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Bond-based peridynamics: A tale of two Poisson's ratios

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Abstract This paper explores the restrictions imposed by bond-based peridynamics, particularly with respect to plane strain and plane stress models. We begin with a review of the derivations in [2] wherein for isotropic materials a Poisson's ratio restriction of $\frac{1}{4}$ for plane strain and $\frac{1}{3}$ for plane stress is deduced. Next, we show Cauchy's relations are an intrinsic limitation of bond-based peridynamics and specialize this result to plane strain and plane stress models, generalizing the results of [2] and demonstrating the Poisson's ratio restrictions in [2] are simply a consequence of Cauchy's relations. We conclude with a discussion of the validity of peridynamic plane strain and plane stress models formulated from two-dimensional bond-based peridynamic models.

Keywords bond-based peridynamics · Cauchy's relations · plane strain · plane stress

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1 Intoduction

Peridynamics was developed as an alternative to classical continuum mechanics for the modeling of material failure and damage [7,9]. In peridynamics, spatial derivatives are replaced by integral operators which, unlike derivatives, are well-defined at discontinuities. This in turn allows material failure and damage to naturally develop within the solution of a peridynamic problem. As a nonlocal theory, peridynamics can be quite computationally expensive. To remedy this, when certain conditions are met, it is sometimes possible to approximate a three-dimensional model with a two-dimensional model [12], thus significantly reducing computational expenses. In classical linear elasticity, two common such model reductions are plane strain and plane stress. Peridynamic formulations for classical plane strain and plane stress have been presented in several works [1,2,3,12,4,6]. Peridynamic models can be classified as state-based [9] or bond-based [7]. In bond-based peridynamics, it is commonly stated that isotropic plane strain models only correspond to materials with a Poisson's ratio of $\frac{1}{4}$ while isotropic plane stress models only apply to materials with a Poisson's ratio of $\frac{1}{3}$. The origin of this claim can be traced to [2]. In this work, we explore the validity of this claim and present a generalization for anisotropic materials.

2 Constraints imposed on Poisson's ratio by bond-based peridynamics for isotropic plane strain and plane stress

In this section, we explore the assertion presented in [2] of a Poisson's ratio restriction of $\frac{1}{4}$ for plane strain and $\frac{1}{3}$ for plane stress in isotropic bond-based peridynamics. In order to deduce these restrictions, which we derive below, one equates the strain energy density of an isotropic material in two-dimensional bond-based peridynamics to the strain energy density of an isotropic material in classical planar linear elasticity for two strain states: the uniform normal strain (*cf.* Figure 1a),

$$\varepsilon_{11} = \varepsilon_{22} = s_0 \text{ and } \varepsilon_{12} = 0, \tag{1}$$

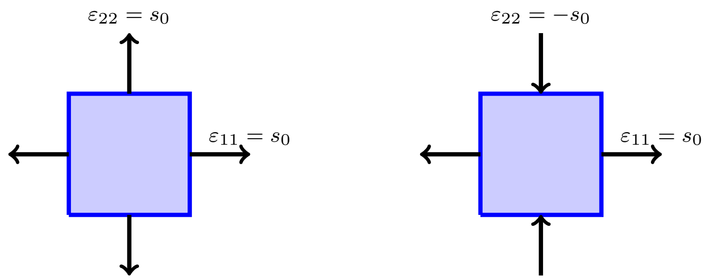
and the uniform shear strain (*cf.* Figure 1b),

$$\varepsilon_{11} = -\varepsilon_{22} = s_0 \text{ and } \varepsilon_{12} = 0, \tag{2}$$

where ε_{ij} are the components of the infinitesimal strain tensor and s_0 is a constant. In this section, for brevity, we simply state the strain energy densities for each strain state; however, in Appendix A we provide the corresponding derivations.

The strain energy density in classical elasticity is

$$W^C = \frac{1}{2}\sigma_{ij}\varepsilon_{ij}, \tag{3}$$



(a) Uniform normal strain. (b) Uniform shear strain.

