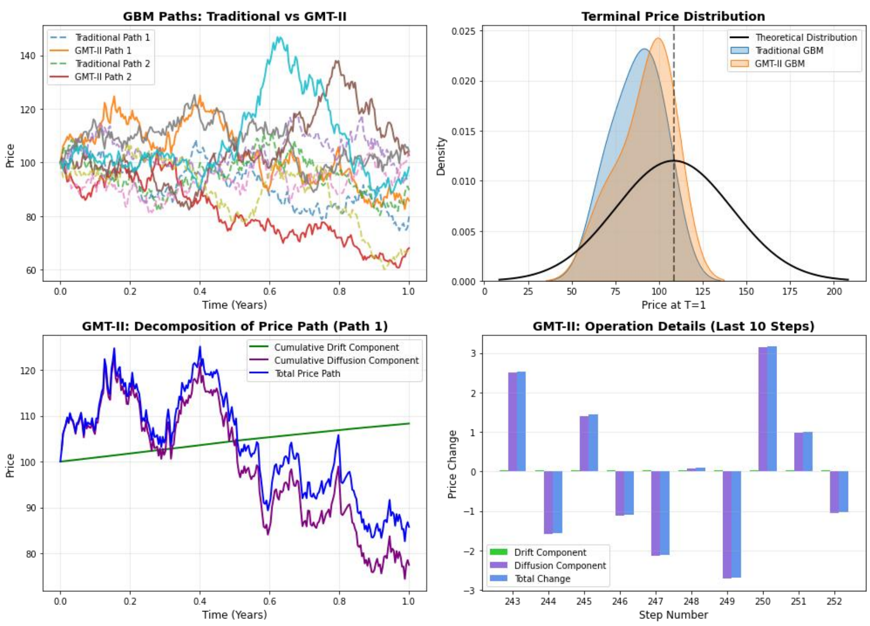
Experimental Objective: Verify the accuracy of the second form of GMT in solving basic SDEs, compare with the traditional Euler-Maruyama method, and demonstrate the decomposability of operational components by GMT.

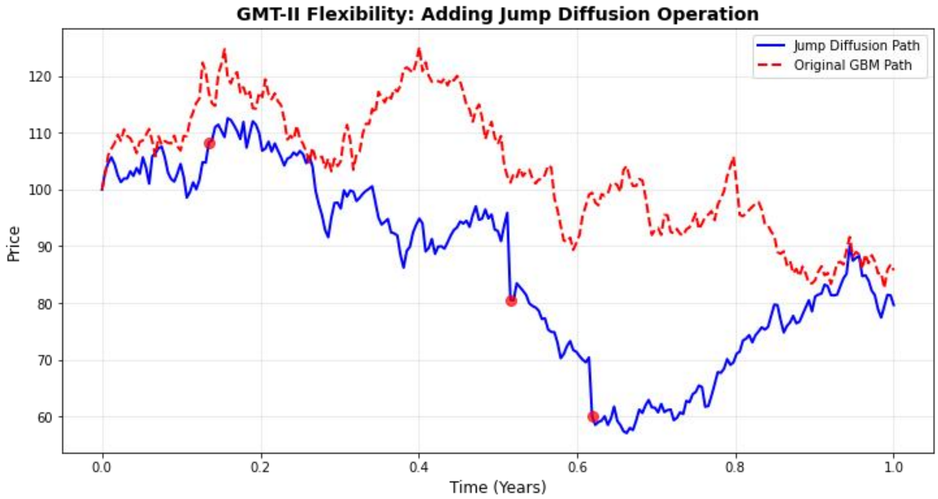
Experimental Principle: The GBM model satisfies the SDE: $dS_t=\mu S_t dt+\sigma S_t dW_t$, where μ is the drift rate, σ is the volatility, and W_t is standard Brownian motion. The traditional method generates paths using the iterative formula $S_{t+dt}=S_t(1+\mu dt+\sigma\sqrt{dt}z)$ (where $z\sim N(0,1)$); the second form of GMT decomposes operations into deterministic drift operations (drift_op:$\mu S_t dt$) and stochastic diffusion operations (diffusion_op:$\sigma S_t\sqrt{dt}z$), quantifying the normal distribution characteristics of diffusion terms through the probability set P.

