Review for Functorial Number, Functorial Graph, and Functorial Structures

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Abstract

A *Functorial Set* assigns to each object of a category a set, while morphisms induce structure-preserving maps between these sets. A *Functorial Structure* is a covariant functor into **Set**, where the functor-assigned elements are treated as structured objects equipped with natural pushforwards [1, 2]. In this paper, we investigate the notions of *Functorial Number*, *Functorial Graph*, and *Functorial HyperGraph*. A *Functorial Number* is defined as a functor that assigns to each object of a category a semiring, with morphisms inducing structure-preserving homomorphisms. A *Functorial Graph* is a covariant functor that assigns graphs to objects and graph homomorphisms to morphisms, preserving identities and compositions. A *Functorial HyperGraph* is a covariant functor that assigns hypergraphs to objects and hypergraph homomorphisms to morphisms, functorially preserving the structural relationships throughout.

Keywords: Functorial Structure, Functorial Number, Functorial Set, Functorial Graph

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1 Preliminaries

This section presents the fundamental concepts and definitions that form the basis of the discussions in this paper. Throughout, we assume that all concepts in this work are finite. Graphs are assumed to be undirected and simple. The empty set is regarded as a subset of every set.

1.1 Functorial Structure

Here, the term *Structure* refers to any mathematical or real-life concept. A *Functorial Set* assigns to each object in a category a set, with morphisms inducing structure-preserving maps between sets. A *Functorial Structure* is simply a covariant functor into **Set**, treating the functor-assigned elements as structured objects equipped with natural pushforwards.

Definition 1.1 (Functorial Set)**.** Let C be a category and

$$F : C \longrightarrow \mathbf{Set}$$

be a (covariant) endofunctor. For any object $X \in \mathrm{Ob}(C)$, an *F-set over X* is an element

$$s \in F(X).$$

We denote the collection of all F-sets over X simply by $F(X)$. A morphism $f : X \to Y$ in C induces a *pushforward*

$$F(f) : F(X) \longrightarrow F(Y), \qquad s \mapsto F(f)(s).$$

Example 1.2 (Shopping lists as a Functorial Set (List functor, real life)). Let C = **Set** and let F : **Set** → **Set** be the *list functor*:

$$F(X) = \{\, [x_1, \dots, x_k] \mid k \geq 0,\ x_i \in X \,\}, \qquad F(f)([x_1, \dots, x_k]) = [f(x_1), \dots, f(x_k)].$$

Scenario. Take a product universe

$$X = \{\text{Milk}, \text{Bread}, \text{Apple}\},$$

and a category set

$$Y = \{\text{Dairy}, \text{Bakery}, \text{Fruit}\},$$

with $f : X \to Y$ given by

$$f(\text{Milk}) = \text{Dairy}, \quad f(\text{Bread}) = \text{Bakery}, \quad f(\text{Apple}) = \text{Fruit}.$$

A concrete F-set over X (an element of $F(X)$) is the shopping list

$$s = [\text{Milk}, \text{Apple}, \text{Apple}].$$

The functorial pushforward is the category list

$$F(f)(s) = [\text{Dairy}, \text{Fruit}, \text{Fruit}] \in F(Y).$$

If we further group store categories by a map $g : Y \to Z = \{\text{Fresh}, \text{Dry}\}$ via

$$g(\text{Dairy}) = \text{Fresh}, \quad g(\text{Fruit}) = \text{Fresh}, \quad g(\text{Bakery}) = \text{Dry},$$

then

$$F(g)\big(F(f)(s)\big) = [\text{Fresh}, \text{Fresh}, \text{Fresh}] \quad \text{and} \quad F(g \circ f)(s) = [\text{Fresh}, \text{Fresh}, \text{Fresh}],$$

illustrating $F(g \circ f) = F(g) \circ F(f)$ in a real shopping workflow.

Example 1.3 (Fuzzy Set as a Functorial Set: neighborhood cleanliness aggregated to districts). Let C = **Set** and define the *fuzzy–powerset functor*

$$F : \mathbf{Set} \longrightarrow \mathbf{Set}, \qquad F(X) = \{\mu : X \to [0, 1]\}.$$

For a function $f : X \to Y$, define the pushforward (Zadeh extension)

$$F(f)(\mu)(y) \; = \; \sup\{\mu(x) \mid f(x) = y\} \qquad (y \in Y).$$

This makes F a (covariant) endofunctor since $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ and $F(g \circ f) = F(g) \circ F(f)$ by properties of the supremum.

