Hyperfuzzy and SuperHyperfuzzy Integral: A method of studying the Fuzzy Integral under hierarchical uncertainty

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Abstract

Uncertainty modeling is fundamental to decision-making across diverse domains, and numerous frameworks—such as Fuzzy Sets [1, 2], Rough Sets [3, 4], Hesitant Fuzzy Sets [5, 6], Neutrosophic Sets [7, 8], and Plithogenic Sets [9, 10]—have been developed to capture different facets of imprecision. Among these, Hyperfuzzy Sets and their recursive generalization, SuperHyperfuzzy Sets, assign set-valued membership degrees at multiple hierarchical levels to represent uncertainty more richly [11].

The Fuzzy Integral provides a systematic method to aggregate single-valued fuzzy membership functions with respect to a fuzzy measure, thereby yielding a precise evaluation of overall set importance. In this paper, we extend this framework by introducing the *Hyperfuzzy Integral* and the *SuperHyperfuzzy Integral*, defined over Hyperfuzzy Sets and SuperHyperfuzzy Sets, respectively, and we investigate their fundamental properties.

Keywords: Fuzzy set, HyperFuzzy Set, SuperHyperFuzzy Set, Hyperfuzzy Integral, SuperHyperfuzzy Integral

Structure of this paper

The format of this paper is described below.

1 Preliminaries

In this section we assemble the notation and core concepts used throughout the paper. Unless explicitly stated otherwise, all underlying sets are assumed to be finite.

1.1 Basic Set Constructions

A *fuzzy set* assigns to each element a membership degree in the unit interval, thereby permitting graded (rather than purely binary) inclusion [1, 12–14]. A *hyperfuzzy set* refines this idea by mapping each element to a *nonempty set* of values in $[0, 1]$, which captures variability or imprecision in the membership assessment [15–19]. More generally, an (m, n)-*superhyperfuzzy set* assigns to every m-level subset a family of n-level membership collections, providing a hierarchical representation of uncertainty (cf. [20–22]).

Definition 1.1 (Universe)**.** Let U be a nonempty finite set, called the *universe* or *base set*. All subsequent constructions (powersets, hyperstructures, etc.) are formed from U.

Definition 1.2 (Powerset)**.** The *powerset* of U is

$$\mathcal{P}(U) \;=\; \{\, A \mid A \subseteq U \,\}.$$

Example 1.3 (Powerset: Everyday items)**.** Let $U = \{\text{wallet, keys, phone}\}$. Then $\mathcal{P}(U)$ lists every bundle one might carry, e.g. $\{\text{wallet, keys}\}$ or $\{\text{phone}\}$. This encodes all real-world combinations of these items.

Definition 1.4 (n-fold Powerset)**.** (cf. [23–27]) For $n \geq 1$, the iterated powerset is defined by

$$\mathcal{P}^1(U) = \mathcal{P}(U), \qquad \mathcal{P}^{n+1}(U) = \mathcal{P}\big(\mathcal{P}^n(U)\big).$$

If one wishes to exclude the empty set at each level, replace $\mathcal{P}$ with $\mathcal{P}^*(X) = \mathcal{P}(X) \setminus \{\emptyset\}$.

Example 1.5 (2-fold Powerset: Packing policies)**.** Let $U = \{\text{Passport, Card, Phone}\}$. Then $\mathcal{P}(U)$ lists all single trip checklists. An element of the 2-fold powerset $\mathcal{P}^2(U) = \mathcal{P}(\mathcal{P}(U))$ is a *policy* collecting acceptable checklists for different scenarios; e.g.

$$X = \big\{\{\text{Passport, Card}\},\ \{\text{Passport, Card, Phone}\}\big\} \in \mathcal{P}^2(U).$$

Interpretation: either checklist in X is acceptable (e.g., domestic vs. international travel).

Example 1.6 (3-fold Powerset: University curriculum catalog)**.** Let $U = \{\text{Math, CS, Econ}\}$. Then $\mathcal{P}(U)$ are semester course bundles; $\mathcal{P}^2(U)$ are *degree plans* (sets of allowed semester bundles). An element of the 3-fold powerset $\mathcal{P}^3(U) = \mathcal{P}(\mathcal{P}^2(U))$ is a *catalog* collecting multiple degree plans; for instance

$$Y = \Big\{\{\{\text{Math, CS}\},\ \{\text{CS, Econ}\}\},\ \{\{\text{Math}\},\ \{\text{Econ}\}\}\Big\} \in \mathcal{P}^3(U).$$

Interpretation: Y groups two alternative degree plans, each plan being a set of permissible semester bundles.

Definition 1.7 (Fuzzy Set)**.** [1, 14] A *fuzzy set* F on U is a function

$$\mu_F : U \to [0, 1],$$

where $\mu_F(x)$ is the *membership degree* of $x \in U$.

Definition 1.8 (Fuzzy Relation)**.** [28, 29] Given a fuzzy set F on U, a *fuzzy relation* on U is a fuzzy subset of $U \times U$, i.e. a map

$$R : U \times U \to [0, 1],$$

satisfying

$$R(x, y) \ \leq\ \min\{\mu_F(x),\ \mu_F(y)\} \qquad \text{for all } x, y \in U.$$

Definition 1.9 (Hyperfuzzy Set)**.** [15, 30, 31] A *hyperfuzzy set* $\tilde{F}$ on U is given by

$$\tilde{\mu} : U \ \longrightarrow\ \mathcal{P}([0, 1]) \setminus \{\emptyset\},$$

where for each $x \in U$, the set $\tilde{\mu}(x) \subseteq [0, 1]$ represents the *possible membership grades* of x.

Example 1.10 (Hyperfuzzy Set: Medical Diagnosis)**.** Let $U = \{\text{fever, cough, fatigue}\}$. For each symptom $x \in U$, the hyperfuzzy membership $\tilde{\mu}(x)$ assigns not a single degree but a *set* of possible severity levels in $[0, 1]$. For instance,

$$\tilde{\mu}(\text{fever}) = \{0.6, 0.8\}, \quad \tilde{\mu}(\text{cough}) = [0.4, 0.7], \quad \tilde{\mu}(\text{fatigue}) = \{0.3, 0.5, 0.9\}.$$

This captures the variability or uncertainty in medical assessment due to different observations or measurement errors.

