A Linear Model of Optimal Control with One-dimensional Control and State Variables

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Abstract

We consider discrete-time infinite-horizon optimal control problems with a linear objective function. For absolutely convergent linear optimal control problems, we prove the existence of a solution, the necessity of a Euler and transversality conditions for a solution and the sufficiency of competitive condition and a different transversality condition for a solution. We show that the satisfaction of a "functional equation of dynamic programming" is necessary and sufficient for a trajectory to solve the optimization problem. Under the additional assumption, namely, "concave in pay-offs for the control variable", being satisfied by absolutely convergent linear optimal control problems, we show that the optimal value functions are concave and continuous. We obtain closed form solutions for such problems under the assumption that there is a state transition function that is strictly increasing and strictly concave in the gap variable and satisfy mild interiority conditions.

Keywords: optimal control, discrete-time, infinite horizon, linear objective function, dynamic programming

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1. Introduction:

While programming in linear spaces originates in the work of Hurwicz (1958), its use in economics- particularly programming in infinite dimensional linear spaces- has primarily been in the context of infinite horizon dynamic optimization, whose abstract analysis is best known in the work of Gale (1968). There may have been others who discussed specific examples of dynamic optimizations prior to the work of Gale (1967), but the "reduced form model" that is discussed in the survey paper of Mitra (2000) is very much in the tradition of Gale (1967). A more recent survey of dynamic optimization can be found in chapter 5 of Sorger (2015). The continuous version of the reduced form model of dynamic optimization is what is known as "calculus of variation". A generalization of calculus of variation is what is known as optimal control theory. Blot and Hayek (2014) presents a discussion of "infinite horizon optimal control in the discrete-time framework". Our purpose here is to begin an investigation of infinite horizon optimal control in the discrete-time framework, assuming that the objective function is linear.

While linearity of the objective function may appear to be a special case of a significantly more general non-linear context, this "supposed generality" is often based on the strict-concavity (if not the existence of non-zero second derivatives of the instantaneous pay-off functions) which become “inapplicable” once linearity of the objective function is assumed. Thus, linearity is not a special case of the theory of discrete-time dynamic optimization as it has in reality been applied in economics. The discrete-time infinite horizon reduced form model with linear objective function has been discussed in Lahiri (2025b, Lahiri 2025c). However, the reduced form model becomes a special case of linear optimal control, once at any point in time, for a given pair of current and future values of the state variable, we allow for the possibility of more than one value of the control variable to be compatible with the pair, and the choice of the control variable affects the value of the objective function.

In order to emphasize concepts without imposing a huge mathematical load, we assume that both the state variable and the control variable are real-valued, instead of being multi-dimensional as in Blot and Hayek (2014). Even under such simplification, the model enjoys considerable “traction” for the purpose of proving results that are conceptually powerful.

In section 2 we develop the framework of analysis. In section 3 we define the optimality criteria and also define a subclass of linear optimal control problems for which our first result in section 4 is about the existence of a (optimal) solution. We refer to the sub-class of problems for which a solution is shown to exist as “absolutely convergent linear optimal control problems”. The next result says that an “Euler-type” condition as well as a version of “transversality condition” are necessary conditions for a solution and the last result in section 4 proves that the “competitive condition” along with a different “transversality condition” is sufficient for optimality.

The existence of the optimal value function is an immediate consequence of the existence result in section 4. Our first result in section 5, shows that the optimal value function satisfies the “functional equation of dynamic programming” and that the satisfaction of the “functional equation of dynamic programming” is necessary and sufficient for a trajectory to solve the optimization problem. None of the results obtained up to this point in the context of the general model, require the convexity of time-dependent two-period constraint sets, consisting of feasible triplets of a value of the current state, a value of the control variable and a value of the future state. However, our second result in section 5 concerning the concavity and continuity of the optimal value function, requires that the two-period constraint sets satisfy “some kind” of convexity property. We refer to the property that we require the linear optimal control problem to satisfy as “concave in pay-offs for the control variable”**.**

In sections 6 and 7, we consider absolutely convergent linear optimal control problems with a state transition function which is a strictly increasing and strictly concave function of the difference between the state variable and control variable for the current period. The difference may be viewed as a “gap variable”. The value of the control variable is constrained to lie in the closed interval, whose left-hand end point is zero and the right-hand end point is the minimum of the current value of the state variable and an affine function of the same current value. Under “interiority conditions” for both state and control variables, we are able to obtain explicit formulas for the evolution of state variable and control variable. Although the definition of the state transition function requires the function to be either an affine function of the square root of the gap variable or a quadratic function of the gap variable, the model is quite general since the state transition function depends on additive and multiplicative parameters.

As shown in Lahiri (2025d), the calculus of polynomials required in this paper, can be developed using the ordered-field properties of the real number system and no more.

2. Framework of Analysis:

Let $X = [0, b] \subset \mathbb{R}$ (the set of real numbers), with $b > 0$ be such that **set of available alternatives** at any time period is a non-empty subset of $X \times X$. Given a current realization $x \in X$ of the state variable that was chosen in the immediately previous time-period, a typical alternative that is "chosen" during the current period is an $u \in X$, where u is the value of the control variable chosen for the current period. Based on the realization (x, u) during the current period a value y for the state variable that is realized in subsequent period with $y \in X$ is determined and an instantaneous pay-off is realized by the decision maker.

With $\mathbb{N}$ denoting the set of natural number (i.e., the set of strictly positive integers) let $\mathbb{N}^0$ denote $\mathbb{N} \cup \{0\}$, i.e., the set of non-negative integers. Time is measured in discrete periods $t \in \mathbb{N}^0$. At each time-period 't' an alternative (state variable-control variable pair) is realized, and the chosen alternative is denoted by $(x_t, u_t) \in X \times X$. While x_t is "inherited" in the current period, u_t is chosen during the current period.

At each time-period $t \in \mathbb{N}^0$, $\Omega_t \subset X \times X \times X$ is the **two-period constraint set at time-period** t, satisfying the following properties:

(i) Ω_t is a non-empty and closed subset of $X \times X \times X$.

(ii) For all $x \in X$, $\{(u, y) \in X \times X | (x, u, y) \in \Omega_t\} \neq \phi$ and $(x, u, y), (x, u, z) \in \Omega_t$ implies $y = z$.

For $t \in \mathbb{N}^0$, $(x, u, y) \in \Omega_t$ can be interpreted in the following manner: given that $x \in X$ is the realization of the state variable at time-period t, it is possible to choose the value $u \in X$ for the control at time period t with the pair (x, u) determining the value $y \in X$ of the state variable at time period t + 1 uniquely.

For all $(x, t) \in X \times \mathbb{N}^0$, let $\Omega_t(x) = \{(u, y) \in X \times X | (x, u, y) \in \Omega_t\}$.

By (ii) for all $(x, t) \in X \times \mathbb{N}^0$, $\Omega_t(x) \neq \phi$ and thus by (i) $\Omega_t(x)$ is a non-empty and closed subset of $X \times X$.

For $x \in X$, let $\mathcal{F}(x) = \{<(x_t, u_t) | t \in \mathbb{N}^0> | (x_t, u_t, x_{t+1}) \in \Omega_t, t \in \mathbb{N}^0, x_0 = x\}$.

We will (whenever necessary) refer to an infinite sequence $<(x_t, u_t) | t \in \mathbb{N}^0> \in \mathcal{F}(x)$ as a **trajectory starting at (from)** x.

Clearly, $\mathcal{F}(x)$ is non-empty for all $x \in X$.

Let $<(p_1^{(t)}, p_2^{(t)}) | t \in \mathbb{N}^0>$ be a sequence of pairs of real numbers. If x is the realization of the state variable at time-period t and u is the choice of the control variable at time period 't', the instantaneous pay-off received by the decision-maker at time period 't' is $p_1^{(t)}x + p_2^{(t)}u$.

We shall refer to the array $<((p_1^{(t)}, p_2^{(t)}), \Omega_t) | t \in \mathbb{N}^0>$ as the **linear optimal control (LOC) problem.**

If we allow for the reasonable possibility that for some t$\in \mathbb{N}^0$, $p_2^{(t)} \neq 0$ and there exists (x, u, y), (x, v, y) $\in \Omega_t$ with u $\neq$ v, then the corresponding LOC problem is an example of "several values of the control variable" chosen at time-period t, being compatible with each value of the state variables for time-periods t and t+1, thereby illustrating that our model of linear optimal control discussed here is a generalization of the linear dynamic optimization model discussed in Lahiri (2025b, 2025c), the latter being motivated by the reduced form model in Mitra (2000) and Sorger (2015).

For what follows we assume that <(($p_1^{(t)}$, $p_2^{(t)}$), Ω_t)| t$\in\mathbb{N}^0$> is a given LOC problem. As and when necessary, we will impose additional assumptions on this LOC problem.

3. Optimal solution:

We will now consider the following optimization problem denoted **OPT**:

Given x$\in$X, Maximize $\sum_{t=0}^{\infty} [p_1^{(t)} x_t + p_2^{(t)} u_t]$ subject to the infinite sequence satisfying the constraints: (x_t, u_t, x_{t+1})$\in \Omega_t$, t$\in \mathbb{N}^0$, x_0 = x.

Note 3.1: The exact mathematical interpretation of the expression (formula) $\sum_{t=0}^{\infty} [p_1^{(t)} x_t + p_2^{(t)} u_t]$ is $\lim_{T\to\infty} (\sum_{t=0}^{T} [p_1^{(t)} x_t + p_2^{(t)} u_t])$. Thus, the problem we are concerned with here is in the domain of asymptotic analysis, which is very different from infinite dimensional analysis.

Let $\mathcal{S}$(x) = {<x_t|t$\in\mathbb{N}^0$>$\in\mathcal{F}$(x)| $\sum_{t=0}^{\infty} [p_1^{(t)} x_t + p_2^{(t)} u_t] \geq \sum_{t=0}^{\infty} [p_1^{(t)} y_t + p_2^{(t)} v_t]$ for all <(y_t, v_t)|t$\in\mathbb{N}^0$>$\in$ $\mathcal{F}$(x)}.

$\mathcal{S}$(x) is the **set of solutions starting from** x **for** OPT.

For all T$\in\mathbb{N}^0$, and x$\in$X, let $\mathcal{F}^T$(x) = {<(x_t, u_t) |t ≥ T>| (x_t, u_t, x_{t+1})$\in$ Ω_t for all t ≥ T and x_T = x}.

For T$\in\mathbb{N}^0$ and y$\in$X, <(x_t, u_t)|t ≥T>$\in$ $\mathcal{F}^T$(x) may be referred to as a **trajectory starting at (from)** x **at time-period** T.

It is easy to see that for all T$\in\mathbb{N}^0$ and x$\in$X, $\mathcal{F}^T$(x) is non-empty.

Given (x, T)$\in$X×$\mathbb{N}^0$, we will denote the following optimization problem by **OPT-T**:

Maximize $\sum_{t=T}^{\infty} [p_1^{(t)} y_t + p_2^{(t)} v_t]$ subject to $< (y_t, v_t)|$ t ≥ T $>\in\mathcal{F}^T(x)$

For (x, T)$\in$X×$\mathbb{N}^0$, $\mathcal{S}^T$(x) is the **set of solutions starting from** x **for** OPT-T, i.e., $\mathcal{S}^T$(x)

$= \underset{<(x_t,u_t)|\, t\geq T>\in\mathcal{F}^T(x)}{\text{argmax}} \sum_{t=T}^{\infty} [p_1^{(t)} x_t + p_2^{(t)} u_t]$.

Clearly, $\mathcal{F}^0$(x) = $\mathcal{F}$(x) and $\mathcal{S}^0$(x) = $\mathcal{S}$(x) for all x$\in$X.

For all T$\in\mathbb{N}^0$ the correspondence h^T: X →→X×X defined by h^T(x) = $\underset{(u,y)\in\Omega_T(x)}{\text{argmax}} (p_2^{(T)} u + V^{T+1}(y))$ is said to be the **optimal period-T decision rule**.

As in Mitra (2000), Sorger (2015) and Lahiri (2025b) (among numerous others) we, in the rest of this paper be concerned with an optimality criterion that requires the following "Absolute Convergence" condition.

The LOC problem <(($p_1^{(t)}$, $p_2^{(t)}$), Ω_t)| t$\in \mathbb{N}^0$> is said to satisfy **Absolute Convergence** if for i$\in$ {1,2}, $\sum_{t=0}^{\infty}|p_i^{(t)}| < +\infty$.