Fig. 1: Uniform strain states used to relate elastic and microelastic constants.

where σ_{ij} are the components of the stress tensor and Einstein summation convention is employed for repeated indices.

Alternatively for peridynamics, in [2] the case of an isotropic bond-based prototype microelastic brittle (PMB) material model [8] was considered. In that case, the strain energy density is given by

$$W^P = \frac{1}{4}\int_{\mathcal{H}} cs^2 \|\boldsymbol{\xi}\| d\boldsymbol{\xi}, \tag{4}$$

where c is a microelastic stiffness constant, $\boldsymbol{\xi} := \mathbf{x}' - \mathbf{x}$ is the bond, $\boldsymbol{\eta} := \mathbf{u}(\mathbf{x}', t) - \mathbf{u}(\mathbf{x}, t)$ is the relative displacement of the bond $\boldsymbol{\xi}$, $s := \frac{\|\boldsymbol{\xi}+\boldsymbol{\eta}\| - \|\boldsymbol{\xi}\|}{\|\boldsymbol{\xi}\|}$ is the bond stretch, and $\mathcal{H}$ is a peridynamic neighborhood.

Under the uniform normal strain (1) in an isotropic material, the strain energy densities for classical plane strain, classical plane stress, and two-dimensional bond-based peridynamics (based on (4)) are, respectively,

$$W_1^{C\varepsilon} = \frac{Es_0^2}{(1+\nu)(1-2\nu)}, \; W_1^{C\sigma} = \frac{Es_0^2}{1-\nu}, \text{ and } W_1^P = \frac{c\pi\delta^3 s_0^2}{6}. \tag{5}$$

Similarly, under the uniform shear strain (2) in an isotropic material, the strain energy densities for classical plane strain, classical plane stress, and two-dimensional bond-based peridynamics (based on (4)) are, respectively,

$$W_2^{C\varepsilon} = \frac{Es_0^2}{1+\nu}, \; W_2^{C\sigma} = \frac{Es_0^2}{1+\nu}, \text{ and } W_2^P = \frac{c\pi\delta^3 s_0^2}{12}. \tag{6}$$

In order to ensure agreement between the classical isotropic plane strain model and the two-dimensional bond-based peridynamic model (4), we equate $W_1^{C\varepsilon}$ and W_1^P in (5) as well as $W_2^{C\varepsilon}$ and W_2^P in (6) to find, respectively,

$$c = \frac{6E}{\pi(1+\nu)(1-2\nu)\delta^3} \text{ and } c = \frac{12E}{\pi(1+\nu)\delta^3} \Rightarrow \nu = \frac{1}{4}. \tag{7}$$

Thus, the strain energy density for the isotropic two-dimensional bond-based peridynamic model (4) can only agree with the strain energy density for isotropic classical plane strain when the material has a Poisson's ratio of $\nu = \frac{1}{4}$. Similarly, in order to ensure agreement between the classical isotropic plane stress model and the two-dimensional bond-based peridynamic model (4), we equate $W_1^{C\sigma}$ and W_1^P in (5) as well as $W_2^{C\sigma}$ and W_2^P in (6) to find, respectively,

$$c = \frac{6E}{\pi(1-\nu)\delta^3} \text{ and } c = \frac{12E}{\pi(1+\nu)\delta^3} \Rightarrow \nu = \frac{1}{3}. \tag{8}$$

Thus, the strain energy density for the isotropic two-dimensional bond-based peridynamic model (4) can only agree with the strain energy density for isotropic classical plane stress when the material has a Poisson's ratio of $\nu = \frac{1}{3}$.

3 Constraints imposed on the elasticity tensor by bond-based peridynamics

In this section, we develop a generalization of the Poisson's ratio constraints presented in Section 2. Specifically, we do not limit the discussion to two dimensions and we allow anisotropy within the model.

