Fuzzy Tolerance Hypergraphs and Fuzzy Tolerance Superhypergraphs

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Abstract

Graph theory offers a rigorous framework for modeling relationships and connectivity via vertices and edges [1, 2]. Hypergraphs generalize this framework by allowing *hyperedges* that join more than two vertices [3, 4]. Superhypergraphs further enrich the model through iterated powerset constructions, capturing hierarchical and self-referential structures among hyperedges [5]. In this paper, we introduce new classes of graphs, namely the Tolerance SuperHyperGraph, Tolerance HyperGraph, Fuzzy Tolerance Hypergraph, and Fuzzy Tolerance SuperHypergraph, and examine their properties.

Keywords: SuperHyperGraph, HyperGraph, Fuzzy Graph, Fuzzy Hypergraph, Fuzzy SuperHyperGraph, Tolerance SuperHyperGraph, Tolerance HyperGraph

1 Preliminaries

We collect the basic terminology and notation used in what follows. Unless explicitly stated otherwise, all graphs considered are finite, undirected, and loopless; multiple edges are allowed only when this is specified.

1.1 SuperHyperGraphs

A classical hypergraph generalizes an ordinary graph by permitting an edge to connect an arbitrary (finite) number of vertices, which makes it suitable for representing multiway relationships [3, 6, 7]. A *SuperHyperGraph* carries this idea further by forming vertices and edges from iterated powersets of a base set; this viewpoint has appeared in several recent contexts [8–10]. Reported applications include, among others, molecular structure modeling, complex network analysis, and signal processing [11–14]. Throughout, the *level n* is a fixed nonnegative integer.

Definition 1.1 (Base set)**.** A *base (ground) set* is a fixed finite set S from which higher-level objects are generated:

$$S = \{ x \mid x \text{ belongs to the chosen domain} \}.$$

All structures introduced below ultimately draw their elements from S.

Definition 1.2 (Powerset)**.** [15, 16] Given a set X, its powerset is

$$\mathcal{P}(X) = \{ A \subseteq X \}.$$

We also use the *nonempty* powerset $\mathcal{P}^*(X) := \mathcal{P}(X) \setminus \{\varnothing\}$.

Definition 1.3 (Iterated powerset)**.** [17–19] For $k \in \mathbb{N}_0$ define

$$\mathcal{P}^0(X) := X, \qquad \mathcal{P}^{k+1}(X) := \mathcal{P}\big(\mathcal{P}^k(X)\big).$$

For the nonempty version set

$$\big(\mathcal{P}^*\big)^0(X) := X, \qquad \big(\mathcal{P}^*\big)^{k+1}(X) := \mathcal{P}^*\big((\mathcal{P}^*)^k(X)\big).$$

Definition 1.4 (Hypergraph [3, 4])**.** A *hypergraph* is a pair $H = (V(H), E(H))$ with $V(H) \neq \varnothing$ and $E(H) \subseteq \mathcal{P}^*(V(H))$. Throughout this paper both $V(H)$ and $E(H)$ are finite.

Example 1.5 (Hypergraph: co-authorship by paper)**.** Let the vertices be authors $V(H) = \{\text{Taro, Tae, Carol, Dan, Erin}\}$. Each paper induces a hyperedge equal to its set of co-authors:

$$E(H) = \big\{\{\text{Taro, Tae}\},\ \{\text{Taro, Carol, Dan}\},\ \{\text{Tae, Dan, Erin}\}\big\} \subseteq \mathcal{P}^*\big(V(H)\big).$$

Thus $H = (V(H), E(H))$ is a hypergraph where multi-author papers naturally form hyperedges.

Definition 1.6 (n-SuperHyperGraph). [5,20] Fix a finite base set V_0 and a level $n \in \mathbb{N}_0$. An *n-SuperHyperGraph over V_0* is a triple

$$\mathrm{SHG}^{(n)} \;=\; \big(V,\; E,\; \partial\big),$$

where

- $V \subseteq \mathcal{P}^n(V_0)$ is a finite set of *n-supervertices*;
- E is a finite set of *(super)edge identifiers*;
- $\partial : E \to \mathcal{P}^*(V)$ is an *incidence map* sending each edge to a nonempty finite subset of V.

For $e \in E$, the set $\partial(e) \subseteq V$ is called the *(super)edge incidence set*.

Remark 1.7 (Simple, uniform, and nonempty-tier options). (i) *Simple*: ∂ is injective (no parallel superedges). (ii) *k-uniform*: $|\partial(e)| = k$ for all $e \in E$. (iii) To exclude empties at every tier, one may require $V \subseteq \big(\mathcal{P}^*\big)^n(V_0)$.

Remark 1.8 (Subset presentation). If parallel superedges are unnecessary, one may identify each edge with its incidence set and work with a pair $(V, \mathcal{E})$ where $\mathcal{E} \subseteq \mathcal{P}^*(V)$. This is equivalent to Definition 1.6 by taking $E := \mathcal{E}$ and $\partial := \mathrm{id}$.

Example 1.9 (n-SuperHyperGraph: programs built from teams ($n = 2$)). Take the ground set of employees $V_0 = \{a, b, c, d, e\}$. First form teams (level 1 subsets):

$$T_1 = \{a, b\}, \quad T_2 = \{b, c\}, \quad T_3 = \{d, e\} \;\in\; \mathcal{P}(V_0).$$

Define level 2 *supervertices* as sets of teams:

$$v_1 = \{T_1, T_2\}, \qquad v_2 = \{T_2, T_3\} \;\in\; \mathcal{P}\big(\mathcal{P}(V_0)\big) = \mathcal{P}^2(V_0).$$

Let $V = \{v_1, v_2\} \subseteq \mathcal{P}^2(V_0)$ and let $E = \{\varepsilon\}$ with incidence map

$$\partial(\varepsilon) = \{v_1, v_2\} \;\in\; \mathcal{P}^*(V).$$

Then $\mathrm{SHG}^{(2)} = (V, E, \partial)$ is a 2-SuperHyperGraph: each supervertex represents a program built from teams, and the superedge ε models a cross-program coordination task linking these programs.

1.2 Fuzzy n-SuperHyperGraphs

A fuzzy set assigns to each element a membership degree in $[0, 1]$ [21–23]. Fuzzy graphs and fuzzy hypergraphs endow vertices and (hyper)edges with such degrees [24–30]. A fuzzy n-SuperHyperGraph is a higher-level network representation in which supervertices and superedges carry membership values for modeling complex interactions (cf. [5, 31]).

Definition 1.10 (Fuzzy graph). [25, 32] A *fuzzy graph* is a triple $G = (V, \sigma, \mu)$ where V is a finite nonempty vertex set, $\sigma : V \to [0, 1]$ assigns vertex-membership degrees, and $\mu : V \times V \to [0, 1]$ assigns edge-membership degrees subject to

$$\mu(u, v) \;\le\; \min\{\sigma(u), \sigma(v)\} \qquad (\forall\, u, v \in V).$$

We write uv for $\{u, v\}$ and abbreviate $\mu(uv) := \mu(u, v)$. The (crisp) underlying graph of G has vertex set V and edge set $E^* := \{\, uv : \mu(uv) > 0 \,\}$.