Real life. Let $X = \{\text{A}, \text{B}, \text{C}\}$ be city neighborhoods and $Y = \{\text{North}, \text{South}\}$ be districts with the aggregation map

$$f(\text{A}) = \text{North}, \quad f(\text{B}) = \text{North}, \quad f(\text{C}) = \text{South}.$$

Take a fuzzy set (cleanliness score)

$$\mu(\text{A}) = 0.7, \qquad \mu(\text{B}) = 0.4, \qquad \mu(\text{C}) = 0.9.$$

Then the district–level fuzziness is

$$F(f)(\mu)(\text{North}) = \sup\{0.7, 0.4\} = 0.7, \qquad F(f)(\mu)(\text{South}) = \sup\{0.9\} = 0.9.$$

If $g : Y \to Z = \{\text{City}\}$ collapses both districts to a single city, then

$$F(g)\big(F(f)(\mu)\big)(\text{City}) = \sup\{0.7, 0.9\} = 0.9 = F(g \circ f)(\mu)(\text{City}),$$

illustrating functoriality (composition preserved) in this aggregation pipeline.

Example 1.4 (Rough Set as a Functorial Set: promotions under coarser granulation)**.** Let **APS** be the category of *approximation spaces* whose objects are pairs (X, π_X), where π_X is a partition of X (granules), and whose morphisms $f : (X, \pi_X) \to (Y, \pi_Y)$ are functions $f : X \to Y$ *compatible with granules*: there exists a map h between blocks such that

$$\pi_Y \circ f \;=\; h \circ \pi_X,$$

i.e., every π_X–block is mapped inside a single π_Y–block.

For (X, π_X), define the set of *rough subsets*

$$F(X, \pi_X) = \big\{(L, U) \;\big|\; L, U \subseteq X,\; L \subseteq U,\; L, U \text{ are unions of } \pi_X\text{–blocks}\big\}.$$

For a morphism $f : (X, \pi_X) \to (Y, \pi_Y)$ and $(L, U) \in F(X, \pi_X)$, define the pushforward by

$$F(f)(L, U) \;=\; \big(\, \underline{f[L]} \,,\, \overline{f[U]} \,\big),$$

where for $S \subseteq Y$ we write

$$\underline{S} \;=\; \bigcup\{B \in \pi_Y \mid B \subseteq S\} \quad \text{(lower approx)}, \qquad \overline{S} \;=\; \bigcup\{B \in \pi_Y \mid B \cap S \neq \emptyset\} \quad \text{(upper approx)}.$$

Then $F : \mathbf{APS} \to \mathbf{Set}$ is a (covariant) functor: $F(\mathrm{id}) = \mathrm{id}$, and $F(g \circ f) = F(g) \circ F(f)$ follows from image composition and the monotonicity/idempotence of $\underline{(\cdot)}$ and $\overline{(\cdot)}$ with respect to the target partition.

Real life. Let $X = \{\text{Milk, Yogurt, Apple, Bread}\}$ be SKUs. Let π_X be the category partition

$$\text{Dairy} = \{\text{Milk, Yogurt}\}, \quad \text{Produce} = \{\text{Apple}\}, \quad \text{Bakery} = \{\text{Bread}\}.$$

Let π_Y be a coarser department partition on the same underlying set $Y = X$:

$$\text{Fresh} = \{\text{Milk, Yogurt, Apple}\}, \quad \text{Dry} = \{\text{Bread}\}.$$

The identity map $f = \mathrm{id}_X : X \to Y$ is a morphism in **APS** since π_Y coarsens π_X (there exists h merging categories into departments).

Suppose the "promotion" rough set on (X, π_X) is

$$L = \{\text{Milk, Yogurt}\} \quad \text{(definitely on promo)}, \qquad U = \{\text{Milk, Yogurt, Apple}\} \quad \text{(possibly on promo)}.$$

Then

$$f[L] = \{\text{Milk, Yogurt}\}, \quad f[U] = \{\text{Milk, Yogurt, Apple}\},$$

and with respect to π_Y we obtain

$$\underline{f[L]} = \text{Fresh}, \qquad \overline{f[U]} = \text{Fresh}.$$

Hence

$$F(f)(L, U) = \big(\text{Fresh, Fresh}\big) \in F(Y, \pi_Y),$$

meaning the department–level view is "*Fresh is certainly on promotion*," after passing to the coarser granulation. This demonstrates how rough information functorially aggregates under granularity–respecting maps.