If <(($p_1^{(t)}$, $p_2^{(t)}$), Ω_t)| t$\in \mathbb{N}^0$> satisfies Absolute Convergence, then we may refer to <(($p_1^{(t)}$, $p_2^{(t)}$), Ω_t)| t$\in \mathbb{N}^0$> as an **Absolutely Convergent LOC** (AC-LOC) **problem**.

Let <(($p_1^{(t)}$, $p_2^{(t)}$), Ω_t)| t$\in \mathbb{N}^0$> be an AC-LOC problem. Thus, for all sequence <(x_t, u_t)|t$\in \mathbb{N}^0$> with (x_t, u_t)$\in$ X×X for all t$\in \mathbb{N}^0$, it must be the case that $\lim_{t\to\infty}|p_1^{(t)}x_t| = 0$, $\lim_{t\to\infty}|p_2^{(t)}u_t| = 0$, $\sum_{t=0}^{\infty}|p_1^{(t)}x_t| \in [0, \mathrm{b}\sum_{t=0}^{\infty}|p_1^{(t)}|]$ and $\sum_{t=0}^{\infty}|p_1^{(t)}u_t| \in [0, \mathrm{b}\sum_{t=0}^{\infty}|p_2^{(t)}|]$.

Let M = max{$\mathrm{b}\sum_{t=0}^{\infty}|p_1^{(t)}|$, $\mathrm{b}\sum_{t=0}^{\infty}|p_2^{(t)}|$} < +∞.

Thus, for all sequence <(x_t, u_t)|t$\in \mathbb{N}^0$> with (x_t, u_t)$\in$ X×X for all t$\in \mathbb{N}^0$, it must be the case that $|\sum_{t=0}^{\infty}[p_1^{(t)}x_t + p_2^{(t)}u_t]| \leq \sum_{t=0}^{\infty}|p_1^{(t)}x_t + p_2^{(t)}u_t| \leq \sum_{t=0}^{\infty}[|p_1^{(t)}x_t| + |p_2^{(t)}u_t|] \leq 2M$

4. Existence of solution, "Euler type" condition, competitive condition, and transversality condition:

The proof of the following proposition is analogous to the proof of proposition 4.1 in Lahiri (2025b).

Proposition 4.1: Let <(($p_1^{(t)}$, $p_2^{(t)}$), Ω_t)| t$\in \mathbb{N}^0$> be an AC-LOC problem. $\mathcal{S}^T$(x) ≠ ϕ for all (x, T)$\in$ X×$\mathbb{N}^0$. Hence, the optimal period-T decision rule h^T is non-empty valued for all T$\in \mathbb{N}^0$.

Proof: Let (x, T)$\in$ X×$\mathbb{N}^0$.

$\Omega_t \subset$ X×X×X = [0, b]×[0, b]×[0, b] for all t$\in \mathbb{N}^0$ and hence as noted in section 5, for all sequence <(x_t, u_t)|t$\in \mathbb{N}^0$> with (x_t, u_t)$\in$ X×X for all t$\in \mathbb{N}^0$, it must be the case that $|\sum_{t=0}^{\infty}[p_1^{(t)}x_t + p_2^{(t)}u_t]| \leq \sum_{t=0}^{\infty}|p_1^{(t)}x_t + p_2^{(t)}u_t| \leq \sum_{t=0}^{\infty}[|p_1^{(t)}x_t| + |p_2^{(t)}u_t|] \leq 2M$.

In particular, for all sequence <(x_t, u_t)|t$\in \mathbb{N}^0$> with (x_t, u_t)$\in$ X×X for all t$\in \mathbb{N}^0$, it must be the case that $|\sum_{t=T}^{\infty}[p_1^{(t)}x_t + p_2^{(t)}u_t]| \leq \sum_{t=T}^{\infty}|p_1^{(t)}x_t + p_2^{(t)}u_t| \leq \sum_{t=T}^{\infty}[|p_1^{(t)}x_t| + |p_2^{(t)}u_t|] \leq 2M$.

Let <(y_t, v_t)|t ≥ T>$\in \mathcal{F}^T$(x). Clearly, $-2M \leq \sum_{t=T}^{\infty}[p_1^{(t)}y_t + p_2^{(t)}v_t] \leq 2M$.

Thus, S(x) = sup {$\sum_{t=T}^{\infty}[p_1^{(t)}y_t + p_2^{(t)}v_t]$|<($y_t$, v_t)|t ≥ T>$\in \mathcal{F}^T$(x)} < +∞.

Thus, there is a sequence of infinite sequences <<($x_t^{(n)}$, $u_t^{(n)}$)| t ≥ T>| n$\in \mathbb{N}$> in $\mathcal{F}^T$(x) such that for all n$\in \mathbb{N}$, S(x) - $\frac{1}{n} < \sum_{t=T}^{\infty}[p_1^{(t)}x_t^{(n)} + p_2^{(t)}u_t^{(n)}]$

Since, $<(x_{T+1}^{(n)}, u_{T+1}^{(n)})|\, n\in \mathbb{N}>$ is a sequence in the closed and bounded set X×X, it has a convergent subsequence $<(x_{T+1}^{N_1(n)}, u_{T+1}^{N_1(n)})|n\in \mathbb{N}>$ converging to $(x_{T+1}^{0}, u_{T+1}^{0})\in$ X×X.

Further, $(x, u_T^{(n)}, x_{T+1}^{N_1(n)})\in \Omega_T$ for all $n\in\mathbb{N}$ implies $(x, u_T^0, x_{T+1}^0)\in \Omega_T$ since Ω_T is closed.

Consider the sequence $<(x_{T+2}^{N_1(n)}, u_{T+2}^{N_1(n)})|n\in \mathbb{N}>$.

By an argument similar to the one in the previous step, it has a convergent subsequence $<(x_{T+2}^{N_2(n)}, u_{T+2}^{N_2(n)})|\, n\in \mathbb{N}>$ converging to $(x_{T+2}^{0}, u_{T+2}^{0})\in$ X×X.

Since, $<(x_{T+1}^{N_2(n)}, u_{T+1}^{N_2(n)})|\, n\in \mathbb{N}>$ is a subsequence of the convergent subsequence $<(x_{T+1}^{N_1(n)}, u_{T+1}^{N_1(n)})|\, n\in \mathbb{N}>$, it must be the case that $<(x_{T+1}^{N_2(n)}, u_{T+1}^{N_2(n)}|\, n\in \mathbb{N}>$ converges to $(x_{T+1}^{0}, u_{T+1}^{0})$.

The sequence $<(x_{T+1}^{N_2(n)}, u_{T+1}^{N_2(n)}, x_{T+2}^{N_2(n)})|\, n\in \mathbb{N}>$ is in Ω_{T+1} and converges to $(x_{T+1}^{0}, u_{T+1}^{0}, x_{T+2}^{0})$. Since Ω_{T+1} is closed $(x_{T+1}^{0}, u_{T+1}^{0}, u_{T+2}^{0})\in \Omega_{T+1}$.

Having obtained convergent subsequences $<(x_{T+\tau-1}^{N_\tau(n)}, u_{T+\tau-1}^{N_\tau(n)}, x_{T+\tau}^{N_\tau(n)})|\, n\in \mathbb{N}>\in \Omega_{T+\tau-1}$ for all $\tau = 1, \ldots, t$ for some $t \geq 1$, let $<(x_{T+t}^{N_{t+1}(n)}, u_{T+t}^{N_{t+1}(n)}, x_{T+t}^{N_{t+1}(n)}|\, n\in \mathbb{N}>$ be a convergent subsequence of the sequence $<(x_{T+t}^{N_t(n)}, u_{T+t}^{N_t(n)}, x_{T+t+1}^{N_t(n)}|\, n\in \mathbb{N}>$ in the closed and bounded set X×X×X, converging to $(x_{T+t}^{0}, u_{T+t}^{0}, x_{T+t+1}^{0})\in$ X×X×X. Since, $<(x_{T+t}^{N_t(n)}, u_{T+t}^{N_t(n)}, x_{T+t+1}^{N_t(n)}|\, n\in \mathbb{N}>\in \Omega_{T+t}$ and Ω_{T+t} is a closed subset of X×X×X, it must be the case that $(x_{T+t}^{0}, u_{T+t}^{0}, x_{T+t+1}^{0})\in \Omega_{T+t}$.

Consider the sequence $<(x_{T+t}^{0}, u_{T+t}^{0}|t\in \mathbb{N}^0>$. Since, $(x_{T+t}^{0}, u_{T+t}^{0}, x_{T+t+1}^{0})\in \Omega_t$ for all $t\in \mathbb{N}^0$, where $x_T^0 = x$, it must be the case that $<x_{T+t}^{0}|t\in \mathbb{N}^0 >\in \mathcal{F}(x)$.

Thus, $\sum_{t=0}^{\infty} [p_1^{(T+t)} x_{T+t}^{0} + p_2^{(T+t)} u_{T+t}^{0}] \leq S(x)$.

We wish to show that $\sum_{t=0}^{\infty} [p_1^{(T+t)} x_{T+t}^{0} + p_2^{(T+t)} u_{T+t}^{0}] = S(x)$.

Since, for all $t\in \mathbb{N}^0$, $<x_{T+\tau}^{N_\tau(n)}|\, n\in \mathbb{N}>$ converges to $x_{T+\tau}^{0}$ and $(x_{T+\tau}^{N_\tau(n)}, u_{T+\tau}^{N_\tau(n)}, x_{T+\tau+1}^{N_\tau(n)})\in \Omega_{T+\tau}$, for all $\tau \leq t$ and $n\in \mathbb{N}$, ***given*** $\varepsilon > 0$, for all $t\in \mathbb{N}$, there exists a sequence $<n_t|t\in \mathbb{N}>$ in $\mathbb{N}$ satisfying $n_{t+1} > n_t$ for all $t\in \mathbb{N}$ such that for all $n\in \mathbb{N}$ with $n \geq n_t$ and $\tau \leq t$, $(x_{T+\tau}^{N_\tau(n)}, u_{T+\tau}^{N_\tau(n)}, x_{T+\tau+1}^{N_\tau(n)})\in \Omega_{T+\tau}$ and $|p_1^{(T+\tau)} x_{T+\tau}^{0} + p_2^{(T+\tau)} u_{T+\tau}^{0} - p_1^{(T+\tau)} x_{T+\tau}^{N_t(n)} - p_2^{(T+\tau)} u_{T+\tau}^{N_t(n)}| < \frac{\varepsilon}{8}(\frac{1}{2})^t$, i.e., $p_1^{(T+\tau)} x_{T+\tau}^{N_t(n)} + p_2^{(T+\tau)} u_{T+\tau}^{N_t(n)} + \frac{\varepsilon}{8}(\frac{1}{2})^t > p_1^{(T+\tau)} x_{T+\tau}^{0} + p_2^{(T+\tau)} u_{T+\tau}^{0} > p_1^{(T+\tau)} x_{T+\tau}^{N_t(n)} + p_2^{(T+\tau)} u_{T+\tau}^{N_t(n)} - \frac{\varepsilon}{8}(\frac{1}{2})^t$.

Note that $\sum_{\tau=0}^{t} \frac{\varepsilon}{8}(\frac{1}{2})^t \leq \sum_{\tau=0}^{t} \frac{\varepsilon}{8}(\frac{1}{2})^\tau = \frac{\varepsilon}{8}\sum_{\tau=0}^{t-1} (\frac{1}{2})^\tau = \frac{\varepsilon}{4}(1- (\frac{1}{2})^t)$ and $x_T^{N_t(n)} = x$ for all $(t, n)\in \mathbb{N}\times \mathbb{N}$.

Thus, for all $t\in \mathbb{N}$, $\sum_{\tau=0}^{t} [p_1^{(T+\tau)} x_{T+\tau}^{0} + p_2^{(T+\tau)} u_{T+\tau}^{0}] > \sum_{\tau=0}^{t} [p_1^{(T+\tau)} x_{T+\tau}^{N_t(n)} + p_2^{(T+\tau)} u_{T+\tau}^{N_t(n)}] - \frac{\varepsilon}{4}(1- (\frac{1}{2})^t)$ for all $n \geq n_t$.

For all t∈ℕ: $\sum_{\tau=0}^{t}[p_1^{(T+\tau)}x_{T+\tau}^{N_t(n)} + p_2^{(T+\tau)}u_{T+\tau}^{N_t(n)}]$> S(x) - $\frac{1}{N_t(n)}$ -$\sum_{\tau=t+1}^{\infty}[p_1^{(T+\tau)}x_{T+\tau}^{N_t(n)} + p_2^{(T+\tau)}u_{T+\tau}^{N_t(n)}]$ for all n ≥ n_t.

