HybridFunctorial Structure and MultiFunctorial Structure

Takaaki Fujita [1] *
[1] Independent Researcher, Tokyo, Japan.
Email: Takaaki.fujita060@gmail.com

Abstract

A *Functorial Structure* is defined as a covariant functor $F : C \to \mathbf{Set}$, assigning sets to objects and functions to morphisms, ensuring functoriality [1]. In this paper, we introduce and formally define two new concepts: the HybridFunctorial Structure and the MultiFunctorial Structure. A HybridFunctorial Structure combines two functors on the same category, linked by a natural transformation, ensuring consistent pushforward compatibility. A MultiFunctorial Structure involves multiple functors indexed by a preorder, coherently related via natural transformations, forming compatible families with functorial consistency.

Keywords: Functorial Structure, HybridFunctorial Structure, MultiFunctorial Structure

Structure of this paper

The format of this paper is described below.

1 Preliminaries

In this section we collect the notation and basic notions used later. Unless explicitly stated otherwise, all underlying sets are *finite*.

1.1 Functorial Structure

A *Functorial Structure* is defined as a covariant functor $F : C \to \mathbf{Set}$, assigning sets to objects and functions to morphisms, ensuring functoriality [1]. A Functorial Set can represent various frameworks such as Fuzzy Set [2–4], Plithogenic Set [5, 6], Hesitant Fuzzy Set [7, 8], Intuitionstic Fuzzy Set [9, 10], Neutrosophic Set [11–13], Rough Set [14, 15], Soft Set [16, 17], Multiset [18], and Probabilistic Set [19].

Definition 1.1 (Functorial Set)**.** [1] Let C be a category and

$$F : C \longrightarrow \mathbf{Set}$$

be a (covariant) endofunctor. For any object $X \in \mathrm{Ob}(C)$, an *F-set over X* is an element

$$s \in F(X).$$

We denote the collection of all F-sets over X simply by $F(X)$. A morphism $f : X \to Y$ in C induces a *pushforward*

$$F(f) : F(X) \longrightarrow F(Y), \qquad s \mapsto F(f)(s).$$

Example 1.2 (Real-world Functorial Set: Project team selection with address-book unification)**.** Let $C = \mathbf{FinSet}$ (finite sets and functions) and take the covariant functor $F = \mathcal{P}$ (powerset) defined by $F(X) = \mathcal{P}(X)$ and, for $f : X \to Y$,

$$F(f) : \mathcal{P}(X) \to \mathcal{P}(Y), \qquad F(f)(S) = f[S] = \{\, f(x) \mid x \in S \,\}.$$

Here, an *F-set over X* (Def. 1.1) is simply a chosen subset $S \subseteq X$.

Interpretation. Let $X = \{\text{Hiroko}, \text{Yutaka}, \text{Cara}\}$ be employees at Site A, $Y = \{\text{a123}, \text{b234}, \text{c345}\}$ be corporate master IDs, and $f : X \to Y$ be the ID-link: $f(\text{Hiroko}) = \text{a123}$, $f(\text{Yutaka}) = \text{b234}$, $f(\text{Cara}) = \text{c345}$. A concrete F-set $s \in F(X)$ is a *project team*:

$$s = \{\text{Hiroko}, \text{Cara}\} \in \mathcal{P}(X).$$

The pushforward (image) along f is

$$F(f)(s) = \{\text{a123}, \text{c345}\} \in \mathcal{P}(Y),$$

which is the same team expressed in corporate IDs. If $g : Y \to Z$ later remaps IDs to anonymized tokens, then functoriality gives $F(g \circ f)(s) = F(g)\big(F(f)(s)\big)$, i.e., anonymization after unification equals direct anonymization of the original team.

Definition 1.3 (Functorial Structure)**.** [1] Let C be a category. A *Functorial Structure* on C is simply a covariant functor

$$F : C \longrightarrow \mathbf{Set}.$$

For each object $X \in \mathrm{Ob}(C)$, an element

$$s \in F(X)$$

is called an *F-structure on X*. Every morphism $f\colon X \to Y$ in C induces a *pushforward*

$$F(f)\colon F(X) \longrightarrow F(Y), \qquad s \longmapsto F(f)(s),$$

and the usual functoriality conditions $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ and $F(g \circ f) = F(g) \circ F(f)$ hold.

Example 1.4 (Real-world Functorial Structure: Inventory aggregation from SKUs to categories (finite multiset functor))**.** Let $C = \mathbf{FinSet}$ and define the covariant functor $F : C \to \mathbf{Set}$ by

$$F(X) = \mathbb{N}^{(X)} = \{\, m : X \to \mathbb{N} \text{ with finite support} \,\},$$

(viewed as finite multisets on X). For a function $f : X \to Y$, define

$$F(f) : \mathbb{N}^{(X)} \to \mathbb{N}^{(Y)}, \qquad \big(F(f)(m)\big)(y) := \sum_{x \in f^{-1}(y)} m(x).$$

Thus $F(f)$ *pushes forward* counts by summing over fibers. One checks $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ and $F(g \circ f) = F(g) \circ F(f)$ by associativity/commutativity of $+$.

Interpretation. Let $X = \{\text{sku}_1, \text{sku}_2, \text{sku}_3, \text{sku}_4\}$ and $Y = \{\text{CatA}, \text{CatB}\}$ with a classification $f : X \to Y$:

$$f(\text{sku}_1) = \text{CatA}, \quad f(\text{sku}_2) = \text{CatA}, \quad f(\text{sku}_3) = \text{CatB}, \quad f(\text{sku}_4) = \text{CatB}.$$

A concrete F-structure $m \in F(X)$ is an inventory multiset (counts per SKU):

$$m(\text{sku}_1) = 5, \;\; m(\text{sku}_2) = 2, \;\; m(\text{sku}_3) = 7, \;\; m(\text{sku}_4) = 3.$$

The pushforward to category-level stock is

$$\big(F(f)(m)\big)(\text{CatA}) = \sum_{x \in f^{-1}(\text{CatA})} m(x) = m(\text{sku}_1) + m(\text{sku}_2) = 5 + 2 = 7,$$
$$\big(F(f)(m)\big)(\text{CatB}) = \sum_{x \in f^{-1}(\text{CatB})} m(x) = m(\text{sku}_3) + m(\text{sku}_4) = 7 + 3 = 10.$$

Functoriality (explicit check). If $g : Y \to Z = \{\text{All}\}$ sends both categories to All, then

$$\big(F(g) \circ F(f)\big)(m)(\text{All}) = F(g)\big(F(f)(m)\big)(\text{All}) = 7 + 10 = 17,$$

while pushing forward in one step yields

$$F(g \circ f)(m)(\text{All}) = \sum_{x \in X} m(x) = 5 + 2 + 7 + 3 = 17,$$

hence $F(g \circ f) = F(g) \circ F(f)$ on m as required.