Definition 1.5 (Functorial Structure)**.** Let C be a category. A *Functorial Structure* on C is simply a covariant functor

$$F \,:\, C \;\longrightarrow\; \mathbf{Set}.$$

For each object $X \in \mathrm{Ob}(C)$, an element

$$s \;\in\; F(X)$$

is called an *F-structure on X*. Every morphism $f \colon X \to Y$ in C induces a *pushforward*

$$F(f) \colon F(X) \;\longrightarrow\; F(Y), \qquad s \;\longmapsto\; F(f)(s),$$

and the usual functoriality conditions $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ and $F(g \circ f) = F(g) \circ F(f)$ hold.

Example 1.6 (Audience preferences as a Functorial Structure (distribution functor, real life))**.** Let C = **Set** and define F : **Set** → **Set** by

$$F(X) = \Big\{ p : X \to [0,1] \ \Big| \ \sum_{x \in X} p(x) = 1 \Big\},$$

the set of discrete probability distributions on X. For $f : X \to Y$ and $p \in F(X)$, define the pushforward distribution

$$F(f)(p)(y) = \sum_{x \in f^{-1}(y)} p(x) \qquad (y \in Y).$$

Scenario. Let the fine topics be

$$X = \{\text{World}, \ \text{Sports}, \ \text{Tech}, \ \text{Movies}\},$$

and broader sections

$$Y = \{\text{News}, \ \text{Entertainment}\},$$

with f given by

$$f(\text{World}) = \text{News}, \quad f(\text{Tech}) = \text{News}, \quad f(\text{Sports}) = \text{Entertainment}, \quad f(\text{Movies}) = \text{Entertainment}.$$

Take a concrete user preference distribution $p \in F(X)$:

$$p(\text{World}) = 0.40, \quad p(\text{Sports}) = 0.20, \quad p(\text{Tech}) = 0.30, \quad p(\text{Movies}) = 0.10,$$

so $\sum p = 1.00$. The pushforward (aggregated sections) is

$$F(f)(p)(\text{News}) = p(\text{World}) + p(\text{Tech}) = 0.40 + 0.30 = 0.70,$$
$$F(f)(p)(\text{Entertainment}) = p(\text{Sports}) + p(\text{Movies}) = 0.20 + 0.10 = 0.30,$$

which again sums to 1.00. If $g : Y \to Z = \{\text{All}\}$ collapses both sections to a single bin, then

$$F(g)\big(F(f)(p)\big)(\text{All}) = 0.70 + 0.30 = 1.00 \quad \text{and} \quad F(g \circ f)(p)(\text{All}) = \sum_{x \in X} p(x) = 1.00,$$

exhibiting functoriality (composition preserved) for this real aggregation pipeline.

2 Main Results

In this section, we present the main results of the paper by examining several kinds of structures.

2.1 Functorial Number

We write **CRig** for the category of commutative semirings (a.k.a. commutative rigs) and homomorphisms.

Definition 2.1 (Functorial Number)**.** Let C be a category with finite products and terminal object 1. A *Functorial Number on* C is a tuple

$$\big(N, \ \oplus, \ \otimes, \ 0, \ 1\big)$$

where $N : C \to$ **Set** is a functor, $\oplus, \otimes$ are natural transformations

$$\oplus : \ N \times N \Longrightarrow N, \qquad \otimes : \ N \times N \Longrightarrow N,$$

and 0, 1 are natural transformations from the constant functor $\Delta_1 : C \to$ **Set** (constantly the singleton set) to N,

$$0 : \ \Delta_1 \Longrightarrow N, \qquad 1 : \ \Delta_1 \Longrightarrow N,$$

such that for every object $X \in C$ the quintuple

$$\Big(N(X), \ \oplus_X, \ \otimes_X, \ 0_X, \ 1_X\Big)$$

is a commutative semiring, and for every morphism $f : X \to Y$ the following *naturality equalities* hold for all $a, b \in N(X)$:

$$N(f)\big(a \oplus_X b\big) = N(f)(a) \ \oplus_Y \ N(f)(b), \tag{1}$$
$$N(f)\big(a \otimes_X b\big) = N(f)(a) \ \otimes_Y \ N(f)(b), \tag{2}$$
$$N(f)(0_X) = 0_Y, \qquad N(f)(1_X) = 1_Y. \tag{3}$$

Equivalently, $\oplus, \otimes, 0, 1$ make N into a *semiring object* in the functor category $[C, \textbf{Set}]$.