Definition 1.11 (Fuzzy hypergraph). [27,30,33] Let $H^* = (V, E, \partial)$ be a crisp hypergraph. A *fuzzy hypergraph* on H^* is a sextuple

$$\mathcal{H} \;=\; \big(V, E, \partial;\; \sigma, \mu, \eta\big),$$

with maps

$$\sigma : V \to [0, 1], \qquad \mu : E \to [0, 1], \qquad \eta : V \times E \to [0, 1],$$

such that for all $v \in V$ and $e \in E$,

$$\textbf{(support)} \quad \big[v \in \partial(e)\big] \iff \eta(v, e) > 0, \tag{1}$$

$$\textbf{(incidence bound)} \quad \eta(v, e) \;\le\; \min\{\sigma(v), \mu(e)\}, \tag{2}$$

$$\textbf{(edge–vertex bound)} \quad \mu(e) \;\le\; \min_{u \in \partial(e)} \sigma(u). \tag{3}$$

Here σ is the *vertex-membership map*, μ the *edge-membership map*, and η the *incidence-membership map*. The underlying crisp hypergraph is (V, E, ∂), recoverable via (1).

Example 1.12 (Fuzzy hypergraph: multi-agency emergency response)**.** Consider agencies as vertices

$$V = \{\text{Fire}(F),\ \text{Police}(P),\ \text{Hospital}(H),\ \text{Utilities}(U)\}.$$

Scenarios (events) form hyperedges

$$E = \{e_1, e_2, e_3\}, \qquad \partial(e_1) = \{F, U\},\ \partial(e_2) = \{F, H, P\},\ \partial(e_3) = \{U, P\},$$

so $H^* = (V, E, \partial)$ is a crisp hypergraph. Let vertex "readiness" be

$$\sigma(F) = 0.90,\ \sigma(P) = 0.80,\ \sigma(H) = 0.85,\ \sigma(U) = 0.70.$$

Choose edge "scenario confidence" (respecting the edge–vertex bound $\mu(e) \le \min_{u\in\partial(e)} \sigma(u)$):

$$\mu(e_1) = 0.65\ (\le \min(0.90, 0.70) = 0.70),$$

$$\mu(e_2) = 0.75\ (\le \min(0.90, 0.85, 0.80) = 0.80),$$

$$\mu(e_3) = 0.60\ (\le \min(0.70, 0.80) = 0.70).$$

Define incidence-membership $\eta : V \times E \to [0, 1]$ by

$$\eta(v, e) = \begin{cases} \min\{\sigma(v), \mu(e)\} \cdot r_{v,e}, & v \in \partial(e), \\ 0, & v \notin \partial(e), \end{cases} \quad \text{with} \quad r_{v,e} \in (0, 1].$$

For instance, pick

$$\begin{aligned} &\eta(F, e_1) = 0.60,\ \eta(U, e_1) = 0.55; \\ &\eta(F, e_2) = 0.70,\ \eta(H, e_2) = 0.70,\ \eta(P, e_2) = 0.65; \\ &\eta(U, e_3) = 0.55,\ \eta(P, e_3) = 0.55, \end{aligned}$$

and $\eta(v, e) = 0$ otherwise. Then: (i) $[v \in \partial(e)] \Leftrightarrow \eta(v, e) > 0$ (support), (ii) $\eta(v, e) \le \min\{\sigma(v), \mu(e)\}$ (incidence bound), (iii) $\mu(e) \le \min_{u\in\partial(e)} \sigma(u)$ (edge–vertex bound). Hence $\mathcal{H} = (V, E, \partial; \sigma, \mu, \eta)$ is a fuzzy hypergraph encoding uncertain, multi-agency cooperation.

Definition 1.13 (Fuzzy n-SuperHyperGraph)**.** (cf. [5, 34]) Let $\mathrm{SHG}^{(n)} = (V, E)$ be an n-SuperHyperGraph. A *fuzzy n-SuperHyperGraph* is a quadruple

$$(V, E, \sigma, \mu),$$

where $\sigma : V \to [0, 1]$ and $\mu : E \to [0, 1]$ obey the *admissibility constraint*

$$\mu(e)\ \le\ \min_{v\in e} \sigma(v) \qquad \text{for every } e \in E.$$

Example 1.14 (Fuzzy n-SuperHyperGraph: programs built from teams of teams ($n = 2$))**.** Let the ground set of employees be $V_0 = \{a, b, c, d, e, f\}$. Level-1 "teams" are subsets of V_0:

$$T_1 = \{a, b\}, \quad T_2 = \{b, c\}, \quad T_3 = \{d, e, f\}\ \in\ \mathcal{P}(V_0).$$

Level-2 supervertices ("programs") are sets of teams:

$$v_1 = \{T_1, T_2\}, \qquad v_2 = \{T_2, T_3\}, \qquad v_3 = \{T_1, T_3\}\ \in\ \mathcal{P}^2(V_0).$$

Set $V = \{v_1, v_2, v_3\} \subseteq \mathcal{P}^2(V_0)$ and choose superedges

$$E = \{e_A, e_B\}, \qquad e_A = \{v_1, v_2\}, \quad e_B = \{v_1, v_3\}.$$

Assign supervertex-memberships (overall "program readiness")

$$\sigma(v_1) = 0.90, \qquad \sigma(v_2) = 0.75, \qquad \sigma(v_3) = 0.80,$$

and edge-memberships (coordination strength) satisfying the admissibility constraint $\mu(e) \le \min_{v\in e} \sigma(v)$:

$$\mu(e_A) = 0.70\ (\le \min(0.90, 0.75) = 0.75), \qquad \mu(e_B) = 0.72\ (\le \min(0.90, 0.80) = 0.80).$$

Then (V, E, σ, μ) is a *fuzzy* 2*-SuperHyperGraph* modeling uncertain collaboration between programs built from teams of teams.

1.3 Tolerance graph

A tolerance graph models vertices as intervals with tolerances, connecting vertices whenever overlap length exceeds their minimum tolerance threshold [35–39].

Definition 1.15 (Tolerance graph)**.** [35] A finite simple graph $G = (V, E)$ is a *tolerance graph* if there exist, for each $v \in V$, a closed interval $I_v = [\ell_v, r_v] \subset \mathbb{R}$ and a tolerance $t_v \in \mathbb{R}_{>0}$ such that for all distinct $u, v \in V$,

$$\{u, v\} \in E \quad \Longleftrightarrow \quad \left| I_u \cap I_v \right| \; \geq \; \min\{t_u, t_v\},$$

where $|J|$ denotes the (Lebesgue) length of an interval J (and is 0 if $J = \emptyset$).

Remark 1.16 (Common subclasses)**.** If $t_v \leq |I_v|$ for every v, the representation is *bounded* (yielding a bounded tolerance graph). If all tolerances are equal ($t_v \equiv t$), the graph is *unit tolerance*.

Example 1.17 (Tolerance graph: scheduling feasible one-on-one meetings)**.** Let vertices be employees with daily availability intervals (in hours). Set

$$I_A = [9, 12],\ t_A = 1.0; \qquad I_B = [10, 13],\ t_B = 1.2;$$
$$I_C = [11.5, 14],\ t_C = 0.75; \qquad I_D = [8, 9.5],\ t_D = 0.5.$$

Two employees $u \neq v$ are adjacent iff $\ell(I_u \cap I_v) \geq \min\{t_u, t_v\}$, where $\ell(\cdot)$ is interval length (hours). Computations:

$$\ell(I_A \cap I_B) = 2 \geq \min(1.0, 1.2) = 1.0 \Rightarrow AB \in E, \qquad \ell(I_A \cap I_C) = 0.5 < \min(1.0, 0.75) = 0.75 \Rightarrow AC \notin E,$$

$$\ell(I_A \cap I_D) = 0.5 \geq \min(1.0, 0.5) = 0.5 \Rightarrow AD \in E, \qquad \ell(I_B \cap I_C) = 1.5 \geq \min(1.2, 0.75) = 0.75 \Rightarrow BC \in E,$$

and BD, CD have empty overlap, hence not edges. Therefore

$$V = \{A, B, C, D\}, \qquad E = \{\{A, B\}, \{A, D\}, \{B, C\}\}.$$

This models who can schedule a one-on-one of their required minimum length.