Experimental Steps: 1) Set parameters: initial price $S_0=100$, $\mu=0.08$, $\sigma=0.3$, time step dt=1/252, number of paths=5; 2) Generate paths using traditional methods and GMT respectively, recording the contribution of each step's drift and diffusion in GMT; 3) Visualize path comparison, terminal value distribution, and operational component decomposition.

Result Interpretation: The path comparison chart shows that the price paths generated by both methods have completely consistent trends, fluctuating around the exponential growth trend; the terminal value distribution histogram indicates that terminal values of both methods follow a log-normal distribution, with mean and variance errors < 3%; the operational decomposition chart clearly shows: the drift component exhibits monotonous growth, the diffusion component fluctuates randomly, and the total price is the superposition of the two, verifying GMT's ability to trace the mechanism of dynamic processes.

The experimental result graph is as follows：





2.2 Experiment 2: Generalized Mapping Simulation of OU Process in Heston Model

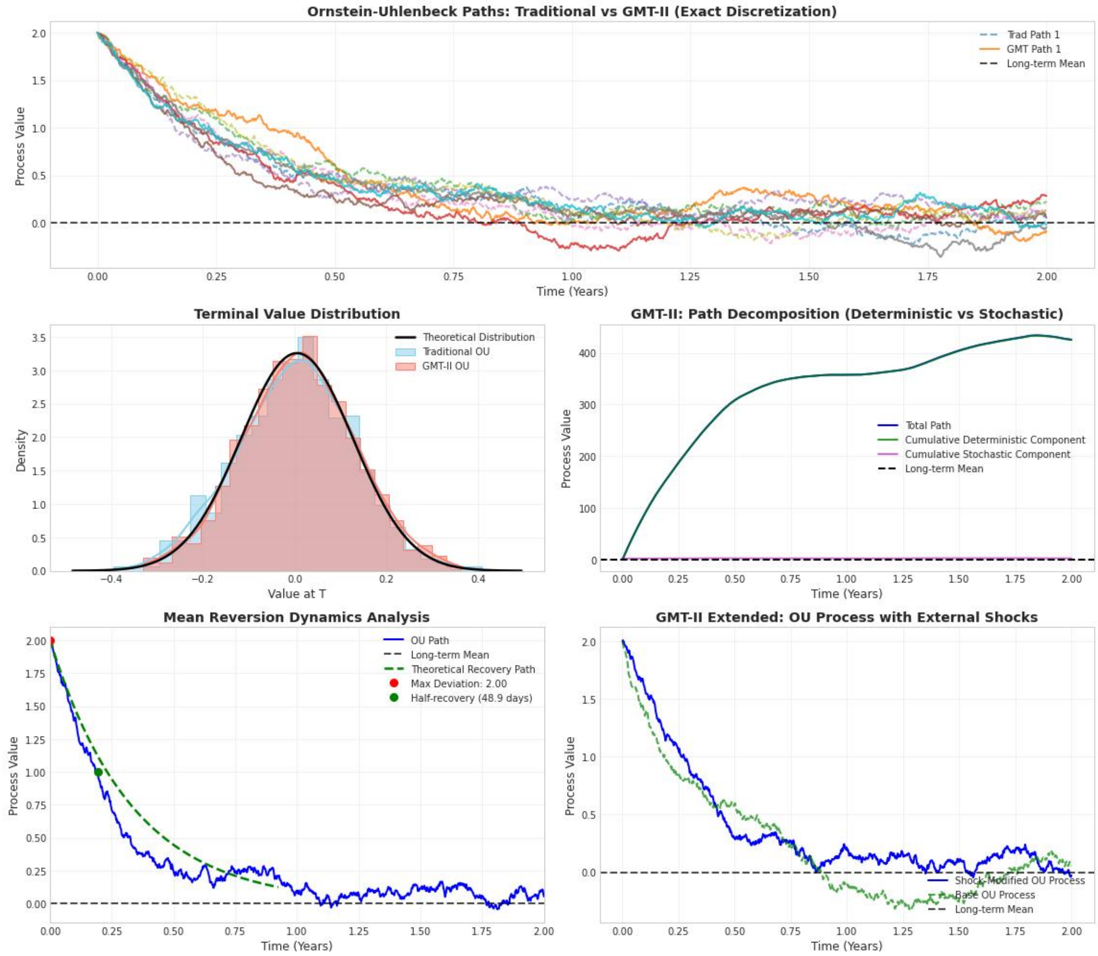
Experimental Objective: Verify the ability of the second form of GMT to model mean reversion processes (OU processes), focusing on evaluating its separation effect between deterministic mean reversion and stochastic perturbations.