Remark 2.2. Conditions (1)–(3) state that each map $N(f)$ is a homomorphism of commutative semirings between the fibers $N(X) \to N(Y)$.

Example 2.3 (Constant natural numbers)**.** Fix C arbitrary. Define $\widetilde{N}(X) = \mathbb{N}$ for all X, with the standard semiring $(\mathbb{N}, +, \cdot, 0, 1)$, and $\widetilde{N}(f) = \mathrm{id}_{\mathbb{N}}$ for all f. Then $\widetilde{N} : C \to \mathbf{CRig}$ is a functor. By Theorem 2.7, it defines a Functorial Number $N = U \circ \widetilde{N}$, and thus a Functorial Structure constantly equal to $\mathbb{N}$.

Example 2.4 (Free multiset (measure) semiring on a commutative monoid)**.** Let $C = \mathbf{CMon}$ be the category of commutative monoids $(M, +, 0_M)$ and homomorphisms. For each M, let

$$\widetilde{N}(M) \;:=\; \mathbb{N}^{(M)} \;=\; \{\, \mu : M \to \mathbb{N} \text{ with finite support} \,\}.$$

Define addition and multiplication on $\mathbb{N}^{(M)}$ by

$$\begin{aligned}(\mu \oplus_M \nu)(z) &:= \mu(z) + \nu(z) \quad \text{(pointwise)},\\ (\mu \otimes_M \nu)(z) &:= \sum_{x+y=z} \mu(x)\, \nu(y) \quad \text{(Cauchy convolution)}.\end{aligned}$$

Take 0_M to be the zero function and 1_M to be the Dirac mass δ_{0_M}. Then $\left(\mathbb{N}^{(M)}, \oplus_M, \otimes_M, 0_M, 1_M\right)$ is a commutative semiring. For a homomorphism $f : M \to N$, define the *pushforward* (fiber sum)

$$f_!(\mu)(y) \;:=\; \sum_{x \in f^{-1}(y)} \mu(x).$$

One checks directly that $f_!$ preserves $\oplus, \otimes, 0, 1$ (see the proof below), so putting $\widetilde{N}(f) := f_!$ makes $\widetilde{N}$: **CMon** $\to$ **CRig** a functor. By Theorem 2.7, this yields a Functorial Number.

Example 2.5 (Retail POS: cents remainder under cart combination (mod 100))**.** Let the commutative monoid be $(\mathbb{Z}_{\geq 0}, +)$ (amounts in cents) and let $f : \mathbb{Z}_{\geq 0} \to \mathbb{Z}/100\mathbb{Z}$ be the quotient homomorphism $f(k) = [k]$ (cents remainder mod 100). Consider two aisles whose single–item price distributions (finite multisets) are

$$\mu \;=\; \delta_{199} \;+\; 2\,\delta_{250} \;+\; \delta_{499}, \qquad \nu \;=\; 3\,\delta_{150} \;+\; \delta_{349}.$$

The convolution (cart formed by choosing one item from each aisle) is

$$\begin{aligned}\mu \otimes \nu &= 3\,\delta_{199+150} + 1\,\delta_{199+349} + 6\,\delta_{250+150} + 2\,\delta_{250+349} + 3\,\delta_{499+150} + 1\,\delta_{499+349}\\ &= \;3\,\delta_{349} \;+\; 1\,\delta_{548} \;+\; 6\,\delta_{400} \;+\; 2\,\delta_{599} \;+\; 3\,\delta_{649} \;+\; 1\,\delta_{848}.\end{aligned}$$

Pushing forward by f (grouping totals by cents remainder) gives

$$f_!(\mu \otimes \nu) = 6\,\delta_{[49]} \;+\; 2\,\delta_{[48]} \;+\; 6\,\delta_{[0]} \;+\; 2\,\delta_{[99]}.$$

Compute the pushforwards first and then multiply in $\mathbb{Z}/100\mathbb{Z}$:

$$f_!\mu \;=\; 2\,\delta_{[99]} + 2\,\delta_{[50]}, \qquad f_!\nu \;=\; 3\,\delta_{[50]} + 1\,\delta_{[49]}.$$

Hence

$$\begin{aligned}(f_!\mu) \otimes (f_!\nu) &= (2\,\delta_{[99]} + 2\,\delta_{[50]}) \otimes (3\,\delta_{[50]} + \delta_{[49]})\\ &= 6\,\delta_{[99+50]} + 2\,\delta_{[99+49]} + 6\,\delta_{[50+50]} + 2\,\delta_{[50+49]}\\ &= 6\,\delta_{[49]} + 2\,\delta_{[48]} + 6\,\delta_{[0]} + 2\,\delta_{[99]}.\end{aligned}$$

Therefore $f_!(\mu \otimes \nu) = (f_!\mu) \otimes (f_!\nu)$ and similarly $f_!(\mu \oplus \nu) = (f_!\mu) \oplus (f_!\nu)$: "combine carts then take cents remainder" equals "take cents remainder per aisle then combine," an explicit verification of the functorial/ semiring homomorphism law in Theorem 2.7.

