# A Linear Model of Optimal Control with One-dimensional Control and State Variables


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**Abstract**

We consider discrete-time infinite-horizon optimal control problems with a linear objective function. For absolutely convergent linear optimal control problems, we prove the existence of a solution, the necessity of a Euler and transversality conditions for a solution and the sufficiency of competitive condition and a different transversality for a solution. We show that the optimal value functions satisfy a "functional equation of dynamic programming" and that the satisfaction of this functional equation is necessary and sufficient for a trajectory to solve the optimization problem. Under the additional assumption, namely, "concave in pay-offs from the control variable", being satisfied by absolutely convergent linear optimal control problems, we show that the optimal value functions are concave and continuous. We obtain closed form solutions for such problems under the assumption that there is a state transition function that is strictly increasing and strictly concave in the gap variable and satisfy mild interiority conditions.



**Keywords:** optimal control, discrete-time, infinite horizon, linear objective function, dynamic programming

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**JEL Codes:** C44, C61.


## 1. Introduction:

While programming in linear spaces originates in the work of Hurwicz (1958), its use in economics- particularly programming in infinite dimensional linear spaces- has primarily been in the context of infinite horizon dynamic optimization, whose abstract analysis is best known in the work of Gale (1968). There may have been others who discussed specific examples of dynamic optimizations prior to the work of Gale (1968), but the "reduced form model" that is discussed in the survey paper of Mitra (2000) is very much in the tradition of Gale (1968). A more recent survey of dynamic optimization can be found in chapter 5 of Sorger (2015). The continuous version of the reduced form model of dynamic optimization is what is known as "calculus of variation". A generalization of calculus of variation is what is known as optimal control theory. Blot and Hayek (2014) presents a discussion of "infinite horizon optimal control in the discrete-time framework". Our purpose here is to begin an investigation of infinite horizon optimal control in the discrete-time framework, assuming that the objective function is linear.

While linearity of the objective function may appear to be a special case of a significantly more general context, this "supposed generality" is often based on the strict-concavity (if not the existence of non-zero second derivatives of the instantaneous pay-off functions) which become "inapplicable" once linearity of the objective function is assumed. Thus, linearity is not a special case of the theory of discrete-time dynamic optimization as it has in reality been applied in economics. The discrete-time infinite horizon reduced form model with linear objective function has been discussed in Lahiri (2025b, Lahiri 2025c). However, the reduced form model becomes a special case of optimal control, once at any point in time, for a given pair of current and future values of the state variable, more than one value of the control variable is compatible with the pair, and the choice of the control variable affects the value of the objective function. This, for instance is the case discussed in section three for "linear optimal control problems with linear constraints", when the evolution of the state-variable at times, is independent of the chosen control variable.

In order to emphasize concepts without imposing a huge mathematical load, we assume that both the state variable and the control variable are real-valued, instead of being multi-dimensional as in Blot and Hayek (2014). Even under such simplification, the model enjoys considerable "traction" for the purpose of proving results that are conceptually powerful.

In section 2 we develop the framework of analysis. In section 3, we present the model of linear optimal control with all constraints being linear equalities. This model is an example of the model we develop in section 2. We provide optimality results for finite horizon optimization problems with linear constraints. Such results are meant for meaningful approximations of infinite horizon problems. In section 4 we define the optimality criteria that we are concerned with and also define a subclass of linear optimal control problems for which a (optimal) solution exists, the latter being our first result in section 5. We refer to the sub-class of problems for which a solution is shown to exist as "absolutely convergent linear optimal control problems". A corollary of the existence result provides a "closed form" solution of period-wise optimal decision rules for linear optimal control problems whose state-transition is defined by a single linear equation that does not depend on the control variable and the control variable cannot assume a value that exceeds the current value of the state variable. The next result says that an "Euler-type" condition as well as a version of "transversality condition" are necessary conditions for a solution and the last result in section 5 proves that the "competitive condition" along with a different "transversality condition" is sufficient for optimality.

The existence of the optimal value function is an immediate consequence of the existence result in section 5. Our first result in section 6, shows that the optimal value function satisfies the "functional equation of dynamic programming" and that the satisfaction of the "functional equation of dynamic programming" is necessary and sufficient for a trajectory to solve the optimization problem. None of the results obtained up to this point in the context of the general model, require the convexity of time-dependent two-period constraint sets, consisting of feasible triplets of a value of the current state, a value of the control variable and a value of the future state. In the case of linear constraints- which is an example- convexity of the two-period constraint sets, is an endogenous property of the model. However, our second result in section 6 concerning the concavity and continuity of the optimal value function, requires that the two-period constraint sets satisfy "some kind" of convexity property. We refer to the property that we require the linear optimal control problem to satisfy as "concave in pay-offs from the control variable".

In sections 7 and 8, we consider absolutely convergent linear optimal control problems with a state transition function which is a strictly increasing and strictly concave function of the difference between the state variable and control variable for the current period. The difference may be viewed as a “gap variable”. The value of the control variable is constrained to lie in the closed interval, whose left-hand end point is zero and the right-hand end point is the minimum of the current value of the state variable and an affine function of the same current value. Under “interiority conditions” for both state and control variables, we are able to obtain explicit formulas for the evolution of state variable and control variable. Although the definition of the state transition function requires the function to be either an affine function of the square root of the gap variable or a quadratic function of the gap variable, the model is quite general since the state transition function depends on additive and multiplicative parameters.

As shown in Lahiri (2025d), the calculus of polynomials required in this paper, can be developed using the ordered-field properties of the real number system and no more.

**2. Framework of Analysis:**

Let $X = [0, b] \subset \mathbb{R}$ (the set of real numbers), with $b > 0$ be such that **set of available alternatives** at any time period is a non-empty subset of $X\times X$. Given a current realization $x\in X$ of the state variable that was chosen in the immediately previous time-period, a typical alternative that is “chosen” during the current period is an ordered pair $(u, y)\in X\times X$, where u is the value of the control variable chosen for the current period and ‘y’ is the value of the state variable that will be realized (as an inheritance) in the immediately next period. Based on the realization (x, u) during the current period an instantaneous pay-off is realized by the decision maker.

With $\mathbb{N}$ denoting the set of natural number (i.e., the set of strictly positive integers) let $\mathbb{N}^0$ denote $\mathbb{N}\cup\{0\}$, i.e., the set of non-negative integers. Time is measured in discrete periods $t\in\mathbb{N}^0$. At each time-period ‘t’ an alternative (state variable-control variable pair) is realized, and the chosen alternative is denoted by $(x_t, u_t)\in X\times X$. While $x_t$ is an “inheritance” in the current period, $u_t$ is chosen during the current period.

At each time-period $t\in\mathbb{N}^0$, $\Omega_t \subset X\times X\times X$ is the **two-period constraint set at time-period** t, satisfying the following properties:

(i) $\Omega_t$ is a non-empty and closed subset of $X\times X\times X$.

(ii) For all $x\in X$, $\{(u, y)\in X\times X|\ (x, u, y)\in\Omega_t\} \neq \phi$.

For $t\in\mathbb{N}^0$, $(x, u, y)\in\Omega_t$ can be interpreted in the following manner: given that $x\in X$ is the realization of the state variable at time-period t, it is possible to choose the pair $(u, y)\in X\times X$ at time-period t.

For all $(x, t)\in X\times\mathbb{N}^0$, let $\Omega_t(x) = \{(u, y)\in X\times X|\ (x, u, y)\in \Omega_t\}$.

By (ii) for all $(x, t)\in X\times\mathbb{N}^0$, $\Omega_t(x) \neq \phi$ and thus by (i) $\Omega_t(x)$ is a non-empty and closed subset of $X\times X$.

For $x\in X$, let $\mathcal{F}(x) = \{<(x_t, u_t)|t\in\mathbb{N}^0>|\ (x_t, u_t, x_{t+1})\in\Omega_t, t\in\mathbb{N}^0, x_0 = x\}$.

We will (whenever necessary) refer to an infinite sequence $<(x_t, u_t)|t\in\mathbb{N}^0 >\in\mathcal{F}(x)$ as a **trajectory starting at (from)** x.

Clearly, $\mathcal{F}(x)$ is non-empty for all $x\in X$.

Let $<(p_1^{(t)}, p_2^{(t)}) \mid t\in\mathbb{N}^0>$ be a sequence of pairs of real numbers. If x is the realization of the state variable at time-period t and u is the choice of the control variable at time period 't', the instantaneous pay-off received by the decision-maker at time period 't' is $p_1^{(t)}x + p_2^{(t)}u$.

We shall refer to the array $<((p_1^{(t)}, p_2^{(t)}), \Omega_t)| t\in\mathbb{N}^0>$ as the **linear optimal control (LOC) problem**.

**For what follows we assume that $<((p_1^{(t)}, p_2^{(t)}), \Omega_t)| t\in\mathbb{N}^0>$ is a given LOC problem. As and when necessary, we will impose additional assumptions on this LOC problem.**

**3. Linear optimal control problems with linear constraints:**

A very important and interesting type of LOC problem is one in which for all $t\in\mathbb{N}^0$, there exists a matrix $[A^t \mid B^t]$ with a finite number of rows and two-columns and a point $(c^t, d^t, e^t)\in\mathbb{R}^3$ such that for all $(x, t)\in X\times\mathbb{N}^0$, $(u, y)\in \Omega_t(x)$ <u>if and only if</u> [$(u, y)\in X\times X$, $u \leq A_i^t + B_i^t x$ where for every i the ordered pair $(A_i^t|B_i^t)$ is the i$^{th}$ row of the matrix $[A^t \mid B^t]$ and $y = c^t + d^t x + e^t u$]. We shall refer to such an LOC problem as a **linear optimal control problem with linear constraints (or an LOC-LC problem).**

We will represent an LOC-LC problem by $<((p_1^{(t)}, p_2^{(t)}), [A^t \mid B^t], (c^t,d^t,e^t))| t\in\mathbb{N}^0>$.

In the case of a LOC-LC it is easy to see that, for all $t\in\mathbb{N}^0$, $\Omega_t$ is a non-empty, closed and "convex" subset of $X\times X\times X$. Thus, in the case of a LOC-LC, for all $x\in X$, $\mathcal{F}(x)$ is a non-empty and convex set.

The number of rows of the matrix $[A^t \mid B^t]$ may vary with the time-period.

Since for all $(x, t)\in X\times\mathbb{N}^0$ it is the case that $\Omega_t(x) \neq \phi$ it must be that for all $t\in\mathbb{N}^0$, $\Omega_t(0) \neq \phi$, $\Omega_t(b) \neq \phi$ and both $\Omega_t(0)$ as well as $\Omega_t(b)$ are closed and convex subsets of $[0, b]\times [0, b]$. Thus, it must be the case that for all $t\in\mathbb{N}^0$, $A_i^t \geq 0$ and $A_i^t + B_i^t b \geq 0$ for all i, i.e., $\min\{A_i^t| (A_i^t|B_i^t)$ is the i$^{th}$ row of $[A^t \mid B^t]\} \geq 0$ and $\min\{A_i^t + B_i^t b| (A_i^t|B_i^t)$ is the i$^{th}$ row of $[A^t \mid B^t]\}\geq 0$.

Since for $x\in(0, b)$, $x = (1-\frac{x}{b})0 + \frac{x}{b}b$, it follows from above that for all $x\in(0, b)$, $\min\{A_i^t + B_i^t x| (A_i^t|B_i^t)$ is the i$^{th}$ row of $[A^t \mid B^t]\}\geq 0$

Similarly, $\Omega_t(0) \neq \phi$ implies that there exists $u\in[0, \min\{A_i^t| (A_i^t|B_i^t)$ is the i$^{th}$ row of $[A^t \mid B^t]\}]$ satisfying $u \leq b$ and $c^t + e^t u\in[0, b]$.

$\Omega_t(b) \neq \phi$ implies that there exists $v\in[0, \min\{A_i^t + B_i^t b| (A_i^t|B_i^t)$ is the i$^{th}$ row of $[A^t \mid B^t]\}]$ satisfying $v \leq b$ and $c^t + d^t b + e^t v\in[0, b]$.

Since for $x\in(0, b)$, $x = (1-\frac{x}{b})0 + \frac{x}{b}b$, if for $u\in[0, \min\{A_i^t| (A_i^t|B_i^t)$ is the i$^{th}$ row of $[A^t \mid B^t]\}$ satisfying $u \leq b$ we have $c^t + e^t u\in[0, b]$ and for $v\in[0, \min\{A_i^t + B_i^t b| (A_i^t|B_i^t)$ is the i$^{th}$ row of

$[A^t \mid B^t]\}]$ satisfying $v \leq b$ we have $c^t + d^t b + e^t v \in [0, b]$, then for $w = (1-\frac{x}{b})u + \frac{x}{b}v$, we get $c^t + d^t x + e^t v \in [0, b]$.

Thus, for a LOC-LC problem the following conditions are required for all $t \in \mathbb{N}^0$:

(i) $\min\{A_i^t | (A_i^t | B_i^t)$ is the $i^{th}$ row of $[A^t \mid B^t]\} \geq 0$.

(ii) $\min\{A_i^t + B_i^t b | (A_i^t | B_i^t)$ is the $i^{th}$ row of $[A^t \mid B^t]\} \geq 0$.

(iii) There exists $u \in [0, \min\{A_i^t | (A_i^t | B_i^t)$ is the $i^{th}$ row of $[A^t \mid B^t]\}]$ satisfying $u \leq b$ and $c^t + e^t u \in [0, b]$.

(iv) For some $v \in [0, \min\{A_i^t + B_i^t b | (A_i^t | B_i^t)$ is the $i^{th}$ row of $[A^t \mid B^t]\}]$ satisfying $v \leq b$, we have $c^t + d^t b + e^t v \in [0, b]$.

Of particular interest is a **(1×1)-LOC-LC problem** in which for all $t \in \mathbb{N}^0$, the matrix $[A^t \mid B^t]$ has a single row given by the ordered pair $(a^t, b^t)$.

For a (1×1)-LOC-LC, for all $(x, t) \in X \times \mathbb{N}^0$: $\Omega_t(x) = \{(u, y) \in X \times X|$ such that $u \leq a^t + b^t x$ and $y = d^t + e^t x + f^t u\}$.

We will represent a (1×1)-LOC-LC problem by $<((p_1^{(t)}, p_2^{(t)}), (a^t, b^t), (c^t, d^t, e^t)| t \in \mathbb{N}^0>$.

For a (1×1)-LOC-LC the following conditions are required for all $t \in \mathbb{N}^0$:

(i) $a^t \geq 0$.

(ii) $a^t + b^t b \geq 0$.

(iii) For some $u \in [0, a^t]$ satisfying $u \leq b$, $c^t + e^t u \in [0, b]$.