Example 1.5 (Real-world Fuzzy Functorial Structure: product interest from SKUs → categories → global)**.** Let C = **FinSet** (finite sets and functions). Define a covariant functor

$$F : C \longrightarrow \mathbf{Set}, \qquad F(X) = [0,1]^X = \{\, \mu : X \to [0,1] \,\},$$

whose elements are *fuzzy sets* (membership functions) [2,20] on X. For a function $f : X \to Y$, define the pushforward $F(f) : F(X) \to F(Y)$ by the *direct-image aggregator*

$$\big(F(f)(\mu)\big)(y) \;:=\; \sup\{\, \mu(x) \mid x \in X,\ f(x) = y \,\}, \quad \text{with } \sup\emptyset := 0.$$

Functoriality holds (on any $z \in Z$) because for $X \xrightarrow{f} Y \xrightarrow{g} Z$,

$$\begin{aligned}\big(F(g\circ f)(\mu)\big)(z) &= \sup_{x:\, g(f(x))=z} \mu(x) = \sup_{y:\, g(y)=z}\ \sup_{x:\, f(x)=y} \mu(x)\\ &= \Big(F(g)\big(F(f)(\mu)\big)\Big)(z),\end{aligned}$$

and clearly $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$.

Interpretation (concrete, real-life). Let $X = \{\mathrm{sku}_1, \mathrm{sku}_2, \mathrm{sku}_3, \mathrm{sku}_4\}$ be individual products, $Y = \{\mathrm{CatA}, \mathrm{CatB}\}$ be product categories, and $Z = \{\mathrm{All}\}$ a single global bucket. Let $f : X \to Y$ classify SKUs to categories and $g : Y \to Z$ send both categories to All:

$$f(\mathrm{sku}_1) = f(\mathrm{sku}_2) = \mathrm{CatA}, \qquad f(\mathrm{sku}_3) = f(\mathrm{sku}_4) = \mathrm{CatB}, \qquad g(\mathrm{CatA}) = g(\mathrm{CatB}) = \mathrm{All}.$$

A fuzzy set $\mu \in F(X)$ encodes *customer interest* scores for each SKU:

$$\mu(\mathrm{sku}_1) = 0.80, \quad \mu(\mathrm{sku}_2) = 0.30, \quad \mu(\mathrm{sku}_3) = 0.60, \quad \mu(\mathrm{sku}_4) = 0.20.$$

Pushing forward along f aggregates to category-level interest by taking the maximum on each fiber:

$$\big(F(f)(\mu)\big)(\mathrm{CatA}) = \max\{0.80, 0.30\} = 0.80, \qquad \big(F(f)(\mu)\big)(\mathrm{CatB}) = \max\{0.60, 0.20\} = 0.60.$$

Pushing forward again along g yields the global interest score

$$\Big(F(g)\big(F(f)(\mu)\big)\Big)(\mathrm{All}) = \max\{0.80, 0.60\} = 0.80.$$

Direct aggregation in one step agrees by functoriality:

$$\big(F(g \circ f)(\mu)\big)(\mathrm{All}) = \max\{0.80, 0.30, 0.60, 0.20\} = 0.80 = \Big(F(g) \circ F(f)\Big)(\mu)(\mathrm{All}).$$

Practical reading. F turns any mapping of items to coarser groupings into a consistent rule for **fuzzy** (graded) summaries:

$$\text{SKU scores} \xrightarrow{\text{max -aggregate along } f} \text{Category scores} \xrightarrow{\text{max -aggregate along } g} \text{Global score},$$

and the result is invariant to whether one aggregates stepwise or in a single pass.

Example 1.6 (Real-world Neutrosophic Functorial Structure: clinical triage from patients → wards → hospital)**.** Let C = **FinSet** (finite sets and functions). Define a covariant functor

$$F : C \longrightarrow \mathbf{Set}, \qquad F(X) = \big([0,1]^3\big)^X = \Big\{ \nu : X \to [0,1]^3 \Big\},$$

where for $x \in X$ we write $\nu(x) = (T_X(x), I_X(x), F_X(x))$ as the *neutrosophic triple* (truth/indeterminacy/falsity degrees) for a fixed predicate (e.g. "influenza suspicion"). For a function $f : X \to Y$, define the pushforward componentwise by fiberwise suprema:

$$\big(F(f)(\nu)\big)(y) \;=\; \Big(\sup_{x:\, f(x)=y} T_X(x),\ \sup_{x:\, f(x)=y} I_X(x),\ \sup_{x:\, f(x)=y} F_X(x) \Big), \quad \sup\emptyset := 0.$$

Then $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ is immediate, and for $X \xrightarrow{f} Y \xrightarrow{g} Z$ and $z \in Z$,

$$\begin{aligned}
\big(F(g\circ f)(\nu)\big)(z) &= \Big(\sup_{x:\, g(f(x))=z} T_X(x),\ \sup_{x:\, g(f(x))=z} I_X(x),\ \sup_{x:\, g(f(x))=z} F_X(x)\Big) \\
&= \Big(\sup_{y:\, g(y)=z}\ \sup_{x:\, f(x)=y} T_X(x),\ \sup_{y:\, g(y)=z}\ \sup_{x:\, f(x)=y} I_X(x),\ \sup_{y:\, g(y)=z}\ \sup_{x:\, f(x)=y} F_X(x)\Big) \\
&= \Big(F(g)\big(F(f)(\nu)\big)\Big)(z)
\end{aligned}$$

since $\sup_y \sup_{x\in f^{-1}(y)}(\cdot) = \sup_x(\cdot)$; hence $F(g \circ f) = F(g) \circ F(f)$.