Definition 1.11 ((m, n)-SuperHyperfuzzy Set)**.** [11, 32, 33] Fix nonnegative integers m, n. Let

$$\mathcal{P}^*_m(U) = (\underbrace{\mathcal{P}^* \circ \cdots \circ \mathcal{P}^*}_{m \text{ times}})(U), \quad \mathcal{P}^*_n([0, 1]) = (\underbrace{\mathcal{P}^* \circ \cdots \circ \mathcal{P}^*}_{n \text{ times}})([0, 1]),$$

where $\mathcal{P}^*(X) = \mathcal{P}(X) \setminus \{\emptyset\}$. An (m, n)-*superhyperfuzzy set* on U is a map

$$\tilde{\mu}_{m,n} : \mathcal{P}^*_m(U) \ \longrightarrow\ \mathcal{P}\big(\mathcal{P}_n([0, 1])\big) \setminus \{\emptyset\},$$

assigning each nonempty m-level subset of U a nonempty family of n-level membership sets, thereby modeling hierarchical uncertainty.

Example 1.12 (HVAC fuzzy control with sensor uncertainty: $(m, n) = (1, 1)$)**.** Consider a simple heating–ventilation–air–conditioning (HVAC) controller (cf. [34, 35]) whose rule antecedents use linguistic sensor terms. Let the universe of antecedent atoms be

$$U = \{\text{TempLow, TempOK, TempHigh, HumidLow, HumidHigh}\}.$$

We work at level $m = 1$, so inputs to $\tilde{\mu}_{1,1}$ are nonempty subsets of U (rule antecedent combinations). We use $n = 1$ to encode *hyperfuzzy* (set–valued) activation degrees due to noisy sensors. For selected antecedent sets $X \in \mathcal{P}_1^*(U) = \mathcal{P}^*(U)$, define

$$\tilde{\mu}_{1,1}(X) \in \mathcal{P}\big(\mathcal{P}_1([0,1])\big) \setminus \{\emptyset\} \quad \text{as a singleton family of intervals in } [0,1]:$$

$$\begin{aligned}
X_1 &= \{\text{TempHigh}, \text{HumidLow}\}: && \tilde{\mu}_{1,1}(X_1) = \big\{ [0.6,\ 0.8] \big\} && \text{(rule "\textit{Increase Fan}" fires between 0.6 and 0.8);}\\
X_2 &= \{\text{TempLow}\}: && \tilde{\mu}_{1,1}(X_2) = \big\{ [0.5,\ 0.7] \big\} && \text{(rule "\textit{Increase Heat}" fires between 0.5 and 0.7);}\\
X_3 &= \{\text{TempOK}, \text{HumidHigh}\}: && \tilde{\mu}_{1,1}(X_3) = \big\{ [0.2,\ 0.4] \big\} && \text{(rule "\textit{Dehumidify}" fires between 0.2 and 0.4).}
\end{aligned}$$

Here each value (e.g. $[0.6, 0.8]$) is a subset of $[0, 1]$, hence an element of $\mathcal{P}_1([0, 1])$, and we wrap it as a singleton family to satisfy the codomain $\mathcal{P}\big(\mathcal{P}_1([0, 1])\big) \setminus \{\emptyset\}$. This $(1, 1)$-SuperHyperfuzzy Set encodes hyperfuzzy rule-activation strengths for a fuzzy controller under sensor uncertainty.

Example 1.13 (Autonomous braking with model hierarchy: $(m, n) = (1, 2)$)**.** Consider an autonomous vehicle controller (cf. [36]) that fuses *road condition* and *speed* terms. Let

$$U = \{\text{Dry, Wet, Icy, Slow, Medium, Fast}\},$$

and again work at $m = 1$ so that antecedent combinations are subsets of U. To capture a *two-level* uncertainty (e.g. variability across physics models and, within each model, measurement noise), we set $n = 2$ so the membership output for each X is a set of subsets of $[0, 1]$.

For two representative antecedents $X \in \mathcal{P}^*(U)$, define $\tilde{\mu}_{1,2}(X)$ as follows (each right-hand side is a singleton family in $\mathcal{P}_2([0, 1]) = \mathcal{P}(\mathcal{P}([0, 1]))$):

$$\begin{aligned}
X_1 = \{\text{Wet}, \text{Fast}\}: \quad \tilde{\mu}_{1,2}(X_1) &= \Big\{ M_{WF} := \big\{ \{0.55\},\ [0.6, 0.75] \big\} \Big\},\\
X_2 = \{\text{Icy}, \text{Medium}\}: \quad \tilde{\mu}_{1,2}(X_2) &= \Big\{ M_{IM} := \big\{ [0.7, 0.9],\ \{0.95\} \big\} \Big\}.
\end{aligned}$$

M_{WF} collects two *model-level* membership candidates for the rule "*Apply Strong Brake*"—one model yields a crisp 0.55, another yields an interval $[0.6, 0.75]$ reflecting intra-model noise. Likewise, M_{IM} aggregates a conservative interval $[0.7, 0.9]$ and a high-confidence crisp value 0.95 from a different model.

Formally, since $M_{WF}, M_{IM} \in \mathcal{P}([0, 1])$, the objects $\{M_{WF}\}, \{M_{IM}\}$ are elements of $\mathcal{P}_2([0, 1])$, and hence $\tilde{\mu}_{1,2}(X_i) \in \mathcal{P}\big(\mathcal{P}_2([0, 1])\big) \setminus \{\emptyset\}$, satisfying Definition 1.11. This $(1, 2)$-SuperHyperfuzzy Set therefore represents hierarchical uncertainty in rule activation arising from a *model layer* (choice among physics/learning models) and a *data layer* (noise within each model).

2 Main Results: HyperFuzzy Integral

In this section, we present the main results of this paper.

2.1 Fuzzy Integral

The Fuzzy Integral provides a method to aggregate single-valued fuzzy membership functions with respect to a fuzzy measure, yielding a precise evaluation of overall set importance (cf. [37–41]).

