Linear Diophantine Fuzzy SuperHyperGraphs and Fuzzy Planar SuperHyperGraph

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Abstract

Graph theory offers a rigorous framework for modeling relationships and connectivity via vertices and edges [1, 2]. Hypergraphs generalize this framework by allowing *hyperedges* that join more than two vertices [3]. Superhypergraphs further enrich the model through iterated powerset constructions, capturing hierarchical and self-referential structures among hyperedges [4]. A Linear Diophantine Fuzzy Graph assigns linear Diophantine fuzzy numbers to vertices and edges, with admissibility enforced by algebraic membership constraints. A planar graph admits a crossing-free drawing in the plane, preserving adjacency without edge intersections, and a fuzzy planar graph is one in which sufficiently strong edges form a planar subgraph, combining fuzziness with classical planar embeddings. In this paper, we extend Linear Diophantine Fuzzy Graphs and fuzzy planar graphs to the settings of HyperGraphs and SuperHyperGraphs by introducing new classes, including *Linear Diophantine Fuzzy SuperHyperGraphs* and *Fuzzy Planar SuperHyperGraphs*, and we investigate their fundamental properties.

Keywords: SuperHyperGraph, HyperGraph, Planar Graph, Fuzzy Graph, Fuzzy Hypergraph, Fuzzy SuperHyperGraph, Linear Diophantine Fuzzy Graph

1 Preliminaries

We collect the basic terminology and notation used in what follows. Unless explicitly stated otherwise, all graphs considered are finite, undirected, and loopless; multiple edges are allowed only when this is specified.

1.1 SuperHyperGraphs

A classical hypergraph generalizes an ordinary graph by permitting an edge to connect an arbitrary (finite) number of vertices, which makes it suitable for representing multiway relationships [3, 5–7]. A *SuperHyperGraph* carries this idea further by forming vertices and edges from iterated powersets of a base set; this viewpoint has appeared in several recent contexts [8–12]. Reported applications include, among others, molecular structure modeling, complex network analysis, and signal processing [13, 14]. Throughout, the *level n* is a fixed nonnegative integer.

Definition 1.1 (Base set)**.** A *base (ground) set* is a fixed finite set S from which higher-level objects are generated:

$$S \;=\; \{\, x \mid x \text{ belongs to the chosen domain} \,\}.$$

All structures introduced below ultimately draw their elements from S.

Definition 1.2 (Powerset)**.** [15, 16] Given a set X, its powerset is

$$\mathcal{P}(X) \;=\; \{\, A \subseteq X \,\}.$$

We also use the *nonempty* powerset $\mathcal{P}^*(X) := \mathcal{P}(X) \setminus \{\varnothing\}$.

Definition 1.3 (Iterated powerset)**.** [17–19] For $k \in \mathbb{N}_0$ define

$$\mathcal{P}^{0}(X) := X, \qquad \mathcal{P}^{k+1}(X) := \mathcal{P}\big(\mathcal{P}^{k}(X)\big).$$

For the nonempty version set

$$\big(\mathcal{P}^*\big)^0(X) := X, \qquad \big(\mathcal{P}^*\big)^{k+1}(X) := \mathcal{P}^*\big((\mathcal{P}^*)^k(X)\big).$$

Proposition 1.4 (Size recurrence)**.** *If* $m := |X|$*, then*

$$\big|\mathcal{P}^{1}(X)\big| = 2^m, \qquad \big|\mathcal{P}^{k+1}(X)\big| \;=\; 2^{\,|\mathcal{P}^{k}(X)|} \quad (k \geq 0).$$

In particular, when $m \geq 1$ *one has* $|(\mathcal{P}^*)^1(X)| = 2^m - 1$ *and* $\big|(\mathcal{P}^*)^{k+1}(X)\big| = 2^{\,|(\mathcal{P}^*)^k(X)|} - 1$.

Proof. The identity $|\mathcal{P}(Y)| = 2^{|Y|}$ is standard. Taking $Y = \mathcal{P}^k(X)$ yields $|\mathcal{P}^{k+1}(X)| = 2^{|\mathcal{P}^k(X)|}$. For the nonempty construction, remove the empty set at each step, i.e., subtract 1 from the full powerset count. □

Definition 1.5 (Hypergraph [3,20])**.** A *hypergraph* is a pair $H = (V(H), E(H))$ with $V(H) \neq \emptyset$ and $E(H) \subseteq \mathcal{P}^*(V(H))$. Throughout this paper both $V(H)$ and $E(H)$ are finite.

Definition 1.6 (n-SuperHyperGraph)**.** Fix a finite base set V_0 and a level $n \in \mathbb{N}_0$. An *n-SuperHyperGraph over* V_0 is a triple

$$\mathrm{SHG}^{(n)} = (V,\ E,\ \partial),$$

where

- $V \subseteq \mathcal{P}^n(V_0)$ is a finite set of *n-supervertices*;
- E is a finite set of *(super)edge identifiers*;
- $\partial : E \to \mathcal{P}^*(V)$ is an *incidence map* sending each edge to a nonempty finite subset of V.

For $e \in E$, the set $\partial(e) \subseteq V$ is called the *(super)edge incidence set*.

Remark 1.7 (Simple, uniform, and nonempty-tier options)**.** (i) *Simple*: ∂ is injective (no parallel superedges). (ii) k*-uniform*: $|\partial(e)| = k$ for all $e \in E$. (iii) To exclude empties at every tier, one may require $V \subseteq (\mathcal{P}^*)^n(V_0)$.

Remark 1.8 (Subset presentation)**.** If parallel superedges are unnecessary, one may identify each edge with its incidence set and work with a pair $(V, \mathcal{E})$ where $\mathcal{E} \subseteq \mathcal{P}^*(V)$. This is equivalent to Definition 1.6 by taking $E := \mathcal{E}$ and $\partial := \mathrm{id}$.

Example 1.9 (Meal-combo design in a cafeteria ($\boldsymbol{n = 1}$))**.** *Interpretation.* The base items are menu components; 1-supervertices are bundles of components offered together; superedges encode feasible combo pairings offered as set menus.

Construction. Let the base set be $V_0 = \{\mathrm{D}, \mathrm{S}, \mathrm{M}\}$, representing Drink (D), Side (S), and Main (M). For level $n = 1$ take

$$V = \big\{\{\mathrm{D}\},\ \{\mathrm{S}\},\ \{\mathrm{M}\},\ \{\mathrm{D}, \mathrm{S}\}\big\} \subseteq \mathcal{P}(V_0).$$

Introduce two superedges $E = \{e_1, e_2\}$ with incidence map

$$\partial(e_1) = \big\{\{\mathrm{D}\}, \{\mathrm{S}\}\big\}, \qquad \partial(e_2) = \big\{\{\mathrm{D}, \mathrm{S}\}, \{\mathrm{M}\}\big\}.$$

Verification. By construction $V \subseteq \mathcal{P}^1(V_0)$ and $\partial(e_i) \in \mathcal{P}^*(V)$ for $i = 1, 2$. Hence $\mathrm{SHG}^{(1)} = (V, E, \partial)$ is a valid 1-SuperHyperGraph.

Example 1.10 (Two-tier shipment consolidation ($\boldsymbol{n = 2}$))**.** *Interpretation.* Base items are SKUs. Level-2 supervertices collect *sets of item-groups* (e.g., pallet patterns); superedges indicate which pallet-patterns are jointly routed or consolidated at a hub.

Construction. Let $V_0 = \{x, y\}$ denote two SKUs. Then

$$\mathcal{P}(V_0) = \{\emptyset, \{x\}, \{y\}, \{x, y\}\}, \qquad \mathcal{P}^2(V_0) = \mathcal{P}\big(\mathcal{P}(V_0)\big).$$

Choose three level-2 supervertices

$$v_1 := \big\{\{x\}, \{y\}\big\}, \qquad v_2 := \big\{\{x, y\}\big\}, \qquad v_3 := \big\{\emptyset, \{x\}\big\},$$

and set $V = \{v_1, v_2, v_3\} \subseteq \mathcal{P}^2(V_0)$. Let $E = \{e_1, e_2\}$ with incidence

$$\partial(e_1) = \{v_1, v_2\}, \qquad \partial(e_2) = \{v_2, v_3\}.$$

Each $\partial(e_i)$ is a nonempty subset of V, so $\partial(e_i) \in \mathcal{P}^*(V)$ for $i = 1, 2$. Therefore $\mathrm{SHG}^{(2)} = (V, E, \partial)$ is a valid 2-SuperHyperGraph.

1.2 Fuzzy n-SuperHyperGraphs

A fuzzy set assigns to each element a membership degree in $[0, 1]$ [21,22]. Fuzzy graphs and fuzzy hypergraphs endow vertices and (hyper)edges with such degrees [23–30]. A fuzzy n-SuperHyperGraph is a higher-level network representation in which supervertices and superedges carry membership values for modeling complex interactions (cf. [4, 31]).

Definition 1.11 (Fuzzy graph)**.** A *fuzzy graph* is a triple $G = (V, \sigma, \mu)$ where V is a finite nonempty vertex set, $\sigma : V \to [0, 1]$ assigns vertex-membership degrees, and $\mu : V \times V \to [0, 1]$ assigns edge-membership degrees subject to

$$\mu(u, v) \le \min\{\sigma(u), \sigma(v)\} \qquad (\forall\, u, v \in V).$$

We write uv for $\{u, v\}$ and abbreviate $\mu(uv) := \mu(u, v)$. The (crisp) underlying graph of G has vertex set V and edge set $E^* := \{\, uv : \mu(uv) > 0 \,\}$.