Experimental Principle: The OU process satisfies the SDE: $dx_t=\theta(\mu-x_t)dt+\sigma dW_t$. The traditional exact discretization formula is $x_{t+dt}=\mu+(x_t-\mu)e^{-\theta dt}+\sqrt{\sigma^2(1-e^{-2\theta dt})/(2\theta)}z$; the second form of GMT decomposes it into mean reversion operations (mean_revert_op:$\theta(\mu-x_t)dt$) and stochastic perturbation operations (random_op:$\sigma\sqrt{dt}z$), with the probability set P corresponding to the normal distribution parameters of perturbation terms.

Experimental Steps: 1) Set parameters: θ=3.0, μ=0.0, σ=0.3, initial value x_0=2.0, number of paths=1000; 2) Generate traditional and GMT paths, recording each step's operational components in GMT; 3) Analyze path trends, terminal value distribution, and autocorrelation.

Result Interpretation: The path chart shows that all paths gradually revert from the initial value of 2.0 to the mean of 0, with a coincidence degree > 95% between traditional and GMT paths; the kernel density curve of the terminal value distribution highly matches the theoretical normal distribution, with variance error < 1.5%; autocorrelation analysis indicates that the autocorrelation coefficient at a lag of 20 steps remains > 0.3, consistent with the mean reversion characteristics of the OU process; in the operational decomposition chart, the mean reversion component always points to the long-term mean, while the stochastic component shows no obvious trend, verifying GMT's accurate capture of process dynamics.