Rather than attempting to match constants between strain energy densities for classical and peridynamic models for specific strain states, we instead consider general infinitesimal deformations. To accomplish this we express the strain energy density of classical linear elasticity in terms of the displacement field. We first recall that in classical linear elasticity the components of the stress and strain tensors are related through a generalized Hooke's Law:

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}, \tag{9}$$

where C_{ijkl} are the components of the fourth-order elasticity tensor[1] $\mathbb{C}$. The elasticity tensor has the minor symmetries $C_{ijkl} = C_{jikl} = C_{ijlk}$ and the major symmetry $C_{ijkl} = C_{klij}$. Substituting (9) into (3), we find

$$\begin{aligned} W^C &= \frac{1}{2}C_{ijkl}\varepsilon_{ij}\varepsilon_{kl} \\ &= \frac{1}{8}C_{ijkl}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right) \\ &= \frac{1}{2}C_{ijkl}\frac{\partial u_i}{\partial x_j}\frac{\partial u_k}{\partial x_l}. \end{aligned} \tag{10}$$

The last equality in (10) is obtained from the minor symmetries of the elasticity tensor.

To consider more general bond-based peridynamic models and compare corresponding strain energy densities with those from classical linear elasticity,

[1] To avoid confusion in later arguments we add a superscript $2D$ or $3D$ to the elasticity tensor $\mathbb{C}$ and its components to refer to a two-dimensional or three-dimensional spatial dimension, respectively.

we employ a linear bond-based peridynamic model with strain energy density given by [7]

$$W^P = \frac{1}{4}\int_{\mathcal{H}} \lambda(\boldsymbol{\xi})(\boldsymbol{\xi}\cdot\boldsymbol{\eta})^2 d\boldsymbol{\xi}. \tag{11}$$

The function $\lambda(\boldsymbol{\xi})$ is commonly referred to as the micromodulus function and determines material response in the bond-based peridynamic model. As in classical linear elasticity, we assume an infinitesimal deformation so that

$$\eta_i \approx \frac{\partial u_i}{\partial x_j}(\mathbf{x}, t)\xi_j.$$

In this case, the peridynamic strain energy density (11) is given by

$$\begin{aligned} W^P &= \frac{1}{4}\int_{\mathcal{H}} \lambda(\boldsymbol{\xi})\xi_i\eta_i\xi_k\eta_k d\boldsymbol{\xi} \\ &\approx \frac{1}{4}\int_{\mathcal{H}} \lambda(\boldsymbol{\xi})\xi_i\xi_j\frac{\partial u_i}{\partial x_j}\xi_k\xi_l\frac{\partial u_k}{\partial x_l} d\boldsymbol{\xi} \\ &= \frac{1}{2}\left(\frac{1}{2}\int_{\mathcal{H}} \lambda(\boldsymbol{\xi})\xi_i\xi_j\xi_k\xi_l d\boldsymbol{\xi}\right)\frac{\partial u_i}{\partial x_j}\frac{\partial u_k}{\partial x_l}. \end{aligned} \tag{12}$$

Equating (10) and (12), we arrive at

$$C_{ijkl} = \frac{1}{2}\int_{\mathcal{H}} \lambda(\boldsymbol{\xi})\xi_i\xi_j\xi_k\xi_l d\boldsymbol{\xi}. \tag{13}$$

Noting that the right-hand side of (13) is invariant under any permutation of the indices i, j, k, and l, we immediately deduce that bond-based peridynamic theory is only applicable to materials whose elasticity tensor $\mathbb{C}$ is completely symmetric. Specifically, in addition to the minor and major symmetries that are intrinsic to the elasticity tensor $\mathbb{C}$, bond-based peridynamic theory imposes the additional symmetry

$$C_{ijkl} = C_{ikjl}. \tag{14}$$

The relations (14) are frequently referred to as Cauchy's relations and a historical account of their origin can be found in [5]. These relations are known to occur in elastic models developed from a molecular theory based on pair potentials between particles [10]. Since bond-based peridynamics employs a pair potential, it is perhaps unsurprising that it is only applicable to materials satisfying Cauchy's relations.