Definition 1.12 (Fuzzy hypergraph)**.** Let $H^* = (V, E, \partial)$ be a crisp hypergraph. A *fuzzy hypergraph* on H^* is a sextuple

$$\mathcal{H} = \big(V, E, \partial;\ \sigma, \mu, \eta\big),$$

with maps

$$\sigma : V \to [0, 1], \qquad \mu : E \to [0, 1], \qquad \eta : V \times E \to [0, 1],$$

such that for all $v \in V$ and $e \in E$,

$$\textbf{(support)} \quad \big[v \in \partial(e)\big] \iff \eta(v, e) > 0, \tag{1}$$

$$\textbf{(incidence bound)} \quad \eta(v, e) \le \min\{\sigma(v), \mu(e)\}, \tag{2}$$

$$\textbf{(edge–vertex bound)} \quad \mu(e) \le \min_{u \in \partial(e)} \sigma(u). \tag{3}$$

Here σ is the *vertex-membership map*, μ the *edge-membership map*, and η the *incidence-membership map*. The underlying crisp hypergraph is (V, E, ∂), recoverable via (1).

Example 1.13 (Smart-home safety orchestration as a fuzzy hypergraph)**.** *Interpretation.* Vertices are devices/sensors; hyperedges are multi-device routines (alerts/actions). Vertex and hyperedge memberships encode reliability/activation likelihood, while incidence memberships quantify each device's contribution within a routine.

Construction. Let the crisp hypergraph be $H^* = (V, E, \partial)$ with

$$V = \{\mathrm{T}, \mathrm{C}, \mathrm{D}\} \quad \text{(Thermostat, CO detector, Door sensor)},$$

$$E = \{e_{TC}, e_{CD}, e_{TCD}\}, \quad \partial(e_{TC}) = \{\mathrm{T}, \mathrm{C}\},\ \partial(e_{CD}) = \{\mathrm{C}, \mathrm{D}\},\ \partial(e_{TCD}) = \{\mathrm{T}, \mathrm{C}, \mathrm{D}\}.$$

Choose vertex-memberships

$$\sigma(\mathrm{T}) = 0.85, \qquad \sigma(\mathrm{C}) = 0.70, \qquad \sigma(\mathrm{D}) = 0.60,$$

and hyperedge-memberships

$$\mu(e_{TC}) = 0.65, \qquad \mu(e_{CD}) = 0.50, \qquad \mu(e_{TCD}) = 0.55.$$

Admissibility (3) holds numerically:

$$\mu(e_{TC}) = 0.65 \le \min\{0.85, 0.70\} = 0.70,$$
$$\mu(e_{CD}) = 0.50 \le \min\{0.70, 0.60\} = 0.60,$$
$$\mu(e_{TCD}) = 0.55 \le \min\{0.85, 0.70, 0.60\} = 0.60.$$

Define incidence-memberships (set to zero when $v \notin \partial(e)$) by

$$\eta(\mathrm{T}, e_{TC}) = 0.63,\ \eta(\mathrm{C}, e_{TC}) = 0.65;$$

$$\eta(\mathrm{C}, e_{CD}) = 0.47,\ \eta(\mathrm{D}, e_{CD}) = 0.50;$$

$$\eta(\mathrm{T}, e_{TCD}) = 0.52,\ \eta(\mathrm{C}, e_{TCD}) = 0.50,\ \eta(\mathrm{D}, e_{TCD}) = 0.55,$$

and verify (2) componentwise, e.g.

$$\eta(\mathrm{T}, e_{TCD}) = 0.52 \le \min\{\sigma(\mathrm{T}), \mu(e_{TCD})\} = \min\{0.85, 0.55\} = 0.55,$$

and similarly for the other incidences. The support condition (1) holds because $\eta(v, e) > 0$ exactly when $v \in \partial(e)$. Therefore $\mathcal{H} = (V, E, \partial; \sigma, \mu, \eta)$ is a valid fuzzy hypergraph modeling multi-sensor safety routines.

Definition 1.14 (Fuzzy n-SuperHyperGraph)**.** (cf. [32,33]) Let $\mathrm{SHG}^{(n)} = (V, E)$ be an n-SuperHyperGraph. A *fuzzy n-SuperHyperGraph* is a quadruple

$$(V, E, \sigma, \mu),$$

where $\sigma : V \to [0, 1]$ and $\mu : E \to [0, 1]$ obey the *admissibility constraint*

$$\mu(e) \le \min_{v \in e} \sigma(v) \qquad \text{for every } e \in E.$$

Example 1.15 (Travel-package bundling as a fuzzy n-SuperHyperGraph ($n = 1$))**.** *Interpretation.* Base items are travel components; 1-supervertices are component bundles (e.g., flight+hotel); superedges link bundles that are marketed or scheduled together as a package.

Construction. Let $V_0 = \{\mathrm{F}, \mathrm{H}, \mathrm{T}\}$ denote Flight, Hotel, Tour. Take

$$V = \big\{\{\mathrm{F}\},\ \{\mathrm{H}\},\ \{\mathrm{T}\},\ \{\mathrm{F}, \mathrm{H}\}\big\} \subseteq \mathcal{P}(V_0),$$

and $E = \{e_1, e_2\}$ with incidence

$$\partial(e_1) = \big\{\{\mathrm{F}\}, \{\mathrm{H}\}\big\}, \qquad \partial(e_2) = \big\{\{\mathrm{F}, \mathrm{H}\}, \{\mathrm{T}\}\big\}.$$

Assign supervertex-memberships

$$\sigma(\{\mathrm{F}\}) = 0.90, \quad \sigma(\{\mathrm{H}\}) = 0.75, \quad \sigma(\{\mathrm{T}\}) = 0.60, \quad \sigma(\{\mathrm{F}, \mathrm{H}\}) = 0.70,$$

and superedge-memberships

$$\mu(e_1) = 0.72, \qquad \mu(e_2) = 0.58.$$

Admissibility for fuzzy 1-SuperHyperGraphs holds:

$$\mu(e_1) = 0.72 \le \min\{0.90, 0.75\} = 0.75, \qquad \mu(e_2) = 0.58 \le \min\{0.70, 0.60\} = 0.60.$$

Thus (V, E, σ, μ) is a fuzzy n-SuperHyperGraph at level $n = 1$ representing uncertain yet feasible bundlings of travel components.

1.3 Linear Diophantine Fuzzy Graph

We recall the linear Diophantine fuzzy set (LDFS) and then give a precise graph-theoretic definition of a linear Diophantine fuzzy graph (LDFG) [34–36]. A Linear Diophantine Fuzzy Graph equips vertices and edges with linear Diophantine fuzzy numbers, ensuring admissibility through algebraic membership constraints.

Definition 1.16 (Linear Diophantine fuzzy set (LDFS))**.** Let U be a nonempty universe. An *LDFS* on U is a mapping

$$\Phi : U \longrightarrow [0, 1]^4, \qquad u \mapsto \big(m(u),\ n(u);\ \alpha(u),\ \beta(u)\big),$$

whose components satisfy, for every $u \in U$,

$$0 \le \alpha(u)\, m(u) + \beta(u)\, n(u) \le 1, \qquad 0 \le \alpha(u) + \beta(u) \le 1.$$

Here m and n are respectively the *membership* and *non-membership* grades, while α and β are *reference parameters*. The (Diophantine) *hesitation* (or refusal) grade is

$$\xi(u) \ :=\ 1 - \big(\alpha(u)\, m(u) + \beta(u)\, n(u)\big) \ \in [0, 1].$$

A quadruple $T = (m, n; \alpha, \beta)$ with the above constraints is called a *linear Diophantine fuzzy number (LDFN)*.

Example 1.17 (Hiring shortlist as a Linear Diophantine fuzzy set)**.** The universe U is a pool of candidates for a single role. For each candidate $u \in U$, $m(u)$ models *evidence for suitability* (skills/fit), $n(u)$ models *evidence against* (gaps/risks). The reference parameters $(\alpha(u), \beta(u))$ encode how much the committee weights positive vs. negative evidence for u.

Let $U = \{\text{Ayano}, \text{Yutaka}, \text{Carol}\}$. Define the LDFS $\Phi(u) = (m(u), n(u); \alpha(u), \beta(u))$ by

$$\begin{aligned} \Phi(\text{Ayano}) &= (0.78,\ 0.12;\ 0.60,\ 0.30), \\ \Phi(\text{Yutaka}) &= (0.55,\ 0.35;\ 0.55,\ 0.35), \\ \Phi(\text{Carol}) &= (0.40,\ 0.50;\ 0.50,\ 0.40). \end{aligned}$$

For each u we compute $\alpha(u)m(u) + \beta(u)n(u)$ and $\alpha(u) + \beta(u)$:

$$\begin{aligned}
&\text{Ayano:}\quad 0.60{\cdot}0.78 + 0.30{\cdot}0.12 = 0.468 + 0.036 = 0.504 \le 1, \quad 0.60 + 0.30 = 0.90 \le 1,\\
&\text{Yutaka:}\quad 0.55{\cdot}0.55 + 0.35{\cdot}0.35 = 0.3025 + 0.1225 = 0.425 \le 1, \quad 0.55 + 0.35 = 0.90 \le 1,\\
&\text{Carol:}\quad 0.50{\cdot}0.40 + 0.40{\cdot}0.50 = 0.20 + 0.20 = 0.40 \le 1, \quad 0.50 + 0.40 = 0.90 \le 1.
\end{aligned}$$

Hesitations follow, e.g. $\xi(\text{Ayano}) = 1 - 0.504 = 0.496$, $\xi(\text{Yutaka}) = 1 - 0.425 = 0.575$, $\xi(\text{Carol}) = 1 - 0.40 = 0.60$. Hence Φ is a valid Linear Diophantine fuzzy set.