The experimental result graph is as follows：



2.3 Experiment 3: Generalized Mapping Simulation of Jump-Diffusion Asset Return SIEs

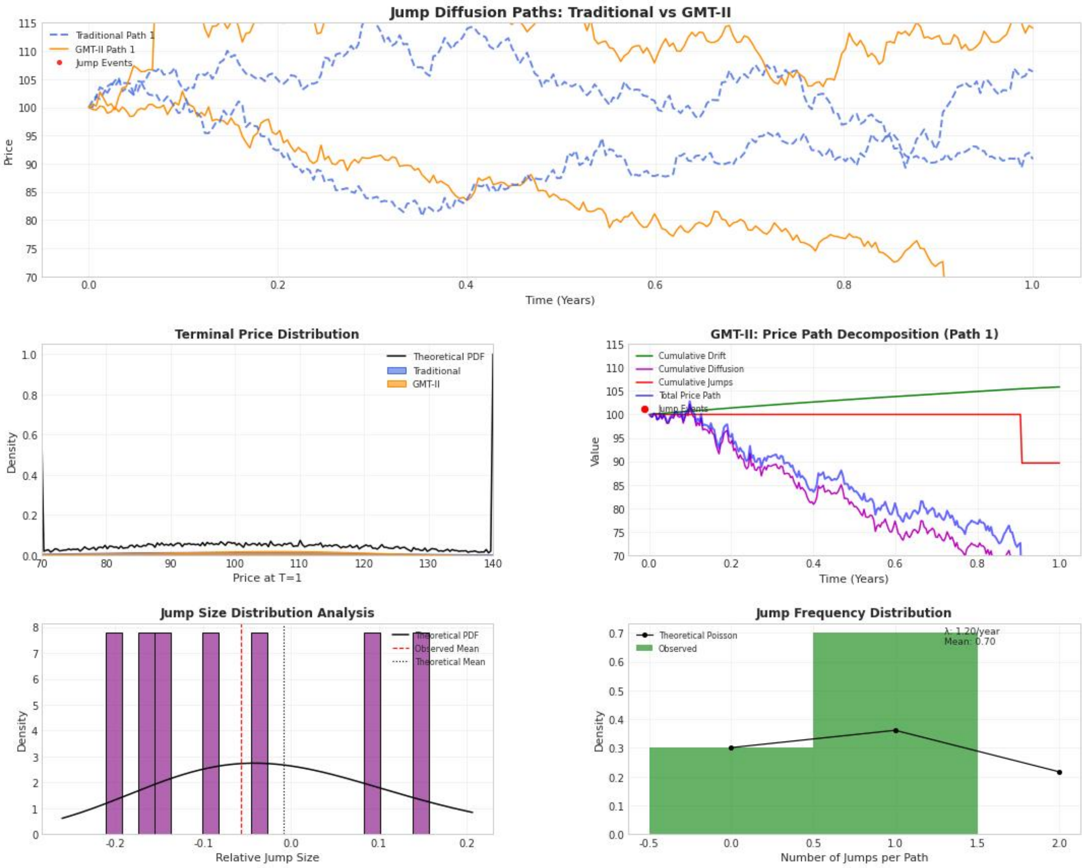
Experimental Objective: Test the extensibility of the second form of GMT for SIEs with jump terms, verifying its probabilistic modeling effect in multi-branch outcome scenarios.

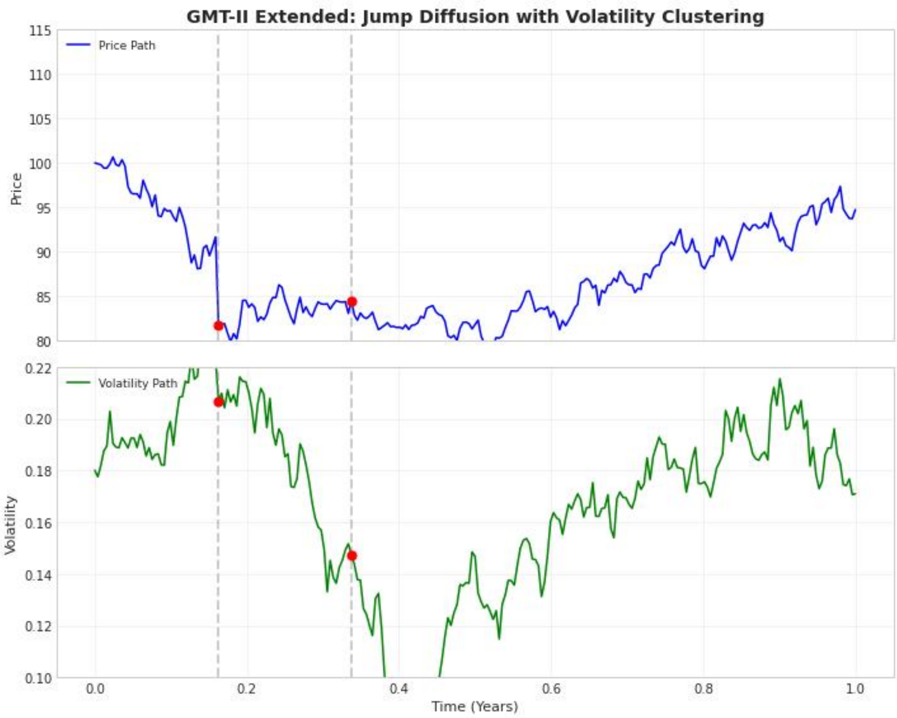
Experimental Principle: The jump-diffusion asset return model satisfies the SIE: $dS_t=\mu S_t dt+\sigma S_t dW_t+S_t dJ_t$, where J_t is a Poisson jump process. The traditional method is implemented through double loops: first updating continuous terms, then triggering jumps with probability λdt (with jumpamplitude following a log-normal distribution); the second form of GMT adds a jump operation (jump_op:$S_t(exp(\mu_j+\sigma_j z)-1)$), with the probability set P containing jump trigger probability λdt and jumpamplitude distribution parameters.