(iv) For some $v \in [0, a^t + b^t b]$ satisfying $v \leq b$, we have $c^t + d^t b + e^t u \in [0, b]$.

If for all $t \in \mathbb{N}^0$, it is the case that $e^t \neq 0$, then for all $t \in \mathbb{N}^0$, $u = \frac{y - c^t - d^t x}{e^t}$, so that $\frac{y - c^t - d^t x}{e^t} \leq a^t + b^t x$.

Thus, for a (1×1)-LOC-LC problem satisfying $e^t \neq 0$ for all $t \in \mathbb{N}^0$ then for all $(x, t) \in X \times \mathbb{N}^0$, there is a single state transition constraint given by $y \in [0, \min\{b, a^t e^t + b^t e^t x + (c^t + d^t x)\}]$ if $e^t > 0$ and $y \in [\max\{0, a^t e^t + b^t e^t x + (c^t + d^t x)\}, b]$ if $e^t < 0$. In this case, $p_1^{(t)} x + p_2^{(t)} u = p_1^{(t)} x + p_2^{(t)} \frac{y - c^t - d^t x}{e^t}$, which is a linear function of x and y. This, model has been discussed in detail in Lahiri (2025c).

Thus, in the case of (1×1)-LOC-LC, our analysis would go beyond what has already been discussed in Lahiri (2025c), only if $e^t = 0$ for some $t \in \mathbb{N}^0$, the latter being a very reasonable possibility. In fact, if for some $t \in \mathbb{N}^0$, $e^t = 0$, then the corresponding (1×1)-LOC-LC is an example of several values of the control variable chosen at time-period t, being compatible with each value of the state variable for time-period t+1, thereby illustrating that our model of linear optimal control discussed here is a generalization of the linear dynamic optimization model discussed in Lahiri (2025b, 2025c), the latter being motivated by the reduced form model in Mitra (2000) and Sorger (2015).

On the other hand, if $e^t = 0$ for all $t \in \mathbb{N}^0$, then for all $x \in X$: $<(x_t, u_t)|t \in \mathbb{N}^0> \in \mathcal{F}(x)$ if and only if $u_t \leq a^t + b^t x_t$ and $x_{t+1} = c^t + d^t x_t$ for all $t \in \mathbb{N}^0$.

Given $(x, T) \in X \times \mathbb{N}$, $<(x_t, u_t)|t \in \mathbb{N}^0> \in \mathcal{F}(x)$ is said to be **T- optimal** if for all $<(y_t, v_t)|t \in \mathbb{N}^0> \in \mathcal{F}(x)$ satisfying $y_{T+1} = x_{T+1}$, $\sum_{\tau=0}^{t}[p_1^{(\tau)}(x_\tau - y_\tau) + p_2^{(\tau)}(u_\tau - v_\tau)] \geq 0$.

The following result is a re-statement of "T-optimal" for LOC-LC problems.

**Proposition 3.1:** Given $x \in X$. $<(x_t, u_t)|t \in \mathbb{N}^0> \in \mathcal{F}(x)$ is "T-optimal" for an LOC-LC problem $<((p_1^{(t)}, p_2^{(t)}), [A^t \mid B^t], (c^t, d^t, e^t))| t \in \mathbb{N}^0>$ if and only if $<(x_\tau, u_\tau)|\tau = 0, 1, \ldots, T>$ solves the following linear programming problem:

Maximize $\sum_{\tau=0}^{T}[p_1^{(\tau)} y_\tau + p_2^{(\tau)} v_\tau]$, subject to $v_\tau \leq A_i^\tau + B_i^\tau y_\tau$ for every i, $y_{\tau+1} = c^\tau + d^\tau y_\tau + e^\tau v_\tau$, $v_\tau \leq b$, $y_\tau \leq b$, $v_\tau \geq 0$, $y_\tau \geq 0$, $\tau = 0, 1, \ldots, T$, $y_{T+1} = x_{T+1}$, $y_0 = x$.

An immediate corollary of proposition 3.1 is the following re-statement of "strongly optimal along the way" for (1×1)-LOC-LC problem.

**Corollary 1 of proposition 3.1:** Given $x \in X$, $<(x_t, u_t)|t \in \mathbb{N}^0> \in \mathcal{F}(x)$ is "T-optimal" for a (1×1)- LOC-LC problem if and only if $<(x_\tau, u_\tau)|\tau = 0, 1, \ldots, T>$ solves the following linear programming problem:

Maximize $\sum_{\tau=1}^{T}[p_1^{(\tau)} y_\tau + p_2^{(\tau)} v_\tau]$, subject to $v_\tau \leq a^\tau + b^\tau y_\tau$, $y_{\tau+1} = c^\tau + d^\tau y_\tau + e^\tau v_\tau$, $v_\tau \leq b$, $y_\tau \leq b$, $v_\tau \geq 0$, $y_\tau \geq 0$, $\tau = 1, \ldots, T$, $y_0 = x$, $y_{T+1} = x_{T+1}$ and $v_0 \geq 0$.

The following result follows immediately from the implications of complementary slackness of a "feasible pair" for the primal and dual of linear programming problems, as discussed in Lahiri (2020).

**Proposition 3.2:** Given $x \in X$, $<(x_t, u_t)|t \in \mathbb{N}^0> \in \mathcal{F}(x)$ is "T-optimal" for a (1×1)- LOC-LC problem if and only if there exists a sequence $<(\alpha_\tau^T, \gamma_\tau^T)| \tau = 0, 1, \ldots, T>$ in $\mathbb{R}_+^2$, a sequence $<\delta_\tau^T| \tau = 1, \ldots, T>$ in $\mathbb{R}_+$ and a sequence $<\beta_\tau^T| \tau = 0, 1, \ldots, T>$ in $\mathbb{R}$ such that:

(i) $\alpha_\tau^T - e^\tau \beta_\tau^T + \gamma_\tau^T \geq p_2^{(\tau)}$, $(\alpha_\tau^T - e^\tau \beta_\tau^T + \gamma_\tau^t - p_2^{(\tau)})u_\tau = 0$, for $\tau = 0, 1, \ldots, T$.

(ii) $- b^\tau \alpha_\tau^T - d^\tau \beta_\tau^T + \beta_{\tau-1}^T + \delta_\tau^T \geq p_1^{(\tau)}$, $(- b^\tau \alpha_\tau^T - d^\tau \beta_\tau^T + \beta_{\tau-1}^T + \delta_\tau^T - p_1^{(\tau)})x_\tau = 0$, for $\tau = 1, \ldots, T$.

(iii) $(u_\tau - b^\tau x_\tau - a^\tau)\alpha_\tau^T = 0$, for $\tau = 0, 1, \ldots, T$.

(iv) $(u_\tau - b)\gamma_\tau^T = 0$, for $\tau = 0, 1, \ldots, T$.

(v) $(x_\tau - b)\delta_\tau^T = 0$, for $\tau = 1, \ldots, T$.

The interesting "problem" in the case of LOC-LC is the characterization of a trajectory that is T-optimal for all $T \in \mathbb{N}^0$.

**4. Optimal solution:**

Following Gale (1967), we introduce the following in the context of LOC problems.

Given $x \in X$ and $<(x_t, u_t)|t \in \mathbb{N}^0>$, $<(y_t, v_t)|t \in \mathbb{N}^0> \in \mathcal{F}(x)$, we say that:

(i) $<(x_t, u_t)|t\in\mathbb{N}^0>$ **overtakes** $<(y_t, v_t)|t\in\mathbb{N}^0>$ if there exists $T\in\mathbb{N}$ such that for all $t \geq T$, $\sum_{\tau=0}^{t}[p_1^{(\tau)}(x_\tau - y_\tau) + p_2^{(\tau)}(u_\tau - v_\tau)] \geq 0$.

(ii) $<(x_t, u_t)|t\in\mathbb{N}^0>$ **catches up with (weakly overtakes)** $<(y_t, v_t)|t\in\mathbb{N}^0>$ if for all $\varepsilon > 0$, there exists $T(\varepsilon)\in\mathbb{N}$ such that for all $t \geq T(\varepsilon)$, $\sum_{\tau=0}^{t}[p_1^{(\tau)}(x_\tau - y_\tau) + p_2^{(\tau)}(u_\tau - v_\tau)] \geq -\varepsilon$.

Given $x\in X$, $<(x_t, u_t)|t\in\mathbb{N}^0>\in\mathcal{F}(x)$ is said to be **strongly optimal along the way** if for all $<(y_t, v_t)|t\in\mathbb{N}^0>\in\mathcal{F}(x)$, it is the case that $<(x_t, u_t)|t\in\mathbb{N}^0>$ "overtakes" $<(y_t, v_t)|t\in\mathbb{N}^0>$.

Given $x\in X$, $<(x_t, u_t)|t\in\mathbb{N}^0>\in \mathcal{F}(x)$ is said to be **weakly optimal along the way** if for all $<(y_t, v_t)|t\in\mathbb{N}^0>\in \mathcal{F}(x)$, it is the case that $<(x_t, u_t)|t\in\mathbb{N}^0>$ "weakly overtakes" $<(y_t, v_t)|t\in\mathbb{N}^0>$.

As in Mitra (2000), Sorger (2015) and Lahiri (2025b) (among numerous others) we will from this section onwards be concerned with a stronger optimality criterion that implies the two optimality conditions of Gale (1967). For what follows we will require the following "Absolute Convergence" condition.

The LOC problem $<((p_1^{(t)}, p_2^{(t)}), \Omega_t)|\ t\in\mathbb{N}^0>$ is said to satisfy **Absolute Convergence** if for $i\in\{1,2\}$, $\sum_{t=0}^{\infty}|p_i^{(t)}| < +\infty$.

If $<((p_1^{(t)}, p_2^{(t)}), \Omega_t)|\ t\in\mathbb{N}^0>$ satisfies Absolute Convergence, then we may refer to $<((p_1^{(t)}, p_2^{(t)}), \Omega_t)|\ t\in\mathbb{N}^0>$ as an **Absolutely Convergent LOC** (AC-LOC) **problem**.

Let $<((p_1^{(t)}, p_2^{(t)}), \Omega_t)|\ t\in\mathbb{N}^0>$ be an AC-LOC problem. Thus, for all sequence $<(x_t, u_t)|t\in\mathbb{N}^0>$ with $(x_t, u_t)\in X\times X$ for all $t\in\mathbb{N}^0$, it must be the case that $\lim_{t\to\infty}|p_1^{(t)}x_t| = 0$, $\lim_{t\to\infty}|p_2^{(t)}u_t| = 0$, $\sum_{t=0}^{\infty}|p_1^{(t)}x_t| \in[0, b\sum_{t=0}^{\infty}|p_1^{(t)}|]$ and $\sum_{t=0}^{\infty}|p_1^{(t)}u_t|\in[0, b\sum_{t=0}^{\infty}|p_2^{(t)}|]$.

Let $M = \max\{b\sum_{t=0}^{\infty}|p_1^{(t)}|, b\sum_{t=0}^{\infty}|p_2^{(t)}|\} < +\infty$.

Thus, for all sequence $<(x_t, u_t)|t\in\mathbb{N}^0>$ with $(x_t, u_t)\in X\times X$ for all $t\in\mathbb{N}^0$, it must be the case that $|\sum_{t=0}^{\infty}[p_1^{(t)}x_t + p_2^{(t)}u_t]| \leq \sum_{t=0}^{\infty}|p_1^{(t)}x_t + p_2^{(t)}u_t| \leq \sum_{t=0}^{\infty}[|p_1^{(t)}x_t| + |p_2^{(t)}u_t|] \leq 2M$

Let $<((p_1^{(t)}, p_2^{(t)}), \Omega_t)|\ t\in\mathbb{N}^0>$ be an AC-LOC problem. We will now consider the following optimization problem denoted **OPT**:

Given $x\in X$, Maximize $\sum_{t=0}^{\infty}[p_1^{(t)}x_t + p_2^{(t)}u_t]$, subject to the infinite sequence satisfying the constraints: $(x_t, u_t, x_{t+1})\in\Omega_t$, $t\in\mathbb{N}^0$, $x_0 = x$.

**Note 4.1:** The exact mathematical interpretation of the expression (formula) $\sum_{t=0}^{\infty}[p_1^{(t)}x_t + p_2^{(t)}u_t]$ is $\lim_{T\to\infty}(\sum_{t=0}^{T}[p_1^{(t)}x_t + p_2^{(t)}u_t])$. Thus, the problem we are concerned with here is in the domain of asymptotic analysis, which is very different from infinite dimensional analysis.

Let $\mathcal{S}(x) = \{<x_t|t\in\mathbb{N}^0>\in\mathcal{F}(x)|\ \sum_{t=0}^{\infty}[p_1^{(t)}x_t + p_2^{(t)}u_t] \geq \sum_{t=0}^{\infty}[p_1^{(t)}y_t + p_2^{(t)}v_t]$ for all $<(y_t, v_t)|t\in\mathbb{N}^0>\in \mathcal{F}(x)\}$.

$\mathcal{S}(x)$ is the **set of solutions starting from** x **for** OPT.

For all $T\in\mathbb{N}^0$, and $x\in X$, let $\mathcal{F}^T(x) = \{<(x_t, u_t) | t \geq T>| (x_t, u_t, x_{t+1})\in \Omega_t$ for all $t \geq T$ and $x_T = x\}$.

For $T\in\mathbb{N}^0$ and $y\in X$, $<(x_t, u_t)|t \geq T>\in \mathcal{F}^T(x)$ may be referred to as a **trajectory starting at (from)** x **at time-period** T.

It is easy to see that for all $T\in\mathbb{N}^0$ and $x\in X$, $\mathcal{F}^T(x)$ is non-empty.

Given $(x, T)\in X\times\mathbb{N}^0$, we will denote the following optimization problem by **OPT-T**:

Maximize $\sum_{t=T}^{\infty}[p_1^{(t)}y_t + p_2^{(t)}v_t]$ subject to $< (y_t, v_t)| t \geq T > \in\mathcal{F}^T(x)$

For $(x, T)\in X\times\mathbb{N}^0$, $\mathcal{S}^T(x)$ is the **set of solutions starting from** x **for** OPT-T, i.e., $\mathcal{S}^T(x) = \underset{<(x_t,u_t)|\, t\geq T>\in\mathcal{F}^T(x)}{\operatorname{argmax}} \sum_{t=T}^{\infty}[p_1^{(t)}x_t + p_2^{(t)}u_t]$.

Clearly, $\mathcal{F}^0(x) = \mathcal{F}(x)$ and $\mathcal{S}^0(x) = \mathcal{S}(x)$ for all $y\in X$.