Concrete instance (with explicit numbers). Let $X = \{p_1, p_2, p_3, p_4\}$ be patients, $Y = \{\text{WardA}, \text{WardB}\}$ wards, and $Z = \{\text{Hospital}\}$ a single node. Define

$$f : X \to Y, \quad f(p_1) = f(p_2) = \text{WardA}, \quad f(p_3) = f(p_4) = \text{WardB},$$

$$g : Y \to Z,\ g(\text{WardA}) = g(\text{WardB}) = \text{Hospital}.$$

Let $\nu \in F(X)$ encode triage assessments for "influenza suspicion" as

$$\begin{aligned}
\nu(p_1) &= (T, I, F) = (0.90,\ 0.10,\ 0.00), \\
\nu(p_2) &= (0.40,\ 0.30,\ 0.20), \\
\nu(p_3) &= (0.50,\ 0.40,\ 0.10), \\
\nu(p_4) &= (0.10,\ 0.20,\ 0.80).
\end{aligned}$$

Pushforward to wards (componentwise maxima over each fiber of f):

$$\begin{aligned}
\big(F(f)(\nu)\big)(\text{WardA}) &= \Big(\max\{0.90, 0.40\},\ \max\{0.10, 0.30\},\ \max\{0.00, 0.20\}\Big) \\
&= (0.90,\ 0.30,\ 0.20), \\
\big(F(f)(\nu)\big)(\text{WardB}) &= \Big(\max\{0.50, 0.10\},\ \max\{0.40, 0.20\},\ \max\{0.10, 0.80\}\Big) \\
&= (0.50,\ 0.40,\ 0.80).
\end{aligned}$$

Pushforward to the hospital via g (max across wards):

$$\begin{aligned}
\Big(F(g)\big(F(f)(\nu)\big)\Big)(\text{Hospital}) &= \Big(\max\{0.90, 0.50\},\ \max\{0.30, 0.40\},\ \max\{0.20, 0.80\}\Big) \\
&= (0.90,\ 0.40,\ 0.80).
\end{aligned}$$

Direct one-step pushforward agrees by functoriality:

$$\begin{aligned}
\big(F(g \circ f)(\nu)\big)(\text{Hospital}) &= \Big(\max\{0.90, 0.40, 0.50, 0.10\},\ \max\{0.10, 0.30, 0.40, 0.20\},\ \max\{0.00, 0.20, 0.10, 0.80\}\Big) \\
&= (0.90,\ 0.40,\ 0.80) = \Big(F(g) \circ F(f)\Big)(\nu)(\text{Hospital}).
\end{aligned}$$

Reading. F assigns to each population a neutrosophic assessment and pushes it forward along any regrouping map (patients → wards → hospital) by fiberwise maxima, giving consistent, order-invariant summaries of truth/indeterminacy/falsity degrees.

2 Main Results

This section presents the main results of the paper.

2.1 HybridFunctorial Structure

We couple two functors on the same base by a natural transformation and package their *compatible pairs* into a single functor.

Definition 2.1 (HybridFunctorial Structure)**.** Fix a category C. A *HybridFunctorial Structure* on C is a triple (F, G, τ) where $F, G\colon C \longrightarrow \mathbf{Set}$ are covariant functors and $\tau\colon F \Rightarrow G$ is a natural transformation, i.e., for every morphism $f\colon X \longrightarrow Y$ in C,

$$G(f) \circ \tau_X \;=\; \tau_Y \circ F(f).$$

Its *underlying hybrid functor* $H_{F,G,\tau}\colon C \longrightarrow \mathbf{Set}$ is defined on objects by

$$H_{F,G,\tau}(X) \;:=\; \big\{(x, y) \in F(X) \times G(X) \;\big|\; y = \tau_X(x)\big\},$$

and on a morphism $f\colon X \longrightarrow Y$ by

$$H_{F,G,\tau}(f)(x, y) \;:=\; \big(F(f)(x),\; G(f)(y)\big).$$

Example 2.2 (Concrete hybrid on $C = \mathbf{FinSet}$)**.** Let $F = \mathcal{P}$ and $G = \mathcal{P}^2$ be the power set and double power set functors. Define $\tau_X\colon \mathcal{P}(X) \longrightarrow \mathcal{P}^2(X)$ by $\tau_X(A) = \{A\}$. For any $f\colon X \longrightarrow Y$,

$$G(f)\big(\tau_X(A)\big) \;=\; \mathcal{P}^2(f)\big(\{A\}\big) \;=\; \{\mathcal{P}(f)(A)\} \;=\; \tau_Y\big(\mathcal{P}(f)(A)\big),$$

so τ is natural. Hence $(\mathcal{P}, \mathcal{P}^2, \tau)$ is a HybridFunctorial Structure and $H_{F,G,\tau}(X) = \{(A, \{A\}) \mid A \subseteq X\}$.

Example 2.3 (Real-world HybridFunctorial Structure: E-commerce logistics)**.** Fix a category C whose objects are delivery zones Z, each equipped with a set of postal addresses A_Z and a fixed finite set of package types Pkg. A morphism $f : Z \longrightarrow Z'$ is specified by an address remapping function $p_f : A_Z \longrightarrow A_{Z'}$; composition is function composition.

Define two covariant functors $F, G : C \longrightarrow \mathbf{Set}$ by

$$F(Z) = \mathcal{P}_{\text{fin}}(A_Z \times \text{Pkg}), \qquad F(f)(S) = \{\, \big(p_f(a),\ p\big) \mid (a, p) \in S\,\},$$

$$G(Z) = \mathcal{P}_{\text{fin}}(A_Z \times \text{Pkg} \times \{\text{deliver}\}), \qquad G(f)(T) = \{\, \big(p_f(a),\ p,\ \text{deliver}\big) \mid (a, p, \text{deliver}) \in T\,\}.$$

Define a natural transformation $\tau : F \Rightarrow G$ by

$$\tau_Z(S) = \{\, \big(a,\ p,\ \text{deliver}\big) \mid (a, p) \in S\,\}.$$

Then (F, G, τ) is a *HybridFunctorial Structure*. Naturality holds objectwise: for any $f : Z \longrightarrow Z'$ and $S \in F(Z)$,

$$G(f)\big(\tau_Z(S)\big) = \big\{\, (p_f(a), p, \text{deliver}) \;\big|\; (a, p) \in S \big\} = \tau_{Z'}\big(F(f)(S)\big).$$

Concrete instance. Let $A_Z = \{\alpha, \beta\}$, $A_{Z'} = \{\gamma, \delta\}$, $\text{Pkg} = \{\text{S}, \text{M}\}$, and $p_f(\alpha) = \gamma$, $p_f(\beta) = \delta$. For $S = \{(\alpha, \text{S}), (\beta, \text{M})\}$,

$$\tau_Z(S) = \{(\alpha, \text{S}, \text{deliver}), (\beta, \text{M}, \text{deliver})\},$$

$$G(f)\big(\tau_Z(S)\big) = \{(\gamma, \text{S}, \text{deliver}), (\delta, \text{M}, \text{deliver})\} = \tau_{Z'}\big(\{(\gamma, \text{S}), (\delta, \text{M})\}\big) = \tau_{Z'}\big(F(f)(S)\big).$$

Example 2.4 (HybridFunctorial Structure: list-to-multiset tally is natural)**.** Let $C = \mathbf{FinSet}$. Define two functors $F, G : C \longrightarrow \mathbf{Set}$ by

$$F(X) = \text{List}(X) := \bigsqcup_{n \geq 0} X^n \quad \text{and} \quad G(X) = \mathbb{N}^{(X)} = \{\, m : X \longrightarrow \mathbb{N} \text{ with finite support}\,\}.$$