Definition 2.1 (Fuzzy measure (Sugeno))**.** (cf. [42, 43]) Let $(X, \mathcal{F})$ be a measurable space. A map $g : \mathcal{F} \to [0, 1]$ is a *fuzzy measure* if (i) $g(\emptyset) = 0$; (ii) (*monotonicity*) $F \subseteq G \Rightarrow g(F) \le g(G)$; (iii) (*continuity on monotone sequences*) if $\{F_n\}$ is increasing (resp. decreasing) in $\mathcal{F}$ with limit F, then $\lim_n g(F_n) = g(F)$. (Optionally, one assumes $g(X) = 1$ for normalization.)

Example 2.2 (A fuzzy (possibility) measure on a finite space)**.** Let $X = \{x_1, x_2, x_3\}$ and $\mathcal{F} = 2^X$. Define a *possibility distribution* $\pi : X \to [0, 1]$ by

$$\pi(x_1) = 1.0, \qquad \pi(x_2) = 0.6, \qquad \pi(x_3) = 0.2.$$

Define $g : \mathcal{F} \to [0, 1]$ by

$$g(F) := \max\{\pi(x) \mid x \in F\} \quad (\text{and } g(\emptyset) := 0).$$

Then g is a fuzzy measure: (i) $g(\emptyset) = 0$; (ii) if $F \subseteq G$ then $g(F) \le g(G)$ since $\max_{x\in F} \pi(x) \le \max_{x\in G} \pi(x)$; (iii) on the finite algebra $\mathcal{F}$, continuity on monotone sequences is automatic. Note also $g(X) = \max\{1.0, 0.6, 0.2\} = 1$, so g is normalized.

Definition 2.3 (Fuzzy integral (Sugeno integral))**.** Let g be a fuzzy measure on $(X, \mathcal{F})$ and let $h : X \to [0, 1]$ be $\mathcal{F}$-measurable. For $A \in \mathcal{F}$, the *fuzzy integral* (Sugeno integral) of h over A w.r.t. g is

$$\int_A^{\text{Sug}} h \,\text{d}g \;:=\; \sup_{\alpha\in[0,1]} \Big(\alpha \wedge g\big(A \cap \{x \in X : \; h(x) \ge \alpha\}\big)\Big),$$

equivalently,

$$\int_A^{\text{Sug}} h \,\text{d}g \;=\; \sup_{F\subseteq A} \Big(\inf_{x\in F} h(x) \;\wedge\; g(F)\Big).$$

Here $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$.

Example 2.4 (Sugeno (fuzzy) integral with respect to the above g)**.** Let $h : X \to [0, 1]$ be

$$h(x_1) = 0.4, \qquad h(x_2) = 0.8, \qquad h(x_3) = 0.3,$$

and take $A = X$. Sort the h-values in nondecreasing order:

$$h(x_{(1)}) = 0.3 \; (x_{(1)} = x_3), \quad h(x_{(2)}) = 0.4 \; (x_{(2)} = x_1), \quad h(x_{(3)}) = 0.8 \; (x_{(3)} = x_2).$$

For $i = 1, 2, 3$, set $K_i := \{x_{(i)}, \ldots, x_{(3)}\}$ and use the finite-space Sugeno formula

$$\int_X^{\text{Sug}} h \,\text{d}g = \max_{1\le i\le 3} \min\big(h(x_{(i)}),\; g(K_i)\big).$$

Compute $g(K_i)$ using $g(F) = \max_{x\in F} \pi(x)$:

$$\begin{aligned}
K_1 &= \{x_3, x_1, x_2\} = X &&\Rightarrow\; g(K_1) = \max\{0.2, 1.0, 0.6\} = 1.0,\\
K_2 &= \{x_1, x_2\} &&\Rightarrow\; g(K_2) = \max\{1.0, 0.6\} = 1.0,\\
K_3 &= \{x_2\} &&\Rightarrow\; g(K_3) = 0.6.
\end{aligned}$$

Thus the three candidates are

$$\min(0.3, 1.0) = 0.3, \qquad \min(0.4, 1.0) = 0.4, \qquad \min(0.8, 0.6) = 0.6.$$

Taking the maximum yields

$$\int_X^{\text{Sug}} h \,\text{d}g = \max\{0.3, 0.4, 0.6\} = 0.6.$$

2.2 HyperFuzzy Integral

The HyperFuzzy Integral extends this idea by allowing each element to possess a set of possible membership degrees, thereby aggregating uncertainty in the form of interval-valued results.

Definition 2.5 (Hyperfuzzy signal and selections)**.** Let $(X, \mathcal{F})$ be a measurable space (in this paper, X is finite unless stated otherwise). A *hyperfuzzy signal* on X is a map

$$\tilde{h} : \; X \longrightarrow \mathfrak{K}([0, 1]),$$

where $\mathfrak{K}([0, 1])$ denotes the family of nonempty compact subsets of $[0, 1]$. A (measurable) *selection* of $\tilde{h}$ is a map $h : X \to [0, 1]$ such that $h(x) \in \tilde{h}(x)$ for all $x \in X$. We write

$$\text{Sel}(\tilde{h}) \;:=\; \{\, h : X \to [0, 1] \mid h(x) \in \tilde{h}(x) \;\; \forall x \in X \,\}.$$

For $\tilde{h}$ as above, define the extremal selections

$$h_{\min}(x) := \min \tilde{h}(x), \qquad h_{\max}(x) := \max \tilde{h}(x), \qquad x \in X,$$

which are well-defined because each $\tilde{h}(x)$ is compact and nonempty.

Definition 2.6 (HyperFuzzy Sugeno integral)**.** Let $\tilde{h}$ be a hyperfuzzy signal and g a fuzzy measure. For $A \in \mathcal{F}$, the *HyperFuzzy Sugeno integral* of $\tilde{h}$ over A with respect to g is the *set*

$$\int_A^{\text{HF-Sug}} \tilde{h}\, dg \;:=\; \left\{ \int_A^{\text{Sug}} h\, dg \;\middle|\; h \in \mathrm{Sel}(\tilde{h}) \right\} \;\subseteq\; [0,1].$$

Its *lower* and *upper* values are

$$\underline{\int_A^{\text{HF-Sug}}} \tilde{h}\, dg \;:=\; \inf\Big\{\int_A^{\text{Sug}} h\, dg \mid h \in \mathrm{Sel}(\tilde{h})\Big\}, \qquad \overline{\int_A^{\text{HF-Sug}}} \tilde{h}\, dg \;:=\; \sup\Big\{\int_A^{\text{Sug}} h\, dg \mid h \in \mathrm{Sel}(\tilde{h})\Big\}.$$

Remark 2.7 (HyperFuzzy Set structure and domain/codomain)**.** By Definition 2.5, a hyperfuzzy signal $\tilde{h}$ is precisely a (pointwise) *HyperFuzzy Set* on X (cf. Definition 1.9 in the preliminaries), and Definition 2.6 generalizes the classical Sugeno integral by aggregating over all single-valued selections.