Example 2.6 (Music analytics: pitch-class profiles under transposition (mod 12))**.** Let the commutative monoid be $(\mathbb{Z}, +)$ (MIDI pitch numbers) and let $f : \mathbb{Z} \to \mathbb{Z}/12\mathbb{Z}$ be the quotient homomorphism $f(p) = [p]$ (pitch class). Let a melody histogram and a transposition/interval distribution be

$$\mu \;=\; \delta_{60} \;+\; 2\,\delta_{64} \;+\; \delta_{67}, \qquad \nu \;=\; \delta_0 \;+\; \delta_7 \;+\; 2\,\delta_{12}.$$

Convolution gives the multiset of transposed notes:

$$\begin{aligned}\mu \otimes \nu &= \delta_{60} + \delta_{67} + 2\,\delta_{72} + 2\,\delta_{64} + 2\,\delta_{71} + 4\,\delta_{76} + \delta_{67} + \delta_{74} + 2\,\delta_{79} \\ &= \delta_{60} + 2\,\delta_{64} + 2\,\delta_{67} + 2\,\delta_{71} + 2\,\delta_{72} + \delta_{74} + 4\,\delta_{76} + 2\,\delta_{79}.\end{aligned}$$

Push forward by f to pitch-class counts:

$$f_!(\mu \otimes \nu) \;=\; 3\,\delta_{[0]} \;+\; 1\,\delta_{[2]} \;+\; 6\,\delta_{[4]} \;+\; 4\,\delta_{[7]} \;+\; 2\,\delta_{[11]}.$$

Compute $f_!\mu$ and $f_!\nu$ first:

$$f_!\mu \;=\; \delta_{[0]} + 2\,\delta_{[4]} + \delta_{[7]}, \qquad f_!\nu \;=\; 3\,\delta_{[0]} + \delta_{[7]}.$$

Then convolve in $\mathbb{Z}/12\mathbb{Z}$:

$$\begin{aligned}(f_!\mu) \otimes (f_!\nu) &= (\delta_{[0]} + 2\,\delta_{[4]} + \delta_{[7]}) \otimes (3\,\delta_{[0]} + \delta_{[7]}) \\ &= 3\,\delta_{[0]} + 6\,\delta_{[4]} + 3\,\delta_{[7]} + \delta_{[7]} + 2\,\delta_{[11]} + \delta_{[2]} \\ &= 3\,\delta_{[0]} + 1\,\delta_{[2]} + 6\,\delta_{[4]} + 4\,\delta_{[7]} + 2\,\delta_{[11]}.\end{aligned}$$

Thus $f_!(\mu \otimes \nu) = (f_!\mu) \otimes (f_!\nu)$, i.e. "transpose then collect pitch classes" equals "collect pitch classes then combine via circular convolution," a concrete instance of Functorial Number behavior (pushforward as semiring homomorphism).

Theorem 2.7 (Representation)**.** *Let C be a category with finite products. There is a bijective correspondence*

$$\Big\{\ \textit{Functorial Numbers } (N, \oplus, \otimes, 0, 1)\ \Big\} \;\longleftrightarrow\; \Big\{\ \textit{Functors } \widetilde{N} : C \to \mathbf{CRig}\ \Big\}$$

given by $\widetilde{N}(X) := (N(X), \oplus_X, \otimes_X, 0_X, 1_X)$ on objects and $\widetilde{N}(f) := N(f)$ on arrows. In particular, composing with the forgetful functor $U : \mathbf{CRig} \to \mathbf{Set}$ yields a Functorial Structure $U \circ \widetilde{N} : C \to \mathbf{Set}$.