For $f : X \longrightarrow Y$ and a list $L = [x_1, \ldots, x_n] \in F(X)$, set

$$F(f)(L) := [\, f(x_1), \ldots, f(x_n)\,],$$

and, for $m \in G(X)$, define the multiset pushforward by summing on fibers,

$$\big(G(f)(m)\big)(y) := \sum_{x \in f^{-1}(y)} m(x) \qquad (y \in Y).$$

Define a natural transformation $\tau : F \Rightarrow G$ by the *tally map*

$$\big(\tau_X(L)\big)(x) := \#\{\, i \in \{1, \dots, n\} \mid x_i = x \,\}.$$

Naturality. For $f : X \to Y$ and $y \in Y$,

$$\begin{aligned}\big(G(f) \circ \tau_X\big)(L)(y) &= \sum_{x \in f^{-1}(y)} \#\{\, i \mid x_i = x \,\} = \#\{\, i \mid f(x_i) = y \,\} \\ &= \big(\tau_Y \circ F(f)\big)(L)(y),\end{aligned}$$

so $G(f) \circ \tau_X = \tau_Y \circ F(f)$.

Concrete numbers. Let $X = \{a, b, c\}$, $Y = \{u, v\}$ with $f(a) = u$, $f(b) = f(c) = v$, and

$$L = [a, b, b, c].$$

Then

$$\tau_X(L)(a, b, c) = (1, 2, 1).$$

Pushing forward the multiset,

$$\big(G(f) \circ \tau_X\big)(L)(u) = 1, \qquad \big(G(f) \circ \tau_X\big)(L)(v) = 2 + 1 = 3.$$

On the other hand

$$F(f)(L) = [u, v, v, v] \;\Rightarrow\; \tau_Y\big(F(f)(L)\big)(u, v) = (1, 3),$$

which matches $(1, 3)$ above. Therefore (F, G, τ) is a HybridFunctorial Structure.

Example 2.5 (HybridFunctorial Structure: fuzzy set to θ-cut is natural)**.** Fix a threshold $\theta \in (0, 1]$ and take $C = \mathbf{FinSet}$. Define

$$F(X) = [0, 1]^X, \qquad \big(F(f)(\mu)\big)(y) := \sup\{\, \mu(x) \mid x \in X,\ f(x) = y \,\},$$

$$G(X) = \mathcal{P}(X), \qquad G(f)(S) = f[S] = \{\, f(x) \mid x \in S \,\}.$$

Let $\tau : F \Rightarrow G$ map each membership $\mu \in [0, 1]^X$ to its θ-cut:

$$\tau_X(\mu) := \{\, x \in X \mid \mu(x) \geq \theta \,\}.$$

Naturality. For $f : X \to Y$,

$$\begin{aligned}G(f)\big(\tau_X(\mu)\big) &= \{\, f(x) \mid \mu(x) \geq \theta \,\} \\ &= \{\, y \in Y \mid \exists x\ (f(x) = y \ \wedge\ \mu(x) \geq \theta) \,\} \\ &= \{\, y \in Y \mid \sup_{x : f(x) = y} \mu(x) \geq \theta \,\} = \tau_Y\big(F(f)(\mu)\big).\end{aligned}$$

Concrete numbers. Let $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2\}$ with $f(x_1) = y_1$, $f(x_2) = f(x_3) = y_2$, and

$$\mu(x_1) = 0.40,\ \mu(x_2) = 0.85,\ \mu(x_3) = 0.60, \quad \theta = 0.60.$$

Then $\tau_X(\mu) = \{x_2, x_3\}$ and thus $G(f)(\tau_X(\mu)) = \{y_2\}$. Also

$$\big(F(f)(\mu)\big)(y_1) = \sup\{0.40\} = 0.40, \quad \big(F(f)(\mu)\big)(y_2) = \sup\{0.85, 0.60\} = 0.85,$$

so $\tau_Y\big(F(f)(\mu)\big) = \{y_2\}$, which coincides with $G(f)\big(\tau_X(\mu)\big)$.

Proposition 2.6 (Well-definedness and functoriality of $H_{F,G,\tau}$)**.** *With notation as above, $H_{F,G,\tau}$ is a well-defined functor $C \to$ **Set**.*

Proof. Well-definedness: take $(x, y) \in H_{F,G,\tau}(X)$, so $y = \tau_X(x)$. Then

$$G(f)\big(y\big) \;=\; G(f)\big(\tau_X(x)\big) \;\overset{(*)}{=}\; \tau_Y\big(F(f)(x)\big),$$

where $(*)$ is the naturality equality $G(f)\circ\tau_X = \tau_Y\circ F(f)$. Hence $H_{F,G,\tau}(f)(x,y) = \big(F(f)(x),\ \tau_Y(F(f)(x))\big) \in H_{F,G,\tau}(Y)$.

Functoriality: for $X \xrightarrow{f} Y \xrightarrow{g} Z$,

$$\begin{aligned} H_{F,G,\tau}(g\circ f)(x,y) &= \big(F(g\circ f)(x),\ G(g\circ f)(y)\big) \\ &= \big(F(g)(F(f)(x)),\ G(g)(G(f)(y))\big) \\ &= H_{F,G,\tau}(g)(F(f)(x),\ G(f)(y)) \\ &= \big(H_{F,G,\tau}(g)\circ H_{F,G,\tau}(f)\big)(x,y), \end{aligned}$$

and $H_{F,G,\tau}(\mathrm{id}_X) = \mathrm{id}_{H_{F,G,\tau}(X)}$ is immediate. □

Theorem 2.7 (Hybrid generalizes Functorial)**.** *There is a fully faithful embedding*

$$\mathbf{Fun}(C,\mathbf{Set}) \hookrightarrow \mathbf{HyFun}(C,\mathbf{Set}), \qquad F \longmapsto (F,F,\mathrm{id}_F),$$

where $\mathbf{HyFun}(C,\mathbf{Set})$ *is the category whose objects are* (F,G,τ) *and whose morphisms* $(\alpha,\beta)\colon (F,G,\tau)\to(F',G',\tau')$ *are pairs of natural transformations* $\alpha\colon F\Rightarrow F'$, $\beta\colon G\Rightarrow G'$ *satisfying the* hybrid compatibility

$$\tau'_X\circ\alpha_X \;=\; \beta_X\circ\tau_X \quad \textit{for all } X\in \mathrm{Ob}(C).$$

Proof. Define $J\colon \mathbf{Fun}(C,\mathbf{Set})\to\mathbf{HyFun}(C,\mathbf{Set})$ by $J(F) = (F,F,\mathrm{id}_F)$ on objects and $J(\eta) = (\eta,\eta)$ on morphisms $\eta\colon F\Rightarrow F'$. For compatibility,

$$\tau'_X\circ\eta_X \;=\; \mathrm{id}_{F'(X)}\circ\eta_X \;=\; \eta_X \;=\; \eta_X\circ\mathrm{id}_{F(X)} \;=\; \eta_X\circ\tau_X.$$

Thus J is a functor. Fullness and faithfulness are immediate since $\mathrm{Nat}(F,F')\ni\eta\longleftrightarrow(\eta,\eta)\in\mathrm{Hom}_{\mathbf{HyFun}}(J(F),J(F'))$ is a bijection. □

2.2 MultiFunctorial Structure

We now allow finitely many functors with *coherent* natural transformations among them.