For all $T\in\mathbb{N}^0$ the correspondence $h^T$: $X \to\to X\times X$ defined by $h^T(x) = \underset{(u,y)\in\Omega_T(x)}{\operatorname{argmax}} (p_2^{(T)}u + V^{(T+1)}(y))$ is said to be the **optimal period-T decision rule**.

**5. Existence of solution, "Euler type" condition, competitive condition, and transversality condition:**

The proof of the following proposition is analogous to the proof of proposition 4.1 in Lahiri (2025b).

**Proposition 5.1:** Let $<((p_1^{(t)}, p_2^{(t)}), \Omega_t)| t\in\mathbb{N}^0>$ be an AC-LOC problem. $\mathcal{S}^T(x) \neq \phi$ for all $(x, T)\in X\times\mathbb{N}^0$. Hence, the optimal period-T decision rule $h^T$ is non-empty valued for all for all $T\in\mathbb{N}^0$.

**Proof:** Let $(x, T)\in X\times\mathbb{N}^0$.

$\Omega_t \subset X\times X\times X = [0, b]\times[0, b]\times[0, b]$ for all $t\in\mathbb{N}^0$ and hence as noted in section 5, for all sequence $<(x_t, u_t)|t\in\mathbb{N}^0>$ with $(x_t, u_t)\in X\times X$ for all $t\in\mathbb{N}^0$, it must be the case that $|\sum_{t=0}^{\infty}[p_1^{(t)}x_t + p_2^{(t)}u_t]| \leq \sum_{t=0}^{\infty} |p_1^{(t)}x_t + p_2^{(t)}u_t| \leq \sum_{t=0}^{\infty}[|p_1^{(t)}x_t| + |p_2^{(t)}u_t|] \leq 2M$.

In particular, for all sequence $<(x_t, u_t)|t\in\mathbb{N}^0>$ with $(x_t, u_t)\in X\times X$ for all $t\in\mathbb{N}^0$, it must be the case that $|\sum_{t=T}^{\infty}[p_1^{(t)}x_t + p_2^{(t)}u_t]| \leq \sum_{t=T}^{\infty} |p_1^{(t)}x_t + p_2^{(t)}u_t| \leq \sum_{t=T}^{\infty}[|p_1^{(t)}x_t| + |p_2^{(t)}u_t|] \leq 2M$.

Let $<(y_t, v_t)|t \geq T>\in\mathcal{F}^T(x)$. Clearly, $-2M \leq \sum_{t=T}^{\infty}[p_1^{(t)}y_t + p_2^{(t)}v_t] \leq 2M$.

Thus, $S(x) = \sup \{\sum_{t=T}^{\infty}[p_1^{(t)}y_t + p_2^{(t)}v_t]|<(y_t, v_t)|t \geq T>\in\mathcal{F}^T(x)\} < +\infty$.

Thus, there is a sequence of infinite sequences $<<(x_t^{(n)}, u_t^{(n)})| t \geq T>| n\in\mathbb{N}>$ in $\mathcal{F}^T(x)$ such that for all $n\in\mathbb{N}$, $S(x) - \frac{1}{n} < \sum_{t=T}^{\infty}[p_1^{(t)}x_t^{(n)} + p_2^{(t)}u_t^{(n)}]$

Since, $<(x_{T+1}^{(n)}, u_{T+1}^{(n)})| n\in\mathbb{N}>$ is a sequence in the closed and bounded set $X\times X$, it has a convergent subsequence $<(x_{T+1}^{N_1(n)}, u_{T+1}^{N_1(n)})|n\in\mathbb{N}>$ converging to $(x_{T+1}^0, u_{T+1}^0)\in X\times X$.

Further, (x, $u_T^{(n)}$, $x_{T+1}^{N_1(n)}$)∈$\Omega_T$ for all n∈ℕ implies (x, $u_T^0$, $x_{T+1}^0$)∈$\Omega_T$ since $\Omega_T$ is closed.

Consider the sequence <($x_{T+2}^{N_1(n)}$, $u_{T+2}^{N_1(n)}$)|n∈ℕ>.

By an argument similar to the one in the previous step, it has a convergent subsequence <($x_{T+2}^{N_2(n)}$, $u_{T+2}^{N_2(n)}$)| n∈ℕ> converging to ($x_{T+2}^0, u_{T+2}^0$)∈X×X.

Since, <($x_{T+1}^{N_2(n)}$, $u_{T+1}^{N_2(n)}$)| n∈ℕ> is a subsequence of the convergent subsequence <($x_{T+1}^{N_1(n)}$, $u_{T+1}^{N_1(n)}$)| n∈ℕ>, it must be the case that <($x_{T+1}^{N_2(n)}$, $u_{T+1}^{N_2(n)}$| n∈ℕ> converges to ($x_{T+1}^0$, $u_{T+1}^0$).

The sequence <($x_{T+1}^{N_2(n)}$, $u_{T+1}^{N_2(n)}$, $x_{T+2}^{N_2(n)}$)| n∈ℕ> is in $\Omega_{T+1}$ and converges to ($x_{T+1}^0$, $u_{T+1}^0$, $x_{T+2}^0$). Since $\Omega_{T+1}$ is closed ($x_{T+1}^0$, $u_{T+1}^0$, $u_{T+2}^0$)∈ $\Omega_{T+1}$.

Having obtained convergent subsequences <($x_{T+\tau-1}^{N_\tau(n)}$, $u_{T+\tau-1}^{N_\tau(n)}$, $x_{T+\tau}^{N_\tau(n)}$| n∈ℕ>∈$\Omega_{T+\tau-1}$ for all τ = 1, …, t for some t ≥ 1, let <($x_{T+t}^{N_{t+1}(n)}$, $u_{T+t}^{N_{t+1}(n)}$, $x_{T+t}^{N_{t+1}(n)}$| n∈ℕ> be a convergent subsequence of the sequence <($x_{T+t}^{N_t(n)}$, $u_{T+t}^{N_t(n)}$, $x_{T+t+1}^{N_t(n)}$| n∈ℕ> in the closed and bounded set X×X×X, converging to ($x_{T+t}^0$, $u_{T+t}^0$, $x_{T+t+1}^0$)∈X×X×X. Since, <($x_{T+t}^{N_t(n)}$, $u_{T+t}^{N_t(n)}$, $x_{T+t+1}^{N_t(n)}$| n∈ℕ>∈$\Omega_{T+t}$ and $\Omega_{T+t}$ is a closed subset of X×X×X, it must be the case that ($x_{T+t}^0$, $u_{T+t}^0$, $x_{T+t+1}^0$)∈$\Omega_{T+t}$.

Consider the sequence <($x_{T+t}^0$, $u_{T+t}^0$|t∈$\mathbb{N}^0$>. Since, ($x_{T+t}^0$, $u_{T+t}^0$, $x_{T+t+1}^0$)∈$\Omega_t$ for all t∈$\mathbb{N}^0$, where $x_T^0$ = x, it must be the case that <$x_{T+t}^0$|t∈$\mathbb{N}^0$ >∈$\mathcal{F}$(x).

Thus, $\sum_{t=0}^{\infty}[p_1^{(T+t)}x_{T+t}^0 + p_2^{(T+t)}u_{T+t}^0]$ ≤ S(x).

We wish to show that $\sum_{t=0}^{\infty}[p_1^{(T+t)}x_{T+t}^0 + p_2^{(T+t)}u_{T+t}^0]$ = S(x).

Since, for all t∈$\mathbb{N}^0$, <$x_{T+\tau}^{N_\tau(n)}$| n∈ℕ> converges to $x_{T+\tau}^0$ and ($x_{T+\tau}^{N_\tau(n)}$, $u_{T+\tau}^{N_\tau(n)}$, $x_{T+\tau+1}^{N_\tau(n)}$)∈$\Omega_{T+\tau}$, for all τ ≤ t and n∈ℕ, ***<u>given</u>*** ε > 0, for all t∈ℕ, there exists a sequence <$n_t$|t∈ ℕ> in ℕ satisfying $n_{t+1} > n_t$ for all t∈ℕ such that for all n∈ℕ with n ≥ $n_t$ and τ ≤ t, ($x_{T+\tau}^{N_t(n)}$, $u_{T+\tau}^{N_t(n)}$, $x_{T+\tau+1}^{N_t(n)}$)∈ $\Omega_{T+\tau}$ and $|p_1^{(T+\tau)}x_{T+\tau}^0 + p_2^{(T+\tau)}u_{T+\tau}^0 - p_1^{(T+\tau)}x_{T+\tau}^{N_t(n)} - p_2^{(T+\tau)}u_{T+\tau}^{N_t(n)}| < \frac{\varepsilon}{8}(\frac{1}{2})^t$, i.e., $p_1^{(T+\tau)}x_{T+\tau}^{N_t(n)} + p_2^{(T+\tau)}u_{T+\tau}^{N_t(n)} + \frac{\varepsilon}{8}(\frac{1}{2})^t > p_1^{(T+\tau)}x_{T+\tau}^0 + p_2^{(T+\tau)}u_{T+\tau}^0 > p_1^{(T+\tau)}x_{T+\tau}^{N_t(n)} + p_2^{(T+\tau)}u_{T+\tau}^{N_t(n)} - \frac{\varepsilon}{8}(\frac{1}{2})^t$.

Note that $\sum_{\tau=0}^{t}\frac{\varepsilon}{8}(\frac{1}{2})^t \le \sum_{\tau=0}^{t}\frac{\varepsilon}{8}(\frac{1}{2})^\tau = \frac{\varepsilon}{8}\sum_{\tau=0}^{t-1}(\frac{1}{2})^\tau = \frac{\varepsilon}{4}(1-(\frac{1}{2})^t)$ and $x_T^{N_t(n)}$ = x for all (t, n)∈ℕ× ℕ.

Thus, for all t∈ℕ, $\sum_{\tau=0}^{t}[p_1^{(T+\tau)}x_{T+\tau}^0 + p_2^{(T+\tau)}u_{T+\tau}^0] > \sum_{\tau=0}^{t}[p_1^{(T+\tau)}x_{T+\tau}^{N_t(n)} + p_2^{(T+\tau)}u_{T+\tau}^{N_t(n)}] - \frac{\varepsilon}{4}(1-(\frac{1}{2})^t)$ for all n ≥ $n_t$.

For all t∈ℕ: $\sum_{\tau=0}^{t}[p_1^{(T+\tau)}x_{T+\tau}^{N_t(n)} + p_2^{(T+\tau)}u_{T+\tau}^{N_t(n)}] > S(x) - \frac{1}{N_t(n)} - \sum_{\tau=t+1}^{\infty}[p_1^{(T+\tau)}x_{T+\tau}^{N_t(n)} + p_2^{(T+\tau)}u_{T+\tau}^{N_t(n)}]$ for all n ≥ $n_t$.

Thus, for all t∈ℕ, $\sum_{\tau=0}^{t}[p_1^{(T+\tau)}x_{T+\tau}^0 + p_2^{(T+\tau)}u_{T+\tau}^0] > \sum_{\tau=0}^{t}[p_1^{(T+\tau)}x_{T+\tau}^{N_t(n)} + p_2^{(T+\tau)}u_{T+\tau}^{N_t(n)}] - \frac{\varepsilon}{8}(1-(\frac{1}{2})^t) > S(x) - \frac{1}{N_t(n)} - \sum_{\tau=t+1}^{\infty}[p_1^{(T+\tau)}x_{T+\tau}^{N_t(n)} + p_2^{(T+\tau)}u_{T+\tau}^{N_t(n)}] - \frac{\varepsilon}{4}(1-(\frac{1}{2})^t)$ for all n ≥ $n_t$.

Thus, for all t∈ℕ, $\sum_{\tau=0}^{t}[p_1^{(T+\tau)}x_{T+\tau}^0 + p_2^{(T+\tau)}u_{T+\tau}^0] + \sum_{\tau=t+1}^{\infty}[p_1^{(T+\tau)}x_{T+\tau}^{N_t(n)} + p_2^{(T+\tau)}u_{T+\tau}^{N_t(n)}] >$ S(x) - $\frac{1}{N_t(n)}$ - $\frac{\varepsilon}{4}$(1- $(\frac{1}{2})^t$) for all n ≥ $n_t$.

By hypothesis, $\sum_{t=0}^{\infty}|p_i^{(t)}| < +\infty$ for i∈{1, 2} and for all (t, n)∈ $\mathbb{N}^0\times\mathbb{N}$, both $x_{T+t}^{(n)}$∈[0, b] and $u_{T+t}^{(n)}$∈[0, b].

Thus, for all t∈ ℕ and n ≥ $n_t$, $\sum_{\tau=t+1}^{\infty}[|p_1^{(T+\tau)}x_{T+\tau}^{N_t(n)}| + |p_2^{(T+\tau)}u_{T+\tau}^{N_t(n)}|] = \sum_{\tau=t+1}^{\infty}[|p_1^{(T+\tau)}||x_{T+\tau}^{N_t(n)}| + |p_2^{(T+\tau)}||u_{T+\tau}^{N_t(n)}|]$.

For all t∈ℕ and n∈ℕ, $x_t^{(n)}$∈[0, b] and $u_t^{(n)}$∈[0, b] <u>implies</u> for all t∈ℕ and n ≥ $n_t$, $b\sum_{\tau=t+1}^{\infty}[|p_1^{(T+\tau)}| + |p_2^{(T+\tau)}|] \geq \sum_{\tau=t+1}^{\infty}[|p_1^{(T+\tau)}||x_{T+\tau}^{N_t(n)}| + |p_2^{(T+\tau)}||u_{T+\tau}^{N_t(n)}|]$.

Further, for all t∈ℕ and n ≥ $n_t$, $\sum_{\tau=t+1}^{\infty}[|p_1^{(T+\tau)}||x_{T+\tau}^{N_t(n)}| + |p_2^{(T+\tau)}||u_{T+\tau}^{N_t(n)}|] \geq \sum_{\tau=t+1}^{\infty}[p_1^{(T+\tau)}x_{T+\tau}^{N_t(n)} + p_2^{(T+\tau)}u_{T+\tau}^{N_t(n)}]$.

Thus, for all t∈ℕ, $\sum_{\tau=0}^{t}[p_1^{(T+\tau)}x_{T+\tau}^0 + p_2^{(T+\tau)}u_{T+\tau}^0] + b\sum_{\tau=t+1}^{\infty}[|p_1^{(T+\tau)}| + |p_2^{(T+\tau)}|] \geq \sum_{\tau=0}^{t}[p_1^{(T+\tau)}x_{T+\tau}^0 + p_2^{(T+\tau)}u_{T+\tau}^0] + \sum_{\tau=t+1}^{\infty}[p_1^{(T+\tau)}x_{T+\tau}^{N_t(n)} + p_2^{(T+\tau)}u_{T+\tau}^{N_t(n)}] >$ S(x) - $\frac{1}{N_t(n)}$ - $\frac{\varepsilon}{4}$(1- $(\frac{1}{2})^t$) for all n ≥ $n_t$.