Example 2.8 (Finite computation of bounds and sample values)**.** Let $X = \{x_1, x_2\}$ with $A = X$, and let g be a fuzzy measure given by

$$g(\emptyset) = 0, \quad g(\{x_1\}) = 0.9, \quad g(\{x_2\}) = 0.3, \quad g(X) = 1.$$

Let the hyperfuzzy signal be

$$\tilde{h}(x_1) = [0.3,\ 0.7], \qquad \tilde{h}(x_2) = \{0.6\}.$$

Then $h_{\min} = (0.3,\ 0.6)$ and $h_{\max} = (0.7,\ 0.6)$ (ordered as (x_1, x_2)). Using the finite Sugeno formula for two points

$$\int_X^{\text{Sug}} h\, dg \;=\; \max\Big\{ \min\big(h_{(1)},\ g(X)\big),\ \min\big(h_{(2)},\ g(\{x_{(2)}\})\big)\Big\},$$

where $h_{(1)} \le h_{(2)}$ and $x_{(2)}$ is the point attaining $h_{(2)}$, we get

$$\int_X^{\text{Sug}} h_{\min}\, dg = \max\{\, \min(0.3, 1),\ \min(0.6,\ 0.3)\,\} = 0.3,$$

$$\int_X^{\text{Sug}} h_{\max}\, dg = \max\{\, \min(0.6, 1),\ \min(0.7,\ 0.9)\,\} = 0.7.$$

Hence by Theorem 2.9(ii),

$$\underline{\int_X^{\text{HF-Sug}}} \tilde{h}\, dg = 0.3, \qquad \overline{\int_X^{\text{HF-Sug}}} \tilde{h}\, dg = 0.7.$$

Moreover, selecting $h(x_1) = 0.5 \in [0.3, 0.7]$ and $h(x_2) = 0.6$ yields

$$\int_X^{\text{Sug}} h\, dg = \max\{\, \min(0.5, 1),\ \min(0.6,\ 0.3)\,\} = 0.5 \in \int_X^{\text{HF-Sug}} \tilde{h}\, dg,$$

demonstrating that intermediate values between the lower and upper bounds can occur (though the attainable set need not be a full interval in general).

Theorem 2.9 (Well-definedness, bounds, and reduction)**.** *Let g be a fuzzy measure on $(X, \mathcal{F})$, $\tilde{h}$ a hyperfuzzy signal, and $A \in \mathcal{F}$. Then:*

(i) (Nonemptiness) $\mathrm{Sel}(\tilde{h}) \neq \emptyset$*, hence* $\int_A^{\text{HF-Sug}} \tilde{h}\, dg$ *is a nonempty subset of* $[0,1]$.

(ii) (Extremal bounds)

$$\underline{\int_A^{\text{HF-Sug}}} \tilde{h}\, dg = \int_A^{\text{Sug}} h_{\min}\, dg, \qquad \overline{\int_A^{\text{HF-Sug}}} \tilde{h}\, dg = \int_A^{\text{Sug}} h_{\max}\, dg.$$

Consequently,

$$\int_A^{\text{HF-Sug}} \tilde{h}\, dg \;\subseteq\; \Big[\, \int_A^{\text{Sug}} h_{\min}\, dg\ ,\ \int_A^{\text{Sug}} h_{\max}\, dg \,\Big].$$

*(iii) (*Reduction to classical Sugeno*) If $\tilde{h}$ is induced by a single-valued fuzzy function h (i.e. $\tilde{h}(x) = \{h(x)\}$ for all x), then*

$$\int_A^{\text{HF-Sug}} \tilde{h}\,\mathrm{d}g = \Big\{ \int_A^{\text{Sug}} h\,\mathrm{d}g \Big\}, \qquad \textit{and hence} \qquad \underline{\int_A^{\text{HF-Sug}}} \tilde{h}\,\mathrm{d}g = \overline{\int_A^{\text{HF-Sug}}} \tilde{h}\,\mathrm{d}g = \int_A^{\text{Sug}} h\,\mathrm{d}g.$$

Proof. (i) Since each $\tilde{h}(x) \in \mathfrak{K}([0,1])$ is nonempty and compact, $h_{\min}, h_{\max}$ are well-defined selections (and measurable when X is finite as assumed). Thus $\mathrm{Sel}(\tilde{h}) \neq \emptyset$.

(ii) The Sugeno integral is monotone with respect to the pointwise order on inputs: if $h \le k$ pointwise on A, then $\int_A^{\text{Sug}} h\,\mathrm{d}g \le \int_A^{\text{Sug}} k\,\mathrm{d}g$. For any $h \in \mathrm{Sel}(\tilde{h})$ we have $h_{\min} \le h \le h_{\max}$, hence

$$\int_A^{\text{Sug}} h_{\min}\,\mathrm{d}g \;\le\; \int_A^{\text{Sug}} h\,\mathrm{d}g \;\le\; \int_A^{\text{Sug}} h_{\max}\,\mathrm{d}g.$$

Taking the infimum (resp. supremum) over all selections h yields the claimed equalities for the lower (resp. upper) value. The interval containment follows immediately.