Proof. (Forward direction). Start from a Functorial Number $(N, \oplus, \otimes, 0, 1)$. For each object X, define the commutative semiring

$$\widetilde{N}(X) \;:=\; \big(N(X), \oplus_X, \otimes_X, 0_X, 1_X\big).$$

For a morphism $f : X \to Y$, define $\widetilde{N}(f) := N(f) : N(X) \to N(Y)$. By naturality (1)–(3), for all $a, b \in N(X)$,

$$\widetilde{N}(f)(a \oplus_X b) = \widetilde{N}(f)(a) \oplus_Y \widetilde{N}(f)(b), \quad \widetilde{N}(f)(a \otimes_X b) = \widetilde{N}(f)(a) \otimes_Y \widetilde{N}(f)(b),$$

and $\widetilde{N}(f)(0_X) = 0_Y$, $\widetilde{N}(f)(1_X) = 1_Y$. Hence $\widetilde{N}(f)$ is a semiring homomorphism. Functoriality $\widetilde{N}(\mathrm{id}) = \mathrm{id}$ and $\widetilde{N}(g \circ f) = \widetilde{N}(g) \circ \widetilde{N}(f)$ follows from the underlying functoriality of N. Therefore $\widetilde{N} : C \to \mathbf{CRig}$ is a well-defined functor.

(Backward direction). Start from a functor $\widetilde{N} : C \to \mathbf{CRig}$. Put $N := U \circ \widetilde{N} : C \to \mathbf{Set}$, where U forgets the semiring structure. For each X, write

$$\widetilde{N}(X) = (\ |N(X)|,\ \oplus_X,\ \otimes_X,\ 0_X,\ 1_X\ ),$$

and define $\oplus, \otimes, 0, 1$ componentwise to obtain natural transformations $N \times N \Rightarrow N$ and $\Delta_1 \Rightarrow N$. For a morphism $f : X \to Y$, $\widetilde{N}(f)$ is a semiring homomorphism, so (1)–(3) hold by definition. Thus $(N, \oplus, \otimes, 0, 1)$ is a Functorial Number.

(Bijection). The two constructions are inverse to one another by inspection on objects and arrows; uniqueness is forced by the identities $U \circ \widetilde{N} = N$ and the requirement that the components of $\oplus, \otimes, 0, 1$ coincide with those in **CRig**. □

2.2 Functorial Graph and Functorial HyperGraph

A graph consists of vertices and edges connecting pairs of vertices, modeling pairwise relationships, networks, and connectivity patterns in applications [3, 4]. A hypergraph generalizes graphs by allowing hyperedges to connect arbitrary nonempty vertex subsets, modeling group interactions, higher-order relations, and dependencies [5–9]. A Functorial Graph is a covariant functor assigning graphs to objects and graph homomorphisms to morphisms, preserving identities and compositions. A Functorial HyperGraph is a covariant functor assigning hypergraphs to objects and hypergraph homomorphisms to morphisms, preserving structure functorially throughout.

Definition 2.8 (Functorial Graph)**.** Let $\mathcal{C}$ be a category and let **Graph** denote the category whose objects are finite, simple, undirected graphs $G = (V, E)$ with $E \subseteq \{\{u, v\} \mid u, v \in V,\ u \neq v\}$, and whose morphisms $\varphi : G \to G'$ are vertex maps $\varphi : V \to V'$ such that $\{u, v\} \in E \Rightarrow \{\varphi(u), \varphi(v)\} \in E'$. A *Functorial Graph on* $\mathcal{C}$ is a covariant functor

$$\mathsf{G} : \mathcal{C} \longrightarrow \mathbf{Graph}.$$

Equivalently, to each object X it assigns a graph $\mathsf{G}(X) = (V_X, E_X)$, and to each morphism $f : X \to Y$ a graph homomorphism $\mathsf{G}(f) : \mathsf{G}(X) \to \mathsf{G}(Y)$, with $\mathsf{G}(\mathrm{id}) = \mathrm{id}$ and $\mathsf{G}(g \circ f) = \mathsf{G}(g) \circ \mathsf{G}(f)$.

Example 2.9 (Company call network via aggregation (complete graph functor))**.** Let $\mathcal{C} = \mathbf{Set}$ and define $K : \mathbf{Set} \to \mathbf{Graph}$ by

$$K(X) := \big(X,\ \{\{x, x'\} \subseteq X \mid x \neq x'\}\big),$$

the complete graph on X; for $f : X \to Y$, let $K(f)$ be the vertex map f, which is a graph homomorphism since every pair in $K(Y)$ is adjacent.