Thus, for all t∈ℕ, $\sum_{\tau=0}^{t}[p_1^{(T+\tau)}x_{T+\tau}^0 + p_2^{(T+\tau)}u_{T+\tau}^0] + b\sum_{\tau=t+1}^{\infty}[|p_1^{(T+\tau)}| + |p_2^{(T+\tau)}|] >$ S(x) - $\frac{1}{N_t(n)}$ - $\frac{\varepsilon}{4}$(1- $(\frac{1}{2})^t$) for all n ≥ $n_t$.

Thus, for all t∈ℕ, $\sum_{\tau=0}^{t}[p_1^{(T+\tau)}x_{T+\tau}^0 + p_2^{(T+\tau)}u_{T+\tau}^0] + b\sum_{\tau=t+1}^{\infty}[|p_1^{(T+\tau)}| + |p_2^{(T+\tau)}|] \geq$ S(x) - $\lim_{n\to\infty}\frac{1}{N_t(n)}$ - $\frac{\varepsilon}{8}$(1- $(\frac{1}{2})^t$) = S(x) - $\frac{\varepsilon}{4}$(1- $(\frac{1}{2})^t$), since $\lim_{n\to\infty}\frac{1}{N_t(n)} = 0$.

Thus, for all t∈ℕ, $\sum_{\tau=0}^{t}[p_1^{(T+\tau)}x_{T+\tau}^0 + p_2^{(T+\tau)}u_{T+\tau}^0] + b\sum_{\tau=t+1}^{\infty}[|p_1^{(T+\tau)}| + |p_2^{(T+\tau)}|] \geq$ S(x) - $\frac{\varepsilon}{4}$(1- $(\frac{1}{2})^t$).

Since $\lim_{t\to\infty}\frac{\varepsilon}{4}(1 - (\frac{1}{2})^t) = \frac{\varepsilon}{8}$ and $\lim_{t\to\infty}\sum_{\tau=0}^{t}[p_1^{(T+\tau)}x_{T+\tau}^0 + p_2^{(T+\tau)}u_{T+\tau}^0] = \sum_{\tau=0}^{\infty}[p_1^{(T+\tau)}x_{T+\tau}^0 + p_2^{(T+\tau)}u_{T+\tau}^0]$, we get $\sum_{\tau=0}^{t}[p_1^{(T+\tau)}x_{T+\tau}^0 + p_2^{(T+\tau)}u_{T+\tau}^0] + \lim_{t\to\infty} b\sum_{\tau=t+1}^{\infty}[|p_1^{(T+\tau)}| + |p_2^{(T+\tau)}|] \geq$ S(x) - $\frac{\varepsilon}{4}$.

Since, for all t∈ℕ, $\sum_{\tau=t+1}^{\infty}|p_i^{(\tau)}| < +\infty$ for i∈{1, 2}, it must be the case that $\lim_{t\to\infty} b\sum_{\tau=t+1}^{\infty}[|p_1^{(T+\tau)}| + |p_2^{(T+\tau)}|] = 0$.

Thus, $\sum_{\tau=0}^{\infty}[p_1^{(T+\tau)}x_{T+\tau}^0 + p_2^{(T+\tau)}u_{T+\tau}^0] \geq$ S(x) - $\frac{\varepsilon}{4}$.

Since the above holds for all ε > 0, we get $\sum_{\tau=0}^{\infty}[p_1^{(T+\tau)}x_{T+\tau}^0 + p_2^{(T+\tau)}u_{T+\tau}^0] \geq$ S(x).

Thus, $\sum_{\tau=0}^{\infty}[p_1^{(T+\tau)}x_{T+\tau}^0 + p_2^{(T+\tau)}u_{T+\tau}^0]$ = S(x).

Thus for $\mathcal{S}^T$(x) ≠ ϕ and ($u_T^0$, $x_{T+1}^0$)∈$h^T$(x). Hence, $h^T$(x) ≠ ϕ. Q. E. D.

**Note 5.1:** For the iterative process outlined above we are unaware of any possibility of it "invariably" converging to a strictly increasing sequence <$N_\infty(n)$|n∈ℕ> such that for all t∈$\mathbb{N}^0$, ($x_t^{N_\infty(n)}$, $u_t^{N_\infty(n)}$ $x_{t+1}^{N_\infty(n)}$)∈ $\Omega_t$ for all n∈ℕ and <($x_t^{N_\infty(n)}$, $u_t^{N_\infty(n)}$)| n∈ℕ> converges to ($x_t^0$, $u_t^0$). However, in appendix A.2 (entitled "A Diagonal Process of Cantor") of Blot and Hayek (2014), it seems to be indicated (provided our understanding is correct), that for all t∈$\mathbb{N}^0$, the subsequence <($x_{T+t}^{N_n(n)}$, $u_{T+t}^{N_n(n)}$)| n∈ℕ> derived from the construction in the proof above, converges to ($x_{T+t}^0$, $u_{T+t}^0$).

In section 3, we defined a (1×1)-LOC-LC <(($p_1^{(t)}$, $p_2^{(t)}$), ($a^t$, $b^t$), ($c^t$, $d^t$, $e^t$)| t∈$\mathbb{N}^0$>, such that for all (x, t) ∈X×$\mathbb{N}^0$: $\Omega_t(x)$ = {(u, y)∈X×X| such that u ≤ $a^t + b^t x$ and y = $c^t + d^t x + e^t u$}. We also discussed in section 3, the implications on the parameters that define such an LOC.

An immediate and interesting consequence of proposition 5.1 is the following corollary.

**Corollary 1 of proposition 5.1:** Suppose <(($p_1^{(t)}$, $p_2^{(t)}$), ($a^t$, $b^t$), ($c^t$, $d^t$, $e^t$)| t∈$\mathbb{N}^0$> is an absolutely convergent (1×1)-LOC-LC. If for some T∈$\mathbb{N}^0$ it is the case that $e^t = 0$ and $p_2^{(t)} \neq 0$ for all t ≥ T, then, for all x∈X: <($x_t$, $u_t$)|t ≥ T>∈$\mathcal{S}^T(x)$ <u>if and only if</u> for all t ≥ T: $x_{t+1} = c^t + d^t x_t$ and $u_t = \max\{\frac{p_2^{(t)}}{|p_2^{(t)}|}(a^t + b^t x_t), 0\}$. Thus, for all t ≥ T and x∈X, it must be that $h^t(x) = (\max\{\frac{p_2^{(t)}}{|p_2^{(t)}|}(a^t + b^t x), 0\}, c^t + d^t x)$, so that $h^t$ is continuous on X.

**Proof:** We know from proposition 5.1 that for all x∈X, $\mathcal{S}^T(x) \neq \phi$. Recall that for all (x, t)∈X×X, $a^t + b^t x \geq 0$ and $e_t = 0$ for all t ≥ T implies, $c^t$, $c^t + d^t b \in [0, b]$. The rest follows from the requirements in the statement of this corollary. Q.E.D.

The next result is an "Euler-type" necessary condition for a solution.

**Proposition 5.2:** Let <(($p_1^{(t)}$, $p_2^{(t)}$), $\Omega_t$)| t∈$\mathbb{N}^0$> be an AC-LOC problem. For x∈X, let <($x_t$, $u_t$)|t∈$\mathbb{N}^0$>∈$\mathcal{S}(x)$.

(i) Suppose that for some T∈ℕ, and (v, y)∈$\Omega_{T-1}(x_{T-1})$ it is the case that given ε > 0, there exists <($v_{T+t}$, $y_{T+t}$)|t∈$\mathbb{N}^0$>∈$\mathcal{F}^T$(y) satisfying $|v_{T+t-1} - u_{T+t-1}| \leq \varepsilon|v - u_{T-1}|$ and $|y_{T+t} - x_{T+t}| \leq \varepsilon|y - x_T|$ if y ≠ $x_T$ for all t∈ℕ. <u>Then</u> $p_1^{(T)}(y - x_T) + p_2^{(T-1)}(v - u_{T-1}) \leq 0$.

(ii) $\lim_{t\to\infty}(|p_1^{(t)}|x_t + |p_2^{(t)}|u_t) = 0$.

**Proof:** (i) Suppose that for some T∈ℕ, and (v, y)∈$\Omega_{T-1}(x_{T-1})$ it is the case that given ε > 0, there exists <($v_{T+t}$, $y_{T+t}$)|t∈$\mathbb{N}^0$>∈$\mathcal{F}^T$(y) satisfying $|v_{T+t-1} - u_{T+t-1}| \leq \varepsilon|v - u_{T-1}|$ and $|y_{T+t} - x_{T+t}| \leq \varepsilon|y - x_T|$ if y ≠ $x_T$ for all t∈ℕ.

Towards a contradiction suppose that $p_1^{(T)}(y - x_T) + p_2^{(T-1)}(v - u_{T-1}) > 0$ and let ε > 0 be such that $p_1^{(T)}(y - x_T) + p_2^{(T-1)}(v - u_{T-1}) > \varepsilon(|y - x_1|\sum_{t=T+1}^{\infty}|p_1^{(t)}| + |v - u_{T-1}|\sum_{t=T+1}^{\infty}|p_2^{(t-1)}|)$.

This is possible since we have assumed towards a contradiction that $p_1^{(T)}(y - x_T) + p_2^{(T-1)}(v - u_{T-1}) > 0$.

Let $\langle (z_t, w_t) | t \in \mathbb{N}^0 \rangle \in \mathcal{F}(x)$ such that $z_t = x_t$ for t = 0, …, T-1, $w_{T-1} = v$, $w_t = u_t$ for t = 0, …, T-2 (provided T ≥ 2), $z_T = y$, $z_{t+1} = y_{t+1}$ and $w_t = v_t$ for all t ≥ T.

If for t ≥ T+1, $p_1^{(t)} \geq 0$, then $p_1^{(t)} y_t \geq p_1^{(t)} x_t - \varepsilon p_1^{(t)} |y - x_T|$.

If for t ≥ T+1, $p_1^{(t)} < 0$, then $p_1^{(t)} y_t \geq p_1^{(t)} x_t + \varepsilon p_1^{(t)} |y - x_T|$.

Thus, for t ≥ T+1, $p_1^{(t)} y_t \geq p_1^{(t)} x_t - \varepsilon |p_1^{(t)}| |y - x_T|$

If for t ≥ T+1, $p_2^{(t-1)} \geq 0$, then $p_2^{(t-1)} v_{t-1} \geq p_2^{(t-1)} u_{t-1} - \varepsilon p_2^{(t-1)} |v - u_{T-1}|$.

If for t ≥ T+1, $p_2^{(t-1)} < 0$, then $p_2^{(t-1)} v_{t-1} \geq p_2^{(t-1)} u_{t-1} + \varepsilon p_2^{(t-1)} |v - u_{T-1}|$.

Thus, for t ≥ T+1, $p_2^{(t-1)} v_{t-1} \geq p_2^{(t-1)} u_{t-1} - \varepsilon |p_2^{(t-1)}| |v - u_{T-1}|$.

Case 1: T = 1.

Hence, T – 1 = 0.

Thus, $\sum_{t=0}^{\infty}[p_1^{(t)} z_t + p_2^{(t)} w_t] = p_1^{(0)} x + p_2^{(0)} v + p_1^{(1)} y + p_2^{(1)} v_1 + \sum_{t=2}^{\infty}[p_1^{(t)} y_t + p_2^{(t)} v_t] = p_1^{(0)} x + p_2^{(0)} v + p_1^{(1)} y + \sum_{t=2}^{\infty}[p_1^{(t)} y_t + p_2^{(t-1)} v_{t-1}]$.

$\sum_{t=2}^{\infty}[p_1^{(t)} y_t + p_2^{(t-1)} v_{t-1}] = \sum_{t=2}^{\infty} p_1^{(t)} y_t + \sum_{t=2}^{\infty} p_1^{(t-1)} v_{t-1}$

Thus, $\sum_{t=0}^{\infty}[p_1^{(t)} z_t + p_2^{(t)} w_t] = p_1^{(0)} x + p_2^{(0)} v + p_1^{(1)} y + \sum_{t=2}^{\infty}[p_1^{(t)} y_t + p_2^{(t-1)} v_{t-1}] = p_1^{(0)} x + p_2^{(0)} v + p_1^{(1)} y + \sum_{t=2}^{\infty} p_1^{(t)} y_t + \sum_{t=2}^{\infty} p_1^{(t-1)} v_{t-1}$.

We know that, for t ≥ 2, $p_1^{(t)} y_t \geq p_1^{(t)} x_t - \varepsilon |p_1^{(t)}| |y - x_1|$, and for t ≥ 2, $p_2^{(t-1)} v_{t-1} \geq p_2^{(t-1)} u_{t-1} - \varepsilon |p_2^{(t-1)}| |v - u_0|$.

Therefore, $\sum_{t=2}^{\infty} p_1^{(t)} y_t \geq \sum_{t=2}^{\infty} p_1^{(t)} x_t - \varepsilon |y - x_1| \sum_{t=2}^{\infty} |p_1^{(t)}|$ and $\sum_{t=2}^{\infty} p_2^{(t-1)} v_{t-1} \geq \sum_{t=2}^{\infty} p_2^{(t-1)} u_{t-1} - \varepsilon |v - u_0| \sum_{t=2}^{\infty} |p_2^{(t-1)}|$.

Hence, $\sum_{t=2}^{\infty} p_1^{(t)} y_t + \sum_{t=2}^{\infty} p_2^{(t-1)} v_{t-1} \geq \sum_{t=2}^{\infty} p_1^{(t)} x_t + \sum_{t=2}^{\infty} p_2^{(t-1)} u_{t-1} - \varepsilon(|y - x_1| \sum_{t=2}^{\infty} |p_1^{(t)}| + |v - u_0| \sum_{t=2}^{\infty} |p_2^{(t-1)}|)$.

Thus, $\sum_{t=0}^{\infty}[p_1^{(t)} z_t + p_2^{(t)} w_t] = p_1^{(0)} x + p_2^{(0)} v + p_1^{(1)} y + \sum_{t=2}^{\infty} p_1^{(t)} y_t + \sum_{t=2}^{\infty} p_1^{(t-1)} v_{t-1} \geq p_1^{(0)} x + p_2^{(0)} v + p_1^{(1)} y + \sum_{t=2}^{\infty} p_1^{(t)} x_t + \sum_{t=2}^{\infty} p_2^{(t-1)} u_{t-1} - \varepsilon(|y - x_1| \sum_{t=2}^{\infty} |p_1^{(t)}| + |v - u_0| \sum_{t=2}^{\infty} |p_2^{(t-1)}|)$.