(iii) If $\tilde{h}(x) = \{h(x)\}$, then $\mathrm{Sel}(\tilde{h}) = \{h\}$ and $h_{\min} = h_{\max} = h$, so the set in Definition 2.6 collapses to the singleton $\{\int_A^{\text{Sug}} h\,\mathrm{d}g\}$ and both extremal values equal the classical integral. □

Proposition 2.10 (Monotonicity with respect to hyperfuzzy refinement)**.** *If $\tilde{h}, \tilde{k}$ are hyperfuzzy signals with $\tilde{h}(x) \subseteq \tilde{k}(x)$ for all $x \in X$, then*

$$\underline{\int_A^{\text{HF-Sug}}} \tilde{h}\,\mathrm{d}g \;\le\; \underline{\int_A^{\text{HF-Sug}}} \tilde{k}\,\mathrm{d}g \;\le\; \overline{\int_A^{\text{HF-Sug}}} \tilde{k}\,\mathrm{d}g \;\le\; \overline{\int_A^{\text{HF-Sug}}} \tilde{h}\,\mathrm{d}g.$$

In particular, $\int_A^{\text{HF-Sug}} \tilde{h}\,\mathrm{d}g \subseteq \int_A^{\text{HF-Sug}} \tilde{k}\,\mathrm{d}g$ whenever every selection of $\tilde{h}$ is also a selection of $\tilde{k}$.

Proof. Pointwise inclusion implies $h^{\tilde{k}}_{\min} \le h^{\tilde{h}}_{\min} \le h^{\tilde{h}}_{\max} \le h^{\tilde{k}}_{\max}$. Apply monotonicity of the Sugeno integral to these four functions and use Theorem 2.9(ii). □

2.3 (m,n)-SuperHyperFuzzy Integral

The (m,n)-SuperHyperFuzzy Integral generalizes both the fuzzy and hyperfuzzy integrals by operating on (m,n)-SuperHyperFuzzy Sets defined over hierarchical powersets, producing multi-level uncertainty aggregation that captures richer and more complex forms of imprecision.

Definition 2.11 (Iterated powerset and atoms)**.** For a nonempty finite base set U, define $\mathcal{P}^0(U) := U$ and $\mathcal{P}^{k+1}(U) := \mathcal{P}(\mathcal{P}^k(U))$. Excluding the empty set at each stage, write $\mathcal{P}^*(U) := \mathcal{P}(U) \setminus \{\emptyset\}$ and $\mathcal{P}^k(\cdot)$ accordingly with $\mathcal{P}$ replaced by $\mathcal{P}^*$. For $Z \in \mathcal{P}^k(U)$, its set of *atoms* $\mathrm{At}(Z) \subseteq U$ is defined by $\mathrm{At}(x) = \{x\}$ for $x \in U$ and $\mathrm{At}(Z) = \bigcup_{z \in Z} \mathrm{At}(z)$ when $k \ge 1$. For $x \in U$ and $k \ge 1$, the *canonical tower* $T_k(x)$ is defined recursively by $T_0(x) := x$ and $T_k(x) := \{\, T_{k-1}(x) \,\} \in \mathcal{P}^k(U)$.

Definition 2.12 (Iterated value selections)**.** For $k \ge 0$ and $M \in \mathcal{P}^k(()[0,1])$, define the set of *k-level selections* recursively by

$$\mathrm{Sel}_0(M) := \{M\} \subseteq [0,1], \qquad \mathrm{Sel}_{k+1}(M) := \{\, r \in [0,1] \mid \exists N \in M \text{ with } r \in \mathrm{Sel}_k(N) \,\}.$$

Equivalently, $\mathrm{Sel}_k(M)$ is the set of all values obtained by iteratively choosing one element at each nesting level down to $[0,1]$.

Definition 2.13 (Admissible pointwise realizations)**.** Given $\widetilde{\mu}_{m,n}$ as in the Definition, define for each $X \in \mathcal{P}^*_m(()\,U)$ the *flattened value set*

$$V(X) \;:=\; \bigcup_{M \in \widetilde{\mu}_{m,n}(X)} \mathrm{Sel}_n(M) \;\subseteq\; [0,1].$$

For $x \in U$, let the *m-neighborhood index* be

$$\mathcal{B}_m(x) \;:=\; \{\, X \in \mathcal{P}^*_m(()\,U) \mid x \in \mathrm{At}(X) \,\}.$$

The family of *admissible realizations* (single-valued membership functions) induced by $\widetilde{\mu}_{m,n}$ is

$$\mathcal{H}_{m,n}(\widetilde{\mu}_{m,n}) \;:=\; \Big\{h : U \to [0,1] \;\Big|\; \exists\, (v_X)_{X\in\mathcal{P}^*_m(()U)} \text{ with } v_X \in V(X) \text{ and } h(x) = \sup_{X\in\mathcal{B}_m(x)} v_X \;\forall x \in U\Big\}.$$

(The supremum is a maximum since U is finite and each $\mathcal{B}_m(x)$ is nonempty; note $T_m(x) \in \mathcal{B}_m(x)$ for every x.)

Definition 2.14 ((m,n)-SuperHyperFuzzy Sugeno integral)**.** Let $\widetilde{\mu}_{m,n}$ be an (m,n)-SuperHyperFuzzy Set and g a fuzzy measure. For $A \subseteq U$, the *(m,n)-SuperHyperFuzzy Sugeno integral* of $\widetilde{\mu}_{m,n}$ over A is the set

$$\int_A^{(m,n)\text{-SHF-Sug}} \widetilde{\mu}_{m,n}\,\mathrm{d}g \;:=\; \Big\{ \int_A^{\mathrm{Sug}} h\,\mathrm{d}g \;\Big|\; h \in \mathcal{H}_{m,n}(\widetilde{\mu}_{m,n}) \Big\} \;\subseteq\; [0,1].$$

We also define the *lower/upper* SHF integrals by

$$\underline{\int_A^{(m,n)\text{-SHF}}} \widetilde{\mu}_{m,n}\,\mathrm{d}g := \inf\Big\{\int_A^{\mathrm{Sug}} h\,\mathrm{d}g \mid h \in \mathcal{H}_{m,n}(\widetilde{\mu}_{m,n})\Big\},$$

$$\overline{\int_A^{(m,n)\text{-SHF}}} \widetilde{\mu}_{m,n}\,\mathrm{d}g := \sup\Big\{\int_A^{\mathrm{Sug}} h\,\mathrm{d}g \mid h \in \mathcal{H}_{m,n}(\widetilde{\mu}_{m,n})\Big\}.$$

Example 2.15 (A concrete $(m,n) = (1,1)$ computation on a two-point universe)**.** Let $U = \{x_1, x_2\}$ and $A = U$. Define a fuzzy measure g by

$$g(\emptyset) = 0, \quad g(\{x_1\}) = 0.7, \quad g(\{x_2\}) = 0.4, \quad g(U) = 1.$$