In order to relate (14) to the constraints on Poisson's ratio presented in Section 2, we express (14) in terms of engineering constants. In three dimensions the expressions are fairly cumbersome for the case of full anisotropy. We therefore restrict the discussion to the case of orthotropic symmetry, where we assume the planes of reflection symmetry coincide with the xy-, xz-, and yz-planes. In this case, there are three relevant Cauchy's relations[2],

$$C^{3D}_{1212} = C^{3D}_{1122}, C^{3D}_{1313} = C^{3D}_{1133}, \text{ and } C^{3D}_{2323} = C^{3D}_{2233}. \tag{15}$$

In terms of engineering constants, (15) is given by

$$G_{12} = \frac{E_1 E_2}{\Delta} \left(E_3 \nu_{13} \nu_{23} + E_2 \nu_{12} \right), \tag{16a}$$

$$G_{13} = \frac{E_1 E_2 E_3}{\Delta} \left(\nu_{12} \nu_{23} + \nu_{13} \right), \tag{16b}$$

$$G_{23} = \frac{E_2 E_3}{\Delta} \left(E_2 \nu_{12} \nu_{13} + E_1 \nu_{23} \right), \tag{16c}$$

where E_i are Young's moduli, ν_{ij} are Poisson's ratios, G_{ij} are shear moduli, and

$$\Delta := E_1 E_2 - 2\, E_2 E_3 \nu_{12} \nu_{13} \nu_{23} - E_1 E_3 {\nu_{23}}^2 - {E_2}^2 {\nu_{12}}^2 - E_2 E_3 {\nu_{13}}^2.$$

In two dimensions there is a single Cauchy's relation,

$$C^{2D}_{1122} = C^{2D}_{1212}. \tag{17}$$

Even in the most general two-dimensional case of oblique symmetry, the expression for (17) in terms of engineering constants is relatively simple:

$$G_{12} = \frac{E_2 \nu_{12}}{1 - \nu_{12} \nu_{21} - \eta_{12,11} \eta_{12,22}}, \tag{18}$$

where $\eta_{ij,kk}$ are coefficients of mutual influence of the second type. When we specialize (16) and (18) to the case of isotropy an interesting development materializes. In the case of isotropy in three dimensions, we have

$$E_1 = E_2 = E_3,\ \nu_{12} = \nu_{13} = \nu_{23}, \text{ and } G_{12} = G_{13} = G_{23} = \frac{E_1}{2(1+\nu)}. \tag{19}$$

Imposing (19) on (16), we find $\nu_{12} = \frac{1}{4}$. Alternatively, in the case of isotropy in two dimensions, we have

$$E_1 = E_2,\ \nu_{12} = \nu_{21}, \quad \eta_{12,11} = \eta_{12,22} = 0, \text{ and } G_{12} = \frac{E_1}{2(1+\nu)}. \tag{20}$$

Imposing (20) on (18), we find $\nu_{12} = \frac{1}{3}$.

Consequently, in the case of isotropy, the constraint on Poisson's ratio obtained in Section 2 for plane strain is identical to the constraint imposed by Cauchy's relations in three dimensions. Similarly, in the case of isotropy, the constraint on Poisson's ratio obtained in Section 2 for plane stress is identical to the constraint imposed by Cauchy's relation in two dimensions. As we will see in Section 3.1, these are no mere coincidences.

[2] The other relations,

$$C^{3D}_{1123} = C^{3D}_{1312}, C^{3D}_{2213} = C^{3D}_{2312}, \text{ and } C^{3D}_{3312} = C^{3D}_{2313},$$

are trivially satisfied as each term is zero for orthotropic symmetry.