Now, $p_1^{(0)} x + p_2^{(0)} v + p_1^{(1)} y + \sum_{t=2}^{\infty} p_1^{(t)} x_t + \sum_{t=2}^{\infty} p_2^{(t-1)} u_{t-1} = \sum_{t=0}^{\infty}[p_1^{(t)} x_t + p_2^{(t)} u_t] + p_2^{(0)}(v - u_0) + p_1^{(1)}(y - x_1)$.

Thus, $\sum_{t=0}^{\infty}[p_1^{(t)} z_t + p_2^{(t)} w_t] \geq \sum_{t=0}^{\infty}[p_1^{(t)} x_t + p_2^{(t)} u_t] - \varepsilon(|y - x_1| \sum_{t=2}^{\infty} |p_1^{(t)}| + |v - u_0| \sum_{t=2}^{\infty} |p_2^{(t-1)}|) + p_2^{(0)}(v - u_0) + p_1^{(1)}(y - x_1)$.

However, according to our assumption towards a contradiction, $p_2^{(0)}(v - u_0) + p_1^{(1)}(y - x_1) > \varepsilon(|y - x_1| \sum_{t=2}^{\infty} |p_1^{(t)}| + |v - u_0| \sum_{t=2}^{\infty} |p_2^{(t-1)}|)$.

Thus, $\sum_{t=0}^{\infty}[p_1^{(t)}z_t + p_2^{(t)}w_t] > \sum_{t=0}^{\infty}[p_1^{(t)}x_t + p_2^{(t)}u_t]$, which in conjunction with $<(z_t, w_t)|t\in\mathbb{N}^0>\in\mathcal{F}(x)$ contradicts $<(x_t, u_t)|t\in\mathbb{N}^0>\in\mathcal{S}(x)$.

Thus, it must be that $p_1^{(T)}(y - x_T) + p_2^{(T-1)}(v - u_{T-1}) \leq 0$.

Case 2: T > 1.

Hence, $T - 1 > 0$ and $T - 2 \geq 0$.

Thus, $\sum_{t=0}^{\infty}[p_1^{(t)}z_t + p_2^{(t)}w_t] = \sum_{t=0}^{T-2}[p_1^{(t)}x_t + p_2^{(t)}u_t] + p_1^{(T-1)}x_{T-1} + p_2^{(T-1)}v + p_1^{(T)}y + \sum_{t=T+1}^{\infty}[p_1^{(t)}y_t + p_2^{(t-1)}v_{t-1}]$.

$\sum_{t=T+1}^{\infty}[p_1^{(t)}y_t + p_2^{(t-1)}v_{t-1}] = \sum_{t=T+1}^{\infty}p_1^{(t)}y_t + \sum_{t=T+1}^{\infty}p_1^{(t-1)}v_{t-1}$

Thus, $\sum_{t=0}^{\infty}[p_1^{(t)}z_t + p_2^{(t)}w_t] = \sum_{t=0}^{T-2}[p_1^{(t)}x_t + p_2^{(t)}u_t] + p_1^{(T-1)}x_{T-1} + p_2^{(T-1)}v + p_1^{(T)}y + \sum_{t=T+1}^{\infty}[p_1^{(t)}y_t + p_2^{(t-1)}v_{t-1}] = p_1^{(T-1)}x_{T-1} + p_2^{(T-1)}v + p_1^{(T)}y + \sum_{t=T+1}^{\infty}p_1^{(t)}y_t + \sum_{t=T+1}^{\infty}p_1^{(t-1)}v_{t-1}$.

We know that, for $t \geq T+1$, $p_1^{(t)}y_t \geq p_1^{(t)}x_t - \varepsilon|p_1^{(t)}||y - x_T|$, <u>and</u> for $t \geq T+1$, $p_2^{(t-1)}v_{t-1} \geq p_2^{(t-1)}u_{t-1} - \varepsilon|p_2^{(t-1)}||v - u_{T-1}|$.

Therefore, $\sum_{t=T+1}^{\infty}p_1^{(t)}y_t \geq \sum_{t=T+1}^{\infty}p_1^{(t)}x_t - \varepsilon|y - x_T|\sum_{t=T+1}^{\infty}|p_1^{(t)}|$ and $\sum_{t=T+1}^{\infty}p_2^{(t-1)}v_{t-1} \geq \sum_{t=T+1}^{\infty}p_2^{(t-1)}u_{t-1} - \varepsilon|v - u_{T-1}|\sum_{t=2}^{\infty}|p_2^{(t)}|$.

Hence, $\sum_{t=T+1}^{\infty}p_1^{(t)}y_t + \sum_{t=T+1}^{\infty}p_2^{(t-1)}v_{t-1} \geq \sum_{t=T+1}^{\infty}p_1^{(t)}x_t + \sum_{t=T+1}^{\infty}p_2^{(t-1)}u_{t-1} - \varepsilon(|y - x_T|\sum_{t=T+1}^{\infty}|p_1^{(t)}| + |v - u_{T-1}|\sum_{t=T+1}^{\infty}|p_2^{(t-1)}|)$.

Thus, $\sum_{t=0}^{\infty}[p_1^{(t)}z_t + p_2^{(t)}w_t] = \sum_{t=0}^{T-2}[p_1^{(t)}x_t + p_2^{(t)}u_t] + p_1^{(T-1)}x_{T-1} + p_2^{(T-1)}v + p_1^{(T)}y + \sum_{t=T+1}^{\infty}p_1^{(t)}y_t + \sum_{t=T+1}^{\infty}p_1^{(t-1)}v_{t-1} \geq \sum_{t=0}^{T-2}[p_1^{(t)}x_t + p_2^{(t)}u_t] + p_1^{(T-1)}x_{T-1} + p_2^{(T-1)}v + p_1^{(T)}y + \sum_{t=T+1}^{\infty}p_1^{(t)}x_t + \sum_{t=T+1}^{\infty}p_2^{(t-1)}u_{t-1} - \varepsilon(|y - x_T|\sum_{t=T+1}^{\infty}|p_1^{(t)}| + |v - u_{T-1}|\sum_{t=T+1}^{\infty}|p_2^{(t-1)}|)$.

Now, $\sum_{t=0}^{T-2}[p_1^{(t)}x_t + p_2^{(t)}u_t] + p_1^{(T-1)}x_{T-1} + p_2^{(T-1)}v + p_1^{(T)}y + \sum_{t=T+1}^{\infty}p_1^{(t)}x_t + \sum_{t=T+1}^{\infty}p_2^{(t-1)}u_{t-1} = \sum_{t=0}^{T-2}[p_1^{(t)}x_t + p_2^{(t)}u_t] + p_1^{(T-1)}x_{T-1} + p_2^{(T-1)}u_{T-1} + p_1^{(T)}x_T + \sum_{t=T+1}^{\infty}p_1^{(t)}x_t + \sum_{t=T+1}^{\infty}p_2^{(t-1)}u_{t-1} + (p_2^{(T-1)}v + p_1^{(T)}y) - (p_2^{(T-1)}u_{T-1} + p_1^{(T)}x_T) = \sum_{t=0}^{\infty}[p_1^{(t)}x_t + p_2^{(t)}u_t] + (p_2^{(T-1)}v + p_1^{(T)}y) - (p_2^{(T-1)}u_{T-1} + p_1^{(T)}x_T)$.

Thus, $\sum_{t=0}^{\infty}[p_1^{(t)}z_t + p_2^{(t)}w_t] \geq \sum_{t=0}^{T-2}[p_1^{(t)}x_t + p_2^{(t)}u_t] + p_1^{(T-1)}x_{T-1} + p_2^{(T-1)}v + p_1^{(T)}y + \sum_{t=T+1}^{\infty}p_1^{(t)}x_t + \sum_{t=T+1}^{\infty}p_2^{(t-1)}u_{t-1} - \varepsilon(|y - x_T|\sum_{t=T+1}^{\infty}|p_1^{(t)}| + |v - u_{T-1}|\sum_{t=T+1}^{\infty}|p_2^{(t-1)}|) = \sum_{t=0}^{\infty}[p_1^{(t)}x_t + p_2^{(t)}u_t] + (p_2^{(T-1)}(v - u_{T-1}) + p_1^{(T)}(y - x_T) - \varepsilon(|y - x_T|\sum_{t=T+1}^{\infty}|p_1^{(t)}| + |v - u_{T-1}|\sum_{t=T+1}^{\infty}|p_2^{(t-1)}|)$.

However, according to our assumption towards a contradiction, $(p_2^{(T-1)}(v - u_{T-1}) + p_1^{(T)}(y - x_T) > \varepsilon(|y - x_T|\sum_{t=T+1}^{\infty}|p_1^{(t)}| + |v - u_{T-1}|\sum_{t=T+1}^{\infty}|p_2^{(t-1)}|)$.

Thus, $\sum_{t=0}^{\infty}[p_1^{(t)}z_t + p_2^{(t)}w_t] > \sum_{t=0}^{\infty}[p_1^{(t)}x_t + p_2^{(t)}u_t]$, which in conjunction with $<(z_t, w_t)|t\in\mathbb{N}^0>\in\mathcal{F}(x)$ contradicts $<(x_t, u_t)|t\in\mathbb{N}^0>\in\mathcal{S}(x)$.

Thus, it must be that $p_1^{(T)}(y - x_T) + p_2^{(T-1)}(v - u_{T-1}) \leq 0$.

(ii) Since, $<((p_1^{(t)}, p_2^{(t)}), \Omega_t)| t\in\mathbb{N}^0>$ is an AC-LOC problem, $\lim_{t\to\infty}|p_i^{(t)}| = 0$ for $i\in\{1, 2\}$ and $x_t\in[0, b]$, $u_t\in[0, b]$ for all $t\in\mathbb{N}^0$. Thus, $\lim_{t\to\infty}( |p_1^{(t)}|x_t + |p_2^{(t)}|u_t) = 0$. Q.E.D.

**Note 5.2:** Part (i) of proposition 5.2, is similar in spirit to what is referred to in section 5.2 of Sorger (2015) as the Euler equation. We may refer to part (i) of the proposition above as "Euler condition." The condition $\lim_{t\to\infty}( |p_1^{(t)}|x_t + |p_2^{(t)}|u_t) = 0$ in part (ii) of proposition 5.2 resembles the several versions of "transversality condition" that are available in the literature on dynamic optimization.

The following result, analogous to proposition 4.1 in Mitra (2000), provides a "neat" sufficient condition for a trajectory starting at x to belong to $\mathcal{S}(x)$.

**Proposition 5.3:** If for $<(x_t, u_t)|t\in\mathbb{N}^0>\in\mathcal{F}(x)$ and there is a sequence $<q^{(t)}|t\in\mathbb{N}^0>$ of non-negative real numbers satisfying:

(i) For all $t\in\mathbb{N}^0$: $p_1^{(t)}x_t + p_2^{(t)}u_t + q^{(t+1)}x_{t+1} - q^{(t)}x_t \geq p_1^{(t)}y + p_2^{(t)}u + q^{(t+1)}z - q^{(t)}y$ for all $(y, u, z)\in\Omega_t$,

(ii) $\lim_{t\to\infty} q^{(t)}x_t = 0$,

then, $<(x_t, u_t)|t\in\mathbb{N}^0>\in\mathcal{S}(x)$.

**Proof:** Let $<(y_t, v_t)|t\in\mathbb{N}^0>\in\mathcal{F}(x)$.

Then, (i) implies that for all $t\in\mathbb{N}^0$: $p_1^{(t)}x_t + p_2^{(t)}u_t + q^{(t+1)}x_{t+1} - q^{(t)}x_t \geq p_1^{(t)}y_t + p_2^{(t)}v_t + q^{(t+1)}y_{t+1} - q^{(t)}y_t$.

Thus, for all $t\in\mathbb{N}^0$: $(p_1^{(t)}x_t + p_2^{(t)}u_t) - (p_1^{(t)}y_t + p_2^{(t)}v_t) \geq (q^{(t+1)}y_{t+1} - q^{(t)}y_t) - (q^{(t+1)}x_{t+1} - q^{(t)}x_t)$.

Hence, for all $T\in\mathbb{N}^0$: $\sum_{t=0}^{T}(p_1^{(t)}x_t + p_2^{(t)}u_t)$ - $\sum_{t=0}^{T}(p_1^{(t)}y_t + p_2^{(t)}v_t) \geq (q^{(T+1)}y_{T+1} - q^{(0)}x)$ - $(q^{(T+1)}x_{T+1} - q^{(0)}x) = q^{(T+1)}y_{T+1}$ - $q^{(T+1)}x_{T+1} \geq$ - $q^{(T+1)}x_{T+1}$, since $q^{(T+1)} \geq 0$ and $y_{T+1} \geq 0$ implies $q^{(T+1)}y_{T+1} \geq 0$.

Thus, $\sum_{t=0}^{\infty}(p_1^{(t)}x_t + p_2^{(t)}u_t)$ - $\sum_{t=0}^{\infty}(p_1^{(t)}y_t + p_2^{(t)}v_t) = \lim_{T\to\infty}\sum_{t=0}^{T}(p_1^{(t)}x_t + p_2^{(t)}u_t)$ - $\lim_{T\to\infty}(p_1^{(t)}y_t + p_2^{(t)}v_t) = \lim_{T\to\infty}[\sum_{t=0}^{T}(p_1^{(t)}x_t + p_2^{(t)}u_t) - \sum_{t=0}^{T}(p_1^{(t)}y_t + p_2^{(t)}v_t)] \geq \lim_{T\to\infty} -q^{(T+1)}x_{T+1} = 0$, by (ii). Q.E.D.

**Note 5.3:** Conditions (i), is a variant of what is referred to as the "competitive condition" in Mitra (2000). In Mitra (2000), (ii) is referred to as the transversality condition.

**6. Linear Dynamic Programming:**

By proposition 5.1 we know that $(x, T)\in X\times\mathbb{N}^0$, $\mathcal{S}^T(x) \neq \phi$, and hence for all $T\in\mathbb{N}^0$, there exists a function $V^T$: $X \to\mathbb{R}$ such that for all $x\in X$, $V^T(x) = \sum_{\tau=0}^{\infty}[p_1^{(T+\tau)}x_{T+\tau}+ p_2^{(T+\tau)}u_{T+\tau}]$ for all $<(x_{T+t}, u_{T+t})|t\in\mathbb{N}^0>\in\mathcal{S}^T(x)$. Clearly for all for all $T\in\mathbb{N}^0$, $V^T$ is well defined on X.

We may refer to $V^T$ as the **period-T optimal value function**

We will denote $V^0$ by V. V is said to be the **optimal value function**.