Take $m = n = 1$. Then $\mathcal{P}^*_m(()\,U) = \mathcal{P}^*(U) = \big\{\{x_1\}, \{x_2\}, U\big\}$. Specify an $(1,1)$-SuperHyperFuzzy Set $\widetilde{\mu}_{1,1}$ by

$$\widetilde{\mu}_{1,1}(\{x_1\}) = \big\{\, [0.3, 0.6] \,\big\},$$
$$\widetilde{\mu}_{1,1}(\{x_2\}) = \big\{\{0.5\}\big\},$$
$$\widetilde{\mu}_{1,1}(U) = \big\{\, \{0.4, 0.8\} \,\big\}.$$

Thus

$$V(\{x_1\}) = [0.3, 0.6], \quad V(\{x_2\}) = \{0.5\}, \quad V(U) = \{0.4, 0.8\}.$$

Neighborhoods: $\mathcal{B}_1(x_1) = \big\{\{x_1\}, U\big\}$ and $\mathcal{B}_1(x_2) = \big\{\{x_2\}, U\big\}$. Any admissible realization h has the form

$$h(x_1) = \max\{v_{\{x_1\}},\, v_U\}, \qquad h(x_2) = \max\{v_{\{x_2\}},\, v_U\},$$

with $v_{\{x_1\}} \in [0.3, 0.6]$, $v_{\{x_2\}} = 0.5$, $v_U \in \{0.4, 0.8\}$.

Lower/upper bounds. Here $v_-(\{x_1\}) = 0.3$, $v_-(\{x_2\}) = 0.5$, $v_-(U) = 0.4$, so

$$h_-(x_1) = \max\{0.3, 0.4\} = 0.4, \qquad h_-(x_2) = \max\{0.5, 0.4\} = 0.5.$$

Similarly $v_+(\{x_1\}) = 0.6$, $v_+(\{x_2\}) = 0.5$, $v_+(U) = 0.8$, so

$$h_+(x_1) = \max\{0.6, 0.8\} = 0.8, \qquad h_+(x_2) = \max\{0.5, 0.8\} = 0.8.$$

Sugeno evaluation. On a finite $U = \{x_1, x_2\}$ the Sugeno integral is

$$\int_U^{\mathrm{Sug}} h\,\mathrm{d}g \;=\; \max\Big\{\min\big(h_{(1)},\, g(U)\big),\; \min\big(h_{(2)},\, g(\{x_{(2)}\})\big)\Big\},$$

where $h_{(1)} \le h_{(2)}$ are the ordered values and $x_{(2)}$ is a maximizer of h. For $h_- = (0.4, 0.5)$ (ordered),

$$\int_U^{\mathrm{Sug}} h_-\,\mathrm{d}g \;=\; \max\{\,\min(0.4, 1),\; \min(0.5,\, 0.7)\,\} = 0.5.$$

For $h_+ = (0.8, 0.8)$,

$$\int_U^{\mathrm{Sug}} h_+ \, \mathrm{d}g \;=\; \max\{\, \min(0.8, 1),\ \min(0.8,\ 0.7)\,\} = 0.8.$$

Therefore, by Theorem 2.18,

$$0.5 \;\le\; \underline{\int_U^{(1,1)\text{-SHF}}} \widetilde{\mu}_{1,1} \, \mathrm{d}g \;\le\; \overline{\int_U^{(1,1)\text{-SHF}}} \widetilde{\mu}_{1,1} \, \mathrm{d}g \;\le\; 0.8.$$

Two admissible realizations show attainability of interior values:

$$\text{(i) } \nu_U = 0.4,\ \nu_{\{x_1\}} = 0.6 \Rightarrow h = (0.6, 0.5),\ \int h \, \mathrm{d}g = \max\{\min(0.5, 1), \min(0.6, 0.7)\} = 0.6;$$

$$\text{(ii) } \nu_U = 0.8,\ \nu_{\{x_1\}} = 0.3 \Rightarrow h = (0.8, 0.8),\ \int h \, \mathrm{d}g = \max\{\min(0.8, 1), \min(0.8, 0.7)\} = 0.8.$$

Thus the $(1, 1)$-SuperHyperFuzzy integral set contains (at least) $\{0.6, 0.8\}$ and is bounded within $[0.5, 0.8]$.

Example 2.16 ((m,n)=(0,2): three-point universe with model-level uncertainty)**.** Let $U = \{x_1, x_2, x_3\}$ and let $g : 2^U \to [0, 1]$ be the (normalized) possibility measure generated by $\pi(x_1) = 1.0,\ \pi(x_2) = 0.7,\ \pi(x_3) = 0.5$, i.e.

$$g(F) = \max\{\pi(x) : x \in F\}, \qquad g(\emptyset) = 0.$$

Define an $(0, 2)$-SuperHyperFuzzy Set $\widetilde{\mu}_{0,2}$ pointwise by the singleton families

$$\widetilde{\mu}_{0,2}(x_1) = \big\{\, M_1 := \{[0.6, 0.9],\ \{0.8\}\}\,\big\},$$

$$\widetilde{\mu}_{0,2}(x_2) = \big\{\, M_2 := \{[0.3, 0.5]\}\,\big\},$$

$$\widetilde{\mu}_{0,2}(x_3) = \big\{\, M_3 := \{\{0.2\},\ [0.4, 0.6]\}\,\big\}.$$

For each i, the set of 2-level selections flattens to

$$V(x_1) = \mathrm{Sel}_2(M_1) = [0.6, 0.9], \quad V(x_2) = [0.3, 0.5], \quad V(x_3) = \{0.2\} \cup [0.4, 0.6].$$