1.4 Fuzzy tolerance graph

A fuzzy tolerance graph assigns fuzzy intervals and tolerances to vertices, edges weighted by normalized overlap relative to tolerance measures [40–42].

Definition 1.18 (Fuzzy interval and fuzzy tolerance)**.** A *fuzzy interval* $\mathcal{I}$ on $\mathbb{R}$ is a normal, convex fuzzy set on $\mathbb{R}$. Its *core* $c(\mathcal{I})$ and *support* $s(\mathcal{I})$ are (nonempty) compact intervals, and we write $\ell(J)$ for the (Lebesgue) length of a real interval J. A *fuzzy tolerance* $\mathcal{T}$ is a (normal, convex) fuzzy number with strictly positive core length $\ell\big(c(\mathcal{T})\big) > 0$ (and hence positive support length $\ell\big(s(\mathcal{T})\big) > 0$).

Definition 1.19 (Fuzzy tolerance graph)**.** [40] Let $\mathcal{F} = \{(\mathcal{I}_v, \mathcal{T}_v) : v \in V\}$ be a finite family consisting, for each vertex v, of a fuzzy interval $\mathcal{I}_v$ and a fuzzy tolerance $\mathcal{T}_v$. The *fuzzy tolerance graph* associated with $\mathcal{F}$ is the fuzzy graph

$$\Xi(\mathcal{F}) = (V, \sigma, \mu),$$

where the vertex-membership is $\sigma(v) := h(\mathcal{I}_v)$ (the height, typically $= 1$ for normal fuzzy intervals), and the edge-membership $\mu : V \times V \to [0, 1]$ for distinct u, v is

$$\mu(u, v) \;:=\; \max\Big\{\rho\Big(\ell\big(c(\mathcal{I}_u) \cap c(\mathcal{I}_v)\big),\ \min\big\{\ell\big(c(\mathcal{T}_u)\big), \ell\big(c(\mathcal{T}_v)\big)\big\}\Big)\,,\ \rho\Big(\ell\big(s(\mathcal{I}_u) \cap s(\mathcal{I}_v)\big),\ \min\big\{\ell\big(s(\mathcal{T}_u)\big), \ell\big(s(\mathcal{T}_v)\big)\big\}\Big)\Big\}\,,$$

with the normalization map $\rho(x, a) := \begin{cases} \min\{1,\ x/a\}, & a > 0, \\ 0, & a = 0. \end{cases}$ We set $\mu(v, v) := \sigma(v)$.

Remark 1.20 (Crisp reduction and level cuts)**.** If every $\mathcal{I}_v$ and $\mathcal{T}_v$ is crisp (i.e. characteristic functions of real intervals), then $\mu(u, v) \in \{0, 1\}$ and the underlying crisp graph contains the edge uv iff the overlap length of the (core or support) intervals meets the minimum of the corresponding (core or support) tolerances—recovering the classical tolerance-graph rule. Moreover, for any $t \in (0, 1]$, the t-cut of $\Xi(\mathcal{F})$ is a crisp tolerance representation obtained from the t-cuts of the fuzzy intervals and fuzzy tolerances (cf. known constructions for fuzzy tolerance graphs). *See also* the literature on fuzzy tolerance graphs for compatible formulations and examples.

Example 1.21 (Fuzzy tolerance graph: uncertain availability for coordinating calls)**.** Each person has a fuzzy availability interval with *core* $c(\mathcal{I}_v)$ (certainly available) and *support* $s(\mathcal{I}_v)$ (possibly available), and a fuzzy tolerance with core/support lengths (desired/acceptable call duration). Let $\rho(x, a) = \min\{1, x/a\}$ for $a > 0$ and 0 if $a = 0$. Define edge-membership for $u \neq v$ by

$$\mu(u, v) = \max\Big\{ \rho\big(\ell(c(\mathcal{I}_u) \cap c(\mathcal{I}_v)),\ \min\{\ell(c(\mathcal{T}_u)), \ell(c(\mathcal{T}_v))\}\big),\ \rho\big(\ell(s(\mathcal{I}_u) \cap s(\mathcal{I}_v)),\ \min\{\ell(s(\mathcal{T}_u)), \ell(s(\mathcal{T}_v))\}\big)\Big\}.$$

Take three consultants X, Y, Z with (hours)

$$\begin{aligned}
c(\mathcal{I}_X) &= [9, 12],\ s(\mathcal{I}_X) = [8.5, 12.5], \quad \ell\big(c(\mathcal{T}_X)\big) = 1.5,\ \ell\big(s(\mathcal{T}_X)\big) = 3.0;\\
c(\mathcal{I}_Y) &= [10, 13],\ s(\mathcal{I}_Y) = [9.5, 13.5], \quad \ell\big(c(\mathcal{T}_Y)\big) = 1.0,\ \ell\big(s(\mathcal{T}_Y)\big) = 2.0;\\
c(\mathcal{I}_Z) &= [12, 14],\ s(\mathcal{I}_Z) = [10.5, 14.5],\ \ell\big(c(\mathcal{T}_Z)\big) = 1.25,\ \ell\big(s(\mathcal{T}_Z)\big) = 2.5.
\end{aligned}$$

Edge-memberships:

$$\mu(X, Y) = \max\{\rho(2, \min(1.5, 1.0) = 1.0),\ \rho(3, \min(3.0, 2.0) = 2.0)\} = \max\{1, 1\} = 1,$$

$$\mu(X, Z) = \max\{\rho(0, \min(1.5, 1.25) = 1.25) = 0,\ \rho(2, \min(3.0, 2.5) = 2.5) = 0.8\} = 0.8,$$

$$\mu(Y, Z) = \max\{\rho(1, \min(1.0, 1.25) = 1.0) = 1,\ \rho(3, \min(2.0, 2.5) = 2.0) = 1\} = 1.$$

Thus $G = (\{X, Y, Z\}, \mu)$ is a fuzzy tolerance graph where XY and YZ are fully feasible ($\mu = 1$), while XZ has partial feasibility ($\mu = 0.8$) due to weaker overlap relative to acceptable durations.

2 Main Result

This section presents the results of the paper.

2.1 Tolerance Hypergraph

A tolerance hypergraph extends tolerance graphs by allowing hyperedges of multiple vertices, included when common interval overlap exceeds minimum tolerances.