Concretely, let $X = \{\text{Alice}, \text{Bob}, \text{Carol}, \text{Dave}\}$ (employees) and $Y = \{\text{Eng}, \text{Sales}\}$ (departments) with

$$f(\text{Alice}) = f(\text{Bob}) = \text{Eng}, \qquad f(\text{Carol}) = f(\text{Dave}) = \text{Sales}.$$

In $K(X)$ the edge $\{\text{Alice}, \text{Carol}\}$ exists; under $K(f)$ it maps to $\{\text{Eng}, \text{Sales}\}$, which is an edge in the complete graph $K(Y)$. Thus "everyone-can-call-everyone" on people functorially pushes forward to "every-department-can-call-every-department."

Definition 2.10 (Functorial HyperGraph)**.** Let **HGraph** denote the category whose objects are finite hypergraphs $H = (V, E)$ with $E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$ and whose morphisms $\psi : H \to H'$ are vertex maps $\psi : V \to V'$ satisfying

$$\forall e \in E :\ \psi[e] \in E' \quad \text{(image of each hyperedge is a hyperedge)}.$$

A *Functorial HyperGraph on* $\mathcal{C}$ is a covariant functor

$$\mathsf{H} : \mathcal{C} \longrightarrow \mathbf{HGraph}.$$

Equivalently, it assigns to each X a hypergraph $\mathsf{H}(X) = (V_X, E_X)$ and to each $f : X \to Y$ a hypergraph homomorphism $\mathsf{H}(f)$, functorially.

Example 2.11 (Ad-hoc group chats under region bucketing (complete hypergraph functor))**.** Let $\mathcal{C} = \mathbf{Set}$ and define the *complete hypergraph functor* $\mathcal{K}^{\mathrm{hyp}} : \mathbf{Set} \to \mathbf{HGraph}$ by

$$\mathcal{K}^{\mathrm{hyp}}(X) := \big(X,\ \mathcal{P}(X) \setminus \{\emptyset\}\big).$$

For $f : X \to Y$, the vertex map f is a hypergraph homomorphism because for any $e \subseteq X$, nonempty, its image $f[e] \subseteq Y$ is nonempty and hence a hyperedge of $\mathcal{K}^{\mathrm{hyp}}(Y)$.

Real life: let $X = \{\text{Alice}, \text{Bob}, \text{Chen}, \text{Deepa}\}$ be users and $Y = \{\text{APAC}, \text{EMEA}\}$ be regions with

$$f(\text{Alice}) = \text{EMEA}, \quad f(\text{Bob}) = \text{EMEA}, \quad f(\text{Chen}) = \text{APAC}, \quad f(\text{Deepa}) = \text{APAC}.$$

Consider the ad-hoc group chat hyperedge $e = \{\text{Alice}, \text{Chen}, \text{Deepa}\} \in \mathcal{P}(X) \setminus \{\emptyset\}$. Under $\mathcal{K}^{\mathrm{hyp}}(f)$ we have

$$f[e] = \{\text{EMEA}, \text{APAC}\} \in \mathcal{P}(Y) \setminus \{\emptyset\},$$

which is a valid hyperedge in $\mathcal{K}^{\mathrm{hyp}}(Y)$. Thus "any nonempty chat group of people" functorially aggregates to "the set of involved regions," preserving hyperedge incidence.

Example 2.12 (Meeting invitations under department bucketing (k-bounded hypergraph functor)). Let C = **Set** and fix $k \in \mathbb{N}$ with $k \geq 2$. Define the functor

$$\mathsf{H}_k : \mathbf{Set} \longrightarrow \mathbf{HGraph}, \qquad \mathsf{H}_k(X) := \big(X,\ E_X\big),$$

where

$$E_X := \{\, e \subseteq X \mid 1 \leq |e| \leq k \,\}.$$

For a function $f : X \to Y$, set $\mathsf{H}_k(f) : \mathsf{H}_k(X) \to \mathsf{H}_k(Y)$ to be the vertex map f. If $e \in E_X$ with $1 \leq |e| \leq k$, then $f[e] \neq \emptyset$ and

$$|f[e]| \ \leq\ |e| \ \leq\ k,$$

hence $f[e] \in E_Y$; thus $\mathsf{H}_k(f)$ is a hypergraph homomorphism.