Thus, for all x∈X, V(x) = $p_1^{(0)}x + p_2^{(0)}u_0 + \sum_{\tau=1}^{\infty}[p_1^{(T+\tau)}x_{T+\tau} + p_2^{(T+\tau)}u_{T+\tau}]$, where $<(x_t, u_t)|t\in\mathbb{N}^0> \in \mathcal{S}(x)$.

The following result is analogous to well-known results on dynamic programming available in Mitra (2000), Sorger (2015) and Lahiri (2025b).

**Proposition 6.1:** (i) For all $T\in\mathbb{N}^0$, $V^T$ satisfies the following ***functional equation of dynamic programming***: For all x∈X and $<(x_t, u_t)|t \geq T> \in \mathcal{S}^T(x)$ and t ≥ T: $V^t(x_t) = p_1^{(t)}x_t + p_2^{(t)}u_t + V^{t+1}(x_{t+1}) = p_1^{(t)}x_t + \max_{(u,y)\in\Omega_t(x_t)}\{p_2^{(t)}u + V^{t+1}(y)\}$.

(ii) For all x∈X: $<(x_t, u_t)|t\in\mathbb{N}^0> \in \mathcal{S}(x)$ <u>if and only if</u> $<(x_t, u_t)|t\in\mathbb{N}^0> \in \mathcal{F}(x)$ and for all $T\in\mathbb{N}^0$ it is the case that $V^T(x_T) = p_1^{(T)}x + p_2^{(T)}u_T + V^{T+1}(x_{T+1})$.

**Proof:** (i) Let $T\in\mathbb{N}^0$ and x∈X. Let $<(x_t, u_t)|t \geq T> \in \mathcal{S}^T(x)$.

Thus, $V^T(x) = V^T(x_T) = \sum_{\tau=T}^{\infty}(p_1^{(\tau)}x_\tau + p_2^{(\tau)}u_\tau)$

Towards a contradiction suppose there exists t ≥ T, such that $V^t(x_t) > p_1^{(t)}x_t + p_2^{(t)}u_t + V^{t+1}(x_{t+1})$.

Thus, there exists $<(y_{t+\tau}, v_{t+\tau})|\tau\in\mathbb{N}^0> \in \mathcal{F}^t(x_t)$ such that $y_t = x_t$ and $p_1^{(t)}x_t + p_2^{(t)}v_t + \sum_{\tau=t+1}^{\infty}(p_1^{(\tau)}y_\tau + p_2^{(\tau)}v_\tau) > p_1^{(t)}x_t + p_2^{(t)}u_t + V^{t+1}(x_{t+1}) \geq p_1^{(t)}x_t + p_2^{(t)}u_t + \sum_{\tau=t+1}^{\infty}(p_1^{(\tau)}x_\tau + p_2^{(\tau)}u_\tau)$.

Hence, $p_2^{(t)}v_t + \sum_{\tau=t+1}^{\infty}(p_1^{(\tau)}y_\tau + p_2^{(\tau)}v_\tau) > p_2^{(t)}u_t + V^{t+1}(x_{t+1}) \geq p_2^{(t)}u_t + \sum_{\tau=t+1}^{\infty}(p_1^{(\tau)}x_\tau + p_2^{(\tau)}u_\tau)$.

Clearly, $\{\tau \geq t+1 |\, (v_{\tau-1}, y_\tau) \neq (u_{\tau-1}, x_\tau)\} \neq \phi$.

Let $<(z_\tau, w_\tau)|\, \tau \geq T>$ be such that $z_\tau = x_\tau$ for τ = T, …, t, $z_\tau = y_\tau$ for all τ ≥ t+1, $w_\tau = u_\tau$ for τ = T, …, t-1 (if t > T) and $w_\tau = v_\tau$ for τ ≥ t.

Thus, $<(z_\tau, w_\tau)|\, \tau \geq T> \in \mathcal{F}^T(x)$ and $\sum_{\tau=T}^{\infty}(p_1^{(\tau)}z_\tau + p_2^{(\tau)}w_\tau) = \sum_{\tau=T}^{t}(p_1^{(\tau)}z_\tau + p_2^{(\tau)}w_\tau) + \sum_{\tau=t+1}^{\infty}(p_1^{(\tau)}z_\tau + p_2^{(\tau)}w_\tau)$.

Case 1: t = T.

Then, $\sum_{\tau=T}^{\infty}(p_1^{(\tau)}z_\tau + p_2^{(\tau)}w_\tau) = p_1^{(T)}z_T + p_2^{(T)}w_T + \sum_{\tau=T+1}^{\infty}(p_1^{(\tau)}z_\tau + p_2^{(\tau)}w_\tau) = p_1^{(T)}x_T + p_2^{(T)}v_T + + \sum_{\tau=T+1}^{\infty}(p_1^{(\tau)}y_\tau + p_2^{(\tau)}v_\tau) > p_1^{(T)}x_T + p_2^{(t)}u_t + \sum_{\tau=T+1}^{\infty}(p_1^{(\tau)}x_\tau + p_2^{(\tau)}u_\tau) = \sum_{\tau=T}^{\infty}(p_1^{(\tau)}x_\tau + p_2^{(\tau)}u_\tau)$, i.e., $\sum_{\tau=T}^{\infty}(p_1^{(\tau)}z_\tau + p_2^{(\tau)}w_\tau) > \sum_{\tau=T}^{\infty}(p_1^{(\tau)}x_\tau + p_2^{(\tau)}u_\tau)$, contradicting $<(x_t, u_t)|t \geq T> \in \mathcal{S}^T(x)$.

Case 2: t > T.

Then, $\sum_{\tau=T}^{\infty}(p_1^{(\tau)}z_\tau + p_2^{(\tau)}w_\tau) = \sum_{\tau=T}^{t-1}(p_1^{(\tau)}z_\tau + p_2^{(\tau)}w_\tau) + (p_1^{(t)}z_t + p_2^{(t)}w_t) + \sum_{\tau=t+1}^{\infty}(p_1^{(\tau)}z_\tau + p_2^{(\tau)}w_\tau) = \sum_{\tau=T}^{t-1}(p_1^{(\tau)}x_\tau + p_2^{(\tau)}u_\tau) + (p_1^{(t)}x_t + p_2^{(t)}v_t) + \sum_{\tau=t+1}^{\infty}(p_1^{(\tau)}y_\tau + p_2^{(\tau)}v_\tau) > \sum_{\tau=T}^{t-1}(p_1^{(\tau)}x_\tau + p_2^{(\tau)}u_\tau) + p_1^{(t)}x_t + p_2^{(t)}u_t + \sum_{\tau=t+1}^{\infty}(p_1^{(\tau)}x_\tau + p_2^{(\tau)}u_\tau) = \sum_{\tau=T}^{\infty}(p_1^{(\tau)}x_\tau + p_2^{(\tau)}u_\tau)$, i.e. $\sum_{\tau=T}^{\infty}(p_1^{(\tau)}z_\tau + p_2^{(\tau)}w_\tau) > \sum_{\tau=T}^{\infty}(p_1^{(\tau)}x_\tau + p_2^{(\tau)}u_\tau)$, once again contradicting $<(x_t, u_t)|t \geq T> \in \mathcal{S}^T(x)$.

Thus, it must be the case that, for all $t \geq T$: $V^t(x_t) = p_1^{(t)}x_t + p_2^{(t)}u_t + V^{t+1}(x_{t+1}) = p_1^{(t)}x_t + \max_{(u,y)\in\Omega_t(x_t)}\{p_2^{(t)}u + V^{t+1}(y)\}$, which is the "functional equation of dynamic programming".

(ii) Suppose $<(x_t, u_t)|\mathbb{N}^0> \in \mathcal{S}(x)$. Then from (i) it follows that $<(x_t, u_t)|\mathbb{N}^0> \in \mathcal{F}(x)$ and for all $T \in \mathbb{N}^0$ it is the case that $V^T(x_T) = p_1^{(T)}x + p_2^{(T)}u_T + V^{T+1}(x_{T+1})$.

Now suppose $<x_t|t\in\mathbb{N}^0> \in \mathcal{F}(x)$ and for all $T\in\mathbb{N}^0$: $V^T(x_T) = p_1^{(T)}x_T + p_2^{(T)}u_T + V^{T+1}(x_{T+1})$.

Hence, $V(x) = V^0(x) = p_1^{(0)}x + p_2^{(0)}u_0 + V^1(x_1) = p_1^{(0)}x + p_2^{(0)}u_0 + p_1^{(1)}x_1 + p_2^{(1)}u_1 + V^2(x_T) = \ldots = \sum_{t=0}^{T}(p_1^{(t)}x_t + p_2^{(t)}u_t) + V^{T+1}(x_{T+1})$ for all $T\in\mathbb{N}$.

Since, $|V^T(x_T)| \leq b\sum_{t=T}^{\infty}(|p_1^{(t)}| + |p_2^{(t)}|)$ and since $\lim_{T\to\infty}\sum_{t=T}^{\infty}(|p_1^{(t)}| + |p_2^{(t)}|) = 0$, it follows that $\lim_{T\to\infty} V^{T+1}(x_{T+1}) = 0$.

Thus, $V(x) = \lim_{T\to\infty}\sum_{t=0}^{T}(p_1^{(t)}x_t + p_2^{(t)}u_t) + \lim_{T\to\infty} V^{T+1}(x_{T+1}) = \sum_{t=0}^{\infty}(p_1^{(t)}x_t + p_2^{(t)}u_t)$.

Thus, $<(x_t, u_t)|t\in\mathbb{N}^0> \in \mathcal{S}(x)$. Q.E.D.

**Note 6.1:** Proposition 6.1 and its proof is similar to the second proposition and its proof in section 5 of Lahiri (2025b).

LOC $<((p_1^{(t)}, p_2^{(t)}), \Omega_t)|t\in\mathbb{N}^0>$ is said be **concave in pay-offs from the control variable** if for all $t\in\mathbb{N}^0$, $(x, u, y), (z, v, \tilde{y})\in\Omega_t$ and $\theta\in(0, 1)$, there exists $w\in X$ such that $(\theta x + (1-\theta)z, w, \theta y + (1-\theta)\tilde{y})\in\Omega_t$ and $p_2^{(t)}w \geq \theta p_2^{(t)}u + (1-\theta)p_2^{(t)}v$.

If for all $t\in\mathbb{N}^0$, $\Omega_t$ is convex, then the LOC is concave in pay-offs from the control variable.

Clearly an LOC-LC is concave in pay-offs from the control variable.

The following proposition and its proof is similar to the first proposition and its proof in section 5 of Lahiri (2025b).

**Proposition 6.2:** If $<((p_1^{(t)}, p_2^{(t)}), \Omega_t)|t\in\mathbb{N}^0>$ is an AC-LOC that is concave in pay-offs from the control variable, <u>then</u> for all $T\in\mathbb{N}^0$, $V^T$ is concave and continuous on X.

**Proof:** Let $T\in\mathbb{N}^0$. For all $(x, T), (y, T)\in X\times\mathbb{N}^0$, let $<(x_t, u_t)\ |t \geq T> \in \mathcal{S}^T(x)$ and $<(y_t, v_t)|t \geq T> \in \mathcal{S}^T(y)$. Suppose $\theta\in(0, 1)$ and consider the infinite sequence $<(\theta x_t + (1-\theta)y_t, \theta u_t + (1-\theta)v_t)\ |t \geq T>$.

Since $(x_t, u_t, x_{t+1}) \in \Omega_t$ and $(y_t, v_t, y_{t+1}) \in \Omega_t$ for all $t \geq T$ and the AC-LOC is concave in pay-offs from the control variable, for all $t \geq T$, there exists $w_t \in X$ such that $(\theta x_t + (1-\theta)y_t, w_t, \theta x_{t+1} + (1-\theta)y_{t+1}) \in \Omega_t$ for all $t \geq T$ and $p_2^{(t)} w_t \geq \theta p_2^{(t)} u_t + (1-\theta) p_2^{(t)} v_t$.

Thus, $<(\theta x_t + (1-\theta)y_t, w_t, \theta u_t + (1-\theta)v_t) | t \geq T> \in \mathcal{F}^T(\theta x + (1-\theta)y)$.

Thus, $V^T(\theta x + (1-\theta)y) \geq \sum_{t=T}^{\infty}[p_1^{(t)}(\theta x_t + (1-\theta)y_t) + p_2^{(t)} w_t] \geq \sum_{t=T}^{\infty}[p_1^{(t)}(\theta x_t + (1-\theta)y_t) + p_2^{(t)}(\theta u_t + (1-\theta)v_t) = \theta \sum_{t=T}^{\infty}(p_1^{(t)} x_t + p_2^{(t)} u_t) + (1-\theta)\sum_{t=T}^{\infty}(p_1^{(t)} y_t + p_2^{(t)} v_t) = \theta V^T(x) + (1-\theta)V^T(y)$.

Thus, for all $T \in \mathbb{N}^0$, $V^T$ is a concave function on X.

From corollary 1 of proposition 1 and part (i) of proposition 3 in Lahiri (2025a) it follows that since $V^T$ is concave on X = [0, b], it must be the case that $V^T$ is continuous on (0, b) and both $\lim_{x \to 0} V(x)$ and $\lim_{x \to b} V(x)$ exists and belong to the interval [-2M, 2M] where $M = \max\{b\sum_{t=0}^{\infty} |p_1^{(t)}|, b\sum_{t=0}^{\infty} |p_2^{(t)}|\} < +\infty$.

Further, it must be that $\lim_{x \to 0} V^T(x) \geq V^T(0)$ and $\lim_{x \to b} V^T(x) \geq V^T(b)$.

Now let $x \in X$ and $<x^{(n)} | n \in \mathbb{N}>$ be a sequence in X converging to x.

For each $n \in \mathbb{N}$, let $<(x_{T+t}^{(n)}, u_{T+t}^{(n)}) | t \in \mathbb{N}^0> \in \mathcal{S}(x^{(n)})$.

Since $x_T^{(n)} = x^{(n)}$ for all $n \in \mathbb{N}$, $\lim_{n \to \infty} x_T^{(n)} = x$.

Since, $<(x_T^{(n)}, u_T^{(n)}, x_{T+1}^{(n)}) | n \in \mathbb{N}>$ is a sequence in the closed and bounded set $X \times X \times X$, it has a convergent subsequence $<(x_T^{N_1(n)}, u_T^{N_1(n)}, x_{T+1}^{N_1(n)}) | n \in \mathbb{N}>$ converging to $(x_T^0, u_T^0, x_{T+1}^0) \in X \times X \times X$, where $x_T^0 = x$.

Further, $(x_T^{N_1(n)}, u_T^{N_1(n)}, x_{T+1}^{N_1(n)}) \in \Omega_T$ for all $n \in \mathbb{N}$ implies $(x, u_T^0, x_{T+1}^0) \in \Omega_T$ since $\Omega_T$ is closed.