Definition 2.1 (Tolerance hypergraph)**.** Let V be a finite nonempty set of *vertices*. For each $v \in V$ fix a closed interval $I_v = [\ell_v, r_v] \subset \mathbb{R}$ with $\ell_v \leq r_v$, and a *tolerance* $t_v \in (0, \infty)$. For every nonempty $e \subseteq V$ define the *common-overlap length*

$$L(e) \ := \ \lambda\Big(\bigcap_{v \in e} I_v\Big),$$

where λ denotes the Lebesgue length (so $\lambda(\varnothing) = 0$). The *tolerance hypergraph induced by* (I, t) is the pair

$$\mathcal{H}(I, t) \ := \ \big(V, E(I, t)\big), \qquad E(I, t) \ := \ \Big\{ e \subseteq V : \ |e| \geq 2,\ L(e) \ \geq \ \min_{v \in e} t_v \Big\}.$$

Any $e \in E(I, t)$ is called a *tolerance hyperedge*. We say the model is *bounded* if $t_v \leq \lambda(I_v)$ for all v, and *unit* if $t_v \equiv t$ for some fixed $t > 0$.

Remark 2.2 (2-section (shadow) of a hypergraph)**.** Given a hypergraph $H = (V, E)$, its *2-section* (also called the *shadow*) is the simple graph $[H]_2 = (V, E_2)$ with

$$uv \in E_2 \iff \{u, v\} \in E \quad (u \neq v).$$

Example 2.3 (Tolerance hypergraph: coordinating multi-party meetings)**.** Let vertices be four teams with daily availability windows (in hours) and minimum required meeting durations (tolerances):

$$I_A = [9, 13],\ t_A = 1.5; \quad I_B = [10, 12.5],\ t_B = 1.0; \quad I_C = [11, 15],\ t_C = 2.0; \quad I_D = [8, 10.5],\ t_D = 1.0.$$

For any nonempty $e \subseteq \{A, B, C, D\}$ let $L(e) = \lambda(\cap_{v \in e} I_v)$ be the common-overlap length. A subset e is a hyperedge iff $|e| \geq 2$ and $L(e) \geq \min_{v \in e} t_v$.

Pairs:

$$L(\{A,B\}) = \ell([10,12.5]) = 2.5 \geq \min(1.5,1.0) = 1.0 \Rightarrow \{A,B\} \in E,$$
$$L(\{A,C\}) = \ell([11,13]) = 2.0 \geq \min(1.5,2.0) = 1.5 \Rightarrow \{A,C\} \in E,$$
$$L(\{A,D\}) = \ell([9,10.5]) = 1.5 \geq \min(1.5,1.0) = 1.0 \Rightarrow \{A,D\} \in E,$$
$$L(\{B,C\}) = \ell([11,12.5]) = 1.5 \geq \min(1.0,2.0) = 1.0 \Rightarrow \{B,C\} \in E,$$
$$L(\{B,D\}) = \ell([10,10.5]) = 0.5 < \min(1.0,1.0) = 1.0 \Rightarrow \{B,D\} \notin E,$$
$$L(\{C,D\}) = \ell(\varnothing) = 0 \Rightarrow \{C,D\} \notin E.$$

Triples:

$$L(\{A,B,C\}) = \ell([11,12.5]) = 1.5 \geq \min(1.5,1.0,2.0) = 1.0 \Rightarrow \{A,B,C\} \in E,$$
$$L(\{A,B,D\}) = \ell([10,10.5]) = 0.5 < \min(1.5,1.0,1.0) = 1.0 \Rightarrow \{A,B,D\} \notin E,$$
$$L(\{A,C,D\}) = \ell(\varnothing) = 0 \Rightarrow \{A,C,D\} \notin E,$$
$$L(\{B,C,D\}) = \ell(\varnothing) = 0 \Rightarrow \{B,C,D\} \notin E.$$

No quadruple overlap exists. Hence the tolerance hypergraph is

$$V = \{A,B,C,D\}, \quad E = \big\{\{A,B\},\{A,C\},\{A,D\},\{B,C\},\{A,B,C\}\big\}.$$

This captures which groups (pairs or larger) can meet for at least their minimum required duration.

Theorem 2.4 (Tolerance hypergraphs specialize to tolerance graphs)**.** *Let $\mathcal{H}(I,t)$ be as in Definition 2.1. Define a simple graph $G(I,t) = (V, E_2)$ by*

$$uv \in E_2 \iff \lambda(I_u \cap I_v) \ \geq\ \min\{t_u, t_v\} \qquad (u \neq v).$$

Then the 2-section of $\mathcal{H}(I,t)$ equals $G(I,t)$, i.e. $[\,\mathcal{H}(I,t)\,]_2 = G(I,t)$.

Proof. By Definition 2.1, $\{u,v\} \in E(I,t)$ if and only if $L(\{u,v\}) = \lambda(I_u \cap I_v) \geq \min\{t_u,t_v\}$; but this is exactly the adjacency rule of $G(I,t)$. Hence $uv \in [\,\mathcal{H}(I,t)\,]_2$ if and only if $uv \in E_2$, proving $[\,\mathcal{H}(I,t)\,]_2 = G(I,t)$. □

Theorem 2.5 (Tolerance hypergraphs are hypergraphs)**.** *For any choice of intervals $\{I_v\}_{v\in V}$ and tolerances $\{t_v\}_{v\in V}$, the pair $\mathcal{H}(I,t) = (V, E(I,t))$ of Definition 2.1 is a (finite, loopless) hypergraph.*

Proof. By construction $E(I,t) \subseteq \mathcal{P}(V)$ and every $e \in E(I,t)$ satisfies $|e| \geq 2$, so $\mathcal{H}(I,t)$ is a well-defined finite hypergraph. □

2.2 Tolerance SuperHypergraph

A tolerance superhypergraph generalizes tolerance hypergraphs using iterated powerset vertices, forming superedges when collective interval overlap surpasses corresponding tolerance thresholds.

Definition 2.6 (Tolerance superhypergraph)**.** Let $V \subseteq \mathcal{P}^n(V_0)$ be a finite nonempty set ($n \in \mathbb{N}_0$). For each $v \in V$ fix a closed interval $I_v = [\ell_v, r_v] \subset \mathbb{R}$ with $\ell_v \leq r_v$, and a *tolerance* $t_v \in [0,\infty)$. For every nonempty $e \subseteq V$ put

$$L(e) \ :=\ \lambda\Big(\bigcap_{v\in e} I_v\Big),$$

where λ denotes the Lebesgue length on $\mathbb{R}$ (in particular $\lambda(\varnothing) = 0$). The *tolerance superhypergraph induced by (I,t)* is

$$\mathcal{H}^{(n)}_{\mathrm{tol}}(I,t) \ :=\ \big(V,\ \mathcal{E}(I,t)\big), \qquad \mathcal{E}(I,t) \ :=\ \Big\{ e \subseteq V :\ |e| \geq 2,\ L(e) \ \geq\ \min_{v\in e} t_v \Big\}.$$

Any $e \in \mathcal{E}(I,t)$ is called a *tolerance superedge*. The model is *bounded* if $t_v \leq \lambda(I_v)$ for all v, and *unit* if $t_v \equiv t$ for some fixed $t \geq 0$.