Thus, for all t∈ℕ, $\sum_{\tau=0}^{t}[p_1^{(T+\tau)}x_{T+\tau}^{0} + p_2^{(T+\tau)}u_{T+\tau}^{0}] > \sum_{\tau=0}^{t}[p_1^{(T+\tau)}x_{T+\tau}^{N_t(n)} + p_2^{(T+\tau)}u_{T+\tau}^{N_t(n)}]$ - $\frac{\varepsilon}{8}(1- (\frac{1}{2})^t)$ > S(x) - $\frac{1}{N_t(n)}$ -$\sum_{\tau=t+1}^{\infty}[p_1^{(T+\tau)}x_{T+\tau}^{N_t(n)} + p_2^{(T+\tau)}u_{T+\tau}^{N_t(n)}]$ - $\frac{\varepsilon}{4}(1- (\frac{1}{2})^t)$ for all n ≥ n_t.

Thus, for all t∈ℕ, $\sum_{\tau=0}^{t}[p_1^{(T+\tau)}x_{T+\tau}^{0} + p_2^{(T+\tau)}u_{T+\tau}^{0}] + \sum_{\tau=t+1}^{\infty}[p_1^{(T+\tau)}x_{T+\tau}^{N_t(n)} + p_2^{(T+\tau)}u_{T+\tau}^{N_t(n)}]$ > S(x) - $\frac{1}{N_t(n)}$ - $\frac{\varepsilon}{4}(1- (\frac{1}{2})^t)$ for all n ≥ n_t.

By hypothesis, $\sum_{t=0}^{\infty}|p_i^{(t)}| < +\infty$ for i∈{1, 2} and for all (t, n)∈ $\mathbb{N}^0\times\mathbb{N}$, both $x_{T+t}^{(n)}$∈[0, b] and $u_{T+t}^{(n)}$∈[0, b].

Thus, for all t∈ ℕ and n ≥ n_t,

$\sum_{\tau=t+1}^{\infty}[|p_1^{(T+\tau)}x_{T+\tau}^{N_t(n)}| + |p_2^{(T+\tau)}u_{T+\tau}^{N_t(n)}|]=\sum_{\tau=t+1}^{\infty}[|p_1^{(T+\tau)}|x_{T+\tau}^{N_t(n)} + |p_2^{(T+\tau)}|u_{T+\tau}^{N_t(n)}]$.

For all t∈ℕ and n∈ℕ, $x_t^{(n)}$∈[0, b] and $u_t^{(n)}$∈[0, b] implies for all t∈ℕ and n ≥ n_t,

$b\sum_{\tau=t+1}^{\infty}[|p_1^{(T+\tau)}| + |p_2^{(T+\tau)}|] \geq \sum_{\tau=t+1}^{\infty}[|p_1^{(T+\tau)}|x_{T+\tau}^{N_t(n)} + |p_2^{(T+\tau)}|u_{T+\tau}^{N_t(n)}]$.

Further, for all t∈ℕ and n ≥ n_t, $\sum_{\tau=t+1}^{\infty}[|p_1^{(T+\tau)}|x_{T+\tau}^{N_t(n)} + |p_2^{(T+\tau)}|u_{T+\tau}^{N_t(n)}] \geq$

$\sum_{\tau=t+1}^{\infty}[p_1^{(T+\tau)}x_{T+\tau}^{N_t(n)} + p_2^{(T+\tau)}u_{T+\tau}^{N_t(n)}]$.

Thus, for all t∈ℕ, $\sum_{\tau=0}^{t}[p_1^{(T+\tau)}x_{T+\tau}^{0} + p_2^{(T+\tau)}u_{T+\tau}^{0}] + b\sum_{\tau=t+1}^{\infty}[|p_1^{(T+\tau)}| + |p_2^{(T+\tau)}|] \geq$

$\sum_{\tau=0}^{t}[p_1^{(T+\tau)}x_{T+\tau}^{0} + p_2^{(T+\tau)}u_{T+\tau}^{0}] + \sum_{\tau=t+1}^{\infty}[p_1^{(T+\tau)}x_{T+\tau}^{N_t(n)} + p_2^{(T+\tau)}u_{T+\tau}^{N_t(n)}]$ > S(x) - $\frac{1}{N_t(n)}$ - $\frac{\varepsilon}{4}(1- (\frac{1}{2})^t)$ for all n ≥ n_t.

Thus, for all t∈ℕ, $\sum_{\tau=0}^{t}[p_1^{(T+\tau)}x_{T+\tau}^{0} + p_2^{(T+\tau)}u_{T+\tau}^{0}] + b\sum_{\tau=t+1}^{\infty}[|p_1^{(T+\tau)}| + |p_2^{(T+\tau)}|]$ > S(x) - $\frac{1}{N_t(n)}$ - $\frac{\varepsilon}{4}(1- (\frac{1}{2})^t)$ for all n ≥ n_t.

Thus, for all t∈ℕ, $\sum_{\tau=0}^{t}[p_1^{(T+\tau)}x_{T+\tau}^{0} + p_2^{(T+\tau)}u_{T+\tau}^{0}] + b\sum_{\tau=t+1}^{\infty}[|p_1^{(T+\tau)}| + |p_2^{(T+\tau)}|]$ ≥ S(x) - $\lim_{n\to\infty}\frac{1}{N_t(n)}$ - $\frac{\varepsilon}{8}(1- (\frac{1}{2})^t)$ = S(x) - $\frac{\varepsilon}{4}(1- (\frac{1}{2})^t)$, since $\lim_{n\to\infty}\frac{1}{N_t(n)} = 0$.

Thus, for all t∈ℕ, $\sum_{\tau=0}^{t}[p_1^{(T+\tau)}x^0_{T+\tau} + p_2^{(T+\tau)}u^0_{T+\tau}] + b\sum_{\tau=t+1}^{\infty}[|p_1^{(T+\tau)}| + |p_2^{(T+\tau)}|] \geq S(x) - \frac{\varepsilon}{4}(1-(\frac{1}{2})^t)$.

Since $\lim_{t\to\infty}\frac{\varepsilon}{4}(1-(\frac{1}{2})^t) = \frac{\varepsilon}{8}$ and $\lim_{t\to\infty}\sum_{\tau=0}^{t}[p_1^{(T+\tau)}x^0_{T+\tau} + p_2^{(T+\tau)}u^0_{T+\tau}] = \sum_{\tau=0}^{\infty}[p_1^{(T+\tau)}x^0_{T+\tau} + p_2^{(T+\tau)}u^0_{T+\tau}]$, we get $\sum_{\tau=0}^{t}[p_1^{(T+\tau)}x^0_{T+\tau} + p_2^{(T+\tau)}u^0_{T+\tau}] + \lim_{t\to\infty} b\sum_{\tau=t+1}^{\infty}[|p_1^{(T+\tau)}| + |p_2^{(T+\tau)}|] \geq S(x) - \frac{\varepsilon}{4}$.

Since, for all t∈ℕ, $\sum_{\tau=t+1}^{\infty}|p_i^{(\tau)}| < +\infty$ for i∈{1, 2}, it must be the case that

$\lim_{t\to\infty} b\sum_{\tau=t+1}^{\infty}[|p_1^{(T+\tau)}| + |p_2^{(T+\tau)}|] = 0$.

Thus, $\sum_{\tau=0}^{\infty}[p_1^{(T+\tau)}x^0_{T+\tau} + p_2^{(T+\tau)}u^0_{T+\tau}] \geq S(x) - \frac{\varepsilon}{4}$.

Since the above holds for all ε > 0, we get $\sum_{\tau=0}^{\infty}[p_1^{(T+\tau)}x^0_{T+\tau} + p_2^{(T+\tau)}u^0_{T+\tau}] \geq S(x)$.

Thus, $\sum_{\tau=0}^{\infty}[p_1^{(T+\tau)}x^0_{T+\tau} + p_2^{(T+\tau)}u^0_{T+\tau}] = S(x)$.

Thus for $\mathcal{S}^T(x) \neq \phi$ and $(u_T^0, x_{T+1}^0) \in h^T(x)$. Hence, $h^T(x) \neq \phi$. Q. E. D.

Note 4.1: For the iterative process outlined above we are unaware of any possibility of it "invariably" converging to a strictly increasing sequence <N∞(n)|n∈ℕ> such that for all t∈$\mathbb{N}^0$, $(x_t^{N_\infty(n)}, u_t^{N_\infty(n)}, x_{t+1}^{N_\infty(n)}) \in \Omega_t$ for all n∈ℕ and <$(x_t^{N_\infty(n)}, u_t^{N_\infty(n)})$| n∈ℕ> converges to (x_t^0, u_t^0). However, in appendix A.2 (entitled "A Diagonal Process of Cantor") of Blot and Hayek (2014), it seems to be indicated (provided our understanding is correct), that for all t∈$\mathbb{N}^0$, the subsequence <$(x_{T+t}^{N_n(n)}, u_{T+t}^{N_n(n)})$| n∈ℕ> derived from the construction in the proof above, converges to (x_{T+t}^0, u_{T+t}^0).

The next result is an "Euler-type" necessary condition for a solution.

Proposition 4.2: Let <$((p_1^{(t)}, p_2^{(t)}), \Omega_t)$| t∈$\mathbb{N}^0$> be an AC-LOC problem. For x∈X, let <(x_t, u_t)|t∈$\mathbb{N}^0$>∈$\mathcal{S}(x)$.

(i) Suppose that for some T∈ℕ, and (v, y)∈$\Omega_{T-1}(x_{T-1})$ it is the case that given ε > 0, there exists <(v_{T+t}, y_{T+t})|t∈$\mathbb{N}^0$>∈$\mathcal{F}^T(y)$ satisfying $|v_{T+t-1} - u_{T+t-1}| \leq \varepsilon|v - u_{T-1}|$ and $|y_{T+t} - x_{T+t}| \leq \varepsilon|y - x_T|$ if $y \neq x_T$ for all t∈ℕ. <u>Then</u> $p_1^{(T)}(y - x_T) + p_2^{(T-1)}(v - u_{T-1}) \leq 0$.

(ii) $\lim_{t\to\infty}(|p_1^{(t)}|x_t + |p_2^{(t)}|u_t) = 0$.

Proof: (i) Suppose that for some $T \in \mathbb{N}$, and $(v, y) \in \Omega_{T-1}(x_{T-1})$ it is the case that given $\varepsilon > 0$, there exists $<(v_{T+t}, y_{T+t})|t \in \mathbb{N}^0> \in \mathcal{F}^T(y)$ satisfying $|v_{T+t-1} - u_{T+t-1}| \leq \varepsilon|v - u_{T-1}|$ and $|y_{T+t} - x_{T+t}| \leq \varepsilon|y - x_T|$ if $y \neq x_T$ for all $t \in \mathbb{N}$.

Towards a contradiction suppose that $p_1^{(T)}(y - x_T) + p_2^{(T-1)}(v - u_{T-1}) > 0$ and let $\varepsilon > 0$ be such that $p_1^{(T)}(y - x_T) + p_2^{(T-1)}(v - u_{T-1}) > \varepsilon(|y - x_1|\sum_{t=T+1}^{\infty}|p_1^{(t)}| + |v - u_{T-1}|\sum_{t=T+1}^{\infty}|p_2^{(t-1)}|)$.

This is possible since we have assumed towards a contradiction that $p_1^{(T)}(y - x_T) + p_2^{(T-1)}(v - u_{T-1}) > 0$.

Let $<(z_t, w_t)|t \in \mathbb{N}^0> \in \mathcal{F}(x)$ such that $z_t = x_t$ for $t = 0, \ldots, T-1$, $w_{T-1} = v$, $w_t = u_t$ for $t = 0, \ldots, T-2$ (provided $T \geq 2$), $z_T = y$, $z_{t+1} = y_{t+1}$ and $w_t = v_t$ for all $t \geq T$.

If for $t \geq T+1$, $p_1^{(t)} \geq 0$, then $p_1^{(t)}y_t \geq p_1^{(t)}x_t - \varepsilon p_1^{(t)}|y - x_T|$.

If for $t \geq T+1$, $p_1^{(t)} < 0$, then $p_1^{(t)}y_t \geq p_1^{(t)}x_t + \varepsilon p_1^{(t)}|y - x_T|$.

Thus, for $t \geq T+1$, $p_1^{(t)}y_t \geq p_1^{(t)}x_t - \varepsilon|p_1^{(t)}||y - x_T|$

If for $t \geq T+1$, $p_2^{(t-1)} \geq 0$, then $p_2^{(t-1)}v_{t-1} \geq p_2^{(t-1)}u_{t-1} - \varepsilon p_2^{(t-1)}|v - u_{T-1}|$.