Definition 2.8 (MultiFunctorial Structure)**.** Fix a category C and a finite nonempty index set I equipped with a reflexive and transitive relation $\le$ (a finite preorder). A *MultiFunctorial Structure* on C indexed by $(I,\le)$ is:

- a family of covariant functors $F_i\colon C\to\mathbf{Set}$ for $i\in I$;
- for every $i\le j$ in I, a natural transformation $\tau_{ij}\colon F_i\Rightarrow F_j$,

such that the *coherence axioms* hold:

$$\tau_{ii} = \mathrm{id}_{F_i} \quad \text{and} \quad \tau_{jk}\circ\tau_{ij} = \tau_{ik} \quad \text{whenever } i\le j\le k.$$

Define the *underlying multi-functor* $M\colon C\to\mathbf{Set}$ by

$$M(X) := \Big\{(x_i)_{i\in I}\in\prod_{i\in I}F_i(X)\ \Big|\ \forall\, i\le j:\ x_j = \tau_{ij,X}(x_i)\Big\},$$

and for $f\colon X\to Y$,

$$M(f)\big((x_i)_{i\in I}\big) := \big(F_i(f)(x_i)\big)_{i\in I}.$$

Example 2.9 (Concrete multi on $C = \mathbf{FinSet}$)**.** Let $I = \{1,2,3\}$ with $1\le 2\le 3$. Take $F_1 = \mathrm{Id}$, $F_2 = \mathcal{P}$, $F_3 = \mathcal{P}^2$. Define $\tau_{12,X}(x) = \{x\}$, $\tau_{23,X}(A) = \{A\}$, and $\tau_{13,X} = \tau_{23,X}\circ\tau_{12,X}$, i.e., $\tau_{13,X}(x) = \{\{x\}\}$. Coherence holds since

$$\tau_{23,X}\big(\tau_{12,X}(x)\big) = \{\,\{x\}\,\} = \tau_{13,X}(x), \qquad \tau_{ii} = \mathrm{id}\,.$$

For $f\colon X\to Y$ and $(x,\ A,\ B)\in M(X)$ with $A = \{x\}$, $B = \{A\}$,

$$\begin{aligned} M(f)(x,A,B) &= \big(f(x),\ \mathcal{P}(f)(A),\ \mathcal{P}^2(f)(B)\big) \\ &= \big(f(x),\ \{f(x)\},\ \{\{f(x)\}\}\big), \end{aligned}$$

and the compatibility equalities are preserved because $\mathcal{P}(f)(\{x\}) = \{f(x)\}$ and $\mathcal{P}^2(f)(\{A\}) = \{\mathcal{P}(f)(A)\}$.

Example 2.10 (Real-world HybridFunctorial Structure: E-commerce logistics)**.** Fix a category C whose objects are delivery zones Z, each equipped with a set of postal addresses A_Z and a fixed finite set of package types Pkg. A morphism $f : Z \to Z'$ is specified by an address remapping function $p_f : A_Z \to A_{Z'}$; composition is function composition.

Define two covariant functors $F, G : C \to \mathbf{Set}$ by

$$F(Z) = \mathcal{P}_{\text{fin}}(A_Z \times \text{Pkg}), \qquad F(f)(S) = \{\, \big(p_f(a),\ p\big) \mid (a, p) \in S \,\},$$

$$G(Z) = \mathcal{P}_{\text{fin}}(A_Z \times \text{Pkg} \times \{\text{deliver}\}), \qquad G(f)(T) = \{\, \big(p_f(a),\ p,\ \text{deliver}\big) \mid (a, p, \text{deliver}) \in T \,\}.$$

Define a natural transformation $\tau : F \Rightarrow G$ by

$$\tau_Z(S) = \{\, \big(a,\ p,\ \text{deliver}\big) \mid (a, p) \in S \,\}.$$

Then (F, G, τ) is a *HybridFunctorial Structure*. Naturality holds objectwise: for any $f : Z \to Z'$ and $S \in F(Z)$,

$$G(f)\big(\tau_Z(S)\big) = \Big\{\, (p_f(a), p, \text{deliver}) \ \Big|\ (a, p) \in S \,\Big\} = \tau_{Z'}\big(F(f)(S)\big).$$

Concrete instance. Let $A_Z = \{\alpha, \beta\}$, $A_{Z'} = \{\gamma, \delta\}$, Pkg $= \{\text{S}, \text{M}\}$, and $p_f(\alpha) = \gamma$, $p_f(\beta) = \delta$. For $S = \{(\alpha, \text{S}), (\beta, \text{M})\}$,

$$\tau_Z(S) = \{(\alpha, \text{S}, \text{deliver}), (\beta, \text{M}, \text{deliver})\},$$

$$G(f)\big(\tau_Z(S)\big) = \{(\gamma, \text{S}, \text{deliver}), (\delta, \text{M}, \text{deliver})\} = \tau_{Z'}\big(\{(\gamma, \text{S}), (\delta, \text{M})\}\big) = \tau_{Z'}\big(F(f)(S)\big).$$

Example 2.11 (Real-world MultiFunctorial Structure: Healthcare cohort → buckets → counts)**.** Let C be the category whose objects are hospitals H with a patient set P_H and fixed label sets: age brackets B and diagnosis codes D. A morphism $f : H \to H'$ is given by a referral map $r_f : P_H \to P_{H'}$ that *preserves labels*: for every $p \in P_H$,

$$\text{Age}_{H'}\big(r_f(p)\big) = \text{Age}_H(p) \in B, \qquad \text{Diag}_{H'}\big(r_f(p)\big) = \text{Diag}_H(p) \in D.$$