Having obtained convergent subsequences $<(x_{T+\tau-1}^{N_\tau(n)}, u_{T+\tau-1}^{N_\tau(n)}, x_{T+\tau}^{N_\tau(n)} | n \in \mathbb{N}> \in \Omega_{T+\tau-1}$ for all $\tau = 1, \ldots, t$ for some $t \geq 1$, let $<(x_{T+t}^{N_{t+1}(n)}, u_{T+t}^{N_{t+1}(n)}, x_{T+t}^{N_{t+1}(n)} | n \in \mathbb{N}>$ be a convergent subsequence of the sequence $<(x_{T+t}^{N_t(n)}, u_{T+t}^{N_t(n)}, x_{T+t+1}^{N_t(n)} | n \in \mathbb{N}>$ in the closed and bounded set $X \times X \times X$, converging to $(x_{T+t}^0, u_{T+t}^0, x_{T+t+1}^0) \in X \times X \times X$. Since, $<(x_{T+t}^{N_t(n)}, u_{T+t}^{N_t(n)}, x_{T+t+1}^{N_t(n)} | n \in \mathbb{N}> \in \Omega_{T+t}$ and $\Omega_{T+t}$ is a closed subset of $X \times X \times X$, it must be the case that $(x_{T+t}^0, u_{T+t}^0, x_{T+t+1}^0) \in \Omega_{T+t}$.

Since, for all $t \in \mathbb{N}^0$, $<(x_{T+\tau}^{N_\tau(n)}, u_{T+\tau}^{N_\tau(n)}, x_{T+\tau+1}^{N_\tau(n)} | n \in \mathbb{N}>$ converges to $(x_{T+\tau}^0, u_{T+\tau}^0, x_{T+\tau+1}^0) \in \Omega_{T+\tau}$, for all $\tau \leq t$ and $n \in \mathbb{N}$, ***given*** $\varepsilon > 0$, for all $t \in \mathbb{N}$, there exists a sequence $<n_t | t \in \mathbb{N}>$ in $\mathbb{N}$ satisfying $n_{t+1} > n_t$ for all $t \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq n_t$ and $\tau \leq t$, $(x_{T+\tau}^{N_t(n)}, u_{T+\tau}^{N_t(n)}, x_{T+\tau+1}^{N_t(n)}) \in \Omega_{T+\tau}$ and $|p_1^{(T+\tau)} x_{T+\tau}^0 + p_2^{(T+\tau)} u_{T+\tau}^0 - p_1^{(T+\tau)} x_{T+\tau}^{N_t(n)} - p_2^{(T+\tau)} u_{T+\tau}^{N_t(n)}| < \frac{\varepsilon}{8}(\frac{1}{2})^t$, i.e., $p_1^{(T+\tau)} x_{T+\tau}^{N_t(n)} + p_2^{(T+\tau)} u_{T+\tau}^{N_t(n)} + \frac{\varepsilon}{8}(\frac{1}{2})^t > p_1^{(T+\tau)} x_{T+\tau}^0 + p_2^{(T+\tau)} u_{T+\tau}^0 > p_1^{(T+\tau)} x_{T+\tau}^{N_t(n)} + p_2^{(T+\tau)} u_{T+\tau}^{N_t(n)} - \frac{\varepsilon}{8}(\frac{1}{2})^t$.

Note that $\sum_{\tau=0}^{t} \frac{\varepsilon}{8} (\frac{1}{2})^{t} \leq \sum_{\tau=0}^{t} \frac{\varepsilon}{8} (\frac{1}{2})^{\tau} = \frac{\varepsilon}{4}(1- (\frac{1}{2})^{t+1})$.

Thus, for all t∈ ℕ, $\sum_{\tau=0}^{t}(p_1^{(T+\tau)} x_{T+\tau}^0 + p_2^{(T+\tau)} u_{T+\tau}^0) > \sum_{\tau=0}^{t}(p_1^{(T+\tau)} x_{T+\tau}^{N_t(n)} + p_2^{(T+\tau)} u_{T+\tau}^{N_t(n)})$- $\frac{\varepsilon}{4}(1- (\frac{1}{2})^{t+1})$ for all n ≥ $n_t$.

Thus, for all t∈ℕ, $\sum_{\tau=0}^{t}(p_1^{(T+\tau)} x_{T+\tau}^0 + p_2^{(T+\tau)} u_{T+\tau}^0) > \sum_{\tau=0}^{t}(p_1^{(T+\tau)} x_{T+\tau}^{N_t(n)} + p_2^{(T+\tau)} u_{T+\tau}^{N_t(n)})$- $\frac{\varepsilon}{4}(1- (\frac{1}{2})^{t+1})$ = V($x^{N_t(n)}$) -$\sum_{t=\tau+1}^{\infty}(p_1^{(T+\tau)} x_{T+\tau}^{N_t(n)} + p_2^{(T+\tau)} u_{T+\tau}^{N_t(n)})$ - $\frac{\varepsilon}{4}(1- (\frac{1}{2})^{t+1})$ for all n ≥ $n_t$.

Thus, for all t∈ℕ, $V^T(x) \geq \sum_{\tau=0}^{\infty}(p_1^{(T+\tau)} x_{T+\tau}^0 + p_2^{(T+\tau)} u_{T+\tau}^0) = \sum_{\tau=0}^{t}(p_1^{(T+\tau)} x_{T+\tau}^0 + p_2^{(T+\tau)} u_{T+\tau}^0) + \sum_{\tau=t+1}^{\infty}(p_1^{(T+\tau)} x_{T+\tau}^0 + p_2^{(T+\tau)} u_{T+\tau}^0)$ > V($x^{N_t(n)}$) -$\sum_{t=\tau+1}^{\infty}(p_1^{(T+\tau)} x_{T+\tau}^{N_t(n)} + p_2^{(T+\tau)} u_{T+\tau}^{N_t(n)})$ - $\frac{\varepsilon}{4}(1- (\frac{1}{2})^{t+1})$ + $\sum_{\tau=t+1}^{\infty}(p_1^{(T+\tau)} x_{T+\tau}^0 + p_2^{(T+\tau)} u_{T+\tau}^0)$ for all n ≥ $n_\tau$.

Thus, for all t∈ℕ and n ≥ $n_t$, $V^T(x) + \sum_{\tau=t+1}^{\infty}(p_1^{(T+\tau)} x_{T+\tau}^{N_t(n)} + p_2^{(T+\tau)} u_{T+\tau}^{N_t(n)}) >$ V($x^{N_t(n)}$) - $\frac{\varepsilon}{4}(1- (\frac{1}{2})^{t+1})$ + $\sum_{\tau=t+1}^{\infty}(p_1^{(T+\tau)} x_{T+\tau}^0 + p_2^{(T+\tau)} u_{T+\tau}^0)$.

By hypothesis, $\sum_{t=0}^{\infty} |p_i^{(t)}| < +\infty$ for i∈{1, 2} and for all (t, n)∈ $\mathbb{N}^0$×ℕ, both $x_{T+t}^{(n)}$∈[0, b] and $u_{T+t}^{(n)}$∈[0, b].

Thus, for all t∈ℕ and n ≥ $n_t$, b$\sum_{\tau=t+1}^{\infty}[|p_1^{(T+\tau)}| + |p_2^{(T+\tau)}|] \geq \sum_{\tau=t+1}^{\infty}[|p_1^{(T+\tau)}| x_{T+\tau}^{N_t(n)} + |p_2^{(T+\tau)}| u_{T+\tau}^{N_t(n)}]$.

Further, for all t∈ℕ and n ≥ $n_t$, $\sum_{\tau=t+1}^{\infty}[|p_1^{(T+\tau)}| x_{T+\tau}^{N_t(n)} + |p_2^{(T+\tau)}| u_{T+\tau}^{N_t(n)}] \geq \sum_{\tau=t+1}^{\infty}[p_1^{(T+\tau)} x_{T+\tau}^{N_t(n)} + p_2^{(T+\tau)} u_{T+\tau}^{N_t(n)}]$.

Thus, for all t∈ℕ and n ≥ $n_t$, $V^T(x)$ + b$\sum_{\tau=t+1}^{\infty}[|p_1^{(T+\tau)}| + |p_2^{(T+\tau)}|] \geq V^T(x) + \sum_{\tau=t+1}^{\infty}[|p_1^{(T+\tau)}| x_{T+\tau}^{N_t(n)} + |p_2^{(T+\tau)}| u_{T+\tau}^{N_t(n)}] \geq V^T(x) + \sum_{\tau=t+1}^{\infty}[|p_1^{(T+\tau)} x_{T+\tau}^{N_t(n)} + p_2^{(T+\tau)} u_{T+\tau}^{N_t(n)}|]$.

Thus, for all t∈ℕ and n ≥ $n_t$, $V^T(x)$ + b$\sum_{\tau=t+1}^{\infty}[|p_1^{(T+\tau)}| + |p_2^{(T+\tau)}|] \geq V^T(x) + \sum_{\tau=t+1}^{\infty}(p_1^{(T+\tau)} x_{T+\tau}^{N_t(n)} + p_2^{(T+\tau)} u_{T+\tau}^{N_t(n)}) > V^T(x^{N_t(n)})$ - $\frac{\varepsilon}{4}(1- (\frac{1}{2})^{t+1})$ + $\sum_{\tau=t+1}^{\infty}(p_1^{(T+\tau)} x_{T+\tau}^0 + p_2^{(T+\tau)} u_{T+\tau}^0)$.

Thus, for all t∈ℕ, $V^T(x)$ + b$\sum_{\tau=t+1}^{\infty}[|p_1^{(T+\tau)}| + |p_2^{(T+\tau)}|] \geq \limsup_{n\to\infty} V^T(x^{N_t(n)})$ - $\frac{\varepsilon}{4}(1- (\frac{1}{2})^{t+1})$ + $\sum_{\tau=t+1}^{\infty}(p_1^{(T+\tau)} x_{T+\tau}^0 + p_2^{(T+\tau)} u_{T+\tau}^0)$.

Thus, for all t∈ℕ, $V^T(x)$ + b$\sum_{\tau=t+1}^{\infty}[|p_1^{(T+\tau)}| + |p_2^{(T+\tau)}|] \geq \limsup_{n\to\infty} V^T(x^{N_t(n)})$ - $\frac{\varepsilon}{4}(1- (\frac{1}{2})^{t+1})$ + $\sum_{\tau=t+1}^{\infty}(p_1^{(T+\tau)} x_{T+\tau}^0 + p_2^{(T+\tau)} u_{T+\tau}^0)$.

By hypothesis, $\sum_{t=0}^{\infty} |p_i^{(t)}| < +\infty$ for i∈{1, 2}.

Thus, $\lim_{t\to\infty} b \sum_{\tau=t+1}^{\infty}[|p_1^{(T+\tau)}| + |p_2^{(T+\tau)}|] = 0$.

Also, $\lim_{t\to\infty} \frac{\varepsilon}{4}(1 - (\frac{1}{2})^{t+1}) = \frac{\varepsilon}{4}$ and $\lim_{t\to\infty} \sum_{\tau=t+1}^{\infty}(p_1^{(T+\tau)} x_{T+\tau}^0 + p_2^{(T+\tau)} u_{T+\tau}^0) = 0$.

Thus, $V^T(x) = V^T(x) + \lim_{t\to\infty} b\sum_{\tau=t+1}^{\infty}[|p_1^{(T+\tau)}| + |p_2^{(T+\tau)}|] \geq \lim_{t\to\infty}\limsup_{n\to\infty} V^T(x^{N_t(n)})$ - $\lim_{t\to\infty}\frac{\varepsilon}{8}(1 - (\frac{1}{2})^{t+1}) + \lim_{t\to\infty}\sum_{\tau=t+1}^{\infty}(p_1^{(T+\tau)}x_{T+\tau}^0 + p_2^{(T+\tau)}u_{T+\tau}^0) = \lim_{\tau\to\infty}\limsup_{n\to\infty} V^T(x^{N_\tau(n)}) - \frac{\varepsilon}{4}$.

Thus, $V^T(x) \geq \lim_{\tau\to\infty}\limsup_{n\to\infty} V^T(x^{N_\tau(n)}) - \frac{\varepsilon}{4}$.

The above being true for all $\varepsilon > 0$, we get $V^T(x) \geq \lim_{t\to\infty}\limsup_{n\to\infty} V^T(x^{N_t(n)})$.

However, $\lim_{n\to\infty} V^T(x^{(n)})$ exists implies $\lim_{n\to\infty} V^T(x^{(n)}) = \limsup_{n\to\infty} V^T(x^{N_t(n)})$ for all $t\in\mathbb{N}$.

Thus, $V^T(x) \geq \lim_{t\to\infty}\limsup_{n\to\infty} V^T(x^{N_t(n)}) = V^T(x) \geq \lim_{t\to\infty}\lim_{n\to\infty} V^T(x^{(n)}) = \lim_{n\to\infty} V^T(x^{(n)})$, i.e., $V^T(x) \geq \lim_{n\to\infty} V^T(x^{(n)})$.

This combined with the continuity of $V^T$ on (0, b), $\lim_{x\to b} V^T(x) \geq V^T(b)$ and $\lim_{x\to 0} V^T(x) \geq V^T(0)$ implies $\lim_{x\to b} V^T(x) = V^T(b)$, $\lim_{x\to 0} V^T(x) = V^T(0)$ and hence the continuity of $V^T$ on X. Q.E.D.

**7. Linear Optimal Control with Linear and Square-Root Constraints:**

The following is an example of a linear optimal control problem that can be relevant for intertemporal economics.

For all $t\in\mathbb{N}^0$, there exists a pair of real numbers $(a_t, b_t)$ satisfying $a_t \geq 0$ and $a_t + b_t b \geq 0$ and an ordered pair of non-negative real numbers $(c_t, d_t)$ satisfying $d_t > 0$ and $c_t < b$, such that for all $x\in X$, $\Omega_t(x) = \{(u, y)|\ u\in[0, \min\{x, a_t + b_t x\}]$ and $y = \min\{b, c_t + d_t\sqrt{(x - u)}\}\}$.

Since for all $t\in\mathbb{N}^0$, $c_t < b$ it must be the case that for all $x\in X$, [u = x implies $y = \min\{b, c_t\} = c_t$.

Note that the function $\xi \mapsto c_t + d_t\sqrt{\xi}$ on $\mathbb{R}_+$ assumes the value $c_t$ at $\xi = 0$ is strictly increasing on $\mathbb{R}_+$ is twice continuously differentiable on $\mathbb{R}_{++}$ with its first derivative being strictly positive and its second derivative being strictly negative at all points in $\mathbb{R}_{++}$. Further, the first derivative diverges to $+\infty$ as $\xi$ tends to zero.