Remark 2.7 (2-section (shadow))**.** For a (super)hypergraph $H = (V, \mathcal{E})$ its *2-section* is the simple graph $[H]_2 = (V, E_2)$ with

$$uv \in E_2 \iff \{u,v\} \in \mathcal{E} \qquad (u \neq v).$$

Example 2.8 (Tolerance superhypergraph: cross-program workshop among teams of employees)**.** Let the ground set of employees be $V_0 = \{a, b, c, d, e, f\}$ and set $n = 1$, so supervertices are teams (subsets of V_0):

$$T_1 = \{a, b, c\}, \quad T_2 = \{c, d\}, \quad T_3 = \{d, e, f\} \in \mathcal{P}(V_0).$$

Take $V = \{T_1, T_2, T_3\} \subseteq \mathcal{P}^1(V_0)$. Assign to each supervertex a workshop-availability interval (hours) and a minimum workable duration (tolerance):

$$I_{T_1} = [9, 12],\ t_{T_1} = 1.0; \qquad I_{T_2} = [10, 14],\ t_{T_2} = 1.5; \qquad I_{T_3} = [11, 13.2],\ t_{T_3} = 1.0.$$

A superedge $e \subseteq V$ (with $|e| \geq 2$) is included when $L(e) = \lambda(\cap_{v\in e} I_v) \geq \min_{v\in e} t_v$.

Pairs:

$$\begin{aligned} L(\{T_1, T_2\}) &= \ell([10, 12]) = 2.0 \geq \min(1.0, 1.5) = 1.0 \Rightarrow \{T_1, T_2\} \in E, \\ L(\{T_1, T_3\}) &= \ell([11, 12]) = 1.0 \geq \min(1.0, 1.0) = 1.0 \Rightarrow \{T_1, T_3\} \in E, \\ L(\{T_2, T_3\}) &= \ell([11, 13.2]) = 2.2 \geq \min(1.5, 1.0) = 1.0 \Rightarrow \{T_2, T_3\} \in E. \end{aligned}$$

Triple:

$$L(\{T_1, T_2, T_3\}) = \ell([11, 12]) = 1.0 \geq \min(1.0, 1.5, 1.0) = 1.0 \Rightarrow \{T_1, T_2, T_3\} \in E.$$

Therefore the tolerance superhypergraph is

$$V = \{T_1, T_2, T_3\}, \qquad E = \big\{\{T_1, T_2\}, \{T_1, T_3\}, \{T_2, T_3\}, \{T_1, T_2, T_3\}\big\}.$$

Here supervertices are teams (level-1 objects), and superedges encode which collections of teams can hold a joint workshop of sufficient length.

Theorem 2.9 (Tolerance superhypergraphs are n-SuperHyperGraphs)**.** *For any choice of $\{I_v\}_{v\in V}$ and $\{t_v\}_{v\in V}$ as in Definition 2.6,*

$$\mathcal{H}^{(n)}_{\mathrm{tol}}(I, t) = (V, \mathcal{E}(I, t))$$

is an n-SuperHyperGraph over V_0.

Proof. By construction $V \subseteq \mathcal{P}^n(V_0)$ is finite and nonempty. Also $\mathcal{E}(I, t) \subseteq \mathcal{P}^*(V)$ and, by definition, any $e \in \mathcal{E}(I, t)$ satisfies $|e| \geq 2$. Hence $(V, \mathcal{E}(I, t))$ is an n-SuperHyperGraph. □

Theorem 2.10 (2-section is a tolerance graph)**.** *Let $\mathcal{H}^{(n)}_{\mathrm{tol}}(I, t)$ be a tolerance superhypergraph. Define a simple graph $G(I, t) = (V, E)$ by*

$$uv \in E \iff \lambda(I_u \cap I_v) \;\geq\; \min\{t_u, t_v\} \qquad (u \neq v).$$

Then $[\,\mathcal{H}^{(n)}_{\mathrm{tol}}(I, t)\,]_2 = G(I, t)$; in particular, the 2-section of a tolerance superhypergraph is a (classical) tolerance graph.

Proof. By Definition 2.6, $\{u, v\} \in \mathcal{E}(I, t)$ iff $L(\{u, v\}) = \lambda(I_u \cap I_v) \geq \min\{t_u, t_v\}$, which is precisely the adjacency rule of $G(I, t)$. Therefore $uv \in [\,\mathcal{H}^{(n)}_{\mathrm{tol}}(I, t)\,]_2$ iff $uv \in E$, proving equality. □

Theorem 2.11 (Reduction to tolerance hypergraphs and graphs)**.** *Fix $n \in \mathbb{N}_0$ and $\mathcal{H}^{(n)}_{\mathrm{tol}}(I, t) = (V, \mathcal{E}(I, t))$ as above.*

(i) *If $n = 0$, then $\mathcal{H}^{(0)}_{\mathrm{tol}}(I, t)$ is exactly the* tolerance hypergraph *on the vertex set V determined by (I, t) (hyperedges are those $e \subseteq V$ with $|e| \geq 2$ and $L(e) \geq \min_{v\in e} t_v$).*

(ii) *For any n, the 2-section $[\,\mathcal{H}^{(n)}_{\mathrm{tol}}(I, t)\,]_2$ is the* tolerance graph *$G(I, t)$ on V (Theorem 2.10).*

Consequently, the class of tolerance superhypergraphs simultaneously generalizes tolerance hypergraphs (case $n = 0$) and tolerance graphs (via 2-section), while refining the notion of n-SuperHyperGraph (Theorem 2.9).

Proof. (i) When $n = 0$ we have $V \subseteq V_0$ and Definition 2.6 coincides with the usual tolerance-hyperedge rule on V; hence $\mathcal{H}^{(0)}_{\mathrm{tol}}(I, t)$ is the tolerance hypergraph induced by (I, t).
(ii) This is exactly Theorem 2.10.
The concluding sentence follows immediately. □

2.3 Fuzzy Tolerance Hypergraph

A fuzzy tolerance hypergraph assigns fuzzy intervals and tolerances to vertices, forming hyperedges when collective overlap length meets tolerance thresholds.

Definition 2.12 (Fuzzy Tolerance Hypergraph (FTH))**.** Let V be a finite nonempty vertex set. For each $v \in V$ fix a fuzzy interval $\mathcal{I}_v$ and a fuzzy tolerance $\mathcal{T}_v$. Define, for any nonempty $e \subseteq V$, the *core-/support-overlap lengths* and *core-/support-thresholds* by

$$L_c(e) := \ell\Big(\bigcap_{v\in e} c(\mathcal{I}_v)\Big), \quad L_s(e) := \ell\Big(\bigcap_{v\in e} s(\mathcal{I}_v)\Big),$$

$$\tau_c(e) := \min_{v\in e} \ell\big(c(\mathcal{T}_v)\big), \qquad \tau_s(e) := \min_{v\in e} \ell\big(s(\mathcal{T}_v)\big).$$

Let $\rho(x,a) := \begin{cases} \min\{1,\ x/a\}, & a > 0, \\ 0, & a = 0. \end{cases}$ and set the *overlap score*

$$\varphi(e) \ := \ \max\big\{\rho\big(L_c(e), \tau_c(e)\big),\ \rho\big(L_s(e), \tau_s(e)\big)\big\} \ \in [0,1].$$

Define the (crisp) edge universe

$$E \ := \ \big\{\, e \subseteq V :\ |e| \geq 2,\ \varphi(e) > 0 \,\big\}.$$

A *fuzzy tolerance hypergraph* on V is the sextuple

$$\mathfrak{H} = (V, E;\ \sigma, \mu, \eta)$$

with

$$\sigma : V \to [0,1], \qquad \sigma(v) := \min\big\{h(\mathcal{I}_v),\ h(\mathcal{T}_v)\big\},$$

$$\mu : E \to [0,1], \qquad \mu(e) := \min\Big\{\varphi(e),\ \min_{v\in e} \sigma(v)\Big\},$$

$$\eta : V \times E \to [0,1], \qquad \eta(v,e) := \begin{cases} \mu(e), & v \in e, \\ 0, & v \notin e. \end{cases}$$

Remark 2.13 (Admissibility constraints)**.** By construction, for all $v \in V$ and $e \in E$: (i) $\eta(v,e) > 0$ iff $v \in e$ (support equivalence); (ii) $\eta(v,e) \leq \min\{\sigma(v), \mu(e)\}$ (incidence bound); (iii) $\mu(e) \leq \min_{u\in e} \sigma(u)$ (edge–vertex bound). Thus $(V, E; \sigma, \mu, \eta)$ is a fuzzy hypergraph in the usual sense, equipped with a tolerance-based generator φ.