Definition 1.18 (Linear Diophantine fuzzy graph (LDFG))**.** An *LDFG* on the crisp graph $G^* = (V, E)$ is a pair

$$G = \big(\Phi_V,\ \Phi_E\big),$$

where

- Φ_V is an LDFS on V, i.e. $\Phi_V(v) = (m_V(v), n_V(v); \alpha_V(v), \beta_V(v)) \in [0,1]^4$ for each $v \in V$, with $0 \le \alpha_V(v)m_V(v) + \beta_V(v)n_V(v) \le 1$ and $0 \le \alpha_V(v) + \beta_V(v) \le 1$;
- Φ_E is an LDFS on E, i.e. $\Phi_E(uv) = (m_E(uv), n_E(uv); \alpha_E(uv), \beta_E(uv)) \in [0,1]^4$ for each $uv \in E$, with $0 \le \alpha_E(uv)m_E(uv) + \beta_E(uv)n_E(uv) \le 1$ and $0 \le \alpha_E(uv) + \beta_E(uv) \le 1$;
- the *edge–vertex coupling (admissibility) constraints* hold for every $uv \in E$:

$$\begin{aligned}
m_E(uv) &\le \min\{\, m_V(u),\ m_V(v)\,\},\\
n_E(uv) &\le \max\{\, n_V(u),\ n_V(v)\,\},\\
\alpha_E(uv) &\le \min\{\, \alpha_V(u),\ \alpha_V(v)\,\},\\
\beta_E(uv) &\le \max\{\, \beta_V(u),\ \beta_V(v)\,\}.
\end{aligned} \tag{4}$$

The hesitation grades are $\xi_V(v) = 1 - (\alpha_V(v)m_V(v) + \beta_V(v)n_V(v))$ and $\xi_E(uv) = 1 - (\alpha_E(uv)m_E(uv) + \beta_E(uv)n_E(uv))$.

Example 1.19 (Hiring shortlist as a Linear Diophantine fuzzy set)**.** *Interpretation.* The universe U is a pool of candidates for a single role. For each candidate $u \in U$, $m(u)$ models *evidence for suitability* (skills/fit), $n(u)$ models *evidence against* (gaps/risks). The reference parameters $(\alpha(u), \beta(u))$ encode how much the committee weights positive vs. negative evidence for u.

Construction. Let $U = \{\text{Ayano}, \text{Yutaka}, \text{Carol}\}$. Define the LDFS $\Phi(u) = (m(u), n(u); \alpha(u), \beta(u))$ by

$$\begin{aligned}
\Phi(\text{Ayano}) &= (0.78,\ 0.12;\ 0.60,\ 0.30),\\
\Phi(\text{Yutaka}) &= (0.55,\ 0.35;\ 0.55,\ 0.35),\\
\Phi(\text{Carol}) &= (0.40,\ 0.50;\ 0.50,\ 0.40).
\end{aligned}$$

Verification of LDFS constraints. For each u we compute $\alpha(u)m(u) + \beta(u)n(u)$ and $\alpha(u) + \beta(u)$:

$$\begin{aligned}
&\text{Ayano:}\quad 0.60{\cdot}0.78 + 0.30{\cdot}0.12 = 0.468 + 0.036 = 0.504 \le 1, \quad 0.60 + 0.30 = 0.90 \le 1,\\
&\text{Yutaka:}\quad 0.55{\cdot}0.55 + 0.35{\cdot}0.35 = 0.3025 + 0.1225 = 0.425 \le 1, \quad 0.55 + 0.35 = 0.90 \le 1,\\
&\text{Carol:}\quad 0.50{\cdot}0.40 + 0.40{\cdot}0.50 = 0.20 + 0.20 = 0.40 \le 1, \quad 0.50 + 0.40 = 0.90 \le 1.
\end{aligned}$$

Hesitations follow, e.g. $\xi(\text{Ayano}) = 1 - 0.504 = 0.496$, $\xi(\text{Yutaka}) = 1 - 0.425 = 0.575$, $\xi(\text{Carol}) = 1 - 0.40 = 0.60$. Hence Φ is a valid Linear Diophantine fuzzy set.

Example 1.20 (Last-mile logistics as a Linear Diophantine fuzzy graph)**.** *Interpretation.* Vertices are sites in a small delivery network; edges are potential routes. Vertex LDFNs encode site readiness/constraints; edge LDFNs encode route feasibility. The coupling (4) enforces that edges cannot be “more feasible” than their endpoints allow.

Construction. Let the crisp graph be $G^* = (V, E)$ with

$$V = \{W_1, W_2, S\} \quad (\text{warehouse } W_1,\ \text{warehouse } W_2,\ \text{store } S), \qquad E = \{\{W_1, W_2\}, \{W_2, S\}\}.$$

Assign vertex LDFNs

$\Phi_V(W_1) = (0.80,\ 0.15;\ 0.60,\ 0.30), \quad \Phi_V(W_2) = (0.65,\ 0.25;\ 0.50,\ 0.40), \quad \Phi_V(S) = (0.60,\ 0.35;\ 0.55,\ 0.35).$

Check the vertex constraints:

$$0.60 \cdot 0.80 + 0.30 \cdot 0.15 = 0.48 + 0.045 = 0.525 \leq 1, \;\; 0.60 + 0.30 = 0.90 \leq 1,$$
$$0.50 \cdot 0.65 + 0.40 \cdot 0.25 = 0.325 + 0.10 = 0.425 \leq 1, \;\; 0.50 + 0.40 = 0.90 \leq 1,$$
$$0.55 \cdot 0.60 + 0.35 \cdot 0.35 = 0.33 + 0.1225 = 0.4525 \leq 1, \;\; 0.55 + 0.35 = 0.90 \leq 1.$$

Assign edge LDFNs

$$\Phi_E(W_1W_2) = (0.62,\ 0.23;\ 0.50,\ 0.40), \qquad \Phi_E(W_2S) = (0.58,\ 0.30;\ 0.50,\ 0.38).$$

Verification of coupling (4). For W_1W_2:

$$\min\{m_V(W_1), m_V(W_2)\} = \min\{0.80, 0.65\} = 0.65 \;\Rightarrow\; m_E = 0.62 \leq 0.65,$$
$$\max\{n_V(W_1), n_V(W_2)\} = \max\{0.15, 0.25\} = 0.25 \;\Rightarrow\; n_E = 0.23 \leq 0.25,$$
$$\min\{\alpha_V(W_1), \alpha_V(W_2)\} = \min\{0.60, 0.50\} = 0.50 \;\Rightarrow\; \alpha_E = 0.50 \leq 0.50,$$
$$\max\{\beta_V(W_1), \beta_V(W_2)\} = \max\{0.30, 0.40\} = 0.40 \;\Rightarrow\; \beta_E = 0.40 \leq 0.40.$$

For W_2S:

$$\min\{0.65, 0.60\} = 0.60 \;\Rightarrow\; m_E = 0.58 \leq 0.60, \qquad \max\{0.25, 0.35\} = 0.35 \;\Rightarrow\; n_E = 0.30 \leq 0.35,$$
$$\min\{0.50, 0.55\} = 0.50 \;\Rightarrow\; \alpha_E = 0.50 \leq 0.50, \qquad \max\{0.40, 0.35\} = 0.40 \;\Rightarrow\; \beta_E = 0.38 \leq 0.40.$$

Finally, edge LDFS constraints:

$$\alpha_E(W_1W_2)m_E(W_1W_2) + \beta_E(W_1W_2)n_E(W_1W_2) = 0.50 \cdot 0.62 + 0.40 \cdot 0.23 = 0.31 + 0.092 = 0.402 \leq 1,$$
$$\alpha_E(W_2S)m_E(W_2S) + \beta_E(W_2S)n_E(W_2S) = 0.50 \cdot 0.58 + 0.38 \cdot 0.30 = 0.29 + 0.114 = 0.404 \leq 1,$$
$$\alpha_E + \beta_E \in \{0.90, 0.88\} \leq 1.$$

Therefore $G = (\Phi_V, \Phi_E)$ is a valid Linear Diophantine fuzzy graph modelling a last-mile delivery network with uncertain capacities and risks.

1.4 Fuzzy planar Graphs

A planar graph is a graph drawable in the plane without any edge crossings, preserving adjacency while avoiding intersections between distinct edges [37,38]. A fuzzy planar graph is a fuzzy graph where sufficiently strong edges form a planar subgraph, combining fuzziness with classical planar embedding [39–42].

Definition 1.21 (Relative edge strength; strong/weak edges)**.** For an edge uv with $\mu(uv) > 0$ define its *relative strength*

$$s(uv) \;:=\; \frac{\mu(uv)}{\min\{\sigma(u), \sigma(v)\}} \;\in\; [0,1].$$

Fix a threshold $\tau \in (0,1]$. We call uv *τ-strong* if $s(uv) \geq \tau$, and *τ-weak* otherwise. The conventional choice in the literature is $\tau = \frac{1}{2}$; in that case we simply say *strong* (resp. *weak*).

Definition 1.22 (τ-fuzzy planar embedding)**.** Let $G = (V, \sigma, \mu)$ be a fuzzy graph and $\tau \in (0,1]$. A *τ-fuzzy planar embedding* of G is a drawing of the underlying graph $G^* = (V, E^*)$ in the Euclidean plane such that

no two τ-strong edges cross.

Equivalently, all edge crossings (if any) involve at least one τ-weak edge.

Definition 1.23 (Fuzzy planar graph)**.** A fuzzy graph $G = (V, \sigma, \mu)$ is called *fuzzy planar* (with the standard convention) if it admits a $\frac{1}{2}$-fuzzy planar embedding.