If $y < b$, then it must be the case that $y = c_t + d_t\sqrt{(x - u)}$ so that $\sqrt{(x - u)} = \frac{y-c_t}{d_t}$, i.e., $u = x - (\frac{y-c_t}{d_t})^2$.

A LOC problem $<((p_1^{(t)}, p_2^{(t)}), \Omega_t)|\ t\in\mathbb{N}^0>$ is said to be a **linear optimal control problem with linear and square-root constraints (LOC-LSRC)** if all $(x, t)\in X\times\mathbb{N}^0$, $\Omega_t(x) = \{(u, y)|\ u\in[0, \min\{x, a_t + b_t x\}]$ and $y = \min\{b, c_t + d_t\sqrt{(x - u)}\}\}$.

For a LOC-LSRC if $<(x_t, u_t)|t\in\mathbb{N}^0>\in\mathcal{F}(x)$, then $u_t = x_t - (\frac{x_{t+1}-c_t}{d_t})^2$ if $x_{t+1} < b$.

**Proposition 7.1:** Suppose $<((p_1^{(t)}, p_2^{(t)}), \Omega_t)|t\in\mathbb{N}^0>$ is an LOC-LSRC with $a_t = 0$, $b_t = 1$ and $p_2^{(t)} \neq 0$ for all $t\in\mathbb{N}^0$. Suppose $x\in X$ and $<(x_t, u_t)|t\in\mathbb{N}^0>\in\mathcal{S}(x)$ satisfies $0 < u_t < x_t < b$ for all $t\in\mathbb{N}^0$. Then, for all $t\in\mathbb{N}^0$, $x_{t+1} = c_t + \frac{(d_t)^2(p_2^{(t+1)} + p_1^{(t+1)})}{2p_2^{(t)}}$ and $u_t = x_t - (\frac{p_2^{(t+1)} + p_1^{(t+1)}}{2p_2^{(t)}})^2$.

**Proof:** Under the conditions of proposition 7.1 and the assumption that $<(x_t, u_t)|t\in\mathbb{N}^0>\in\mathcal{S}(x)$, it follows that $<(y_t, v_t)|t\in\mathbb{N}^0>\in\mathcal{F}(x)$, where $y_\tau = x_\tau$ for all $\tau \neq t+1$, $v_\tau = y_\tau$ for all $\tau \notin \{t, t+1\}$, $v_t = x_t - (\frac{y_{t+1}-c_t}{d_t})^2$ and $v_{t+1} = y_{t+1} - (\frac{x_{t+2}-c_{t+1}}{d_{t+1}})^2$ so long as $y_{t+1}$ belongs to a sufficiently small interval around $x_{t+1}$.

Thus, for all $t\in\mathbb{N}^0$, $(u_t, x_{t+1})$ is a local maximizer of the optimization problem: Maximize $p_2^{(t)}u + (p_2^{(t+1)}+ p_1^{(t+1)})y$, subject to $u = x_t - (\frac{y-c_t}{d_t})^2$.

Thus, for all $t\in\mathbb{N}^0$, $x_{t+1}$ is a local maximizer of the optimization problem: Maximize $(p_2^{(t+1)}+ p_1^{(t+1)})y - p_2^{(t)}(\frac{y-c_t}{d_t})^2$

The function $y|\rightarrow (p_2^{(t+1)}+ p_1^{(t+1)})y - p_2^{(t)}(\frac{y-c_t}{d_t})^2)$, is continuously differentiable in a neighborhood of $x_{t+1}$. Thus, its derivative at $y = x_{t+1}$ must be zero.

The derivative of the function $y|\rightarrow p_1^{(t+1)}y - p_2^{(t)}(\frac{y-c_t}{d_t})^2)$ in a small interval around $x_{t+1}$ is $(p_2^{(t+1)}+ p_1^{(t+1)}) - 2p_2^{(t)}(\frac{y-c_t}{(d_t)^2})$.

Thus, $(p_2^{(t+1)}+ p_1^{(t+1)}) - 2p_2^{(t)}(\frac{x_{t+1}-c_t}{(d_t)^2}) = 0$ and hence $x_{t+1} = c_t + \frac{(d_t)^2(p_2^{(t+1)}+ p_1^{(t+1)})}{2p_2^{(t)}}$.

Further, $u_t = x_t - (\frac{x_{t+1}-c_t}{(d_t)^2})^2 = x_t - (\frac{p_2^{(t+1)}+ p_1^{(t+1)}}{2p_2^{(t)}})^2$. Q.E.D.

**Note 7.1:** For $t\in\mathbb{N}^0$ and $(x_t, u, y)\in\Omega_t$, $u = x_t$ implies $y = \min \{b, c_t\} = c_t$.

The right-hand derivative of the function $y|\rightarrow (p_2^{(t+1)}+ p_1^{(t+1)})y - p_2^{(t)}(\frac{y-c_t}{(d_t)^2})^2$ at $y = c_t$ is $(p_2^{(t+1)}+ p_1^{(t+1)})$.

If u is decreased slightly from its initial value of $x_t$, then the value of y goes up slightly leading to an increase in the value of the function, if $(p_2^{(t+1)}+ p_1^{(t+1)}) > 0$.

Thus, assuming $(p_2^{(t+1)}+ p_1^{(t+1)}) > 0$ for all $t\in\mathbb{N}^0$ may be consistent with the premises of proposition 7.1, so that the proposition is not "vacuously true", i.e., a trajectory satisfying the premises in the statement of proposition 7.1 exists.

**Note 7.2:** Proposition 7.1 is a characterization of a "specific type" of solution provided such a solution exists. It does not provide us with enough information to define optimal decision rules, which are functions on X.

## 8. Linear Optimal Control with Linear and Quadratic Constraints:

Yet another example of a linear optimal control problem that can be relevant for intertemporal economics is the following.

For all $t\in\mathbb{N}^0$, there exists a pair of real number $(a_t, b_t)$ satisfying $a_t \geq 0$ and $a_t + b_t b \geq 0$ and an ordered triplet of non-negative real numbers $(c_t, d_t, e_t)$ satisfying $d_t > 0$, $e_t > 0$ and $c_t < b \leq \frac{d_t}{2e_t}$, such that for all x∈X, $\Omega_t(x)$ = {(u, y)| u∈[0, min {x , $a_t + b_t x$}] and y = min {b, $c_t + d_t(x- u) - e_t(x-u)^2$}}.

Since for all t∈$\mathbb{N}^0$, $c_t < b$ it must be the case that for all x∈X, [u∈{x - $\frac{d_t}{e_t}$, x} implies y = min {b, $c_t$} = $c_t$.

Note that the graph of the function $\xi \mapsto c_t + d_t\xi - e_t\xi^2$ on $\mathbb{R}_+$ is an inverted parabola that assumes the value $c_t$ at ξ = 0, attains its maximum value $c_t + \frac{(d_t)^2}{4e_t} > 0$ at $\xi = \frac{d_t}{2e_t} < \frac{d_t + \sqrt[2]{(d_t)^2+4e_tc_t}}{2e_t}$ and assumes the value zero at $\xi = \frac{d_t + \sqrt[2]{(d_t)^2+4e_tc_t}}{2e_t}$. The maximum value of the function on $\mathbb{R}_+$ is strictly greater than $c_t$.

If y < b, then it must be the case that $y = c_t + d_t(x- u) - e_t(x-u)^2$.

A LOC problem <(($p_1^{(t)}$, $p_2^{(t)}$), $\Omega_t$)| t∈$\mathbb{N}^0$> is said to be a **linear optimal control problem with linear and quadratic constraints (LOC-LQC)** if all (x, t)∈X×$\mathbb{N}^0$, $\Omega_t(x)$ = {(u, y)| u∈[0, min {x , $a_t + b_t x$}] and y = min {b, $c_t + d_t(x- u) - e_t(x-u)^2$}}.

**Note 8.1:** For a LOC-LQC if <($x_t$, $u_t$)|t∈$\mathbb{N}^0$>∈$\mathcal{F}$(x), then $x_{t+1} = c_t + d_t(x_t - u_t) - e_t(x_t-u_t)^2$ if $x_{t+1} < b$.

Further, $x_{t+1}$∈[$c_t$, b]. If $x_{t+1} = c_t$ then $u_t$∈{$x_t$, $x_t - \frac{d_t}{e_t}$} for all t∈$\mathbb{N}^0$. However, $u_t \le x_t \le b \le \frac{d_t}{2e_t} < \frac{d_t}{e_t}$ implies $u_t = x_t - \frac{d_t}{e_t} < 0$, which is not possible since $u_t \ge 0$. Thus, if $x_{t+1} = c_t$ then $u_t = x_t$.

Also, note that on the set of {u∈[0, $x_t$]| $c_t + d_t(x_t - u) - e_t(x_t-u)^2 < b$}, the function $u \mapsto c_t + d_t(x_t- u) - e_t(x_t-u)^2$ is a strictly increasing and strictly concave function of u. The fact that it is strictly increasing function of u on {u∈[0, $x_t$]| $c_t + d_t(x_t - u) - e_t(x_t-u)^2 < b$} implies that if $y = c_t + d_t(x_t - u) - e_t(x_t-u)^2$ for any u in this set, then $u = x_t - \frac{d_t + \sqrt[2]{(d_t)^2+4e_t(c_t-y)}}{2e_t}$, i.e., $u \neq x_t - \frac{d_t - \sqrt[2]{(d_t)^2+4e_t(c_t-y)}}{2e_t}$.

**Proposition 8.1:** Suppose <(($p_1^{(t)}$, $p_2^{(t)}$), $\Omega_t$)|t∈$\mathbb{N}^0$> is an LOC-LQC with $a_t = 0$, $b_t = 1$, $p_1^{(t+1)} + p_2^{(t+1)} > 0$ and $(d_t)^2 + 4e_t(c_t - b) > 0$, for all t∈$\mathbb{N}^0$. Suppose x∈X and <($x_t$, $u_t$)|t∈$\mathbb{N}^0$>∈$\mathcal{S}$(x) satisfies $0 < u_t < x_t < b$ for all t∈$\mathbb{N}^0$. Then, for all t∈$\mathbb{N}^0$, $u_t = x_t - \frac{\left(p_1^{(t+1)} + p_2^{(t+1)}\right)d_t - p_2^{(t)}}{2(p_1^{(t+1)} + p_2^{(t+1)})e_t}$

**Proof:** Under the conditions of proposition 8.1 and the assumption that <($x_t$, $u_t$)|t∈$\mathbb{N}^0$>∈$\mathcal{S}$(x), it follows that <($y_t$, $v_t$)|t∈$\mathbb{N}^0$>∈$\mathcal{F}$(x), where $y_\tau = x_\tau$ for all τ ≠ t+1, $v_\tau = u_\tau$ for all τ ∉ {t, t+1}, $y_{t+1} = c_t + d_t(x_t - v_t) - e_t(x_t-v_t)^2$, $x_{t+2} = c_t + d_t(y_{t+1} - v_{t+1}) - e_t(y_{t+1}-v_{t+1})^2$ so long as $v_t$, belongs to a sufficiently small interval of $u_t$, so that $y_{t+1}$ remains in a sufficiently small neighborhood of $x_{t+1}$ and $y_{t+1} < b$.

$x_{t+2} = c_t + d_t(y_{t+1} - v_{t+1}) - e_t(y_{t+1}-v_{t+1})^2$ implies $v_{t+1} = y_{t+1} - \frac{d_{t+1} + \sqrt[2]{(d_{t+1})^2+4e_{t+1}(c_{t+1}-x_{t+2})}}{2e_{t+1}}$.

Hence, if $v_t$ remains in a sufficiently small neighborhood of $u_t$, $v_{t+1}$ remains close to $u_{t+1}$ and satisfies $0 < v_{t+1} < y_{t+1}$.

Thus, for all $t \in \mathbb{N}^0$: ($u_t$, $x_{t+1}$) is a local maximizer of the optimization problem: Maximize $p_2^{(t)}u + (p_1^{(t+1)} + p_2^{(t+1)})y$, subject to $y = c_t + d_t(x_t - u) - e_t(x_t - u)^2$.

Hence, for all $t \in \mathbb{N}^0$: $u_t$ is a local maximizer of the optimization problem: Maximize $p_2^{(t)}u + (p_1^{(t+1)} + p_2^{(t+1)})(c_t + d_t(x_t - u) - e_t(x_t - u)^2)$ which in turn is equivalent to the statement, [for all $t \in \mathbb{N}^0$: $u_t$ is a local maximizer of the optimization problem: Maximize $p_2^{(t)}u + (p_1^{(t+1)} + p_2^{(t+1)})(d_t(x_t - u) - e_t(x_t - u)^2)$].

Since we have assumed $p_1^{(t+1)} + p_2^{(t+1)} > 0$, the function $u \mapsto p_2^{(t)}u + (p_1^{(t+1)} + p_2^{(t+1)})(d_t(x_t - u) - e_t(x_t - u)^2)$ on $[c_t, b]$ is continuously differentiable and locally strictly concave in a neighborhood of $u_t$. Thus, its derivative at $u = u_t$ must be zero.

The derivative of the function $u \mapsto p_2^{(t)}u + (p_1^{(t+1)} + p_2^{(t+1)})(c_t + d_t(x_t - u) - e_t(x_t - u)^2)$ in a small interval around $u_t$ is $p_2^{(t)} - (p_1^{(t+1)} + p_2^{(t+1)})d_t + 2(p_1^{(t+1)} + p_2^{(t+1)})e_t(x_t - u)$

Thus, $p_2^{(t)} - (p_1^{(t+1)} + p_2^{(t+1)})d_t + 2(p_1^{(t+1)} + p_2^{(t+1)})e_t(x_t - u_t) = 0$ implies $x_t - u_t = \frac{(p_1^{(t+1)} + p_2^{(t+1)})d_t - p_2^{(t)}}{2(p_1^{(t+1)} + p_2^{(t+1)})e_t}$ so that $u_t = x_t - \frac{(p_1^{(t+1)} + p_2^{(t+1)})d_t - p_2^{(t)}}{2(p_1^{(t+1)} + p_2^{(t+1)})e_t}$.

Hence $x_{t+1} = c_t + d_t\left(\frac{(p_1^{(t+1)} + p_2^{(t+1)})d_t - p_2^{(t)}}{2(p_1^{(t+1)} + p_2^{(t+1)})e_t}\right) - e_t\left(\frac{(p_1^{(t+1)} + p_2^{(t+1)})d_t - p_2^{(t)}}{2(p_1^{(t+1)} + p_2^{(t+1)})e_t}\right)^2$. Q.E.D.

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