If for $t \geq T+1$, $p_2^{(t-1)} < 0$, then $p_2^{(t-1)}v_{t-1} \geq p_2^{(t-1)}u_{t-1} + \varepsilon p_2^{(t-1)}|v - u_{T-1}|$.

Thus, for $t \geq T+1$, $p_2^{(t-1)}v_{t-1} \geq p_2^{(t-1)}u_{t-1} - \varepsilon\, |p_2^{(t-1)}||v - u_{T-1}|$.

Case 1: T = 1.

Hence, T – 1 = 0.

Thus, $\sum_{t=0}^{\infty}[p_1^{(t)}z_t + p_2^{(t)}w_t] = p_1^{(0)}x + p_2^{(0)}v + p_1^{(1)}y + p_2^{(1)}v_1 + \sum_{t=2}^{\infty}[p_1^{(t)}y_t + p_2^{(t)}v_t] = p_1^{(0)}x + p_2^{(0)}v + p_1^{(1)}y + \sum_{t=2}^{\infty}[p_1^{(t)}y_t + p_2^{(t-1)}v_{t-1}]$.

$\sum_{t=2}^{\infty}[p_1^{(t)}y_t + p_2^{(t-1)}v_{t-1}] = \sum_{t=2}^{\infty}p_1^{(t)}y_t + \sum_{t=2}^{\infty}p_1^{(t-1)}v_{t-1}$

Thus, $\sum_{t=0}^{\infty}[p_1^{(t)}z_t + p_2^{(t)}w_t] = p_1^{(0)}x + p_2^{(0)}v + p_1^{(1)}y + \sum_{t=2}^{\infty}[p_1^{(t)}y_t + p_2^{(t-1)}v_{t-1}] = p_1^{(0)}x + p_2^{(0)}v + p_1^{(1)}y + \sum_{t=2}^{\infty}p_1^{(t)}y_t + \sum_{t=2}^{\infty}p_1^{(t-1)}v_{t-1}$.

We know that, for $t \geq 2$, $p_1^{(t)}y_t \geq p_1^{(t)}x_t - \varepsilon|p_1^{(t)}||y - x_1|$, and for $t \geq 2$, $p_2^{(t-1)}v_{t-1} \geq p_2^{(t-1)}u_{t-1} - \varepsilon\, |p_2^{(t-1)}||v - u_0|$.

Therefore, $\sum_{t=2}^{\infty}p_1^{(t)}y_t \geq \sum_{t=2}^{\infty}p_1^{(t)}x_t - \varepsilon|y - x_1|\sum_{t=2}^{\infty}|p_1^{(t)}|$ and $\sum_{t=2}^{\infty}p_2^{(t-1)}v_{t-1} \geq \sum_{t=2}^{\infty}p_2^{(t-1)}u_{t-1} - \varepsilon|v - u_0|\sum_{t=2}^{\infty}|p_2^{(t-1)}|$.

Hence, $\sum_{t=2}^{\infty} p_1^{(t)} y_t + \sum_{t=2}^{\infty} p_2^{(t-1)} v_{t-1} \geq \sum_{t=2}^{\infty} p_1^{(t)} x_t + \sum_{t=2}^{\infty} p_2^{(t-1)} u_{t-1}$- ε(|y – x_1|$\sum_{t=2}^{\infty} |p_1^{(t)}|$ + |v – u_0|$\sum_{t=2}^{\infty} |p_2^{(t-1)}|$).

Thus, $\sum_{t=0}^{\infty} [p_1^{(t)} z_t + p_2^{(t)} w_t] = p_1^{(0)}$x + $p_2^{(0)}$v + $p_1^{(1)}$y + $\sum_{t=2}^{\infty} p_1^{(t)} y_t + \sum_{t=2}^{\infty} p_1^{(t-1)} v_{t-1} \geq$ $p_1^{(0)}$x + $p_2^{(0)}$v + $p_1^{(1)}$y + $\sum_{t=2}^{\infty} p_1^{(t)} x_t + \sum_{t=2}^{\infty} p_2^{(t-1)} u_{t-1}$- ε(|y – x_1|$\sum_{t=2}^{\infty} |p_1^{(t)}|$ + |v – u_0|$\sum_{t=2}^{\infty} |p_2^{(t-1)}|$).

Now, $p_1^{(0)}$x + $p_2^{(0)}$v + $p_1^{(1)}$y + $\sum_{t=2}^{\infty} p_1^{(t)} x_t + \sum_{t=2}^{\infty} p_2^{(t-1)} u_{t-1} = \sum_{t=0}^{\infty} [p_1^{(t)} x_t + p_2^{(t)} u_t]$ + $p_2^{(0)}$(v – u_0) + $p_1^{(1)}$(y – x_1).

Thus, $\sum_{t=0}^{\infty} [p_1^{(t)} z_t + p_2^{(t)} w_t] \geq \sum_{t=0}^{\infty} [p_1^{(t)} x_t + p_2^{(t)} u_t]$ - ε(|y – x_1|$\sum_{t=2}^{\infty} |p_1^{(t)}|$ + |v – u_0|$\sum_{t=2}^{\infty} |p_2^{(t-1)}|$) + $p_2^{(0)}$(v – u_0) + $p_1^{(1)}$(y – x_1).

However, according to our assumption towards a contradiction, $p_2^{(0)}$(v – u_0) + $p_1^{(1)}$(y – x_1) > ε(|y – x_1|$\sum_{t=2}^{\infty} |p_1^{(t)}|$ + |v – u_0|$\sum_{t=2}^{\infty} |p_2^{(t-1)}|$).

Thus, $\sum_{t=0}^{\infty} [p_1^{(t)} z_t + p_2^{(t)} w_t] > \sum_{t=0}^{\infty} [p_1^{(t)} x_t + p_2^{(t)} u_t]$, which in conjunction with <(z_t, w_t)|t∈ $\mathbb{N}^0$>∈ $\mathcal{F}$(x) contradicts <(x_t, u_t)|t∈ $\mathbb{N}^0$>∈ $\mathcal{S}(x)$.

Thus, it must be that $p_1^{(T)}$(y – x_T) + $p_2^{(T-1)}$(v – u_{T-1}) ≤ 0.

Case 2: T > 1.

Hence, T – 1 > 0 and T – 2 ≥ 0.

Thus, $\sum_{t=0}^{\infty} [p_1^{(t)} z_t + p_2^{(t)} w_t] = \sum_{t=0}^{T-2} [p_1^{(t)} x_t + p_2^{(t)} u_t] + p_1^{(T-1)} x_{T-1} + p_2^{(T-1)}$v + $p_1^{(T)} y$ + $\sum_{t=T+1}^{\infty} [p_1^{(t)} y_t + p_2^{(t-1)} v_{t-1}]$.

$\sum_{t=T+1}^{\infty} [p_1^{(t)} y_t + p_2^{(t-1)} v_{t-1}] = \sum_{t=T+1}^{\infty} p_1^{(t)} y_t + \sum_{t=T+1}^{\infty} p_1^{(t-1)} v_{t-1}$

Thus, $\sum_{t=0}^{\infty} [p_1^{(t)} z_t + p_2^{(t)} w_t] = \sum_{t=0}^{T-2} [p_1^{(t)} x_t + p_2^{(t)} u_t] + p_1^{(T-1)}$$x_{T-1}$ + $p_2^{(T-1)}$v + $p_1^{(T)}$y + $\sum_{t=T+1}^{\infty} [p_1^{(t)} y_t + p_2^{(t-1)} v_{t-1}] = p_1^{(T-1)}$$x_{T-1}$ + $p_2^{(T-1)}$v + $p_1^{(T)}$y + $\sum_{t=T+1}^{\infty} p_1^{(t)} y_t$ + $\sum_{t=T+1}^{\infty} p_1^{(t-1)} v_{t-1}$.

We know that, for t ≥ T+1, $p_1^{(t)}$$y_t$ ≥ $p_1^{(t)}$$x_t$ - ε|$p_1^{(t)}$||y – x_T|, <u>and</u> for t ≥ T+1, $p_2^{(t-1)}$$v_{t-1}$ ≥ $p_2^{(t-1)}$$u_{t-1}$- ε |$p_2^{(t-1)}$||v – u_{T-1}|.

Therefore, $\sum_{t=T+1}^{\infty} p_1^{(t)} y_t \geq \sum_{t=T+1}^{\infty} p_1^{(t)} x_t$ - ε|y – x_T|$\sum_{t=T+1}^{\infty} |p_1^{(t)}|$ and $\sum_{t=T+1}^{\infty} p_2^{(t-1)} v_{t-1} \geq$ $\sum_{t=T+1}^{\infty} p_2^{(t-1)} u_{t-1}$- ε|v – u_{T-1}|$\sum_{t=2}^{\infty} |p_2^{(t)}|$.

Hence, $\sum_{t=T+1}^{\infty} p_1^{(t)} y_t + \sum_{t=T+1}^{\infty} p_2^{(t-1)} v_{t-1} \geq \sum_{t=T+1}^{\infty} p_1^{(t)} x_t + \sum_{t=T+1}^{\infty} p_2^{(t-1)} u_{t-1}$- ε(|y – x_T|$\sum_{t=T+1}^{\infty} |p_1^{(t)}|$ + |v – u_{T-1}|$\sum_{t=T+1}^{\infty} |p_2^{(t-1)}|$).

Thus, $\sum_{t=0}^{\infty} [p_1^{(t)} z_t + p_2^{(t)} w_t] = \sum_{t=0}^{T-2} [p_1^{(t)} x_t + p_2^{(t)} u_t] + p_1^{(T-1)} x_{T-1} + p_2^{(T-1)} v + p_1^{(T)} y + \sum_{t=T+1}^{\infty} p_1^{(t)} y_t + \sum_{t=T+1}^{\infty} p_1^{(t-1)} v_{t-1} \geq \sum_{t=0}^{T-2} [p_1^{(t)} x_t + p_2^{(t)} u_t] + p_1^{(T-1)} x_{T-1} + p_2^{(T-1)} v + p_1^{(T)} y + \sum_{t=T+1}^{\infty} p_1^{(t)} x_t + \sum_{t=T+1}^{\infty} p_2^{(t-1)} u_{t-1}$- ε(|y – x_T|$\sum_{t=T+1}^{\infty} |p_1^{(t)}|$ + |v – u_{T-1}|$\sum_{t=T+1}^{\infty} |p_2^{(t-1)}|$).

Now, $\sum_{t=0}^{T-2} [p_1^{(t)} x_t + p_2^{(t)} u_t] + p_1^{(T-1)} x_{T-1} + p_2^{(T-1)} v + p_1^{(T)} y + \sum_{t=T+1}^{\infty} p_1^{(t)} x_t + \sum_{t=T+1}^{\infty} p_2^{(t-1)} u_{t-1} = \sum_{t=0}^{T-2} [p_1^{(t)} x_t + p_2^{(t)} u_t] + p_1^{(T-1)} x_{T-1} + p_2^{(T-1)} u_{T-1} + p_1^{(T)} x_T + \sum_{t=T+1}^{\infty} p_1^{(t)} x_t + \sum_{t=T+1}^{\infty} p_2^{(t-1)} u_{t-1} + (p_2^{(T-1)} v + p_1^{(T)} y) - (p_2^{(T-1)} u_{T-1} + p_1^{(T)} x_T) = \sum_{t=0}^{\infty} [p_1^{(t)} x_t + p_2^{(t)} u_t] + (p_2^{(T-1)} v + p_1^{(T)} y) - (p_2^{(T-1)} u_{T-1} + p_1^{(T)} x_T)$.

Thus, $\sum_{t=0}^{\infty} [p_1^{(t)} z_t + p_2^{(t)} w_t] \geq \sum_{t=0}^{T-2} [p_1^{(t)} x_t + p_2^{(t)} u_t] + p_1^{(T-1)} x_{T-1} + p_2^{(T-1)} v + p_1^{(T)} y + \sum_{t=T+1}^{\infty} p_1^{(t)} x_t + \sum_{t=T+1}^{\infty} p_2^{(t-1)} u_{t-1}$- ε(|y – x_T|$\sum_{t=T+1}^{\infty} |p_1^{(t)}|$ + |v – u_{T-1}|$\sum_{t=T+1}^{\infty} |p_2^{(t-1)}|$) = $\sum_{t=0}^{\infty} [p_1^{(t)} x_t + p_2^{(t)} u_t] + (p_2^{(T-1)}(v - u_{T-1}) + p_1^{(T)}(y - x_T)$ – ε(|y – x_T|$\sum_{t=T+1}^{\infty} |p_1^{(t)}|$ + |v – u_{T-1}|$\sum_{t=T+1}^{\infty} |p_2^{(t-1)}|$).