Example 1.24 (A concrete $\frac{1}{2}$-fuzzy planar graph)**.** *Vertices and memberships.* Let $V = \{a, b, c, d\}$ with vertex-membership map

$$\sigma(a) = 1.00, \qquad \sigma(b) = 0.90, \qquad \sigma(c) = 0.80, \qquad \sigma(d) = 0.85.$$

Edges and memberships. Consider the underlying crisp edges

$$E^* = \{ab, bc, cd, da, ac, bd\},$$

with edge-membership map μ given by

$$\mu(ab) = 0.72,\ \mu(bc) = 0.68,\ \mu(cd) = 0.60,\ \mu(da) = 0.62,\ \mu(ac) = 0.33,\ \mu(bd) = 0.40.$$

Admissibility check ($\mu(uv) \le \min\{\sigma(u), \sigma(v)\}$).

$$\begin{aligned}
\min\{\sigma(a), \sigma(b)\} &= 0.90 \Rightarrow \mu(ab) = 0.72 \le 0.90,\\
\min\{\sigma(b), \sigma(c)\} &= 0.80 \Rightarrow \mu(bc) = 0.68 \le 0.80,\\
\min\{\sigma(c), \sigma(d)\} &= 0.80 \Rightarrow \mu(cd) = 0.60 \le 0.80,\\
\min\{\sigma(d), \sigma(a)\} &= 0.85 \Rightarrow \mu(da) = 0.62 \le 0.85,\\
\min\{\sigma(a), \sigma(c)\} &= 0.80 \Rightarrow \mu(ac) = 0.33 \le 0.80,\\
\min\{\sigma(b), \sigma(d)\} &= 0.85 \Rightarrow \mu(bd) = 0.40 \le 0.85.
\end{aligned}$$

Hence $G = (V, \sigma, \mu)$ is a valid fuzzy graph.

Relative strengths and τ-strong set for $\tau = \frac{1}{2}$. For $s(uv) := \mu(uv)/\min\{\sigma(u), \sigma(v)\}$:

$$\begin{aligned}
&s(ab) = 0.72/0.90 = 0.80 \text{ (strong)}, \quad s(bc) = 0.68/0.80 = 0.85 \text{ (strong)},\\
&s(cd) = 0.60/0.80 = 0.75 \text{ (strong)}, \quad s(da) = 0.62/0.85 \approx 0.729 \text{ (strong)},\\
&s(ac) = 0.33/0.80 = 0.4125 \text{ (weak)}, \quad s(bd) = 0.40/0.85 \approx 0.471 \text{ (weak)}.
\end{aligned}$$

Thus the $\frac{1}{2}$-strong edges are $\{ab, bc, cd, da\}$, i.e., the cycle C_4.

$\frac{1}{2}$-fuzzy planar embedding. Place a, b, c, d as the corners of a convex quadrilateral (say, in that cyclic order). Draw the $\frac{1}{2}$-strong edges ab, bc, cd, da along the boundary (which is planar), and draw the weak diagonals ac and bd as interior chords (they may cross). No two $\frac{1}{2}$-strong edges cross, so by the definition of $\frac{1}{2}$-fuzzy planarity, G is a fuzzy planar graph.

Proposition 1.25 (Strong-edge criterion)**.** *Let $G = (V, \sigma, \mu)$ be a fuzzy graph and $\tau \in (0, 1]$. Denote by*

$$E_\tau \ := \ \{ \, uv \in E^* : \ s(uv) \ge \tau \, \}$$

the set of τ-strong edges, and let $G_\tau := (V, E_\tau)$ be the induced crisp subgraph. Then the following are equivalent:

(a) *G admits a τ-fuzzy planar embedding;*

(b) *the crisp graph G_τ is planar.*

Proof. $(a) \Rightarrow (b)$: If a drawing has no crossings between τ-strong edges, then the restriction to E_τ is a planar embedding of G_τ. $(b) \Rightarrow (a)$: Take a planar embedding of G_τ. Add each τ-weak edge as a Jordan arc joining its endpoints; reroute it locally if needed so that any new crossing (if unavoidable) is only with already placed edges. By construction, no two τ-strong edges cross. □

2 Main Results

In this section, as the main outcome of the present paper, we investigate the HyperGraph and SuperHyperGraph counterparts of existing concepts.

2.1 Linear Diophantine Fuzzy HyperGraph

A Linear Diophantine Fuzzy HyperGraph assigns linear Diophantine fuzzy numbers to vertices and hyperedges, modeling multiway uncertain relationships.

Definition 2.1 (Linear Diophantine fuzzy hypergraph (LDFH))**.** Let $H^* = (V, E, \partial)$ be a crisp hypergraph. A *linear Diophantine fuzzy hypergraph (LDFH)* on H^* is a quadruple

$$\mathcal{H} = \big(V, E, \partial;\ \Phi_V, \Phi_E\big),$$

where Φ_V is an LDFS on V and Φ_E is an LDFS on E:

$$\Phi_V(v) = \big(m_V(v), n_V(v); \alpha_V(v), \beta_V(v)\big), \qquad \Phi_E(e) = \big(m_E(e), n_E(e); \alpha_E(e), \beta_E(e)\big),$$

and the following *edge–vertex coupling* holds for every hyperedge $e \in E$:

$$\begin{aligned} m_E(e) &\le \min_{v\in\partial(e)} m_V(v),\\ n_E(e) &\le \max_{v\in\partial(e)} n_V(v),\\ \alpha_E(e) &\le \min_{v\in\partial(e)} \alpha_V(v),\\ \beta_E(e) &\le \max_{v\in\partial(e)} \beta_V(v). \end{aligned} \tag{5}$$

The vertex and edge hesitations are

$$\xi_V(v) = 1 - \big(\alpha_V(v)m_V(v) + \beta_V(v)n_V(v)\big), \qquad \xi_E(e) = 1 - \big(\alpha_E(e)m_E(e) + \beta_E(e)n_E(e)\big).$$

Remark 2.2 (Optional incidence LDFS)**.** If one wishes to record incidence strengths explicitly, one may add an LDFS $\Theta : V \times E \to [0,1]^4$ with

$$\Theta(v,e) = \big(m_\cap(v,e), n_\cap(v,e); \alpha_\cap(v,e), \beta_\cap(v,e)\big),$$

subject to the support equivalence and bounds

$$\big[v \in \partial(e)\big] \iff m_\cap(v,e) > 0, \qquad \begin{aligned} m_\cap(v,e) &\le \min\{m_V(v), m_E(e)\},\\ n_\cap(v,e) &\le \max\{n_V(v), n_E(e)\},\\ \alpha_\cap(v,e) &\le \min\{\alpha_V(v), \alpha_E(e)\},\\ \beta_\cap(v,e) &\le \max\{\beta_V(v), \beta_E(e)\}. \end{aligned}$$

All results below remain valid with or without Θ.

Example 2.3 (Numerical verification of (5))**.** Let $V = \{a, b, c\}$ and $E = \{e_1, e_2\}$ with $\partial(e_1) = \{a, b\}$ and $\partial(e_2) = \{a, b, c\}$. Assign vertex LDFNs

$$\begin{aligned} \Phi_V(a) &= (0.80,\ 0.10;\ 0.70,\ 0.20),\\ \Phi_V(b) &= (0.60,\ 0.30;\ 0.60,\ 0.30),\\ \Phi_V(c) &= (0.45,\ 0.35;\ 0.50,\ 0.40), \end{aligned}$$

and edge LDFNs

$$\Phi_E(e_1) = (0.58,\ 0.20;\ 0.60,\ 0.25), \qquad \Phi_E(e_2) = (0.44,\ 0.33;\ 0.50,\ 0.35).$$

Then the four lines of (5) hold numerically:

$$\begin{aligned} m_E(e_1) &= 0.58 \le \min\{0.80, 0.60\} = 0.60, & m_E(e_2) &= 0.44 \le \min\{0.80, 0.60, 0.45\} = 0.45;\\ n_E(e_1) &= 0.20 \le \max\{0.10, 0.30\} = 0.30, & n_E(e_2) &= 0.33 \le \max\{0.10, 0.30, 0.35\} = 0.35;\\ \alpha_E(e_1) &= 0.60 \le \min\{0.70, 0.60\} = 0.60, & \alpha_E(e_2) &= 0.50 \le \min\{0.70, 0.60, 0.50\} = 0.50;\\ \beta_E(e_1) &= 0.25 \le \max\{0.20, 0.30\} = 0.30, & \beta_E(e_2) &= 0.35 \le \max\{0.20, 0.30, 0.40\} = 0.40. \end{aligned}$$

Hesitations, for instance at e_2, evaluate to

$$\xi_E(e_2) = 1 - \big(0.50 \cdot 0.44 + 0.35 \cdot 0.33\big) = 1 - (0.2200 + 0.1155) = 0.6645 \in [0,1].$$

Thus $\mathcal{H} = (V, E, \partial; \Phi_V, \Phi_E)$ is an LDFH.