Fix the preorder $I = \{1, 2, 3\}$ with $1 \le 2 \le 3$ and define three functors:

$$\begin{aligned} F_1(H) &= \mathcal{P}_{\text{fin}}(P_H) && \text{(finite patient cohorts)},\\ F_2(H) &= \mathcal{P}_{\text{fin}}(B \times D) && \text{(finite label buckets)},\\ F_3(H) &= \mathbb{N}^{B \times D} && \text{(count tables on } B \times D\text{)}. \end{aligned}$$

Their actions on $f : H \to H'$ are

$$F_1(f)(S) = \{\, r_f(p) \mid p \in S \,\}, \qquad F_2(f) = \text{id}_{\mathcal{P}_{\text{fin}}(B \times D)}, \qquad F_3(f) = \text{id}_{\mathbb{N}^{B \times D}}\,.$$

Define natural transformations $\tau_{12} : F_1 \Rightarrow F_2$ and $\tau_{23} : F_2 \Rightarrow F_3$ by

$$\tau_{12,H}(S) = \{\, \big(\text{Age}_H(p), \text{Diag}_H(p)\big) \mid p \in S \,\},$$

$$\tau_{23,H}(T) = c_T \in \mathbb{N}^{B \times D}, \quad c_T(b, d) = \#\{\, (b', d') \in T \mid (b', d') = (b, d) \,\}.$$

Coherence holds with $\tau_{11} = \text{id}_{F_1}$, $\tau_{22} = \text{id}_{F_2}$, $\tau_{33} = \text{id}_{F_3}$ and $\tau_{13} = \tau_{23} \circ \tau_{12}$. Naturality is immediate: for $S \in F_1(H)$,

$$\begin{aligned} F_2(f)\big(\tau_{12,H}(S)\big) &= \tau_{12,H}(S) = \{\, (\text{Age}_H(p), \text{Diag}_H(p)) \mid p \in S \,\}\\ &= \{\, (\text{Age}_{H'}(r_f(p)), \text{Diag}_{H'}(r_f(p))) \mid p \in S \,\} = \tau_{12,H'}\big(F_1(f)(S)\big), \end{aligned}$$

and $F_3(f) \circ \tau_{23,H} = \tau_{23,H'} \circ F_2(f)$ since both sides are the identity on $\mathbb{N}^{B \times D}$.

Concrete instance. Let $B = \{30\text{s}, 50\text{s}\}$, $D = \{\text{A}, \text{B}\}$, and in H take three patients p_1, p_2, p_3 with

$$(\text{Age}_H, \text{Diag}_H)(p_1) = (30\text{s}, \text{A}), \ (\text{Age}_H, \text{Diag}_H)(p_2) = (30\text{s}, \text{B}), \ (\text{Age}_H, \text{Diag}_H)(p_3) = (50\text{s}, \text{A}).$$

For the cohort $S = \{p_1, p_2, p_3\}$,

$$\tau_{12,H}(S) = \{(30\text{s}, \text{A}), (30\text{s}, \text{B}), (50\text{s}, \text{A})\},$$

$$\tau_{23,H}\big(\tau_{12,H}(S)\big) = c \in \mathbb{N}^{B \times D}, \quad c(30\text{s}, \text{A}) = 1, \ c(30\text{s}, \text{B}) = 1, \ c(50\text{s}, \text{A}) = 1,$$

and $c(b, d) = 0$ for all other $(b, d) \in B \times D$. If $f : H \to H'$ merely relabels patient identifiers via r_f while preserving (Age, Diag), then the above equalities are unchanged in H' by naturality.

Example 2.12 (MultiFunctorial Structure: fuzzy ladder → α-cut → non-emptiness)**.** Let $C = \mathbf{FinSet}$ and fix a threshold $\theta \in (0,1]$. Take the preorder $I = \{1,2,3\}$ with $1 \le 2 \le 3$ and define

$$F_1(X) = [0,1]^X, \qquad F_2(X) = \mathcal{P}(X), \qquad F_3(X) = \{0,1\}.$$

For $f : X \to Y$, set

$$\big(F_1(f)(\mu)\big)(y) := \sup_{x : f(x) = y} \mu(x), \qquad F_2(f)(S) = f[S], \qquad F_3(f) = \mathrm{id}_{\{0,1\}}.$$

Define natural transformations

$$\tau_{12,X}(\mu) = \{\, x \in X \mid \mu(x) \ge \theta \,\}, \qquad \tau_{23,X}(S) = \begin{cases} 1, & S \neq \emptyset, \\ 0, & S = \emptyset. \end{cases}$$

Set $\tau_{11} = \mathrm{id}_{F_1}$, $\tau_{22} = \mathrm{id}_{F_2}$, $\tau_{33} = \mathrm{id}_{F_3}$, and $\tau_{13} := \tau_{23} \circ \tau_{12}$. Coherence holds by construction. Naturality is immediate from Example 2.5 and the fact that $F_3(f)$ is the identity.

Concrete numbers. Let $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2\}$ with $f(x_1) = y_1$, $f(x_2) = f(x_3) = y_2$, and

$$\mu(x_1) = 0.40, \ \mu(x_2) = 0.85, \ \mu(x_3) = 0.60, \quad \theta = 0.60.$$

Then $\tau_{12,X}(\mu) = \{x_2, x_3\}$, so $\tau_{23,X}\big(\tau_{12,X}(\mu)\big) = 1$. Pushing forward:

$$\big(F_1(f)(\mu)\big)(y_1) = 0.40, \quad \big(F_1(f)(\mu)\big)(y_2) = 0.85,$$

so $\tau_{12,Y}\big(F_1(f)(\mu)\big) = \{y_2\}$ and $\tau_{23,Y}$ of that set is again 1. Thus $\tau_{23} \circ \tau_{12}$ commutes with f, and M is well defined.

Example 2.13 (MultiFunctorial Structure: list → multiset → labeled counts)**.** Fix a finite label set C. Let $\mathcal{C}$ be the category whose objects are pairs (X, ℓ_X) with a labeling map $\ell_X : X \to C$, and whose morphisms $f : (X, \ell_X) \to (Y, \ell_Y)$ are functions $f : X \to Y$ preserving labels:

$$\ell_Y \circ f = \ell_X.$$

Take the preorder $I = \{1,2,3\}$ with $1 \le 2 \le 3$ and define functors

$$F_1(X, \ell_X) = \mathrm{List}(X) = \bigsqcup_{n \ge 0} X^n, \qquad F_2(X, \ell_X) = \mathbb{N}^{(X)}, \qquad F_3(X, \ell_X) = \mathbb{N}^C.$$

For $f : (X, \ell_X) \to (Y, \ell_Y)$ set

$$F_1(f)\big([x_1, \dots, x_n]\big) = [\, f(x_1), \dots, f(x_n) \,], \qquad \big(F_2(f)(m)\big)(y) = \sum_{x \in f^{-1}(y)} m(x), \qquad F_3(f) = \mathrm{id}_{\mathbb{N}^C}.$$