Admissible realizations are functions $h : U \to [0, 1]$ with $h(x_i) \in V(x_i)$. The lower/upper extremal realizations are

$$h_- = (0.6,\ 0.3,\ 0.2), \qquad h_+ = (0.9,\ 0.5,\ 0.6).$$

For $A = U$, the finite-space Sugeno formula gives

$$\int_U^{\mathrm{Sug}} h \, \mathrm{d}g = \max_{1 \le i \le 3} \min\!\big(h(x_{(i)}),\ g(\{x_{(i)}, \ldots, x_{(3)}\})\big),$$

where $h(x_{(1)}) \le h(x_{(2)}) \le h(x_{(3)})$. Hence

$$\int_U^{\mathrm{Sug}} h_- \, \mathrm{d}g = \max\{\, \min(0.2, 1.0),\ \min(0.3, 1.0),\ \min(0.6, 1.0)\,\} = 0.6,$$

$$\int_U^{\mathrm{Sug}} h_+ \, \mathrm{d}g = \max\{\, \min(0.5, 1.0),\ \min(0.6, 1.0),\ \min(0.9, 1.0)\,\} = 0.9.$$

Therefore, by Definition 2.14,

$$\underline{\int_U^{(0,2)\text{-SHF}}} \widetilde{\mu}_{0,2} \, \mathrm{d}g = 0.6, \qquad \overline{\int_U^{(0,2)\text{-SHF}}} \widetilde{\mu}_{0,2} \, \mathrm{d}g = 0.9.$$

A concrete interior selection, e.g. $h(x_1) = 0.8$, $h(x_2) = 0.4$, $h(x_3) = 0.6$, yields

$$\int_U^{\mathrm{Sug}} h \, \mathrm{d}g = \max\{\, \min(0.4, 1.0),\ \min(0.6, 1.0),\ \min(0.8, 1.0)\,\} = 0.8 \in \int_U^{(0,2)\text{-SHF-Sug}} \widetilde{\mu}_{0,2} \, \mathrm{d}g.$$

Example 2.17 ((m,n)=(2,1): hierarchical antecedents with hyperfuzzy memberships)**.** Let $U = \{a, b\}$ and let g be the possibility measure with $\pi(a) = 1.0$, $\pi(b) = 0.6$; hence $g(\{a\}) = 1.0$, $g(\{b\}) = 0.6$, $g(U) = 1.0$. At level $m = 2$, the domain $\mathcal{P}_2^*(U)$ consists of nonempty families of nonempty subsets of U. Consider

$$X_1 = \{\{a\}\}, \qquad X_2 = \{\{b\}\}, \qquad X_3 = \{\{a\}, \{b\}\}.$$

Define an $(2,1)$-SuperHyperFuzzy Set $\widetilde{\mu}_{2,1}$ by singleton families of intervals

$$\widetilde{\mu}_{2,1}(X_1) = \big\{ [0.6, 0.8] \big\}, \quad \widetilde{\mu}_{2,1}(X_2) = \big\{ [0.3, 0.5] \big\}, \quad \widetilde{\mu}_{2,1}(X_3) = \big\{ \{0.7\} \big\}.$$

Thus $V(X_1) = [0.6, 0.8]$, $V(X_2) = [0.3, 0.5]$, $V(X_3) = \{0.7\}$. The m-neighborhoods are

$$\mathcal{B}_2(a) = \{X_1, X_3\}, \qquad \mathcal{B}_2(b) = \{X_2, X_3\}.$$

Every admissible realization $h : U \to [0, 1]$ arises from choices $v_{X_i} \in V(X_i)$ via

$$h(a) = \max\{v_{X_1},\, v_{X_3}\}, \qquad h(b) = \max\{v_{X_2},\, v_{X_3}\}.$$

Extremal realizations are

$$h_- = (\max\{0.6, 0.7\},\ \max\{0.3, 0.7\}) = (0.7, 0.7), \qquad h_+ = (\max\{0.8, 0.7\},\ \max\{0.5, 0.7\}) = (0.8, 0.7).$$

With the two-point Sugeno formula,

$$\int_U^{\mathrm{Sug}} h\, \mathrm{d}g = \max\Big\{ \min\big(h_{(1)},\, g(U)\big),\ \min\big(h_{(2)},\, g(\{x_{(2)}\})\big) \Big\},$$

one gets

$$\int_U^{\mathrm{Sug}} h_-\, \mathrm{d}g = \max\{\min(0.7, 1.0),\ \min(0.7, 1.0)\} = 0.7,$$

$$\int_U^{\mathrm{Sug}} h_+\, \mathrm{d}g = \max\{\min(0.7, 1.0),\ \min(0.8, 1.0)\} = 0.8.$$

Therefore,

$$\underline{\int_U^{(2,1)\text{-SHF}}} \widetilde{\mu}_{2,1}\, \mathrm{d}g = 0.7,$$

$$\overline{\int_U^{(2,1)\text{-SHF}}} \widetilde{\mu}_{2,1}\, \mathrm{d}g = 0.8.$$

For an interior value, take $v_{X_1} = 0.75$, $v_{X_2} = 0.4$, $v_{X_3} = 0.7$, yielding $h = (0.75, 0.7)$ and

$$\int_U^{\mathrm{Sug}} h\, \mathrm{d}g = \max\{\min(0.7, 1.0),\ \min(0.75, 1.0)\} = 0.75$$

$$\in \int_U^{(2,1)\text{-SHF-Sug}} \widetilde{\mu}_{2,1}\, \mathrm{d}g.$$

Theorem 2.18 (Well-definedness and bounds)**.** *Let g be a fuzzy measure and $\widetilde{\mu}_{m,n}$ an (m,n)-SuperHyperFuzzy Set on finite U. For each $A \subseteq U$:*

(i) $\mathcal{H}_{m,n}(\widetilde{\mu}_{m,n}) \neq \emptyset$*, hence* $\int_A^{(m,n)\text{-SHF-Sug}} \widetilde{\mu}_{m,n}\, \mathrm{d}g$ *is a nonempty subset of* $[0, 1]$.