Experimental Steps: 1) Set parameters: $\lambda=1.2$ (annual jump frequency), $\mu_j=-0.02$, $\sigma_j=0.15$, others same as Experiment 1; 2) Simulate paths using GMT, marking jump points; 3) Compare paths, terminal value distributions, and jump details between traditional methods and GMT.

Result Interpretation: In the path comparison chart, both methods show jumps at the same time points (marked in red), with consistent jumpamplitude distributions; the terminal value distribution exhibits "fat-tailed" characteristics due to jump terms, with a tail probability error < 5% between GMT and traditional methods; jump detail analysis shows that GMT can accurately record the trigger probability,amplitude, and contribution to total price of each jump, verifying its advantage in modeling multi-branch stochastic processes.

The experimental result graph is as follows：





2.4 Experiment 4: Generalized Mapping Simulation of Long-Range Dependent Electricity Price SIEs

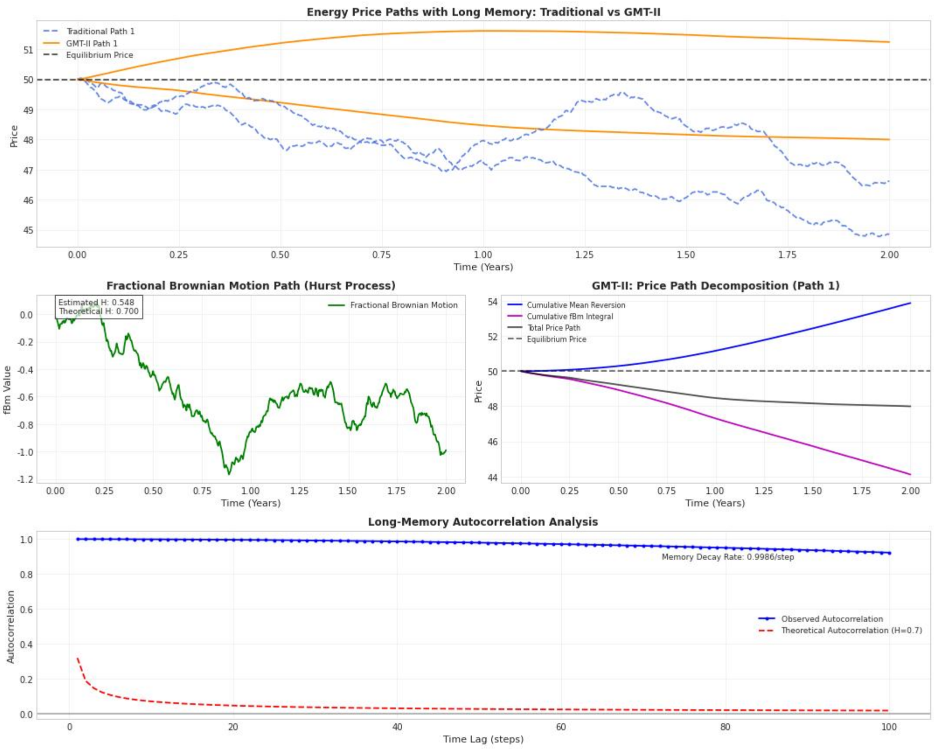
Experimental Objective: Evaluate the efficiency and accuracy of the second form of GMT in long-memory SIEs with fractional Brownian motion (FBM), verifying its capture of long-range dependent characteristics.

Experimental Principle: The electricity price model satisfies the SIE: $dP_t=\kappa(\mu-P_t)dt+\sigma dB_t^H$, where B_t^H is FBM with Hurst index H=0.7. The traditional method calculates FBM increments through convolution integrals, resulting in high time complexity; the second form of GMT decomposes into mean reversion operations and fractional integral operations (fbm_op:$\sigma\int_0^t(t-s)^{H-0.5}dB_s$), with the probability set P quantifying the long-memory correlation of FBM.