Define natural transformations by

$$\big(\tau_{12,X}(L)\big)(x) = \#\{\, i \mid x_i = x \,\}, \qquad \big(\tau_{23,X}(m)\big)(c) = \sum_{x:\ \ell_X(x) = c} m(x) \quad (c \in C),$$

and set $\tau_{11} = \mathrm{id}_{F_1}$, $\tau_{22} = \mathrm{id}_{F_2}$, $\tau_{33} = \mathrm{id}_{F_3}$, $\tau_{13} = \tau_{23} \circ \tau_{12}$. Coherence is automatic. Naturality of τ_{12} is Example 2.4; naturality of τ_{23} follows from $\ell_Y \circ f = \ell_X$:

$$\begin{aligned}\big(F_3(f) \circ \tau_{23,X}\big)(m)(c) &= \sum_{x:\ \ell_X(x) = c} m(x) = \sum_{y:\ \ell_Y(y) = c} \ \sum_{x:\ f(x) = y} m(x) \\ &= \sum_{y:\ \ell_Y(y) = c} \big(F_2(f)(m)\big)(y) = \big(\tau_{23,Y} \circ F_2(f)\big)(m)(c).\end{aligned}$$

Concrete numbers. Let $C = \{\mathrm{A}, \mathrm{B}\}$ and (X, ℓ_X) with

$$X = \{x_1, x_2, x_3\}, \qquad \ell_X(x_1) = \mathrm{A}, \ \ell_X(x_2) = \mathrm{A}, \ \ell_X(x_3) = \mathrm{B}.$$

Take the list $L = [x_1, x_2, x_2, x_3]$. Then

$$\tau_{12,X}(L)(x_1, x_2, x_3) = (1, 2, 1).$$

Aggregate by labels:

$$\tau_{23,X}\big(\tau_{12,X}(L)\big)(\mathrm{A}) = 1 + 2 = 3, \qquad \tau_{23,X}\big(\tau_{12,X}(L)\big)(\mathrm{B}) = 1.$$

Now let (Y, ℓ_Y) with $Y = \{y_1, y_2, y_3\}$ and

$$\ell_Y(y_1) = \mathrm{A}, \quad \ell_Y(y_2) = \mathrm{B}, \quad \ell_Y(y_3) = \mathrm{A},$$

and define the label-preserving map $f : X \to Y$ by $f(x_1) = y_3$ (A→A), $f(x_2) = y_1$ (A→A), $f(x_3) = y_2$ (B→B). Then

$$F_1(f)(L) = [y_3, y_1, y_1, y_2],$$
$$\tau_{12,Y}\big(F_1(f)(L)\big)(y_1, y_2, y_3) = (2, 1, 1),$$

and aggregating on labels gives

$$\tau_{23,Y}\big(\tau_{12,Y}(F_1(f)(L))\big)(\mathrm{A}) = 2 + 1 = 3, \qquad \tau_{23,Y}\big(\tau_{12,Y}(F_1(f)(L))\big)(\mathrm{B}) = 1,$$

which coincides with the previous $(3, 1)$. Hence M is well defined and functorial.

Proposition 2.14 (Well-definedness and functoriality of M)**.** *With notation as above, M is a well-defined functor $C \to$ **Set**.*

Proof. Well-definedness: suppose $(x_i)_i \in M(X)$, so for all $i \le j$, $x_j = \tau_{ij,X}(x_i)$. Then, for any $i \le j$,

$$F_j(f)(x_j) = F_j(f)\big(\tau_{ij,X}(x_i)\big) \;=\; \tau_{ij,Y}\big(F_i(f)(x_i)\big) \quad \text{(naturality of } \tau_{ij}\text{)}.$$

Hence $\big(F_i(f)(x_i)\big)_i \in M(Y)$. Functoriality follows componentwise:

$$M(g \circ f)((x_i)) = \big(F_i(g \circ f)(x_i)\big)_i = \big(F_i(g)\,F_i(f)(x_i)\big)_i = M(g)\big(M(f)((x_i))\big),$$

and $M(\mathrm{id}_X) = \mathrm{id}_{M(X)}$ holds componentwise. □

Theorem 2.15 (Multi generalizes Hybrid and Functorial)**.** *Let C be a category.*

*(i) Every Functorial Structure $F\colon C \to$ **Set** yields a MultiFunctorial Structure by taking $I = \{1\}$, $F_1 = F$, and $M(X) = F(X)$.*

(ii) Every HybridFunctorial Structure (F, G, τ) yields a MultiFunctorial Structure by taking $I = \{1, 2\}$ with order $1 \le 2$, setting $F_1 = F$, $F_2 = G$, $\tau_{11} = \mathrm{id}_F$, $\tau_{22} = \mathrm{id}_G$, and $\tau_{12} = \tau$.

Conversely, any MultiFunctorial Structure with $|I| = 1$ is precisely a Functorial Structure, and any MultiFunctorial Structure with $I = \{1, 2\}$, $1 \le 2$, is precisely a HybridFunctorial Structure.

Proof. (i) Trivial: the only coherence map is $\tau_{11} = \mathrm{id}_F$, and $M(X) = \{x_1 \mid x_1 \in F(X)\} \cong F(X)$.

(ii) With $I = \{1, 2\}$ and $1 \le 2$, the coherence data are exactly $\tau_{11} = \mathrm{id}_F$, $\tau_{22} = \mathrm{id}_G$, and $\tau_{12} = \tau$. Then

$$M(X) = \{(x_1, x_2) \in F(X) \times G(X) \mid x_2 = \tau_X(x_1)\} \;=\; H_{F,G,\tau}(X),$$

and $M(f) = H_{F,G,\tau}(f)$ by Definition 2.1.