However, according to our assumption towards a contradiction, $(p_2^{(T-1)}(v - u_{T-1}) + p_1^{(T)}(y - x_T) >$ ε(|y – x_T|$\sum_{t=T+1}^{\infty} |p_1^{(t)}|$ + |v – u_{T-1}|$\sum_{t=T+1}^{\infty} |p_2^{(t-1)}|$).

Thus, $\sum_{t=0}^{\infty} [p_1^{(t)} z_t + p_2^{(t)} w_t] > \sum_{t=0}^{\infty} [p_1^{(t)} x_t + p_2^{(t)} u_t]$, which in conjunction with <(z_t, w_t)|t∈ $\mathbb{N}^0$>∈ $\mathcal{F}$(x) contradicts <(x_t, u_t)|t∈ $\mathbb{N}^0$>∈ $\mathcal{S}(x)$.

Thus, it must be that $p_1^{(T)}(y - x_T) + p_2^{(T-1)}(v - u_{T-1}) \leq 0$.

(ii) Since, <(($p_1^{(t)}$, $p_2^{(t)}$), Ω_t)| t∈ $\mathbb{N}^0$> is an AC-LOC problem, $\lim_{t\to\infty} |p_i^{(t)}| = 0$ for i∈ {1, 2} and x_t∈ [0, b], u_t∈ [0, b] for all t∈ $\mathbb{N}^0$. Thus, $\lim_{t\to\infty}(|p_1^{(t)}| x_t + |p_2^{(t)}| u_t) = 0$.   Q.E.D.

Note 4.1: Part (i) of proposition 4.2, is similar in spirit to what is referred to in section 5.2 of Sorger (2015) as the Euler equation. We may refer to part (i) of the proposition above as "Euler condition." The condition $\lim_{t\to\infty}(|p_1^{(t)}| x_t + |p_2^{(t)}| u_t) = 0$ in part (ii) of proposition 4.2 resembles the several versions of "transversality condition" that are available in the literature on dynamic optimization.

The following result, analogous to proposition 4.1 in Mitra (2000), provides a "neat" sufficient condition for a trajectory starting at x to belong to $\mathcal{S}$(x).

Proposition 4.3: If for <(x_t, u_t)|t∈ $\mathbb{N}^0$>∈ $\mathcal{F}$(x) and there is a sequence <$q^{(t)}$|t∈ $\mathbb{N}^0$> of non-negative real numbers satisfying:

(i) For all t∈ $\mathbb{N}^0$: $p_1^{(t)}x_t + p_2^{(t)}u_t$ + $q^{(t+1)}x_{t+1}$ – $q^{(t)}x_t$ ≥ $p_1^{(t)}y + p_2^{(t)}u$ + $q^{(t+1)}z$ – $q^{(t)}y$ for all (y, u, z)∈ Ω_t,

(ii) $\lim_{t\to\infty} q^{(t)}x_t = 0$,

then, <(x_t, u_t)|t∈ $\mathbb{N}^0$>∈ $\mathcal{S}$(x).

Proof: Let <(y_t, v_t)|t∈ $\mathbb{N}^0$>∈ $\mathcal{F}(x)$.

Then, (i) implies that for all t∈ $\mathbb{N}^0$: $p_1^{(t)}x_t + p_2^{(t)}u_t$ + $q^{(t+1)}x_{t+1}$ – $q^{(t)}x_t$ ≥ $p_1^{(t)}y_t + p_2^{(t)}v_t$ + $q^{(t+1)}y_{t+1}$ – $q^{(t)}y_t$.

Thus, for all t∈ $\mathbb{N}^0$: $(p_1^{(t)}x_t + p_2^{(t)}u_t)$ – $(p_1^{(t)}y_t + p_2^{(t)}v_t)$ ≥ ($q^{(t+1)}y_{t+1}$ – $q^{(t)}y_t$) – ($q^{(t+1)}x_{t+1}$ – $q^{(t)}x_t$).

Hence, for all T∈ $\mathbb{N}^0$: $\sum_{t=0}^{T}(p_1^{(t)}x_t + p_2^{(t)}u_t)$ - $\sum_{t=0}^{T}(p_1^{(t)}y_t + p_2^{(t)}v_t)$ ≥ ($q^{(T+1)}y_{T+1}$ – $q^{(0)}$x) - ($q^{(T+1)}x_{T+1}$ – $q^{(0)}$x) = $q^{(T+1)}y_{T+1}$ - $q^{(T+1)}x_{T+1}$ ≥ - $q^{(T+1)}x_{T+1}$, since $q^{(T+1)}$ ≥ 0 and y_{T+1} ≥ 0 implies $q^{(T+1)}y_{T+1}$ ≥ 0.

Thus, $\sum_{t=0}^{\infty}(p_1^{(t)}x_t + p_2^{(t)}u_t)$ - $\sum_{t=0}^{\infty}(p_1^{(t)}y_t + p_2^{(t)}v_t)$ = $\lim_{T\to\infty}\sum_{t=0}^{T}(p_1^{(t)}x_t + p_2^{(t)}u_t)$ - $\lim_{T\to\infty}\sum_{t=0}^{T}(p_1^{(t)}y_t + p_2^{(t)}v_t)$ = $\lim_{T\to\infty}[\sum_{t=0}^{T}(p_1^{(t)}x_t + p_2^{(t)}u_t) - \sum_{t=0}^{T}(p_1^{(t)}y_t + p_2^{(t)}v_t)]$ ≥ $\lim_{T\to\infty} - q^{(T+1)}x_{T+1}$ = 0, by (ii). Q.E.D.

Note 4.2: Conditions (i), is a variant of what is referred to as the "competitive condition" in Mitra (2000). In Mitra (2000), (ii) is referred to as the transversality condition.

5. Linear Dynamic Programming:

By proposition 4.1 we know that (x, T)∈ X×$\mathbb{N}^0$, $\mathcal{S}^T$(x) ≠ ϕ, and hence for all T∈ $\mathbb{N}^0$, there exists a function V^T: X →ℝ such that for all x∈ X, V^T(x) = $\sum_{\tau=0}^{\infty}[p_1^{(T+\tau)}x_{T+\tau}$+ $p_2^{(T+\tau)}u_{T+\tau}]$ for all <(x_{T+t}, u_{T+t})|t∈ $\mathbb{N}^0$>∈ $\mathcal{S}^T$(x). Clearly for all for all T∈ $\mathbb{N}^0$, V^T is well defined on X.

We may refer to V^T as the **period-T optimal value function**

We will denote V^0 by V. V is said to be the **optimal value function**.

Thus, for all x∈ X, V(x) = $p_1^{(0)}x + p_2^{(0)}u_0$ + $\sum_{\tau=1}^{\infty}p_1^{(T+\tau)}x_{T+\tau}$+ $p_2^{(T+\tau)}u_{T+\tau}]$, where <($x_t$, u_t)|t∈ $\mathbb{N}^0$>∈ $\mathcal{S}$(x).

The following result is analogous to well-known results on dynamic programming available in Mitra (2000), Sorger (2015) and Lahiri (2025b).

Proposition 5.1: (i) For all T∈ $\mathbb{N}^0$, V^T satisfies the following ***functional equation of dynamic programming***: For all x∈ X and <(x_t, u_t)|t ≥ T>∈ $\mathcal{S}^T$(x) and t ≥ T: $V^t(x_t)$ = $p_1^{(t)}x_t + p_2^{(t)}u_t$ + $V^{t+1}(x_{t+1})$ = $p_1^{(t)}x_t$ + $\max_{(u,y)\in\Omega_t(x_t)}\{p_2^{(t)}u + V^{t+1}(y)\}$.

(ii) For all x∈ X: <(x_t, u_t)|t∈ $\mathbb{N}^0$>∈ $\mathcal{S}$(x) if and only if <(x_t, u_t)|t∈ $\mathbb{N}^0$>∈ $\mathcal{F}$(x) and for all T∈ $\mathbb{N}^0$ it is the case that $V^T(x_T)$ = $p_1^{(T)}x + p_2^{(T)}u_T$ + $V^{T+1}(x_{T+1})$.

Proof: (i) Let $T \in \mathbb{N}^0$ and $x \in X$. Let $<(x_t, u_t)|t \geq T> \in \mathcal{S}^T(x)$.

Thus, $V^T(x) = V^T(x_T) = \sum_{\tau=T}^{\infty} (p_1^{(\tau)} x_\tau + p_2^{(\tau)} u_\tau)$.

Towards a contradiction suppose there exists $t \geq T$, such that $V^t(x_t) > p_1^{(t)} x_t + p_2^{(t)} u_t + V^{t+1}(x_{t+1})$.

Thus, there exists $<(y_{t+\tau}, v_{t+\tau})|\tau \in \mathbb{N}^0> \in \mathcal{F}^t(x_t)$ such that $y_t = x_t$ and $p_1^{(t)} x_t + p_2^{(t)} v_t + \sum_{\tau=t+1}^{\infty} (p_1^{(\tau)} y_\tau + p_2^{(\tau)} v_\tau) > p_1^{(t)} x_t + p_2^{(t)} u_t + V^{t+1}(x_{t+1}) \geq p_1^{(t)} x_t + p_2^{(t)} u_t + \sum_{\tau=t+1}^{\infty} (p_1^{(\tau)} x_\tau + p_2^{(\tau)} u_\tau)$.

Hence, $p_2^{(t)} v_t + \sum_{\tau=t+1}^{\infty} (p_1^{(\tau)} y_\tau + p_2^{(\tau)} v_\tau) > p_2^{(t)} u_t + V^{t+1}(x_{t+1}) \geq p_2^{(t)} u_t + \sum_{\tau=t+1}^{\infty} (p_1^{(\tau)} x_\tau + p_2^{(\tau)} u_\tau)$.

Clearly, $\{\tau \geq t+1 | (v_{\tau-1}, y_\tau) \neq (u_{\tau-1}, x_\tau)\} \neq \phi$.

Let $<(z_\tau, w_\tau)| \tau \geq T>$ be such that $z_\tau = x_\tau$ for $\tau = T, \ldots, t$, $z_\tau = y_\tau$ for all $\tau \geq t+1$, $w_\tau = u_\tau$ for $\tau = T, \ldots, t-1$ (if $t > T$) and $w_\tau = v_\tau$ for $\tau \geq t$.

Thus, $<(z_\tau, w_\tau)| \tau \geq T> \in \mathcal{F}^T(x)$ and $\sum_{\tau=T}^{\infty} (p_1^{(\tau)} z_\tau + p_2^{(\tau)} w_\tau) = \sum_{\tau=T}^{t} (p_1^{(\tau)} z_\tau + p_2^{(\tau)} w_\tau) + \sum_{\tau=t+1}^{\infty} (p_1^{(\tau)} z_\tau + p_2^{(\tau)} w_\tau)$.

Case 1: $t = T$.

Then, $\sum_{\tau=T}^{\infty} (p_1^{(\tau)} z_\tau + p_2^{(\tau)} w_\tau) = p_1^{(T)} z_T + p_2^{(T)} w_T + \sum_{\tau=T+1}^{\infty} (p_1^{(\tau)} z_\tau + p_2^{(\tau)} w_\tau) = p_1^{(T)} x_T + p_2^{(T)} v_T + \sum_{\tau=T+1}^{\infty} (p_1^{(\tau)} y_\tau + p_2^{(\tau)} v_\tau) > p_1^{(T)} x_T + p_2^{(t)} u_t + \sum_{\tau=T+1}^{\infty} (p_1^{(\tau)} x_\tau + p_2^{(\tau)} u_\tau) = \sum_{\tau=T}^{\infty} (p_1^{(\tau)} x_\tau + p_2^{(\tau)} u_\tau)$, i.e., $\sum_{\tau=T}^{\infty} (p_1^{(\tau)} z_\tau + p_2^{(\tau)} w_\tau) > \sum_{\tau=T}^{\infty} (p_1^{(\tau)} x_\tau + p_2^{(\tau)} u_\tau)$, contradicting $<(x_t, u_t)|t \geq T> \in \mathcal{S}^T(x)$.

Case 2: $t > T$.