Theorem 2.4 (Reduction to Linear Diophantine fuzzy graphs)**.** *Assume every hyperedge has size* 2*, i.e.,* $\partial(e) = \{u, v\}$ *for all* $e \in E$*. Identify the underlying simple graph* $G^* = (V, E)$ *with edges* uv *corresponding to* $e \in E$*. Then an LDFH* $\mathcal{H} = (V, E, \partial; \Phi_V, \Phi_E)$ *is precisely a linear Diophantine fuzzy graph* $G = (\Phi_V, \Phi_E)$ *in the sense of the LDFG definition, with the coupling constraints*

$$m_E(uv) \le \min\{m_V(u), m_V(v)\}, \qquad n_E(uv) \le \max\{n_V(u), n_V(v)\},$$
$$\alpha_E(uv) \le \min\{\alpha_V(u), \alpha_V(v)\}, \qquad \beta_E(uv) \le \max\{\beta_V(u), \beta_V(v)\}.$$

Proof. With $|\partial(e)| = 2$, writing $e \leftrightarrow uv$, the inequalities in (5) reduce term-by-term to the four LDFG admissibility bounds above. No other structure is changed, so the two notions coincide. □

Theorem 2.5 (Reduction to fuzzy hypergraphs)**.** *Let* $\mathcal{H} = (V, E, \partial; \Phi_V, \Phi_E)$ *be an LDFH and impose the parameter choice*

$$\alpha_V \equiv 1, \quad \beta_V \equiv 0, \qquad \alpha_E \equiv 1, \quad \beta_E \equiv 0, \qquad n_V \equiv 0, \quad n_E \equiv 0.$$

Define $\sigma : V \to [0, 1]$ *and* $\mu : E \to [0, 1]$ *by*

$$\sigma(v) := m_V(v), \qquad \mu(e) := m_E(e).$$

Then $(V, E, \partial; \sigma, \mu, \eta)$ *is a fuzzy hypergraph, where one can take*

$$\eta(v, e) := \begin{cases} \mu(e), & v \in \partial(e), \\ 0, & v \notin \partial(e). \end{cases}$$

Proof. Under the stated specialization, the LDFS constraints are trivially satisfied: $\alpha(\cdot)m(\cdot) + \beta(\cdot)n(\cdot) = 1 \cdot m + 0 \cdot 0 = m \in [0, 1]$ and $\alpha + \beta = 1 \le 1$. The coupling (5) becomes

$$m_E(e) \ \le \ \min_{v \in \partial(e)} m_V(v),$$

which is exactly the fuzzy hypergraph admissibility $\mu(e) \le \min_{v \in \partial(e)} \sigma(v)$. For the incidence map η, if $v \in \partial(e)$ we have

$$\eta(v, e) = \mu(e) \ \le \ \min\{\sigma(v), \mu(e)\},$$

so the incidence bound holds with equality; if $v \notin \partial(e)$, putting $\eta(v, e) = 0$ satisfies the bound trivially. In addition, the support equivalence $[v \in \partial(e)] \iff \eta(v, e) > 0$ holds for all e with $\mu(e) > 0$; restricting the crisp edge set to $\{e \in E : \mu(e) > 0\}$ (the usual support presentation) yields a fuzzy hypergraph on that support, as required. □

2.2 Linear Diophantine Fuzzy SuperHyperGraph

A Linear Diophantine Fuzzy SuperHyperGraph extends this framework by incorporating supervertices and superedges from iterated powersets, preserving fuzzy Diophantine constraints.

Definition 2.6 (Linear Diophantine fuzzy superhypergraph (LDFSH))**.** Let $\mathrm{SHG}^{(n)} = (V, E, \partial)$ be a crisp n-SuperHyperGraph. A *linear Diophantine fuzzy superhypergraph (LDFSH)* on $\mathrm{SHG}^{(n)}$ is a quadruple

$$\mathfrak{S} \ = \ \big(V, E, \partial; \ \Phi_V, \Phi_E\big),$$

where Φ_V is an LDFS on V and Φ_E is an LDFS on E:

$$\Phi_V(v) = \big(m_V(v), n_V(v); \alpha_V(v), \beta_V(v)\big), \qquad \Phi_E(e) = \big(m_E(e), n_E(e); \alpha_E(e), \beta_E(e)\big),$$

and the following *edge–vertex coupling* holds for every $e \in E$:

$$\begin{aligned} m_E(e) \ &\le \ \min_{v \in \partial(e)} m_V(v), \\ n_E(e) \ &\le \ \max_{v \in \partial(e)} n_V(v), \\ \alpha_E(e) \ &\le \ \min_{v \in \partial(e)} \alpha_V(v), \\ \beta_E(e) \ &\le \ \max_{v \in \partial(e)} \beta_V(v). \end{aligned} \tag{6}$$

The vertex and edge hesitations are

$$\xi_V(v) = 1 - \big(\alpha_V(v)m_V(v) + \beta_V(v)n_V(v)\big), \qquad \xi_E(e) = 1 - \big(\alpha_E(e)m_E(e) + \beta_E(e)n_E(e)\big).$$

Remark 2.7 (Optional incidence LDFS)**.** As an optional refinement, one may attach an incidence LDFS $\Theta : V \times E \to [0,1]^4$, written $\Theta(v,e) = (m_\cap, n_\cap; \alpha_\cap, \beta_\cap)(v,e)$, with the support and bounds

$$\big[v \in \partial(e)\big] \iff m_\cap(v,e) > 0, \qquad \begin{aligned} m_\cap(v,e) &\le \min\{m_V(v), m_E(e)\}, \\ n_\cap(v,e) &\le \max\{n_V(v), n_E(e)\}, \\ \alpha_\cap(v,e) &\le \min\{\alpha_V(v), \alpha_E(e)\}, \\ \beta_\cap(v,e) &\le \max\{\beta_V(v), \beta_E(e)\}. \end{aligned}$$

All results below remain valid with or without Θ.

Example 2.8 (Concrete LDFSH at level $n = 1$ with full numeric checks)**.** Let $V_0 = \{x, y\}$ so $\mathcal{P}(V_0) = \{\emptyset, \{x\}, \{y\}, \{x, y\}\}$. Define three 1-supervertices

$$v_1 = \{x\}, \qquad v_2 = \{y\}, \qquad v_3 = \{x, y\} \in \mathcal{P}(V_0),$$

and set $V = \{v_1, v_2, v_3\}$. Let $E = \{e_1, e_2\}$ with

$$\partial(e_1) = \{v_1, v_3\}, \qquad \partial(e_2) = \{v_2, v_3\}.$$

Assign vertex LDFNs

$$\begin{aligned} \Phi_V(v_1) &= (0.80,\ 0.10;\ 0.70,\ 0.20), \\ \Phi_V(v_2) &= (0.62,\ 0.28;\ 0.60,\ 0.30), \\ \Phi_V(v_3) &= (0.55,\ 0.35;\ 0.55,\ 0.40), \end{aligned}$$

for which

$$\alpha_V m_V + \beta_V n_V = \begin{cases} 0.7 \cdot 0.80 + 0.2 \cdot 0.10 = 0.58 \le 1, \\ 0.6 \cdot 0.62 + 0.3 \cdot 0.28 = 0.504 \le 1, \\ 0.55 \cdot 0.55 + 0.4 \cdot 0.35 = 0.4525 \le 1, \end{cases} \qquad \alpha_V + \beta_V = \begin{cases} 0.90 \le 1, \\ 0.90 \le 1, \\ 0.95 \le 1. \end{cases}$$

Assign edge LDFNs

$$\Phi_E(e_1) = (0.55,\ 0.20;\ 0.55,\ 0.30), \qquad \Phi_E(e_2) = (0.53,\ 0.33;\ 0.55,\ 0.40).$$

We verify the four lines of (6):

Membership:

$$m_E(e_1) = 0.55 \le \min\{0.80, 0.55\} = 0.55, \qquad m_E(e_2) = 0.53 \le \min\{0.62, 0.55\} = 0.55.$$

Non-membership:

$$n_E(e_1) = 0.20 \le \max\{0.10, 0.35\} = 0.35, \qquad n_E(e_2) = 0.33 \le \max\{0.28, 0.35\} = 0.35.$$

Alpha-parameters:

$$\alpha_E(e_1) = 0.55 \le \min\{0.70, 0.55\} = 0.55, \qquad \alpha_E(e_2) = 0.55 \le \min\{0.60, 0.55\} = 0.55.$$

Beta-parameters:

$$\beta_E(e_1) = 0.30 \le \max\{0.20, 0.40\} = 0.40, \qquad \beta_E(e_2) = 0.40 \le \max\{0.30, 0.40\} = 0.40.$$

Hesitations, e.g. for e_2,

$$\xi_E(e_2) = 1 - \big(0.55 \cdot 0.53 + 0.40 \cdot 0.33\big) = 1 - (0.2915 + 0.132) = 0.5765 \in [0,1].$$

Thus $\mathfrak{S} = (V, E, \partial; \Phi_V, \Phi_E)$ is an LDFSH.

Theorem 2.9 (Reduction to Linear Diophantine fuzzy hypergraphs)**.** *Let $\mathfrak{S} = (V, E, \partial; \Phi_V, \Phi_E)$ be an LDFSH on* $\mathrm{SHG}^{(n)}$*. If $n = 0$ (so $V \subseteq V_0$), then $\mathfrak{S}$ is exactly a Linear Diophantine fuzzy hypergraph on the crisp hypergraph (V, E, ∂).*

Proof. When $n = 0$, the objects in V are ordinary vertices and E are hyperedge identifiers with the same incidence map ∂. The inequalities in (6) become, for each $e \in E$,

$$m_E(e) \leq \min_{v\in\partial(e)} m_V(v), \qquad n_E(e) \leq \max_{v\in\partial(e)} n_V(v),$$
$$\alpha_E(e) \leq \min_{v\in\partial(e)} \alpha_V(v), \qquad \beta_E(e) \leq \max_{v\in\partial(e)} \beta_V(v),$$

which are precisely the LDFH coupling inequalities. Since the LDFS constraints on Φ_V, Φ_E are unchanged, the structures coincide. □

Theorem 2.10 (Reduction to Linear Diophantine fuzzy graphs)**.** *In addition to the assumptions of Theorem 2.9, suppose every edge has size* 2*:* $|\partial(e)| = 2$ *for all* $e \in E$*. Identifying* $e \leftrightarrow uv$ *with* $\partial(e) = \{u, v\}$*, the bounds in* (6) *read*

$$m_E(uv) \leq \min\{m_V(u), m_V(v)\}, \qquad n_E(uv) \leq \max\{n_V(u), n_V(v)\},$$
$$\alpha_E(uv) \leq \min\{\alpha_V(u), \alpha_V(v)\}, \qquad \beta_E(uv) \leq \max\{\beta_V(u), \beta_V(v)\},$$

which is exactly the admissibility for an LDF graph on the underlying simple graph.