Conversely, if $|I| = 1$, say $I = \{1\}$, then $M(X) = F_1(X)$ and $M(f) = F_1(f)$. If $I = \{1, 2\}$ with $1 \le 2$, the only nontrivial coherence is a single natural transformation $\tau_{12}\colon F_1 \Rightarrow F_2$, yielding precisely a hybrid triple (F_1, F_2, τ_{12}). □

Proposition 2.16 (Reconstruction from a chosen root index)**.** *Let* $(F_i, \tau_{ij})_{i \le j}$ *be a MultiFunctorial Structure and fix any* $r \in I$*. Define*

$$\alpha_r(X) \colon M(X) \to F_r(X), \qquad \alpha_r\big((x_i)_{i \in I}\big) := x_r,$$

and

$$\beta_r(X) \colon F_r(X) \to M(X), \qquad \beta_r(s) := \big(\tau_{ri,X}(s)\big)_{i \in I}.$$

Then α_r *and* β_r *are natural in* X*, and satisfy*

$$\alpha_r(X) \circ \beta_r(X) = \mathrm{id}_{F_r(X)}, \qquad \beta_r(X) \circ \alpha_r(X) = \mathrm{id}_{M(X)}.$$

Proof. For any $s \in F_r(X)$,

$$\alpha_r(\beta_r(s)) = \alpha_r\big((\tau_{ri,X}(s))_{i \in I}\big) = \tau_{rr,X}(s) = \mathrm{id}_{F_r(X)}(s) = s.$$

For any $(x_i)_i \in M(X)$, the coherence gives $x_i = \tau_{ri,X}(x_r)$ for all i, hence

$$\beta_r(\alpha_r((x_i)_i)) = \beta_r(x_r) = \big(\tau_{ri,X}(x_r)\big)_i = (x_i)_i.$$

Naturality of α_r, β_r follows from naturality of each τ_{ri} and the definition of $M(f)$. □

3 Conclusion

We introduced HybridFunctorial and MultiFunctorial Structures on a fixed base category C, verified the induced functors $H_{F,G,\tau}$ and M are well defined by explicit naturality equations, and proved precise embeddings:

$$\begin{tikzcd}
\mathbf{Fun}(C, \mathbf{Set}) \arrow[r, hook, "F \mapsto (F,F,\mathrm{id})"] & \mathbf{HyFun}(C, \mathbf{Set}) \arrow[r, hook, "(F,G,\tau) \mapsto (I=\{1,2\}, \tau_{12}=\tau)"] & \mathbf{MultiFun}(C, \mathbf{Set}).
\end{tikzcd}$$

Thus MultiFunctorial Structures strictly generalize HybridFunctorial and Functorial Structures.

Funding

No external funding was received for this work.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this work.

Acknowledgments

We thank all colleagues, reviewers, and readers whose comments and questions have greatly improved this manuscript. We are also grateful to the authors of the works cited herein for providing the theoretical foundations that underpin our study. Finally, we appreciate the institutional and technical support that enabled this research.

Data Availability

This paper is theoretical and did not generate or analyze any empirical data. We welcome future studies that apply and test these concepts in practical settings.

Research Integrity

The author confirms that this manuscript is original, has not been published elsewhere, and is not under consideration by any other journal.

Use of Computational Tools

All proofs and derivations were performed manually; no computational software (e.g., Mathematica, SageMath, Coq) was used.

Code Availability

No code or software was developed for this study.

Ethical Approval

This research did not involve human participants or animals, and therefore did not require ethical approval.

Use of Generative AI and AI-Assisted Tools

We use generative AI and AI-assisted tools for tasks such as English grammar checking, and We do not employ them in any way that violates ethical standards.

Supplementary Information

No supplementary materials accompany this paper.

Disclaimer

The ideas presented here are theoretical and have not yet been validated through empirical testing. While we have strived for accuracy and proper citation, inadvertent errors may remain. Readers should verify any referenced material independently. The opinions expressed are those of the authors and do not necessarily reflect the views of their institutions.

References

[1] Takaaki Fujita and Florentin Smarandache. A unified framework for u-structures and functorial structure: Managing super, hyper, superhyper, tree, and forest uncertain over/under/off models. *Neutrosophic Sets and Systems*, 91:337–380, 2025.

[2] Lotfi A Zadeh. Fuzzy sets. *Information and control*, 8(3):338–353, 1965.

[3] Lotfi A Zadeh. Fuzzy logic, neural networks, and soft computing. In *Fuzzy sets, fuzzy logic, and fuzzy systems: selected papers by Lotfi A Zadeh*, pages 775–782. World Scientific, 1996.

[4] John N Mordeson and Premchand S Nair. *Fuzzy graphs and fuzzy hypergraphs*, volume 46. Physica, 2012.

[5] Takaaki Fujita and Arif Mehmood. A study on the effectiveness of contradiction values in upside-down logic and de-plithogenication within plithogenic sets. *Neutrosophic Computing and Machine Learning*, 40(1):28–65, 2025.

[6] S Gomathy, D Nagarajan, S Broumi, and M Lathamaheswari. *Plithogenic sets and their application in decision making*. Infinite Study, 2020.

[7] Vicenç Torra and Yasuo Narukawa. On hesitant fuzzy sets and decision. In *2009 IEEE international conference on fuzzy systems*, pages 1378–1382. IEEE, 2009.

[8] Vicenç Torra. Hesitant fuzzy sets. *International journal of intelligent systems*, 25(6):529–539, 2010.

[9] Krassimir T Atanassov. Circular intuitionistic fuzzy sets. *Journal of Intelligent & Fuzzy Systems*, 39(5):5981–5986, 2020.

[10] Krassimir T Atanassov and G Gargov. *Intuitionistic fuzzy logics*. Springer, 2017.

[11] Said Broumi, Mohamed Talea, Assia Bakali, and Florentin Smarandache. Single valued neutrosophic graphs. *Journal of New theory*, (10):86–101, 2016.

[12] Haibin Wang, Florentin Smarandache, Yanqing Zhang, and Rajshekhar Sunderraman. *Single valued neutrosophic sets*. Infinite study, 2010.

[13] Florentin Smarandache. *Neutrosophic Overset, Neutrosophic Underset, and Neutrosophic Offset. Similarly for Neutrosophic Over-/Under-/Off-Logic, Probability, and Statistics*. Infinite Study, 2016.

[14] Zdzis law Pawlak. Rough sets. *International journal of computer & information sciences*, 11:341–356, 1982.

[15] Said Broumi, Florentin Smarandache, and Mamoni Dhar. Rough neutrosophic sets. *Infinite Study*, 32:493–502, 2014.

[16] Pradip Kumar Maji, Ranjit Biswas, and A Ranjan Roy. Soft set theory. *Computers & mathematics with applications*, 45(4-5):555–562, 2003.

[17] Dmitriy Molodtsov. Soft set theory-first results. *Computers & mathematics with applications*, 37(4-5):19–31, 1999.

[18] Florentin Smarandache, Xiaohong Zhang, and Mumtaz Ali. Algebraic structures of neutrosophic triplets, neutrosophic duplets, or neutrosophic multisets. *Symmetry*, 11(2):171, 2019.

[19] Kaoru Hirota. Concepts of probabilistic sets. *Fuzzy sets and systems*, 5(1):31–46, 1981.

[20] Hans-Jürgen Zimmermann. *Fuzzy set theory—and its applications*. Springer Science & Business Media, 2011.