Then, $\sum_{\tau=T}^{\infty} (p_1^{(\tau)} z_\tau + p_2^{(\tau)} w_\tau) = \sum_{\tau=T}^{t-1} (p_1^{(\tau)} z_\tau + p_2^{(\tau)} w_\tau) + (p_1^{(t)} z_t + p_2^{(t)} w_t) + \sum_{\tau=t+1}^{\infty} (p_1^{(\tau)} z_\tau + p_2^{(\tau)} w_\tau) = \sum_{\tau=T}^{t-1} (p_1^{(\tau)} x_\tau + p_2^{(\tau)} u_\tau) + (p_1^{(t)} x_t + p_2^{(t)} v_t) + \sum_{\tau=t+1}^{\infty} (p_1^{(\tau)} y_\tau + p_2^{(\tau)} v_\tau) > \sum_{\tau=T}^{t-1} (p_1^{(\tau)} x_\tau + p_2^{(\tau)} u_\tau) + p_1^{(t)} x_t + p_2^{(t)} u_t + \sum_{\tau=t+1}^{\infty} (p_1^{(\tau)} x_\tau + p_2^{(\tau)} u_\tau) = \sum_{\tau=T}^{\infty} (p_1^{(\tau)} x_\tau + p_2^{(\tau)} u_\tau)$, i.e. $\sum_{\tau=T}^{\infty} (p_1^{(\tau)} z_\tau + p_2^{(\tau)} w_\tau) > \sum_{\tau=T}^{\infty} (p_1^{(\tau)} x_\tau + p_2^{(\tau)} u_\tau)$, once again contradicting $<(x_t, u_t)|t \geq T> \in \mathcal{S}^T(x)$.

Thus, it must be the case that, for all t ≥ T: $V^t(x_t) = p_1^{(t)}x_t + p_2^{(t)}u_t + V^{t+1}(x_{t+1}) = p_1^{(t)}x_t + \max_{(u,y)\in\Omega_t(x_t)}\{p_2^{(t)}u + V^{t+1}(y)\}$, which is the "functional equation of dynamic programming".

(ii) Suppose $<(x_t, u_t)|\mathbb{N}^0>\in \mathcal{S}(x)$. Then from (i) it follows that $<(x_t, u_t)|\mathbb{N}^0>\in \mathcal{F}(x)$ and for all $T\in \mathbb{N}^0$ it is the case that $V^T(x_T) = p_1^{(T)}x + p_2^{(T)}u_T + V^{T+1}(x_{T+1})$.

Now suppose $<x_t|t\in \mathbb{N}^0>\in \mathcal{F}(x)$ and for all $T\in \mathbb{N}^0$: $V^T(x_T) = p_1^{(T)}x_T + p_2^{(T)}u_T + V^{T+1}(x_{T+1})$.

Hence, $V(x) = V^0(x) = p_1^{(0)}x + p_2^{(0)}u_0 + V^1(x_1) = p_1^{(0)}x + p_2^{(0)}u_0 + p_1^{(1)}x_1 + p_2^{(1)}u_1 + V^2(x_T) =\ldots = \sum_{t=0}^{T}(p_1^{(t)}x_t + p_2^{(t)}u_t) + V^{T+1}(x_{T+1})$ for all $T\in \mathbb{N}$.

Since, $|V^T(x_T)| \leq b\sum_{t=T}^{\infty}(|p_1^{(t)}| + |p_2^{(t)}|)$ and since $\lim_{T\to\infty}\sum_{t=T}^{\infty}(|p_1^{(t)}| + |p_2^{(t)}|) = 0$, it follows that $\lim_{T\to\infty} V^{T+1}(x_{T+1}) = 0$.

Thus, $V(x) = \lim_{T\to\infty}\sum_{t=0}^{T}(p_1^{(t)}x_t + p_2^{(t)}u_t) + \lim_{T\to\infty} V^{T+1}(x_{T+1}) = \sum_{t=0}^{\infty}(p_1^{(t)}x_t + p_2^{(t)}u_t)$.

Thus, $<(x_t, u_t)|t\in \mathbb{N}^0>\in \mathcal{S}(x)$. Q.E.D.

Note 5.1: Proposition 5.1 and its proof is similar to the second proposition and its proof in section 5 of Lahiri (2025b).

LOC $<((p_1^{(t)}, p_2^{(t)}), \Omega_t)|t\in \mathbb{N}^0>$ is said be **concave in pay-offs for the control variable** if for all $t\in \mathbb{N}^0$, (x, u, y), $(z, v, \tilde{y})\in \Omega_t$ and $\theta\in (0, 1)$, there exists $w\in X$ such that $(\theta x + (1-\theta)z, w, \theta y + (1-\theta)\tilde{y})\in \Omega_t$ and $p_2^{(t)}w \geq \theta p_2^{(t)}u + (1-\theta)p_2^{(t)}v$.

If for all $t\in \mathbb{N}^0$, Ω_t is convex, then the LOC is concave in pay-offs for the control variable.

The following proposition and its proof is similar to the first proposition and its proof in section 5 of Lahiri (2025b).

Proposition 5.2: If $<((p_1^{(t)}, p_2^{(t)}), \Omega_t)|t\in \mathbb{N}^0>$ is an AC-LOC that is concave in pay-offs for the control variable, <u>then</u> for all $T\in \mathbb{N}^0$, V^T is concave and continuous on X.

Proof: Let $T\in \mathbb{N}^0$. For all (x, T), $(y, T)\in X\times\mathbb{N}^0$, let $<(x_t, u_t)\ |t \geq T>\in \mathcal{S}^T(x)$ and $<(y_t, v_t)|t \geq T>\in \mathcal{S}^T(y)$. Suppose $\theta\in (0, 1)$ and consider the infinite sequence $<(\theta x_t + (1-\theta)y_t, \theta u_t + (1-\theta)v_t)\ |t \geq T>$.

Since $(x_t, u_t, x_{t+1})\in \Omega_t$ and $(y_t, v_t, y_{t+1})\in \Omega_t$ for all t ≥ T and the AC-LOC is concave in pay-offs for the control variable, for all t ≥ T, there exists $w_t\in X$ such that $(\theta x_t + (1-\theta)y_t, w_t, \theta x_{t+1} + (1-\theta)y_{t+1}) \in \Omega_t$ for all t ≥ T and $p_2^{(t)}w_t \geq \theta p_2^{(t)}u_t + (1-\theta)p_2^{(t)}v_t$.

Thus, $<(\theta x_t + (1-\theta)y_t, w_t, \theta u_t + (1-\theta)v_t)|t \geq T>\in \mathcal{F}^T(\theta x + (1-\theta)y)$.

Thus, $V^T(\theta x + (1-\theta)y) \geq \sum_{t=T}^{\infty}[p_1^{(t)}(\theta x_t + (1-\theta)y_t) + p_2^{(t)}w_t] \geq \sum_{t=T}^{\infty}[p_1^{(t)}\left(\theta x_t + (1-\theta)y_t\right) + p_2^{(t)}(\theta u_t + (1-\theta)v_t) = \theta\sum_{t=T}^{\infty}(p_1^{(t)}x_t + p_2^{(t)}u_t) + (1-\theta)\sum_{t=T}^{\infty}(p_1^{(t)}y_t + p_2^{(t)}v_t) = \theta V^T(x) + (1-\theta)V^T(y)$.

Thus, for all $T \in \mathbb{N}^0$, V^T is a concave function on X.

From corollary 1 of proposition 1 and part (i) of proposition 3 in Lahiri (2025a) it follows that since V^T is concave on X = [0, b], it must be the case that V^T is continuous on (0, b) and both $\lim_{x \to 0} V^T(x)$ and $\lim_{x \to b} V^T(x)$ exists and belong to the interval [-2M, 2M] where M = $\max\{b\sum_{t=0}^{\infty} |p_1^{(t)}|, b\sum_{t=0}^{\infty} |p_2^{(t)}|\} < +\infty$.

Further, it must be that $\lim_{x \to 0} V^T(x) \geq V^T(0)$ and $\lim_{x \to b} V^T(x) \geq V^T(b)$.

Now let $x \in X$ and $<x^{(n)}|n \in \mathbb{N}>$ be a sequence in X converging to x.

For each $n \in \mathbb{N}$, let $<(x_{T+t}^{(n)}, u_{T+t}^{(n)})|\ t \in \mathbb{N}^0> \in \mathcal{S}(x^{(n)})$.

Since $x_T^{(n)} = x^{(n)}$ for all $n \in \mathbb{N}$, $\lim_{n \to \infty} x_T^{(n)} = x$.

Since, $<(x_T^{(n)}, u_T^{(n)}, x_{T+1}^{(n)})|\ n \in \mathbb{N}>$ is a sequence in the closed and bounded set X×X×X, it has a convergent subsequence $<(x_T^{N_1(n)}, u_T^{N_1(n)}, x_{T+1}^{N_1(n)})|n \in \mathbb{N}>$ converging to $(x_T^0, u_T^0, x_{T+1}^0) \in$ X×X×X, where $x_T^0 = x$.

Further, $(x_T^{N_1(n)}, u_T^{N_1(n)}, x_{T+1}^{N_1(n)}) \in \Omega_T$ for all $n \in \mathbb{N}$ implies $(x, u_T^0, x_{T+1}^0) \in \Omega_T$ since Ω_T is closed.

Having obtained convergent subsequences $<(x_{T+\tau-1}^{N_\tau(n)}, u_{T+\tau-1}^{N_\tau(n)}, x_{T+\tau}^{N_\tau(n)})|\ n \in \mathbb{N}> \in \Omega_{T+\tau-1}$ for all τ = 1, …, t for some $t \geq 1$, let $<(x_{T+t}^{N_{t+1}(n)}, u_{T+t}^{N_{t+1}(n)}, x_{T+t}^{N_{t+1}(n)}|\ n \in \mathbb{N}>$ be a convergent subsequence of the sequence $<(x_{T+t}^{N_t(n)}, u_{T+t}^{N_t(n)}, x_{T+t+1}^{N_t(n)}|\ n \in \mathbb{N}>$ in the closed and bounded set X×X×X, converging to $(x_{T+t}^0, u_{T+t}^0, x_{T+t+1}^0) \in$ X×X×X. Since, $<(x_{T+t}^{N_t(n)}, u_{T+t}^{N_t(n)}, x_{T+t+1}^{N_t(n)}|\ n \in \mathbb{N}> \in \Omega_{T+t}$ and Ω_{T+t} is a closed subset of X×X×X, it must be the case that $(x_{T+t}^0, u_{T+t}^0, x_{T+t+1}^0) \in \Omega_{T+t}$.

Since, for all $t \in \mathbb{N}^0$, $<(x_{T+\tau}^{N_\tau(n)}, u_{T+\tau}^{N_\tau(n)}, x_{T+\tau+1}^{N_\tau(n)})|\ n \in \mathbb{N}>$ converges to $(x_{T+\tau}^0, u_{T+\tau}^0, x_{T+\tau+1}^0) \in \Omega_{T+\tau}$, for all $\tau \leq t$ and $n \in \mathbb{N}$, ***given*** $\varepsilon > 0$, for all $t \in \mathbb{N}$, there exists a sequence $<n_t|t \in \mathbb{N}>$ in $\mathbb{N}$ satisfying $n_{t+1} > n_t$ for all $t \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq n_t$ and $\tau \leq t$, $(x_{T+\tau}^{N_\tau(n)}, u_{T+\tau}^{N_\tau(n)}, x_{T+\tau+1}^{N_\tau(n)}) \in \Omega_{T+\tau}$ and $|p_1^{(T+\tau)}x_{T+\tau}^0 + p_2^{(T+\tau)}u_{T+\tau}^0 - p_1^{(T+\tau)}x_{T+\tau}^{N_t(n)} - p_2^{(T+\tau)}u_{T+\tau}^{N_t(n)}| < \frac{\varepsilon}{8}(\frac{1}{2})^t$, i.e., $p_1^{(T+\tau)}x_{T+\tau}^{N_t(n)} + p_2^{(T+\tau)}u_{T+\tau}^{N_t(n)} + \frac{\varepsilon}{8}(\frac{1}{2})^t > p_1^{(T+\tau)}x_{T+\tau}^0 + p_2^{(T+\tau)}u_{T+\tau}^0 > p_1^{(T+\tau)}x_{T+\tau}^{N_t(n)} + p_2^{(T+\tau)}u_{T+\tau}^{N_t(n)} - \frac{\varepsilon}{8}(\frac{1}{2})^t$.

Note that $\sum_{\tau=0}^{t} \frac{\varepsilon}{8}(\frac{1}{2})^t \leq \sum_{\tau=0}^{t} \frac{\varepsilon}{8}(\frac{1}{2})^\tau = \frac{\varepsilon}{4}(1 - (\frac{1}{2})^{t+1})$.