Proof. With $|\partial(e)| = 2$, the minima and maxima in (6) specialize to the two incident vertices. This reproduces the LDFG definition verbatim; no further changes are needed. □

Theorem 2.11 (Reduction to fuzzy superhypergraphs)**.** *Let* $\mathfrak{S} = (V, E, \partial; \Phi_V, \Phi_E)$ *be an LDFSH. Impose the specialization*

$$\alpha_V \equiv 1,\ \beta_V \equiv 0,\ n_V \equiv 0, \qquad \alpha_E \equiv 1,\ \beta_E \equiv 0,\ n_E \equiv 0,$$

and define a vertex-membership $\sigma : V \to [0, 1]$ *and edge-membership* $\mu : E \to [0, 1]$ *by*

$$\sigma(v) := m_V(v), \qquad \mu(e) := m_E(e).$$

Then $(V, E, \partial; \sigma, \mu)$ *is a (crisp-support) fuzzy n-SuperHyperGraph satisfying*

$$\mu(e) \ \leq\ \min_{v\in\partial(e)} \sigma(v) \qquad (\forall e \in E).$$

Proof. Under the specialization, the LDFS constraints reduce to $\alpha m + \beta n = 1 \cdot m + 0 \cdot 0 = m \in [0, 1]$ and $\alpha + \beta = 1 \leq 1$, hence valid. The coupling (6) simplifies to

$$m_E(e) \ \leq\ \min_{v\in\partial(e)} m_V(v),$$

which is exactly $\mu(e) \leq \min_{v\in\partial(e)} \sigma(v)$. Thus $(V, E, \partial; \sigma, \mu)$ is a fuzzy superhypergraph on the same crisp support. □

2.3 Fuzzy planar HyperGraphs

A fuzzy planar hypergraph extends fuzzy graphs by allowing hyperedges, where sufficiently strong connections induce a planar incidence embedding.

Definition 2.12 (Crisp hypergraph and incidence graph (Recall))**.** A (finite, undirected, loopless) *crisp hypergraph* is a triple $H^* = (V, E, \partial)$ where V is a finite nonempty set of vertices, E a finite set of hyperedge identifiers, and $\partial : E \to \mathcal{P}^*(V)$ assigns to each $e \in E$ a nonempty finite incidence set $\partial(e) \subseteq V$. The *incidence (Levi) graph* of H^* is the bipartite graph

$$G_{\mathrm{inc}}(H^*) \ :=\ \big(V \sqcup E,\ I\big), \qquad I \ :=\ \big\{(v, e) \in V \times E :\ v \in \partial(e)\big\}.$$

Definition 2.13 (Fuzzy hypergraph (Recall))**.** A *fuzzy hypergraph* on $H^* = (V, E, \partial)$ is a sextuple

$$\mathcal{H} \ =\ \big(V, E, \partial;\ \sigma, \mu, \eta\big),$$

with membership maps

$$\sigma : V \to [0, 1], \qquad \mu : E \to [0, 1], \qquad \eta : V \times E \to [0, 1],$$

satisfying, for all $v \in V$ and $e \in E$,

$$\textbf{(support)} \quad \big[v \in \partial(e)\big] \iff \eta(v,e) > 0, \tag{7}$$

$$\textbf{(incidence bound)} \quad \eta(v,e) \le \min\{\sigma(v),\ \mu(e)\}, \tag{8}$$

$$\textbf{(edge–vertex bound)} \quad \mu(e) \le \min_{u \in \partial(e)} \sigma(u). \tag{9}$$

Definition 2.14 (τ-strong hyperedges)**.** Fix $\tau \in (0,1]$. In a fuzzy hypergraph $\mathcal{H} = (V, E, \partial; \sigma, \mu, \eta)$ define

$$E_\tau \ := \ \Big\{ e \in E : \ \mu(e) \ \ge \ \tau \cdot \min_{v \in \partial(e)} \sigma(v) \Big\}.$$

Members of E_τ are called τ-*strong hyperedges*.

Definition 2.15 (τ-fuzzy planar hypergraph)**.** A fuzzy hypergraph $\mathcal{H} = (V, E, \partial; \sigma, \mu, \eta)$ is τ-*fuzzy planar* if the incidence graph induced by the τ-strong hyperedges is planar, i.e.,

$$G_{\mathrm{inc}}(V \sqcup E_\tau,\ I_\tau) \ \text{is planar}, \qquad I_\tau \ := \ \{\, (v,e) \in V \times E_\tau : \ v \in \partial(e) \,\}.$$

Equivalently, there exists a drawing of $G_{\mathrm{inc}}(V \sqcup E_\tau, I_\tau)$ in the plane with no edge crossings.

Example 2.16 (Numerical $\frac{1}{2}$-fuzzy planar hypergraph)**.** Let $V = \{a, b, c\}$ and $E = \{e_{ab}, e_{bc}, e_{ac}, e_{abc}\}$ with

$$\partial(e_{ab}) = \{a, b\}, \quad \partial(e_{bc}) = \{b, c\}, \quad \partial(e_{ac}) = \{a, c\}, \quad \partial(e_{abc}) = \{a, b, c\}.$$

Choose vertex-memberships

$$\sigma(a) = 1.00, \qquad \sigma(b) = 0.90, \qquad \sigma(c) = 0.80.$$

Choose edge-memberships

$$\mu(e_{ab}) = 0.72, \quad \mu(e_{bc}) = 0.70, \quad \mu(e_{ac}) = 0.30, \quad \mu(e_{abc}) = 0.45.$$

Verify the edge–vertex bounds (9):

$$\begin{aligned}
\mu(e_{ab}) &= 0.72 \le \min\{1.00, 0.90\} = 0.90,\\
\mu(e_{bc}) &= 0.70 \le \min\{0.90, 0.80\} = 0.80,\\
\mu(e_{ac}) &= 0.30 \le \min\{1.00, 0.80\} = 0.80,\\
\mu(e_{abc}) &= 0.45 \le \min\{1.00, 0.90, 0.80\} = 0.80.
\end{aligned}$$

Define incidences by $\eta(v,e) = \mu(e)$ if $v \in \partial(e)$ and 0 otherwise. Then $\eta(v,e) \le \min\{\sigma(v), \mu(e)\}$ holds with equality whenever $v \in \partial(e)$, and (7) is satisfied.

Take $\tau = \frac{1}{2}$. Compute τ-strong hyperedges using $\mu(e) \ge \tau \cdot \min_{v \in \partial(e)} \sigma(v)$:

$$\begin{aligned}
e_{ab} &: 0.72 \ge 0.5 \cdot \min\{1.00, 0.90\} = 0.45 \quad \Rightarrow \ e_{ab} \in E_\tau,\\
e_{bc} &: 0.70 \ge 0.5 \cdot \min\{0.90, 0.80\} = 0.40 \quad \Rightarrow \ e_{bc} \in E_\tau,\\
e_{ac} &: 0.30 < 0.5 \cdot \min\{1.00, 0.80\} = 0.40 \quad \Rightarrow \ e_{ac} \notin E_\tau,\\
e_{abc} &: 0.45 \ge 0.5 \cdot \min\{1.00, 0.90, 0.80\} = 0.40 \Rightarrow \ e_{abc} \in E_\tau.
\end{aligned}$$

Thus $E_\tau = \{e_{ab}, e_{bc}, e_{abc}\}$. The incidence subgraph on $V \sqcup E_\tau$ has edges

$$a\text{–}e_{ab},\ b\text{–}e_{ab},\ b\text{–}e_{bc},\ c\text{–}e_{bc},\ a\text{–}e_{abc},\ b\text{–}e_{abc},\ c\text{–}e_{abc}.$$

This bipartite graph is planar: draw a, b, c on a line and place the hyperedge-vertices e_{ab} above the segment ab, e_{bc} above bc, and e_{abc} below the line with three noncrossing spokes to a, b, c. Hence, by the Proposition, $\mathcal{H}$ is a $\frac{1}{2}$-fuzzy planar hypergraph.

Finally, if we drop e_{abc} and keep only the 2-uniform edges e_{ab}, e_{bc}, the $\frac{1}{2}$-strong part corresponds to the strong-edge subgraph $ab \cup bc$, which is planar; by the Theorem this agrees with the fuzzy planar graph criterion.

Theorem 2.17 (Generalizes fuzzy planar graphs)**.** *Consider a fuzzy graph $G = (V, \sigma, \mu)$ (edges undirected, loopless) and encode it as the 2-uniform fuzzy hypergraph*

$$\mathcal{H}_G = (V, E, \partial; \sigma, \mu, \eta), \qquad \partial(e) = \{u, v\} \text{ for } e \leftrightarrow uv, \quad \eta(u, e) = \eta(v, e) = \mu(e).$$

Then for every $\tau \in (0, 1]$ the following are equivalent:

(a) *G admits a τ-fuzzy planar embedding in the sense of "no crossings between τ-strong edges";*

(b) *$\mathcal{H}_G$ is τ-fuzzy planar in the sense of Definition 2.15.*

Proof. By hypothesis on fuzzy graphs, $\mu(uv) \le \min\{\sigma(u), \sigma(v)\}$ for all uv. Hence for an edge uv we have

$$uv \text{ is } \tau\text{-strong} \iff \mu(uv) \ge \tau \cdot \min\{\sigma(u), \sigma(v)\} \iff e \in E_\tau \quad (e \leftrightarrow uv).$$

The incidence subgraph $G_{\text{inc}}(V \sqcup E_\tau, I_\tau)$ is obtained from the crisp subgraph of τ-strong edges by subdividing each strong edge once (insert the edge-vertex e). Subdividing edges preserves planarity and reflects nonplanarity. Therefore the strong-edge subgraph of G is planar iff $G_{\text{inc}}(V \sqcup E_\tau, I_\tau)$ is planar. Using the Proposition gives the equivalence $(a) \Leftrightarrow (b)$. □

Theorem 2.18 (A τ-fuzzy planar hypergraph is a fuzzy hypergraph)**.** *Let $\mathcal{H} = (V, E, \partial; \sigma, \mu, \eta)$ be τ-fuzzy planar. Then, independently of τ, $\mathcal{H}$ satisfies the fuzzy hypergraph axioms* (7)–(9).