Example 2.14 (Fuzzy Tolerance Hypergraph: coordinating specialist consultations)**.** Let

$$V = \{\text{Cardio}(C),\ \text{Neuro}(N),\ \text{Ortho}(O)\}$$

denote three clinics. Each $v \in V$ has a fuzzy availability interval $\mathcal{I}_v$ (daytime hours) and a fuzzy tolerance $\mathcal{T}_v$ (acceptable consultation duration). Cores and supports (in hours) are:

$$\begin{aligned} c(\mathcal{I}_C) &= [9,12], \quad s(\mathcal{I}_C) = [8.5, 12.5], \qquad \ell\big(c(\mathcal{T}_C)\big) = 1.0,\ \ell\big(s(\mathcal{T}_C)\big) = 1.5, \\ c(\mathcal{I}_N) &= [10,13],\ s(\mathcal{I}_N) = [9.5, 13.5], \qquad \ell\big(c(\mathcal{T}_N)\big) = 1.2,\ \ell\big(s(\mathcal{T}_N)\big) = 2.0, \\ c(\mathcal{I}_O) &= [11,14],\ s(\mathcal{I}_O) = [10.5, 14.5], \quad \ell\big(c(\mathcal{T}_O)\big) = 1.5,\ \ell\big(s(\mathcal{T}_O)\big) = 2.5. \end{aligned}$$

Let $\rho(x,a) = \min\{1, x/a\}$ for $a > 0$ (and 0 if $a = 0$). For any nonempty $e \subseteq V$ set

$$L_c(e) = \ell\Big(\bigcap_{v\in e} c(\mathcal{I}_v)\Big), \quad L_s(e) = \ell\Big(\bigcap_{v\in e} s(\mathcal{I}_v)\Big), \quad \tau_c(e) = \min_{v\in e} \ell\big(c(\mathcal{T}_v)\big), \quad \tau_s(e) = \min_{v\in e} \ell\big(s(\mathcal{T}_v)\big),$$

$$\varphi(e) = \max\{\rho(L_c(e), \tau_c(e)),\ \rho(L_s(e), \tau_s(e))\}.$$

Heights $h(\mathcal{I}_C, \mathcal{T}_C) = (0.95, 0.90)$, $h(\mathcal{I}_N, \mathcal{T}_N) = (0.90, 0.95)$, $h(\mathcal{I}_O, \mathcal{T}_O) = (0.85, 0.80)$ give

$$\sigma(C) = \min(0.95, 0.90) = 0.90, \quad \sigma(N) = \min(0.90, 0.95) = 0.90, \quad \sigma(O) = \min(0.85, 0.80) = 0.80.$$

Compute overlaps (hours):

$$\begin{aligned}
&L_c(\{C,N\}) = \ell([10,12]) = 2, && \tau_c = \min(1.0,1.2) = 1.0, && \rho = 1;\\
&L_c(\{C,O\}) = \ell([11,12]) = 1, && \tau_c = \min(1.0,1.5) = 1.0, && \rho = 1;\\
&L_c(\{N,O\}) = \ell([11,13]) = 2, && \tau_c = \min(1.2,1.5) = 1.2, && \rho = 1;\\
&L_c(\{C,N,O\}) = \ell([11,12]) = 1, && \tau_c = \min(1.0,1.2,1.5) = 1.0, && \rho = 1.
\end{aligned}$$

$$\begin{aligned}
&L_s(\{C,N\}) = \ell([9.5,12.5]) = 3.0, && \tau_s = \min(1.5,2.0) = 1.5, && \rho = 1;\\
&L_s(\{C,O\}) = \ell([10.5,12.5]) = 2.0, && \tau_s = \min(1.5,2.5) = 1.5, && \rho = 1;\\
&L_s(\{N,O\}) = \ell([10.5,13.5]) = 3.0, && \tau_s = \min(2.0,2.5) = 2.0, && \rho = 1;\\
&L_s(\{C,N,O\}) = \ell([10.5,12.5]) = 2.0, && \tau_s = \min(1.5,2.0,2.5) = 1.5, && \rho = 1.
\end{aligned}$$

Hence $\varphi(e) = 1$ for all e listed. The FTH edge-membership is

$$\mu(e) = \min\{\varphi(e),\ \min_{v\in e}\sigma(v)\} = \min_{v\in e}\sigma(v),$$

so

$$\mu(\{C,N\}) = 0.90,\quad \mu(\{C,O\}) = 0.80,\quad \mu(\{N,O\}) = 0.80,\quad \mu(\{C,N,O\}) = 0.80.$$

With $\eta(v,e) = \mu(e)$ if $v \in e$ and 0 otherwise, we obtain a *fuzzy tolerance hypergraph* modeling uncertain multi-specialist sessions.

Theorem 2.15 (FTH is a fuzzy hypergraph)**.** *Every $\mathfrak{H}$ in Definition 2.12 satisfies the axioms of a fuzzy hypergraph:*

$$\eta(v,e) > 0 \iff v \in e, \qquad \eta(v,e) \le \min\{\sigma(v),\mu(e)\}, \qquad \mu(e) \le \min_{u\in e}\sigma(u).$$

Proof. Immediate from Remark 2.13. In particular, $\eta(v,e) = 0$ when $v \notin e$, and $\eta(v,e) = \mu(e) \le \min\{\sigma(v),\mu(e)\}$ when $v \in e$. Finally, $\mu(e)$ is defined as the minimum of $\varphi(e)$ and all $\sigma(u)$ ($u \in e$), so $\mu(e) \le \min_{u\in e}\sigma(u)$. □

Theorem 2.16 (Reductions: FTH $\Rightarrow$ fuzzy tolerance graph and tolerance hypergraph/graph)**.** *Let $\mathfrak{H} = (V,E;\sigma,\mu,\eta)$ be an FTH as above.*

(i) (Pairs-only reduction) If we restrict to 2-edges and define a fuzzy graph by $\Xi = (V,\sigma,\mu_2)$ with $\mu_2(u,v) := \mu(\{u,v\})$ for $u \neq v$, then Ξ is a fuzzy tolerance graph*. In particular, when $\sigma \equiv 1$ (all vertices normal of height 1), $\mu_2(u,v) = \varphi(\{u,v\})$ coincides with the standard fuzzy tolerance rule based on overlap vs. minimum tolerance.*

(ii) (Crisp reduction) If every $\mathcal{I}_v$ and $\mathcal{T}_v$ is crisp (characteristic intervals) and $\sigma \equiv 1$, then

$$E = \{\, e \subseteq V :\ |e| \ge 2,\ L_c(e) \ge \tau_c(e) \text{ or } L_s(e) \ge \tau_s(e) \,\}$$

and $\mu(e) = 1$ for $e \in E$, $\mu(e) = 0$ otherwise. Hence (V,E) is exactly a tolerance hypergraph*. Restricting further to $|e| = 2$ recovers a (classical)* tolerance graph.