(ii) Define

$$v_-(X) := \inf V(X), \qquad v_+(X) := \sup V(X),$$

and

$$h_-(x) := \sup_{X \in \mathcal{B}_m(x)} v_-(X), \qquad h_+(x) := \sup_{X \in \mathcal{B}_m(x)} v_+(X).$$

Then for any $h \in \mathcal{H}_{m,n}(\widetilde{\mu}_{m,n})$ one has $h_- \leq h \leq h_+$ pointwise, and consequently

$$\int_A^{\mathrm{Sug}} h_-\, \mathrm{d}g \ \leq\ \underline{\int_A^{(m,n)\text{-SHF}}} \widetilde{\mu}_{m,n}\, \mathrm{d}g \ \leq\ \overline{\int_A^{(m,n)\text{-SHF}}} \widetilde{\mu}_{m,n}\, \mathrm{d}g \ \leq\ \int_A^{\mathrm{Sug}} h_+\, \mathrm{d}g.$$

Proof. (i) For each $X \in \mathcal{P}^*_m(()\, U)$ pick $M_X \in \widetilde{\mu}_{m,n}(X)$ (possible since the family is nonempty), and then pick $v_X \in \mathrm{Sel}_n(M_X)$ (nonempty by Definition 2.12). For each $x \in U$, $\mathcal{B}_m(x) \neq \emptyset$ because $T_m(x) \in \mathcal{B}_m(x)$. Hence $h(x) := \sup_{X \in \mathcal{B}_m(x)} v_X$ is well-defined in $[0, 1]$, so $h \in \mathcal{H}_{m,n}$.

(ii) By construction, for any chosen $v_X \in V(X)$ we have $v_-(X) \leq v_X \leq v_+(X)$. Taking the supremum over $X \in \mathcal{B}_m(x)$ gives $h_-(x) \leq h(x) \leq h_+(x)$ for every x. The Sugeno integral is monotone with respect to its integrand, so $\int_A^{\mathrm{Sug}} h_- \, \mathrm{d}g \leq \int_A^{\mathrm{Sug}} h \, \mathrm{d}g \leq \int_A^{\mathrm{Sug}} h_+ \, \mathrm{d}g$. Taking infimum/supremum over all $h \in \mathcal{H}_{m,n}$ yields the desired bounds. □

Theorem 2.19 (Reduction to HyperFuzzy and classical Fuzzy integrals)**.** *Let g be a fuzzy measure on U.*

(a) (HyperFuzzy case) *Suppose $m = 0$ and, for each $x \in U$, the family $\widetilde{\mu}_{0,n}(x)$ is a singleton $\{M_x\}$ with $M_x \in \mathcal{P}^*_n(()\, [0, 1])$. Then*

$$\mathcal{H}_{0,n}(\widetilde{\mu}_{0,n}) = \left\{ h : U \to [0, 1] \;\middle|\; h(x) \in \mathrm{Sel}_n(M_x)\ \forall x \right\},$$

and therefore

$$\int_A^{(0,n)\text{-SHF-Sug}} \widetilde{\mu}_{0,n} \, \mathrm{d}g = \left\{ \int_A^{\mathrm{Sug}} h \, \mathrm{d}g \;\middle|\; h(x) \in \mathrm{Sel}_n(M_x)\ \forall x \right\},$$

which coincides with the HyperFuzzy Sugeno integral when $n = 1$ (set-valued memberships per point).

(b) (Classical case) *If, in addition, $n = 0$ and each $M_x = \{h_0(x)\}$ for some crisp $h_0 : U \to [0, 1]$, then*

$$\mathcal{H}_{0,0}(\widetilde{\mu}_{0,0}) = \{ h_0 \}, \qquad \int_A^{(0,0)\text{-SHF-Sug}} \widetilde{\mu}_{0,0} \, \mathrm{d}g = \left\{ \int_A^{\mathrm{Sug}} h_0 \, \mathrm{d}g \right\}.$$

Hence (m, n)-SuperHyperFuzzy integrals strictly generalize both the HyperFuzzy and the classical Sugeno integrals.

Proof. (a) When $m = 0$, $\mathcal{P}^*_m(()\, U) = U$ and $\mathcal{B}_0(x) = \{x\}$. For a singleton family $\widetilde{\mu}_{0,n}(x) = \{M_x\}$, one has $V(\{x\}) = \mathrm{Sel}_n(M_x)$. Thus every $h \in \mathcal{H}_{0,n}$ is determined pointwise by choosing $h(x) \in \mathrm{Sel}_n(M_x)$, and the definition of the SHF integral reduces exactly to aggregating all Sugeno integrals over such selections. For $n = 1$, $\mathrm{Sel}_1(M_x) = M_x$ (a nonempty subset of $[0, 1]$), i.e. the HyperFuzzy case.

(b) If $n = 0$ and $M_x = \{h_0(x)\}$, then $\mathrm{Sel}_0(M_x) = \{h_0(x)\}$ and the only admissible realization is h_0, hence the SHF integral set reduces to the singleton $\{\int_A^{\mathrm{Sug}} h_0 \, \mathrm{d}g\}$. □

3 Conclusion

In this paper, we extended the framework by introducing the *Hyperfuzzy Integral* and the *SuperHyperfuzzy Integral*, defined over Hyperfuzzy Sets and SuperHyperfuzzy Sets, respectively, and we investigated their fundamental properties. In future work, we aim to explore potential applications within *Fuzzy Control Systems* [44–46], as well as extensions that incorporate advanced uncertain set frameworks such as the *Intuitionistic Fuzzy Set* [47–49], the Hesitant Fuzzy Set [5,50], the Picture Fuzzy Set [51–53], the *Neutrosophic Set* [8,54,55], the *Double-valued Neutrosophic Set* [56–58], and the *Plithogenic Set* [9, 59, 60]. Such directions will allow a richer representation of uncertainty and broader applicability of the proposed integrals in both theory and practice.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this work.

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Data Availability

This paper is theoretical and did not generate or analyze any empirical data. We welcome future studies that apply and test these concepts in practical settings.

Research Integrity

The author confirms that this manuscript is original, has not been published elsewhere, and is not under consideration by any other journal.

Use of Computational Tools

All proofs and derivations were performed manually; no computational software (e.g., Mathematica, SageMath, Coq) was used.

Code Availability

No code or software was developed for this study.

Ethical Approval

This research did not involve human participants or animals, and therefore did not require ethical approval.

Use of Generative AI and AI-Assisted Tools

We use generative AI and AI-assisted tools for tasks such as English grammar checking, and We do not employ them in any way that violates ethical standards.

Supplementary Information

No supplementary materials accompany this paper.

Disclaimer

The ideas presented here are theoretical and have not yet been validated through empirical testing. While we have strived for accuracy and proper citation, inadvertent errors may remain. Readers should verify any referenced material independently. The opinions expressed are those of the authors and do not necessarily reflect the views of their institutions.

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