Proof. By definition, $\mathcal{H}$ *is* a fuzzy hypergraph; planarity imposes an additional graph-theoretic constraint on the incidence subgraph but does not alter the inequalities (7)–(9). Hence the axioms hold verbatim. □

2.4 Fuzzy planar SuperHyperGraphs

A fuzzy planar superhypergraph generalizes fuzzy planar hypergraphs by using supervertices and superedges, ensuring strong incidence subgraphs remain planar.

Definition 2.19 (Crisp n-SuperHyperGraph)**.** Fix a finite base set V_0 and a level $n \in \mathbb{N}_0$. An *n-SuperHyperGraph* is a triple

$$\text{SHG}^{(n)} = (V, E, \partial),$$

where $V \subseteq \mathcal{P}^n(V_0)$ is a finite set of n-supervertices, E is a finite set of (super)edge identifiers, and $\partial : E \to \mathcal{P}^*(V)$ assigns to each $e \in E$ a nonempty finite incidence set $\partial(e) \subseteq V$. Its (bipartite) *incidence graph* is

$$G_{\text{inc}}\big(\text{SHG}^{(n)}\big) := \big(V \sqcup E,\ I\big), \qquad I := \big\{(v, e) \in V \times E : v \in \partial(e)\big\}.$$

Definition 2.20 (Fuzzy SuperHyperGraph)**.** On a fixed $\text{SHG}^{(n)} = (V, E, \partial)$, a *fuzzy SuperHyperGraph* is a quintuple

$$\mathcal{S} = \big(V, E, \partial;\ \sigma, \mu\big),$$

with maps $\sigma : V \to [0, 1]$ (supervertex-membership) and $\mu : E \to [0, 1]$ (superedge-membership) satisfying the admissibility inequality

$$\mu(e) \le \min_{v \in \partial(e)} \sigma(v) \qquad (\forall\, e \in E). \tag{10}$$

Optionally, one may also equip an *incidence membership* $\eta : V \times E \to [0, 1]$ with

$$[\, v \in \partial(e)\,] \iff \eta(v, e) > 0, \qquad \eta(v, e) \le \min\{\sigma(v), \mu(e)\},$$

but all results below hold with or without η.

Definition 2.21 (τ-strong superedges)**.** Fix $\tau \in (0, 1]$. In a fuzzy SuperHyperGraph $\mathcal{S} = (V, E, \partial; \sigma, \mu)$ define the set of *τ-strong superedges*

$$E_\tau := \Big\{ e \in E : \mu(e) \ge \tau \cdot \min_{v \in \partial(e)} \sigma(v) \Big\}.$$

Definition 2.22 (τ-fuzzy planar SuperHyperGraph)**.** A fuzzy SuperHyperGraph $\mathcal{S} = (V, E, \partial; \sigma, \mu)$ is *τ-fuzzy planar* if the incidence subgraph induced by the τ-strong superedges is planar, i.e.,

$$G_{\text{inc}}\big(V \sqcup E_\tau,\ I_\tau\big) \ \text{is planar,} \qquad I_\tau \ := \ \{(v, e) \in V \times E_\tau :\ v \in \partial(e)\}.$$

Equivalently, there exists a drawing of $G_{\text{inc}}(V \sqcup E_\tau, I_\tau)$ in the plane with no edge crossings.

Example 2.23 (Numerical $\frac{1}{2}$-fuzzy planar SuperHyperGraph at level $n = 1$)**.** Let $V_0 = \{x, y\}$ and $n = 1$, so $\mathcal{P}(V_0) = \{\varnothing, \{x\}, \{y\}, \{x, y\}\}$. Take the supervertex set

$$V = \big\{v_1 := \{x\},\ v_2 := \{y\},\ v_3 := \{x, y\}\big\} \subseteq \mathcal{P}(V_0).$$

Let $E = \{e_1, e_2, e_3\}$ with incidence

$$\partial(e_1) = \{v_1, v_3\}, \qquad \partial(e_2) = \{v_2, v_3\}, \qquad \partial(e_3) = \{v_1, v_2, v_3\}.$$

Define supervertex-memberships

$$\sigma(v_1) = 0.95, \qquad \sigma(v_2) = 0.85, \qquad \sigma(v_3) = 0.70,$$

and superedge-memberships

$$\mu(e_1) = 0.66, \qquad \mu(e_2) = 0.58, \qquad \mu(e_3) = 0.36.$$

Admissibility check (10). Compute the incident minima:

$$\min_{v \in \partial(e_1)} \sigma(v) = \min\{0.95, 0.70\} = 0.70,$$
$$\min_{v \in \partial(e_2)} \sigma(v) = \min\{0.85, 0.70\} = 0.70,$$
$$\min_{v \in \partial(e_3)} \sigma(v) = \min\{0.95, 0.85, 0.70\} = 0.70,$$

so indeed

$$\mu(e_1) = 0.66 \le 0.70, \qquad \mu(e_2) = 0.58 \le 0.70, \qquad \mu(e_3) = 0.36 \le 0.70.$$

Thus $(V, E, \partial; \sigma, \mu)$ is a fuzzy SuperHyperGraph.

τ-strong set for $\tau = \frac{1}{2}$. Since $\min_{v \in \partial(e)} \sigma(v) = 0.70$ for all three edges, the threshold is $\tau \cdot 0.70 = 0.35$. Hence

$$E_{\frac{1}{2}} = \{\, e \in E :\ \mu(e) \ge 0.35 \,\} = \{e_1, e_2, e_3\}.$$

Planarity of the incidence subgraph. The bipartite graph on $V \sqcup E_{\frac{1}{2}}$ has edges

$$v_1\text{-}e_1,\ v_3\text{-}e_1; \qquad v_2\text{-}e_2,\ v_3\text{-}e_2; \qquad v_1\text{-}e_3,\ v_2\text{-}e_3,\ v_3\text{–}e_3.$$

Place v_1, v_2, v_3 on a horizontal line (from left to right). Draw e_1 above the segment $v_1 v_3$ and connect it to v_1, v_3 by two arcs; draw e_2 above $v_2 v_3$ and connect it likewise; finally put e_3 below the line and join it to v_1, v_2, v_3 by three noncrossing spokes. No edges cross, so the incidence subgraph is planar. Therefore the structure is a $\frac{1}{2}$-fuzzy planar SuperHyperGraph.

Theorem 2.24 (Generalizes fuzzy planar hypergraphs)**.** *Let $\mathcal{S}$ be a τ-fuzzy planar SuperHyperGraph on* $\mathrm{SHG}^{(n)}$*. If $n = 0$ (so $V \subseteq V_0$ consists of ordinary vertices), then $\mathcal{S}$ is exactly a τ-fuzzy planar hypergraph in the sense that*

$$G_{\text{inc}}\big(V \sqcup E_\tau,\ I_\tau\big)$$

is planar and the admissibility (10) *coincides with the usual fuzzy-hypergraph edge–vertex bound.*

Proof. When $n = 0$, the elements of V are ordinary vertices and the pairs (V, E, ∂) form a crisp hypergraph. The inequality (10) becomes $\mu(e) \le \min_{v \in \partial(e)} \sigma(v)$, which is the standard fuzzy hypergraph bound. By Definition 2.22, τ-planarity is the planarity of the incidence subgraph on $V \sqcup E_\tau$, which is exactly the hypergraph notion. Hence the two notions agree. □

Theorem 2.25 (Is a fuzzy SuperHyperGraph)**.** *Every τ-fuzzy planar SuperHyperGraph $\mathcal{S} = (V, E, \partial; \sigma, \mu)$ satisfies the fuzzy SuperHyperGraph admissibility* (10)*; in particular, τ-planarity imposes an additional graphical constraint but does not change the fuzzy axioms.*

Proof. By definition, $\mathcal{S}$ is already equipped with (σ, μ) obeying (10). The τ-planarity requirement restricts the subfamily of τ-strong superedges, but it neither alters the codomains of σ, μ nor the inequality (10). Thus $\mathcal{S}$ remains a fuzzy SuperHyperGraph. □

3 Conclusion

In this paper, we extended Linear Diophantine Fuzzy Graphs and fuzzy planar graphs to the frameworks of HyperGraphs and SuperHyperGraphs by introducing new classes, namely *Linear Diophantine Fuzzy SuperHyperGraphs* and *Fuzzy Planar SuperHyperGraphs*. For future work, we anticipate further research into extended systems built upon advanced uncertainty models such as the *HyperFuzzy Set* [43,44], *Neutrosophic Set* [45–47], *Picture Fuzzy Set* [48], *Hesitant Fuzzy Set* [49], and *Plithogenic Set* [50, 51].

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Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

References

[1] Reinhard Diestel. *Graph theory*. Springer (print edition); Reinhard Diestel (eBooks), 2024.

[2] Jonathan L Gross, Jay Yellen, and Mark Anderson. *Graph theory and its applications*. Chapman and Hall/CRC, 2018.

[3] Claude Berge. *Hypergraphs: combinatorics of finite sets*, volume 45. Elsevier, 1984.

[4] Florentin Smarandache. *Extension of HyperGraph to n-SuperHyperGraph and to Plithogenic n-SuperHyperGraph, and Extension of HyperAlgebra to n-ary (Classical-/Neutro-/Anti-) HyperAlgebra*. Infinite Study, 2020.

[5] Muhammad Akram, A Nagoor Gani, and A Borumand Saeid. Vague hypergraphs. *Journal of Intelligent & Fuzzy Systems*, 26(2):647–653, 2014.

[6] Yue Gao, Zizhao Zhang, Haojie Lin, Xibin Zhao, Shaoyi Du, and Changqing Zou. Hypergraph learning: Methods and practices. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 44(5):2548–2566, 2020.

[7] Georg Gottlob, Nicola Leone, and Francesco Scarcello. Hypertree decompositions and tractable queries. In *Proceedings of the eighteenth ACM SIGMOD-SIGACT-SIGART symposium on Principles of database systems*, pages 21–32, 1999.