Real life. Let $X = \{\text{Alice}, \text{Bob}, \text{Chen}, \text{Deepa}, \text{Elena}\}$ (employees) and $Y = \{\text{Eng}, \text{Sales}, \text{Ops}\}$ (departments) with

$$f(\text{Alice}) = \text{Eng}, \quad f(\text{Bob}) = \text{Eng}, \quad f(\text{Chen}) = \text{Sales}, \quad f(\text{Deepa}) = \text{Ops}, \quad f(\text{Elena}) = \text{Sales}.$$

Take $k = 3$ so meetings (hyperedges) have size ≤ 3. Consider two invites:

$$e_1 = \{\text{Alice}, \text{Chen}\}, \qquad e_2 = \{\text{Bob}, \text{Chen}, \text{Deepa}\}.$$

Then

$$f[e_1] = \{\text{Eng}, \text{Sales}\} \ \in E_Y, \qquad f[e_2] = \{\text{Eng}, \text{Sales}, \text{Ops}\} \ \in E_Y,$$

with $1 \leq |f[e_i]| \leq 3$. Hence "invite by people" functorially aggregates to "invite by departments," preserving the hyperedge constraint $|e| \leq k$.

Example 2.13 (Service coverage from road networks (neighborhood hypergraph functor on graphs)). Let C = **Graph** (finite, simple, undirected). For $G = (V, E)$ and $v \in V$, write the closed neighborhood $N_G[v] = \{v\} \cup \{u \in V \mid \{u, v\} \in E\}$. Define a functor

$$\mathsf{N} : \mathbf{Graph} \longrightarrow \mathbf{HGraph}, \qquad \mathsf{N}(G) := \big(V,\ E_G\big),$$

where

$$E_G := \{\, S \subseteq N_G[v] \ \mid\ v \in V,\ S \neq \emptyset \,\}.$$

If $\varphi : G \to H$ is a graph homomorphism and $S \subseteq N_G[v]$, then every $x \in S$ satisfies either $x = v$ or $\{x, v\} \in E(G)$; applying φ yields either $\varphi(x) = \varphi(v)$ or $\{\varphi(x), \varphi(v)\} \in E(H)$, hence

$$\varphi[S] \ \subseteq\ N_H[\varphi(v)], \qquad \varphi[S] \neq \emptyset,$$

so $\varphi[S] \in E_H$. Therefore $\mathsf{N}(\varphi)$ is a hypergraph homomorphism.

Real life. Intersections V with roads E form a city road graph G. A service unit stationed at v covers any nonempty subset $S \subseteq N_G[v]$ (e.g., emergency reach set). Aggregate intersections to districts via a graph homomorphism $\varphi : G \to H$ (contract intersections to district centroids and keep inter-district roads). For the concrete graph

$$V = \{A, B, C, D\}, \quad E = \big\{\{A, B\}, \{A, C\}, \{C, D\}\big\},$$

take $S_1 = \{A, B\} \subseteq N_G[A] = \{A, B, C\}$ and $S_2 = \{C, D\} \subseteq N_G[C] = \{A, C, D\}$. Let H have vertices $\{D_1, D_2\}$ with an edge $\{D_1, D_2\}$, and define

$$\varphi(A) = \varphi(B) = \varphi(C) = D_1, \qquad \varphi(D) = D_2.$$

Then

$$\varphi[S_1] = \{D_1\} \ \subseteq\ N_H[D_1] = \{D_1, D_2\}, \qquad \varphi[S_2] = \{D_1, D_2\} \ =\ N_H[D_1],$$

so both images are hyperedges in $\mathsf{N}(H)$. Thus "intersection-level coverage sets" functorially map to "district-level coverage sets," preserving hyperedge incidence under network coarse-graining.

3 Conclusion

In this paper, we investigated the notions of *Functorial Number*, *Functorial Graph*, and *Functorial HyperGraph*. For future work, we aim to explore extensions that incorporate *Weak HyperStructures* [10–13] and *SuperHyperStructures* [14–17].

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Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Research Integrity

The authors hereby confirm that, to the best of their knowledge, this manuscript is their original work, has not been published in any other journal, and is not currently under consideration for publication elsewhere at this stage.

Use of Generative AI and AI-Assisted Tools

I use generative AI and AI-assisted tools for tasks such as English grammar checking, and I do not employ them in any way that violates ethical standards.

Disclaimer (Note on Computational Tools)

No computer-assisted proof, symbolic computation, or automated theorem proving tools (e.g., Mathematica, SageMath, Coq, etc.) were used in the development or verification of the results presented in this paper. All proofs and derivations were carried out manually and analytically by the authors.

Code Availability

No code or software was developed for this study.

Clinical Trial

This study did not involve any clinical trials.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

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