Thus, for all $t \in \mathbb{N}$, $\sum_{\tau=0}^{t} (p_1^{(T+\tau)}x_{T+\tau}^0 + p_2^{(T+\tau)}u_{T+\tau}^0) > \sum_{\tau=0}^{t} (p_1^{(T+\tau)}x_{T+\tau}^{N_t(n)} + p_2^{(T+\tau)}u_{T+\tau}^{N_t(n)}) - \frac{\varepsilon}{4}(1 - (\frac{1}{2})^{t+1})$ for all $n \geq n_t$.

Thus, for all t∈ℕ, $\sum_{\tau=0}^{t}(p_1^{(T+\tau)}x_{T+\tau}^0 + p_2^{(T+\tau)}u_{T+\tau}^0) > \sum_{\tau=0}^{t}(p_1^{(T+\tau)}x_{T+\tau}^{N_t(n)} + p_2^{(T+\tau)}u_{T+\tau}^{N_t(n)}) - \frac{\varepsilon}{4}(1-(\frac{1}{2})^{t+1}) = V(x^{N_t(n)}) - \sum_{t=\tau+1}^{\infty}(p_1^{(T+\tau)}x_{T+\tau}^{N_t(n)} + p_2^{(T+\tau)}u_{T+\tau}^{N_t(n)}) - \frac{\varepsilon}{4}(1-(\frac{1}{2})^{t+1})$ for all $n \geq n_t$.

Thus, for all t∈ℕ, $V^T(x) \geq \sum_{\tau=0}^{\infty}(p_1^{(T+\tau)}x_{T+\tau}^0 + p_2^{(T+\tau)}u_{T+\tau}^0) = \sum_{\tau=0}^{t}(p_1^{(T+\tau)}x_{T+\tau}^0 + p_2^{(T+\tau)}u_{T+\tau}^0) + \sum_{\tau=t+1}^{\infty}(p_1^{(T+\tau)}x_{T+\tau}^0 + p_2^{(T+\tau)}u_{T+\tau}^0) > V(x^{N_t(n)}) - \sum_{t=\tau+1}^{\infty}(p_1^{(T+\tau)}x_{T+\tau}^{N_t(n)} + p_2^{(T+\tau)}u_{T+\tau}^{N_t(n)}) - \frac{\varepsilon}{4}(1-(\frac{1}{2})^{t+1}) + \sum_{\tau=t+1}^{\infty}(p_1^{(T+\tau)}x_{T+\tau}^0 + p_2^{(T+\tau)}u_{T+\tau}^0)$ for all $n \geq n_\tau$.

Thus, for all t∈ℕ and $n \geq n_t$, $V^T(x) > V(x^{N_t(n)}) - \sum_{t=\tau+1}^{\infty}(p_1^{(T+\tau)}x_{T+\tau}^{N_t(n)} + p_2^{(T+\tau)}u_{T+\tau}^{N_t(n)}) - \frac{\varepsilon}{4}(1-(\frac{1}{2})^{t+1}) + \sum_{\tau=t+1}^{\infty}(p_1^{(T+\tau)}x_{T+\tau}^0 + p_2^{(T+\tau)}u_{T+\tau}^0)$.

By hypothesis, $\sum_{t=0}^{\infty}|p_i^{(t)}| < +\infty$ for i∈{1, 2} and for all (t, n)∈ $\mathbb{N}^0\times\mathbb{N}$, both $x_{T+t}^{(n)} \in [0, b]$ and $u_{T+t}^{(n)} \in [0, b]$.

Thus, for all t∈ℕ and $n \geq n_t$, $b\sum_{\tau=t+1}^{\infty}[|p_1^{(T+\tau)}| + |p_2^{(T+\tau)}|] \geq \sum_{\tau=t+1}^{\infty}[|p_1^{(T+\tau)}|x_{T+\tau}^{N_t(n)} + p_2^{(T+\tau)}|u_{T+\tau}^{N_t(n)}]$.

Further, for all t∈ℕ and $n \geq n_t$, $\sum_{\tau=t+1}^{\infty}[|p_1^{(T+\tau)}|x_{T+\tau}^{N_t(n)} + |p_2^{(T+\tau)}|u_{T+\tau}^{N_t(n)}] \geq \sum_{\tau=t+1}^{\infty}[|p_1^{(T+\tau)}x_{T+\tau}^{N_t(n)} + p_2^{(T+\tau)}u_{T+\tau}^{N_t(n)}|]$.

Thus, for all t∈ℕ and $n \geq n_t$, $V^T(x) + b\sum_{\tau=t+1}^{\infty}[|p_1^{(T+\tau)}| + |p_2^{(T+\tau)}|] \geq V^T(x) + \sum_{\tau=t+1}^{\infty}[|p_1^{(T+\tau)}x_{T+\tau}^{N_t(n)} + p_2^{(T+\tau)}u_{T+\tau}^{N_t(n)}|] \geq V^T(x) + \sum_{\tau=t+1}^{\infty}[|p_1^{(T+\tau)}x_{T+\tau}^{N_t(n)} + p_2^{(T+\tau)}u_{T+\tau}^{N_t(n)}|]$.

Thus, for all t∈ℕ and $n \geq n_t$, $V^T(x) + b\sum_{\tau=t+1}^{\infty}[|p_1^{(T+\tau)}| + |p_2^{(T+\tau)}|] \geq V^T(x) + \sum_{\tau=t+1}^{\infty}[p_1^{(T+\tau)}x^{N_t(n)} + p_2^{(T+\tau)}u_{T+\tau}^{N_t(n)}] > V^T(x^{N_t(n)}) - \frac{\varepsilon}{4}(1-(\frac{1}{2})^{t+1}) + \sum_{\tau=t+1}^{\infty}(p_1^{(T+\tau)}x_{T+\tau}^0 + p_2^{(T+\tau)}u_{T+\tau}^0)$.

Thus, for all t∈ℕ, $V^T(x) + b\sum_{\tau=t+1}^{\infty}[|p_1^{(T+\tau)}| + |p_2^{(T+\tau)}|] \geq \limsup_{n\to\infty} V^T(x^{N_t(n)}) - \frac{\varepsilon}{4}(1-(\frac{1}{2})^{t+1}) + \sum_{\tau=t+1}^{\infty}(p_1^{(T+\tau)}x_{T+\tau}^0 + p_2^{(T+\tau)}u_{T+\tau}^0)$.

By hypothesis, $\sum_{t=0}^{\infty}|p_i^{(t)}| < +\infty$ for i∈{1, 2}.

Thus, $\lim_{t\to\infty} b\sum_{\tau=t+1}^{\infty}[|p_1^{(T+\tau)}| + |p_2^{(T+\tau)}|] = 0$.

Also, $\lim_{t\to\infty}\frac{\varepsilon}{4}(1-(\frac{1}{2})^{t+1}) = \frac{\varepsilon}{4}$ and $\lim_{t\to\infty}\sum_{\tau=t+1}^{\infty}(p_1^{(T+\tau)}x_{T+\tau}^0 + p_2^{(T+\tau)}u_{T+\tau}^0) = 0$.

Thus, $V^T(x) = V^T(x) + \lim_{t\to\infty} b\sum_{\tau=t+1}^{\infty}[|p_1^{(T+\tau)}| + |p_2^{(T+\tau)}|] \geq \lim_{t\to\infty}\limsup_{n\to\infty} V^T(x^{N_t(n)})$ - $\lim_{t\to\infty}\frac{\varepsilon}{8}(1-(\frac{1}{2})^{t+1}) + \lim_{t\to\infty}(p_1^{(T+\tau)}x_{T+\tau}^0 + p_2^{(T+\tau)}u_{T+\tau}^0) = \lim_{t\to\infty}\limsup_{n\to\infty} V^T(x^{N_t(n)}) - \frac{\varepsilon}{4}$.

Thus, $V^T(x) \geq \lim_{t\to\infty}\limsup_{n\to\infty} V^T(x^{N_t(n)}) - \frac{\varepsilon}{4}$.

The above being true for all $\varepsilon > 0$, we get $V^T(x) \geq \lim_{t\to\infty}\limsup_{n\to\infty} V^T(x^{N_t(n)})$.

However, $\lim_{n\to\infty} V^T(x^{(n)})$ exists implies $\lim_{n\to\infty} V^T(x^{(n)}) = \limsup_{n\to\infty} V^T(x^{N_t(n)})$ for all $t\in\mathbb{N}$.

Thus, $V^T(x) \geq \lim_{t\to\infty}\limsup_{n\to\infty} V^T(x^{N_t(n)}) = V^T(x) \geq \lim_{t\to\infty}\lim_{n\to\infty} V^T(x^{(n)}) = \lim_{n\to\infty} V^T(x^{(n)})$, i.e., $V^T(x) \geq \lim_{n\to\infty} V^T(x^{(n)})$.

This combined with the continuity of V^T on (0, b), $\lim_{x\to b} V^T(x) \geq V^T(b)$ and $\lim_{x\to 0} V^T(x) \geq V^T(0)$ implies $\lim_{x\to b} V^T(x) = V^T(b)$, $\lim_{x\to 0} V^T(x) = V^T(0)$ and hence the continuity of V^T on X. Q.E.D.

6. Linear Optimal Control with Linear and Square-Root Constraints:

The following is an example of a linear optimal control problem that can be relevant for intertemporal economics.

For all $t\in\mathbb{N}^0$, there exists a pair of real numbers (a_t, b_t) satisfying $a_t \geq 0$ and $a_t + b_t b \geq 0$ and an ordered pair of non-negative real numbers (c_t, d_t) satisfying $d_t > 0$ and $c_t < b$, such that for all $x\in X$, $\Omega_t(x) = \{(u, y) | u\in[0, \min\{x, a_t + b_t x\}]$ and $y = \min\{b, c_t + d_t\sqrt{(x-u)}\}\}$.

Since for all $t\in\mathbb{N}^0$, $c_t < b$ it must be the case that for all $x\in X$, [u = x implies $y = \min\{b, c_t\} = c_t$.

Note that the function $\xi \mapsto c_t + d_t\sqrt{\xi}$ on $\mathbb{R}_+$ assumes the value c_t at $\xi = 0$ is strictly increasing on $\mathbb{R}_+$ is twice continuously differentiable on $\mathbb{R}_{++}$ with its first derivative being strictly positive and its second derivative being strictly negative at all points in $\mathbb{R}_{++}$. Further, the first derivative diverges to $+\infty$ as ξ tends to zero.

If $y < b$, then it must be the case that $y = c_t + d_t\sqrt{(x-u)}$ so that $\sqrt{(x-u)} = \frac{y-c_t}{d_t}$, i.e., $u = x - (\frac{y-c_t}{d_t})^2$.

A LOC problem $<((p_1^{(t)}, p_2^{(t)}), \Omega_t) | t\in\mathbb{N}^0>$ is said to be a **linear optimal control problem with linear and square-root constraints (LOC-LSRC)** if all $(x, t)\in X\times\mathbb{N}^0$, $\Omega_t(x) = \{(u, y) | u\in[0, \min\{x, a_t + b_t x\}]$ and $y = \min\{b, c_t + d_t\sqrt{(x-u)}\}\}$.

For a LOC-LSRC if <(x_t, u_t)|t∈$\mathbb{N}^0$>∈$\mathcal{F}$(x), then $u_t = x_t - (\frac{x_{t+1}-c_t}{d_t})^2$ if $x_{t+1} < b$.

Proposition 6.1: Suppose <(($p_1^{(t)}$, $p_2^{(t)}$), Ω_t)|t∈$\mathbb{N}^0$> is an LOC-LSRC with $a_t = 0$, $b_t = 1$ and $p_2^{(t)} \neq 0$ for all t∈$\mathbb{N}^0$. Suppose x∈X and <(x_t, u_t)|t∈$\mathbb{N}^0$>∈$\mathcal{S}$(x) satisfies $0 < u_t < x_t < b$ for all t∈$\mathbb{N}^0$. Then, for all t∈$\mathbb{N}^0$, $x_{t+1} = c_t + \frac{(d_t)^2(p_2^{(t+1)}+p_1^{(t+1)})}{2p_2^{(t)}}$ and $u_t = x_t - (\frac{p_2^{(t+1)}+p_1^{(t+1)}}{2p_2^{(t)}})^2$.

Proof: Under the conditions of proposition 6.1 and the assumption that <(x_t, u_t)|t∈$\mathbb{N}^0$>∈$\mathcal{S}$(x), it follows that <(y_t, v_t)|t∈$\mathbb{N}^0$>∈$\mathcal{F}$(x), where $y_\tau = x_\tau$ for all $\tau \neq t+1$, $v_\tau = y_\tau$ for all $\tau \notin \{t, t+1\}$, $v_t = x_t - (\frac{y_{t+1}-c_t}{d_t})^2$ and $v_{t+1} = y_{t+1} - (\frac{x_{t+2}-c_{t+1}}{d_{t+1}})^2$ so long as y_{t+1} belongs to a sufficiently small interval around x_{t+1}.