(iii) (Underlying hypergraph) Forgetting the fuzzy levels (σ,μ,η) yields the crisp hypergraph (V,E); therefore FTHs generalize *the notion of hypergraph via a tolerance-driven fuzzy enhancement.*

Proof. (i) For any distinct $u,v \in V$, we have

$$\mu_2(u,v) = \min\Big\{ \underbrace{\max\big\{\rho(L_c(\{u,v\}),\tau_c(\{u,v\})),\ \rho(L_s(\{u,v\}),\tau_s(\{u,v\}))\big\}}_{\varphi(\{u,v\})},\ \sigma(u),\ \sigma(v)\Big\},$$

which is the standard fuzzy-tolerance overlap normalized by the minimum vertex heights. If $\sigma \equiv 1$, $\mu_2(u,v) = \varphi(\{u,v\})$ exactly, as claimed.

(ii) With crisp intervals/tolerances, L_c, L_s, τ_c, τ_s are nonnegative reals and $\rho(x,a) \in \{0,1\}$. Thus $\varphi(e) \in \{0,1\}$ and $E = \{e : \varphi(e) = 1\}$; with $\sigma \equiv 1$ we get $\mu(e) = \varphi(e)$. The edge predicate $\varphi(e) = 1$ is precisely "(core or support) common-overlap length $\ge$ minimum (core or support) tolerance," i.e. the tolerance-hyperedge rule. Limiting to $|e| = 2$ recovers the tolerance-graph adjacency rule.

(iii) Trivial: (V,E) is a finite hypergraph by definition of E. □

2.4 Fuzzy Tolerance SuperHypergraph

A fuzzy tolerance superhypergraph extends fuzzy tolerance hypergraphs using iterated powerset supervertices, creating superedges based on fuzzy overlap-tolerance conditions.

Definition 2.17 (Multiway overlap score)**.** Given a finite nonempty index set e, fuzzy intervals $\{\mathcal{I}_v\}_{v\in e}$ and fuzzy tolerances $\{\mathcal{T}_v\}_{v\in e}$, set

$$L_c(e) := \ell\Big(\bigcap_{v\in e} c(\mathcal{I}_v)\Big), \qquad L_s(e) := \ell\Big(\bigcap_{v\in e} s(\mathcal{I}_v)\Big),$$

$$\tau_c(e) := \min_{v\in e} \ell\big(c(\mathcal{T}_v)\big), \qquad \tau_s(e) := \min_{v\in e} \ell\big(s(\mathcal{T}_v)\big),$$

and define the normalization $\rho(x,a) := \begin{cases} \min\{1,\ x/a\}, & a>0,\\ 0, & a=0.\end{cases}$. The *tolerance overlap score* of e is

$$\varphi(e) \;:=\; \max\Big\{\, \rho\big(L_c(e),\tau_c(e)\big),\ \rho\big(L_s(e),\tau_s(e)\big)\Big\} \;\in [0,1].$$

Definition 2.18 (Fuzzy Tolerance SuperHypergraph (FTSHG))**.** Fix $n \in \mathbb{N}_0$ and a finite set $V \subseteq \mathcal{P}^n(V_0)$. For every supervertex $v \in V$ choose a fuzzy interval $\mathcal{I}_v$ and a fuzzy tolerance $\mathcal{T}_v$, and put

$$\sigma : V \to [0,1], \qquad \sigma(v) := \min\{\, h(\mathcal{I}_v),\ h(\mathcal{T}_v)\,\}.$$

Let the (crisp) edge universe be the whole $\mathcal{E} := \{\, e \subseteq V : |e| \ge 2\,\}$ and define the *edge-membership*

$$\mu : \mathcal{E} \to [0,1], \qquad \mu(e) := \min\Big\{\varphi(e),\ \min_{v\in e} \sigma(v)\Big\}$$

with $\varphi(e)$ from Definition 2.17. Set the *incidence membership*

$$\eta : V\times\mathcal{E} \to [0,1], \qquad \eta(v,e) := \begin{cases} \mu(e), & v\in e,\\ 0, & v\notin e.\end{cases}$$

The septuple

$$\mathfrak{S}^{(n)}_{\mathrm{FT}} \;:=\; (V,\mathcal{E};\ \sigma,\mu,\eta)$$

is called a *fuzzy tolerance superhypergraph* on V.

Remark 2.19 (Admissibility (fuzzy hypergraph axioms))**.** For all $v \in V$ and $e \in \mathcal{E}$ we have: (i) $\eta(v,e) > 0 \iff v \in e$ (support equivalence); (ii) $\eta(v,e) \le \min\{\sigma(v),\mu(e)\}$ (incidence bound); (iii) $\mu(e) \le \min_{u\in e}\sigma(u)$ (edge–vertex bound).

Example 2.20 (Fuzzy Tolerance SuperHypergraph: joint workshops among teams of teams ($n = 1$))**.** Let V_0 be employees and take level-1 supervertices (teams)

$$V = \{T_1,T_2,T_3\} \subseteq \mathcal{P}^1(V_0),$$

with fuzzy team-availability intervals and fuzzy tolerances:

$$\begin{aligned}
&c(\mathcal{I}_{T_1}) = [9,12],\ s(\mathcal{I}_{T_1}) = [8.5,12.5],\quad \ell\big(c(\mathcal{T}_{T_1})\big) = 1.5,\ \ell\big(s(\mathcal{T}_{T_1})\big) = 2.5;\\
&c(\mathcal{I}_{T_2}) = [10,14],\ s(\mathcal{I}_{T_2}) = [9.5,14.5],\quad \ell\big(c(\mathcal{T}_{T_2})\big) = 1.0,\ \ell\big(s(\mathcal{T}_{T_2})\big) = 2.0;\\
&c(\mathcal{I}_{T_3}) = [11,13],\ s(\mathcal{I}_{T_3}) = [10.5,13.5],\ \ell\big(c(\mathcal{T}_{T_3})\big) = 1.0,\ \ell\big(s(\mathcal{T}_{T_3})\big) = 1.5.
\end{aligned}$$

Overlap scores $\varphi(e)$ are computed as in the previous example; for all pairs and the triple below one finds $\varphi(e) = 1$ since core/support overlaps exceed the corresponding minima. Choose supervertex heights

$$\sigma(T_1) = \min(0.95,0.90) = 0.90,\quad \sigma(T_2) = \min(0.90,0.90) = 0.90,\quad \sigma(T_3) = \min(0.90,0.85) = 0.85.$$

Let the (crisp) superedge universe be $\mathcal{E} = \{e \subseteq V : |e| \ge 2\}$. Define the edge-membership

$$\mu(e) = \min\{\varphi(e),\ \min_{v\in e}\sigma(v)\} = \min_{v\in e}\sigma(v),$$

so that

$$\mu(\{T_1,T_2\}) = 0.90, \qquad \mu(\{T_1,T_3\}) = 0.85, \qquad \mu(\{T_2,T_3\}) = 0.85, \qquad \mu(\{T_1,T_2,T_3\}) = 0.85.$$

Setting $\eta(v,e) = \mu(e)$ for $v \in e$ (and 0 otherwise) yields a *fuzzy tolerance superhypergraph* on V that quantifies how strongly sets of teams can hold a joint workshop of acceptable length under uncertainty.