3.1 Constraints imposed on the elasticity tensor by bond-based peridynamics for anisotropic plane strain and plane stress

Given the three-dimensional elasticity tensor $\mathbb{C}^{3D}$, the strain energy density for classical plane strain may be expressed as

$$W^{C\varepsilon} = \frac{1}{2} C^{3D}_{ijkl} \varepsilon_{ij} \varepsilon_{kl}, \tag{21}$$

where, since $\varepsilon_{13} = \varepsilon_{23} = \varepsilon_{33} = 0$ is assumed for plane strain, the summations are over $\{1, 2\}$. Equation (21) is identical to the two-dimensional formulation of (10) when one replaces the two-dimensional elasticity tensor components C^{2D}_{ijkl} with the corresponding three-dimensional components C^{3D}_{ijkl}. Equating (21) with (12), we find

$$C^{3D}_{1212} = C^{3D}_{1122}. \tag{22}$$

Thus, in the case of plane strain, two-dimensional bond-based peridynamics imposes a single Cauchy's relation directly on the three-dimensional elasticity tensor. In terms of engineering constants, (22) is equivalent to (16a) for orthotropic symmetry and simplifies to $\nu = \frac{1}{4}$ in the case of isotropy (*cf.* (19)).

Alternatively, given the three-dimensional elasticity tensor $\mathbb{C}^{3D}$, the strain energy density for classical plane stress may be expressed as[3]

$$W^{C\sigma} = \frac{1}{2} C'_{ijkl} \varepsilon_{ij} \varepsilon_{kl}, \tag{23}$$

where, since $\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$ is assumed for plane stress, the summations are over $\{1, 2\}$ and

$$C'_{ijkl} := C^{3D}_{ijkl} - \frac{C^{3D}_{33ij} C^{3D}_{33kl}}{C^{3D}_{3333}} \tag{24}$$

are the reduced elastic stiffnesses [11]. Equation (23) is identical to the two-dimensional formulation of (10) when one replaces the two-dimensional elasticity tensor components C^{2D}_{ijkl} with the corresponding reduced elastic stiffnesses C'_{ijkl}. Equating (23) with (12), we arrive at $C'_{1212} = C'_{1122}$. From (24) we deduce

$$C^{3D}_{1212} - \frac{(C^{3D}_{3312})^2}{C^{3D}_{3333}} = C^{3D}_{1122} - \frac{C^{3D}_{1133} C^{3D}_{2233}}{C^{3D}_{3333}}. \tag{25}$$

In terms of engineering constants $C'_{ijkl} = C^{2D}_{ijkl}$ and consequently (25) is equivalent to

$$C^{2D}_{1212} = C^{2D}_{1122}. \tag{26}$$

[3] Typically plane stress is applied to thin plate-like structures. This formulation assumes monoclinic symmetry with the plane of reflection symmetry coinciding with the xy-plane. This assumption is essential in keeping the mid-plane of the plate planar under inplane loading [11]. To obtain (23), set $\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$ in (9) and solve for $\varepsilon_{13}, \varepsilon_{23}$, and ε_{33}. Then, substitute the resulting expressions back into the equations for σ_{11}, σ_{22}, and σ_{12} in (9). Lastly, use the resulting stress-strain relationship in (3).

Thus, in the case of plane stress, two-dimensional bond-based peridynamics effectively imposes a single Cauchy's relation directly on the two-dimensional elasticity tensor rather than on the three-dimensional elasticity tensor. In terms of engineering constants, (26) is equivalent to (18) for oblique symmetry and simplifies to $\nu = \frac{1}{3}$ in the case of isotropy (*cf.* (20)).