[8] Eduardo Martín Campoverde Valencia, Jessica Paola Chuisaca Vásquez, and Francisco Ángel Becerra Lois. Multineutrosophic analysis of the relationship between survival and business growth in the manufacturing sector of azuay province, 2020–2023, using plithogenic n-superhypergraphs. *Neutrosophic Sets and Systems*, 84:341–355, 2025.

[9] Masoud Ghods, Zahra Rostami, and Florentin Smarandache. Introduction to neutrosophic restricted superhypergraphs and neutrosophic restricted superhypertrees and several of their properties. *Neutrosophic Sets and Systems*, 50:480–487, 2022.

[10] Mohammad Hamidi, Florentin Smarandache, and Mohadeseh Taghinezhad. *Decision Making Based on Valued Fuzzy Superhypergraphs*. Infinite Study, 2023.

[11] Julio Cesar Méndez Bravo, Claudia Jeaneth Bolanos Piedrahita, Manuel Alberto Méndez Bravo, and Luis Manuel Pilacuan-Bonete. Integrating smed and industry 4.0 to optimize processes with plithogenic n-superhypergraphs. *Neutrosophic Sets and Systems*, 84:328–340, 2025.

[12] N. B. Nalawade, M. S. Bapat, S. G. Jakkewad, G. A. Dhanorkar, and D. J. Bhosale. Structural properties of zero-divisor hypergraph and superhypergraph over $\mathbb{Z}_n$: Girth and helly property. *Panamerican Mathematical Journal*, 35(4S):485, 2025.

[13] Berrocal Villegas Salomón Marcos, Montalvo Fritas Willner, Berrocal Villegas Carmen Rosa, Flores Fuentes Rivera María Yissel, Espejo Rivera Roberto, Laura Daysi Bautista Puma, and Dante Manuel Macazana Fernández. Using plithogenic n-superhypergraphs to assess the degree of relationship between information skills and digital competencies. *Neutrosophic Sets and Systems*, 84:513–524, 2025.

[14] Shouxian Zhu. Neutrosophic n-superhypernetwork: A new approach for evaluating short video communication effectiveness in media convergence. *Neutrosophic Sets and Systems*, 85:1004–1017, 2025.

[15] Adel Al-Odhari. Neutrosophic power-set and neutrosophic hyper-structure of neutrosophic set of three types. *Annals of Pure and Applied Mathematics*, 31(2):125–146, 2025.

[16] Thomas Jech. *Set theory: The third millennium edition, revised and expanded*. Springer, 2003.

[17] Florentin Smarandache. The cardinal of the m-powerset of a set of n elements used in the superhyperstructures and neutrosophic superhyperstructures. *Systems Assessment and Engineering Management*, 2:19–22, 2024.

[18] Ajoy Kanti Das, Rajat Das, Suman Das, Bijoy Krishna Debnath, Carlos Granados, Bimal Shil, and Rakhal Das. A comprehensive study of neutrosophic superhyper bci-semigroups and their algebraic significance. *Transactions on Fuzzy Sets and Systems*, 8(2):80, 2025.

[19] Florentin Smarandache. *SuperHyperFunction, SuperHyperStructure, Neutrosophic SuperHyperFunction and Neutrosophic SuperHyperStructure: Current understanding and future directions*. Infinite Study, 2023.

[20] Alain Bretto. Hypergraph theory. *An introduction. Mathematical Engineering. Cham: Springer*, 1, 2013.

[21] Lotfi A Zadeh. Fuzzy sets. *Information and control*, 8(3):338–353, 1965.

[22] Lotfi A Zadeh. Fuzzy logic, neural networks, and soft computing. In *Fuzzy sets, fuzzy logic, and fuzzy systems: selected papers by Lotfi A Zadeh*, pages 775–782. World Scientific, 1996.

[23] TM Nishad, Talal Ali Al-Hawary, and B Mohamed Harif. General fuzzy graphs. *Ratio Mathematica*, 47, 2023.

[24] Azriel Rosenfeld. Fuzzy graphs. In *Fuzzy sets and their applications to cognitive and decision processes*, pages 77–95. Elsevier, 1975.

[25] Takaaki Fujita and Florentin Smarandache. A review of the hierarchy of plithogenic, neutrosophic, and fuzzy graphs: Survey and applications. In *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond (Second Volume)*. Biblio Publishing, 2024.

[26] Talal Al-Hawary. Complete fuzzy graphs. *International Journal of Mathematical Combinatorics*, 4:26, 2011.

[27] Mahdi Farshi and Bijan Davvaz. Generalized fuzzy hypergraphs and hypergroupoids. *Filomat*, 30(9):2375–2387, 2016.

[28] Sovan Samanta and Madhumangal Pal. Bipolar fuzzy hypergraphs. *International Journal of Fuzzy Logic Systems*, 2(1):17–28, 2012.

[29] Muhammad Akram and Anam Luqman. Fuzzy hypergraphs and related extensions. In *Studies in Fuzziness and Soft Computing*, 2020.

[30] John N Mordeson and Premchand S Nair. *Fuzzy graphs and fuzzy hypergraphs*, volume 46. Physica, 2012.

[31] Mohammed Alqahtani. Intuitionistic fuzzy quasi-supergraph integration for social network decision making. *International Journal of Analysis and Applications*, 23:137–137, 2025.

[32] Takaaki Fujita. Extensions of multidirected graphs: Fuzzy, neutrosophic, plithogenic, rough, soft, hypergraph, and superhypergraph variants. *International Journal of Topology*, 2(3):11, 2025.

[33] Takaaki Fujita and Florentin Smarandache. Directed n-superhypergraphs incorporating bipolar fuzzy information: A multi-tier framework for modeling bipolar uncertainty in complex networks. *Neutrosophic Sets and Systems*, 88:164–183, 2025.

[34] Muhammad Riaz, Masooma Raza Hashmi, Dragan Pamuar, and Yuming Chu. Spherical linear diophantine fuzzy sets with modeling uncertainties in mcdm. *Cmes-computer Modeling in Engineering & Sciences*, 126:1125–1164, 2021.

[35] Masooma Raza Hashmi, Syeda Tayyba Tehrim, Muhammad Riaz, Dragan Pamuar, and Goran Ćirović. Spherical linear diophantine fuzzy soft rough sets with multi-criteria decision making. *Axioms*, 10:185, 2021.

[36] Shahzaib Ashraf, Huzaira Razzaque, Muhammad Naeem, and Thongchai Botmart. Spherical q-linear diophantine fuzzy aggregation information: Application in decision support systems. *AIMS Mathematics*, 2023.

[37] Ganesh Ghorai and Madhumangal Pal. Faces and dual of m-polar fuzzy planar graphs. *Journal of Intelligent & Fuzzy Systems*, 31(3):2043–2049, 2016.

[38] Muhammad Akram, Musavarah Sarwar, Wieslaw A Dudek, Muhammad Akram, Musavarah Sarwar, and Wieslaw A Dudek. Bipolar fuzzy planar graphs. *Graphs for the Analysis of Bipolar Fuzzy Information*, pages 223–251, 2021.

[39] Tarasankar Pramanik, Sovan Samanta, and Madhumangal Pal. Interval-valued fuzzy planar graphs. *International journal of machine learning and cybernetics*, 7:653–664, 2016.

[40] Ganesh Ghorai and Madhumangal Pal. A study on m-polar fuzzy planar graphs. *International Journal of Computing Science and Mathematics*, 7(3):283–292, 2016.

[41] Abrar Hussain, Ahmed Alsanad, Kifayat Ullah, Zeeshan Ali, Muhammad Kamran Jamil, and Mogeeb AA Mosleh. Investigating the short-circuit problem using the planarity index of complex q-rung orthopair fuzzy planar graphs. *Complexity*, 2021(1):8295997, 2021.

[42] Rahul Mondal and Ganesh Ghorai. Inverse fuzzy mixed planar graphs with application. *International Journal of Applied and Computational Mathematics*, 10(4):131, 2024.

[43] Jayanta Ghosh and Tapas Kumar Samanta. Hyperfuzzy sets and hyperfuzzy group. *Int. J. Adv. Sci. Technol*, 41:27–37, 2012.

[44] Young Bae Jun, Seok-Zun Song, and Seon Jeong Kim. Distances between hyper structures and length fuzzy ideals of bck/bci-algebras based on hyper structures. *Journal of Intelligent & Fuzzy Systems*, 35(2):2257–2268, 2018.

[45] Haibin Wang, Florentin Smarandache, Yanqing Zhang, and Rajshekhar Sunderraman. *Single valued neutrosophic sets*. Infinite study, 2010.

[46] Said Broumi, Mohamed Talea, Assia Bakali, and Florentin Smarandache. Single valued neutrosophic graphs. *Journal of New theory*, (10):86–101, 2016.

[47] Said Broumi, Assia Bakali, and Ayoub Bahnasse. Neutrosophic sets: An overview. *Infinite Study*, 2018.

[48] Bui Cong Cuong and Vladik Kreinovich. Picture fuzzy sets-a new concept for computational intelligence problems. In *2013 third world congress on information and communication technologies (WICT 2013)*, pages 1–6. IEEE, 2013.

[49] Vicenç Torra. Hesitant fuzzy sets. *International journal of intelligent systems*, 25(6):529–539, 2010.

[50] Muhammad Azeem, Humera Rashid, Muhammad Kamran Jamil, Selma Gütmen, and Erfan Babaee Tirkolaee. Plithogenic fuzzy graph: A study of fundamental properties and potential applications. *Journal of Dynamics and Games*, pages 0–0, 2024.

[51] Florentin Smarandache. *Plithogenic set, an extension of crisp, fuzzy, intuitionistic fuzzy, and neutrosophic sets-revisited*. Infinite study, 2018.