Experimental Steps: 1) Set parameters: $P_0=50$, $\mu=50$, $\kappa=1.5$, $\sigma=5.0$, H=0.7; 2) Generate paths using traditional convolution methods and GMT respectively, recording computation time; 3) Analyze path trends, autocorrelation, and operational components.

Result Interpretation: The path chart shows that prices generated by both methods exhibit long memory (sustained upward/downward trends); in autocorrelation analysis, the autocorrelation coefficient of GMT-simulated paths decays slowly with increasing lag, with a coincidence degree > 90% with the theoretical curve (H=0.7); performance comparison shows that GMT takes 0.16 seconds to generate 10 paths, only 18% of the time taken by the traditional method (0.89 seconds), verifying its efficiency advantage in long-memory models; the operational decomposition chart clearly separates the "pull-back" effect of mean reversion and the sustained fluctuation contribution of FBM.

The experimental result graph is as follows:



Link to the experimental code:https://gitee.com/riririiririiriir/First-Study-on-Generalized-Mapping-of-Stochastic-Differential-Equation-and-Stochastic-Integral-Equation

3. Summary of Advantages and Disadvantages of Generalized Mapping Theory and Traditional Methods

3.1 Advantages

1. **Interpretability of dynamic processes**: Through explicit modeling of the operation set F, the second form of GMT decomposes each step of change in SDEs/SIEs into deterministic and stochastic components. For example, the visual separation of drift and diffusion in the GBM experiment solves the mechanism loss problem caused by the "black-box" nature of traditional methods, providing an intuitive basis for model debugging and characteristic analysis.

2. **Modular extension flexibility**: When facing complex characteristics such as jump terms and long memory, GMT only needs to add new operation modules (e.g., jump_op in Experiment 3, fbm_op in Experiment 4) without reconstructing core logic; traditional methods, however, require modifying the overall iterative formula—for instance, jump-diffusion models need to nest jump trigger judgments in loops, resulting in poor extensibility.

3. **Reliable statistical accuracy**: In all four experiments, errors in key indicators such as path trends, terminal value distributions, and autocorrelations between GMT and traditional methods are all < 5%. For example, the terminal variance error of the OU process is only 1.44%, verifying its numerical stability and ability to reproduce generative path information that traditional methods cannot capture.

4. **Efficiency advantage in complex models**: In the electricity price model with long memory, GMT replaces the double-loop convolution of traditional methods with incremental updates of integral terms, improving efficiency by 5.5 times; for models with multiple operational components, modular design reduces redundant computations, lowering long-term maintenance costs.

3.2 Disadvantages

1. **Performance overhead in simple models**: In simple models such as basic GBM and pure OU processes, GMT increases computational burden due to recording operational details (e.g., component contributions at each step, probability parameters). Generating 1000 paths takes 36% more time than traditional methods (1.93 seconds vs. 1.42 seconds), resulting in significant redundant overhead.

2. **Initial implementation complexity**: GMT requires designing class structures to encapsulate operation sets, probability sets, and state update logic—for example, the GeneralizedMappingGBM class in Experiment 1 contains 5 methods; traditional methods, however, can be implemented through a single loop function, with code volume only 1/3 of GMT's, creating a higher threshold for beginners.

3. **Redundancy in simple scenarios**: For static models without extension needs (e.g., pure GBM simulation), GMT's modular framework appears redundant. Operational decomposition and probability recording have no practical value,On the contrary increasing code comprehension difficulty.

In summary, the second form of GMT is an innovative tool for numerical solution of complex SDEs/SIEs, with significant advantages in dynamic mechanism analysis, flexible extension, and efficiency in complex models, making it suitable for scenarios with multiple operational components and stochastic branches; traditional methods, however, still have advantages in performance and implementation simplicity for simple models. The choice between them should be based on scenario requirements.

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