Thus, for all t∈$\mathbb{N}^0$, (u_t, x_{t+1}) is a local maximizer of the optimization problem: Maximize $p_2^{(t)}$u + ($p_2^{(t+1)}$+ $p_1^{(t+1)}$)y, subject to $u = x_t - (\frac{y-c_t}{d_t})^2$.

Thus, for all t∈$\mathbb{N}^0$, x_{t+1} is a local maximizer of the optimization problem: Maximize ($p_2^{(t+1)}$+ $p_1^{(t+1)}$)y - $p_2^{(t)}(\frac{y-c_t}{d_t})^2$

The function y|→ ($p_2^{(t+1)}$+ $p_1^{(t+1)}$)y - $p_2^{(t)}(\frac{y-c_t}{d_t})^2$), is continuously differentiable in a neighborhood of x_{t+1}. Thus, its derivative at $y = x_{t+1}$ must be zero.

The derivative of the function y|→ $p_1^{(t+1)}$y - $p_2^{(t)}(\frac{y-c_t}{d_t})^2$) in a small interval around x_{t+1} is ($p_2^{(t+1)}$+ $p_1^{(t+1)}$) - $2p_2^{(t)}(\frac{y-c_t}{(d_t)^2})$.

Thus, ($p_2^{(t+1)}$+ $p_1^{(t+1)}$) - $2p_2^{(t)}(\frac{x_{t+1}-c_t}{(d_t)^2})$ = 0 and hence $x_{t+1} = c_t + \frac{(d_t)^2(p_2^{(t+1)}+p_1^{(t+1)})}{2p_2^{(t)}}$.

Further, $u_t = x_t - (\frac{x_{t+1}-c_t}{(d_t)^2})^2 = x_t - (\frac{p_2^{(t+1)}+p_1^{(t+1)}}{2p_2^{(t)}})^2$. Q.E.D.

Note 6.1: For t∈$\mathbb{N}^0$ and (x_t, u, y)∈Ω_t, $u = x_t$ implies y = min {b, c_t} = c_t.

The right-hand derivative of the function y|→ ($p_2^{(t+1)}$+ $p_1^{(t+1)}$)y - $p_2^{(t)}(\frac{y-c_t}{(d_t)^2})^2$ at $y = c_t$ is ($p_2^{(t+1)}$+ $p_1^{(t+1)}$).

If u is decreased slightly from its initial value of x_t, then the value of y goes up slightly leading to an increase in the value of the function, if ($p_2^{(t+1)}$+ $p_1^{(t+1)}$) > 0.

Thus, assuming ($p_2^{(t+1)}$+ $p_1^{(t+1)}$) > 0 for all t∈$\mathbb{N}^0$ may be consistent with the premises of proposition 6.1, so that the proposition is not "vacuously true", i.e., a trajectory satisfying the premises in the statement of proposition 6.1 exists.

Note 6.2: Proposition 6.1 is a characterization of a "specific type" of solution provided such a solution exists. It does not provide us with enough information to define optimal decision rules, which are functions on X.

7. Linear Optimal Control with Linear and Quadratic Constraints:

Yet another example of a linear optimal control problem that can be relevant for intertemporal economics is the following.

For all $t \in \mathbb{N}^0$, there exists a pair of real number (a_t, b_t) satisfying $a_t \geq 0$ and $a_t + b_t b \geq 0$ and an ordered triplet of non-negative real numbers (c_t, d_t, e_t) satisfying $d_t > 0$, $e_t > 0$ and $c_t < b \leq \frac{d_t}{2e_t}$, such that for all $x \in X$, $\Omega_t(x) = \{(u, y) \mid u \in [0, \min\{x, a_t + b_t x\}]$ and $y = \min\{b, c_t + d_t(x - u) - e_t(x - u)^2\}\}$.

Since for all $t \in \mathbb{N}^0$, $c_t < b$ it must be the case that for all $x \in X$, [$u \in \{x - \frac{d_t}{e_t}, x\}$ implies $y = \min\{b, c_t\} = c_t$.

Note that the graph of the function $\xi \mapsto c_t + d_t\xi - e_t\xi^2$ on $\mathbb{R}_+$ is an inverted parabola that assumes the value c_t at $\xi = 0$, attains its maximum value $c_t + \frac{(d_t)^2}{4e_t} > 0$ at $\xi = \frac{d_t}{2e_t} < \frac{d_t + \sqrt{(d_t)^2 + 4e_t c_t}}{2e_t}$ and assumes the value zero at $\xi = \frac{d_t + \sqrt{(d_t)^2 + 4e_t c_t}}{2e_t}$. The maximum value of the function on $\mathbb{R}_+$ is strictly greater than c_t.

If $y < b$, then it must be the case that $y = c_t + d_t(x - u) - e_t(x - u)^2$.

A LOC problem $<((p_1^{(t)}, p_2^{(t)}), \Omega_t) \mid t \in \mathbb{N}^0>$ is said to be a **linear optimal control problem with linear and quadratic constraints (LOC-LQC)** if all $(x, t) \in X \times \mathbb{N}^0$, $\Omega_t(x) = \{(u, y) \mid u \in [0, \min\{x, a_t + b_t x\}]$ and $y = \min\{b, c_t + d_t(x - u) - e_t(x - u)^2\}\}$.

Note 7.1: For a LOC-LQC if $<(x_t, u_t) \mid t \in \mathbb{N}^0> \in \mathcal{F}(x)$, then $x_{t+1} = c_t + d_t(x_t - u_t) - e_t(x_t - u_t)^2$ if $x_{t+1} < b$.

Further, $x_{t+1} \in [c_t, b]$. If $x_{t+1} = c_t$ then $u_t \in \{x_t, x_t - \frac{d_t}{e_t}\}$ for all $t \in \mathbb{N}^0$. However, $u_t \leq x_t \leq b \leq \frac{d_t}{2e_t} < \frac{d_t}{e_t}$ implies $u_t = x_t - \frac{d_t}{e_t} < 0$, which is not possible since $u_t \geq 0$. Thus, if $x_{t+1} = c_t$ then $u_t = x_t$.

Also, note that on the set of $\{u \in [0, x_t] \mid c_t + d_t(x_t - u) - e_t(x_t - u)^2 < b\}$, the function $u \mapsto c_t + d_t(x_t - u) - e_t(x_t - u)^2$ is a strictly increasing and strictly concave function of u. The fact that it is strictly increasing function of u on $\{u \in [0, x_t] \mid c_t + d_t(x_t - u) - e_t(x_t - u)^2 < b\}$ implies that if $y = c_t + d_t(x_t - u) - e_t(x_t - u)^2$ for any u in this set, then $u = x_t - \frac{d_t + \sqrt{(d_t)^2 + 4e_t(c_t - y)}}{2e_t}$, i.e., $u \neq x_t - \frac{d_t - \sqrt{(d_t)^2 + 4e_t(c_t - y)}}{2e_t}$.

Proposition 7.1: Suppose $<((p_1^{(t)}, p_2^{(t)}), \Omega_t) \mid t \in \mathbb{N}^0>$ is an LOC-LQC with $a_t = 0$, $b_t = 1$, $p_1^{(t+1)} + p_2^{(t+1)} > 0$ and $(d_t)^2 + 4e_t(c_t - b) > 0$, for all $t \in \mathbb{N}^0$. Suppose $x \in X$ and $<(x_t, u_t) \mid t \in \mathbb{N}^0> \in \mathcal{S}(x)$ satisfies $0 < u_t < x_t < b$ for all $t \in \mathbb{N}^0$. Then, for all $t \in \mathbb{N}^0$, $u_t = x_t - \frac{\left(p_1^{(t+1)} + p_2^{(t+1)}\right) d_t - p_2^{(t)}}{2(p_1^{(t+1)} + p_2^{(t+1)}) e_t}$

Proof: Under the conditions of proposition 7.1 and the assumption that $<(x_t, u_t)|t\in \mathbb{N}^0>\in \mathcal{S}(x)$, it follows that $<(y_t, v_t)|t\in \mathbb{N}^0>\in \mathcal{F}(x)$, where $y_\tau = x_\tau$ for all $\tau \neq t+1$, $v_\tau = u_\tau$ for all $\tau \notin \{t, t+1\}$, $y_{t+1} = c_t + d_t(x_t - v_t) - e_t(x_t - v_t)^2$, $x_{t+2} = c_t + d_t(y_{t+1} - v_{t+1}) - e_t(y_{t+1} - v_{t+1})^2$ so long as v_t, belongs to a sufficiently small interval of u_t, so that y_{t+1} remains in a sufficiently small neighborhood of x_{t+1} and $y_{t+1} < b$.

$x_{t+2} = c_t + d_t(y_{t+1} - v_{t+1}) - e_t(y_{t+1} - v_{t+1})^2$ implies $v_{t+1} = y_{t+1} - \frac{d_{t+1}+\sqrt[2]{(d_{t+1})^2+4e_{t+1}(c_{t+1}-x_{t+2})}}{2e_{t+1}}$.

Hence, if v_t remains in a sufficiently small neighborhood of u_t, v_{t+1} remains close to u_{t+1} and satisfies $0 < v_{t+1} < y_{t+1}$.

Thus, for all $t\in \mathbb{N}^0$: (u_t, x_{t+1}) is a local maximizer of the optimization problem: Maximize $p_2^{(t)}u + (p_1^{(t+1)}+ p_2^{(t+1)})y$, subject to $y = c_t + d_t(x_t - u) - e_t(x_t - u)^2$.

Hence, for all $t\in \mathbb{N}^0$: u_t is a local maximizer of the optimization problem: Maximize $p_2^{(t)}u + (p_1^{(t+1)}+ p_2^{(t+1)})(c_t + d_t(x_t - u) - e_t(x_t - u)^2)$ which in turn is equivalent to the statement, [for all $t\in \mathbb{N}^0$: u_t is a local maximizer of the optimization problem: Maximize $p_2^{(t)}u + (p_1^{(t+1)}+ p_2^{(t+1)})(d_t(x_t - u) - e_t(x_t - u)^2)$].

Since we have assumed $p_1^{(t+1)} + p_2^{(t+1)} > 0$, the function $u \mapsto p_2^{(t)}u + (p_1^{(t+1)}+ p_2^{(t+1)})(d_t(x_t - u) - e_t(x_t - u)^2)$ on $[c_t, b]$ is continuously differentiable and locally strictly concave in a neighborhood of u_t. Thus, its derivative at $u = u_t$ must be zero.

The derivative of the function $u \mapsto p_2^{(t)}u + (p_1^{(t+1)}+ p_2^{(t+1)})(c_t + d_t(x_t - u) - e_t(x_t - u)^2)$ in a small interval around u_t is $p_2^{(t)} - (p_1^{(t+1)}+ p_2^{(t+1)})d_t + 2(p_1^{(t+1)}+ p_2^{(t+1)})e_t(x_t - u)$

Thus, $p_2^{(t)} - (p_1^{(t+1)}+ p_2^{(t+1)})d_t + 2(p_1^{(t+1)}+ p_2^{(t+1)})e_t(x_t - u_t) = 0$ implies $x_t - u_t = \frac{\left(p_1^{(t+1)}+ p_2^{(t+1)}\right)d_t - p_2^{(t)}}{2(p_1^{(t+1)}+ p_2^{(t+1)})e_t}$ so that $u_t = x_t - \frac{\left(p_1^{(t+1)}+ p_2^{(t+1)}\right)d_t - p_2^{(t)}}{2(p_1^{(t+1)}+ p_2^{(t+1)})e_t}$.

Hence $x_{t+1} = c_t + d_t\left(\frac{\left(p_1^{(t+1)}+ p_2^{(t+1)}\right)d_t - p_2^{(t)}}{2(p_1^{(t+1)}+ p_2^{(t+1)})e_t}\right) - e_t\left(\frac{\left(p_1^{(t+1)}+ p_2^{(t+1)}\right)d_t - p_2^{(t)}}{2(p_1^{(t+1)}+ p_2^{(t+1)})e_t}\right)^2$. Q.E.D.

Note: In view of a related paper entitled "Infinite Horizon Linear Optimal Control with Linear Constraints" the current revised version of this paper excludes all discussion on linear optimal control problems with linear constraints that was contained in any earlier version.

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