In classical linear elasticity, plane strain and plane stress models are derived from a three-dimensional model of linear elasticity. In the peridynamic plane strain and plane stress models presented in [2], one begins with a two-dimensional peridynamic model and informs it with either the classical plane strain or plane stress model. While this creates peridynamic models which agree with the classical plane strain or plane stress model for infinitesimal smooth deformations, the resulting peridynamic models will not necessarily approximate a three-dimensional bond-based peridynamic model. In fact, as we saw earlier, an isotropic three-dimensional bond-based peridynamic model immediately imposes a Poisson's ratio restriction of $\nu = \frac{1}{4}$ and therefore the peridynamic plane stress model presented in [2], which requires $\nu = \frac{1}{3}$, cannot be a plane stress approximation of an isotropic three-dimensional bond-based peridynamic model. More generally, for anistropic materials, two-dimensional bond-based peridynamic models for plane stress result in the restriction (25), whereas three-dimensional bond-based peridynamic models impose the restrictions (14), in particular (22). Consequently, *a two-dimensional bond-based peridynamic model cannot be a plane stress approximation of a three-dimensional bond-based peridynamic model.* Potentially, the peridynamic plane stress model presented in [2] can be shown to be a plane stress approximation of a three-dimensional state-based peridynamic model as it is not bound by Cauchy's relations. Alternatively, in [12] it was shown that imposing assumptions similar to those assumed for classical plane stress on a three-dimensional bond-based peridynamic model naturally produces a two-dimensional state-based peridynamic model. The resulting state-based peridynamic model has the same restrictions in terms of engineering constants as the three-dimensional peridynamic model it approximates. Moreover, in [12] it was also shown that imposing similar assumptions to those assumed for classical plane strain on a three-dimensional bond-based peridynamic model does produce a two-dimensional bond-based peridynamic model. As opposed to [2], the main benefit of the plane strain and plane stress peridynamic models presented in [12] is that the original three-dimensional bond-based peridynamic model being approximated is known.

4 Conclusions

In this work we explored the limitations of bond-based peridynamics with respect to agreement with classical linear elasticity. We examined the claim posed in [2] for isotropic materials that peridynamic plane strain requires a Poisson's ratio of $\frac{1}{4}$ while peridynamic plane stress requires a Poisson's ratio of $\frac{1}{3}$, and we generalized the analysis to the case of anisotropy. In the general anisotropic setting, we demonstrated that bond-based peridynamics is constrained by Cauchy's relations. Specifically, we deduced a two-dimensional bond-based peridynamic model imposes $C_{1212} = C_{1122}$ for plane strain or $C_{1212} - \frac{(C_{3312})^2}{C_{3333}} = C_{1122} - \frac{C_{1133}C_{2233}}{C_{3333}}$ for plane stress on the three-dimensional elasticity tensor. In particular, we showed the restrictions posed in [2] are simply consequences of Cauchy's relations being imposed on the corresponding plane strain or plane stress elasticity tensor. This analysis demonstrates that a two-dimensional bond-based peridynamic model describing plane stress cannot approximate a three-dimensional bond-based peridynamic model.

A Derivations of strain energy density results from Section 2

In this section, we present derivations of the strain energy density results utilized in Section 2 to derive the Poisson's ratio restrictions in isotropic bond-based peridynamics.

A.1 Strain energy densities for isotropic classical plane strain and plane stress

In classical plane strain, the stress-strain relationship for an isotropic material is given by

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & 0\\ \nu & 1-\nu & 0\\ 0 & 0 & \frac{1}{2}(1-2\nu)\end{bmatrix}\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ 2\varepsilon_{12}\end{bmatrix}. \tag{27}$$

Substituting (27) into (3), we find the strain energy density for an isotropic material in a state of plane strain is given by

$$W^{C\varepsilon} = \frac{E}{2(1+\nu)(1-2\nu)}\left[(1-\nu)(\varepsilon_{11}^2+\varepsilon_{22}^2) + 2\nu\varepsilon_{11}\varepsilon_{22} + 2(1-2\nu)\varepsilon_{12}^2\right]. \tag{28}$$

Substituting the strain states (1) and (2) into (28), we find, respectively,

$$W_1^{C\varepsilon} = \frac{Es_0^2}{(1+\nu)(1-2\nu)} \quad \text{and} \quad W_2^{C\varepsilon} = \frac{Es_0^2}{1+\nu}. \tag{29}$$

Alternatively, in classical plane stress, the stress-strain relationship for an isotropic material is given by

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{bmatrix} = \frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \frac{1}{2}(1-\nu)\end{bmatrix}\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ 2\varepsilon_{12}\end{bmatrix}. \tag{30}$$

Substituting (30) into (3), we find the strain energy density for an isotropic material in a state of plane stress is given by

$$W^{C\sigma} = \frac{E}{2(1-\nu^2)}\left[\varepsilon_{11}^2+\varepsilon_{22}^2 + 2\nu\varepsilon_{11}\varepsilon_{22} + 2(1-\nu)\varepsilon_{12}^2\right]. \tag{31}$$

Substituting the strain states (1) and (2) into (31), we find, respectively,

$$W_1^{C\sigma} = \frac{Es_0^2}{1-\nu} \quad \text{and} \quad W_2^{C\sigma} = \frac{Es_0^2}{1+\nu}. \tag{32}$$

A.2 Strain energy densities for the two-dimensional PMB peridynamic model

We begin by expressing the relative displacement $\boldsymbol{\eta}$ for the two strain states (1) and (2). Under the strain state (1), we find

$$\boldsymbol{\eta} = s_0\langle \xi_1, \xi_2\rangle. \tag{33}$$

Under the strain state (2), we find

$$\boldsymbol{\eta} = s_0\langle \xi_1, -\xi_2\rangle. \tag{34}$$

We only consider the strain energy density at material points within the bulk of the body so that one may suppose the peridynamic neighborhood $\mathcal{H} = B^{2D}_\delta(\mathbf{0})$, i.e., the ball in two dimensions of radius δ centered at the origin. To obtain the strain energy density for the two-dimensional PMB peridynamic model under the strain state (1), we substitute (33) into (4) to find

$$W_1^P = \frac{c}{4}\int_{B^{2D}_\delta(\mathbf{0})} \frac{(\|\boldsymbol{\xi}+\boldsymbol{\eta}\| - \|\boldsymbol{\xi}\|)^2}{\|\boldsymbol{\xi}\|} d\boldsymbol{\xi} = \frac{c}{4}\int_{B^{2D}_\delta(\mathbf{0})} s_0^2 \|\boldsymbol{\xi}\| d\boldsymbol{\xi} = \frac{c\pi\delta^3 s_0^2}{6}. \tag{35}$$

To obtain the strain energy density for the two-dimensional PMB peridynamic model under the strain state (2), we substitute (34) into (4) to find

$$\begin{aligned} W_2^P =& \frac{c}{4}\int_{B_\delta(\mathbf{0})} \frac{(\|\boldsymbol{\xi}+\boldsymbol{\eta}\| - \|\boldsymbol{\xi}\|)^2}{\|\boldsymbol{\xi}\|} d\boldsymbol{\xi} = \frac{c}{4}\int_{B_\delta(\mathbf{0})} \frac{\left(\sqrt{(1+s_0)^2\xi_1^2 + (1-s_0)^2\xi_2^2} - \|\boldsymbol{\xi}\|\right)^2}{\|\boldsymbol{\xi}\|} d\boldsymbol{\xi} \\ =& \frac{c\delta^3}{6}\left(\pi s_0^2 + 2\pi - 4(1+s_0)E\left(1, \frac{2\sqrt{s_0}}{s_0+1}\right)\right). \end{aligned} \tag{36}$$

Here $E(x;k) := \int_0^x \frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}} dt$, i.e., the incomplete elliptic integral of the second kind. In linear elasticity, we are concerned with infinitesimal deformations. Consequently, by noticing the limit

$$\lim_{s_0\to 0} \frac{W_2^P}{s_0^2} = \lim_{s_0\to 0} \frac{c\delta^3}{6}\left(\pi s_0^2 + 2\pi - 4(1+s_0)E\left(1, \frac{2\sqrt{s_0}}{s_0+1}\right)\right)\frac{1}{s_0^2} = \frac{c\pi\delta^3}{12}, \tag{37}$$

we may suppose for infinitesimal deformations that (36) simplifies to

$$W_2^P = \frac{c\pi\delta^3 s_0^2}{12}. \tag{38}$$

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