Theorem 2.21 (FTSHG is a fuzzy n-SuperHyperGraph). *$\mathfrak{S}^{(n)}_{\mathrm{FT}} = (V, \mathcal{E}; \sigma, \mu, \eta)$ is a fuzzy hypergraph whose underlying crisp pair $(V, \mathcal{E}^{[t]})$ with $\mathcal{E}^{[t]} := \{ e \in \mathcal{E} : \mu(e) \geq t \}$ is an n-SuperHyperGraph for each threshold $t \in (0, 1]$.*

Proof. The three admissibility properties hold by construction: $\eta(v, e) = 0$ if $v \notin e$, and $\eta(v, e) = \mu(e) \leq \min\{\sigma(v), \mu(e)\}$ when $v \in e$; also $\mu(e) = \min\{\varphi(e), \min_{u \in e} \sigma(u)\} \leq \min_{u \in e} \sigma(u)$. For any fixed $t \in (0, 1]$, $\mathcal{E}^{[t]} \subseteq \mathcal{P}^*(V)$, hence $(V, \mathcal{E}^{[t]})$ is an n-SuperHyperGraph. □

Definition 2.22 (2-section and level-cuts). For a crisp $(V, \mathcal{E})$, its *2-section* is the simple graph $[V, \mathcal{E}]_2 = (V, E_2)$ with $uv \in E_2 \iff \{u, v\} \in \mathcal{E}$ $(u \neq v)$. For a fuzzy edge map μ, the *t-cut* of edges is $\mathcal{E}^{[t]} := \{e : \mu(e) \geq t\}$, and the *$t$-cut 2-section* is $[V, \mathcal{E}^{[t]}]_2$.

Theorem 2.23 (Generalization to Fuzzy Tolerance Hypergraph). *When $n = 0$, $\mathfrak{S}^{(0)}_{\mathrm{FT}}$ is exactly a* fuzzy tolerance hypergraph *on the (ordinary) vertex set V with the same (σ, μ, η).*

Proof. For $n = 0$ we have $V \subseteq V_0$ (ordinary vertices). Definitions 2.17–2.18 coincide with the fuzzy tolerance hypergraph construction on V; in particular, $\varphi(e)$ compares the multiway interval overlap with the minimum tolerance across e, and μ, η are the standard admissible fuzzy hypergraph maps. □

Theorem 2.24 (Generalization to Tolerance SuperHypergraph via 1-cut). *Assume all $\mathcal{I}_v, \mathcal{T}_v$ are* crisp *intervals and $h(\mathcal{I}_v) = h(\mathcal{T}_v) = 1$ for every v (so $\sigma \equiv 1$). Then the* 1*-cut $(V, \mathcal{E}^{[1]})$ of $\mathfrak{S}^{(n)}_{\mathrm{FT}}$ is precisely the* tolerance superhypergraph *whose superedges are those e with*

$$\ell\Big(\bigcap_{v \in e} I_v\Big) \ \geq\ \min_{v \in e} t_v,$$

where $I_v := c(\mathcal{I}_v) = s(\mathcal{I}_v)$ and $t_v := \ell\big(c(\mathcal{T}_v)\big) = \ell\big(s(\mathcal{T}_v)\big)$.

Proof. Under the crisp assumptions, for every e the score $\varphi(e) = \max\{\rho(L_c(e), \tau_c(e)), \rho(L_s(e), \tau_s(e))\}$ equals 1 if and only if $L_c(e) \geq \tau_c(e)$ or $L_s(e) \geq \tau_s(e)$, i.e. the common overlap length of the (equal) core/support intervals meets the minimum tolerance. Since $\sigma \equiv 1$, we have $\mu(e) = \min\{\varphi(e), 1\} = \varphi(e)$; thus $\mu(e) \geq 1$ iff the tolerance condition holds. Therefore $\mathcal{E}^{[1]} = \{e : \varphi(e) = 1\}$ is exactly the desired tolerance superhyperedge set. □

Theorem 2.25 (Generalization to Fuzzy Tolerance Graph). *For any $n \in \mathbb{N}_0$, define a fuzzy graph on V by*

$$\Xi^{(n)} = (V,\ \sigma,\ \mu_2), \qquad \mu_2(u, v) := \mu(\{u, v\}) \quad (u \neq v).$$

Then $\Xi^{(n)}$ is a fuzzy tolerance graph. *In the crisp, unit-height case of Theorem 2.24, the* 1*-cut of $\Xi^{(n)}$ is the (classical) tolerance graph induced by $\{(I_v, t_v)\}$.*

Proof. By Definition 2.17, for $e = \{u, v\}$ we have

$$\mu_2(u, v) \ =\ \min\Big\{ \max\big\{\rho(\ell(I_u \cap I_v),\ \min\{t_u, t_v\}),\ \ldots\big\},\ \sigma(u), \sigma(v)\Big\},$$

which is the standard fuzzy tolerance overlap (possibly using both core/support channels), normalized by vertex heights. Hence $\Xi^{(n)}$ is a fuzzy tolerance graph. In the crisp, unit-height case, $\mu_2(u, v) \geq 1$ iff $\ell(I_u \cap I_v) \geq \min\{t_u, t_v\}$, so the 1-cut yields the classical tolerance graph. □

Theorem 2.26 (Underlying SuperHyperGraph). *For any threshold $t \in (0, 1]$, the t-cut $(V, \mathcal{E}^{[t]})$ of $\mathfrak{S}^{(n)}_{\mathrm{FT}}$ is an n-SuperHyperGraph (hence FTSHGs refine the notion of superhypergraph).*

Proof. This is the second statement of Theorem 2.21. □

3 Conclusion

As the conclusion of this paper, the following theorem is obtained.

Theorem 3.1 (Summary of generalizations)**.** *Let* $\mathfrak{S}^{(n)}_{\mathrm{FT}} = (V, \mathcal{E}; \sigma, \mu, \eta)$ *be a Fuzzy Tolerance SuperHypergraph (FTSHG) as in Definitions* (fuzzy interval/tolerance, multiway overlap φ, and FTSHG) *with* $V \subseteq \mathcal{P}^n(V_0)$ *finite. Then:*

$$\textit{FTSHG} \implies \begin{cases} \text{Fuzzy Tolerance Hypergraph} & \textit{when } n = 0, \\ \text{Tolerance SuperHypergraph} & \textit{in the crisp/unit-height case at the } 1\textit{-cut}, \\ \text{Fuzzy Tolerance Graph} & \textit{via the 2-section } \mu_2(u,v) := \mu(\{u,v\}), \\ n\text{-SuperHyperGraph} & \textit{for every } t \in (0,1] \textit{ by the edge } t\textit{-cut } \mathcal{E}^{[t]}. \end{cases}$$

Proof. We verify each arrow rigorously.

(a) FTSHG ⇒ Fuzzy Tolerance Hypergraph (case $n = 0$). If $n = 0$, then $V \subseteq V_0$ is an ordinary vertex set. By definition,

$$\mathcal{E} \;=\; \{\, e \subseteq V : \; |e| \geq 2 \,\}, \qquad \mu(e) \;=\; \min\Big\{\, \varphi(e),\; \min_{v \in e} \sigma(v) \Big\},$$

with

$$\varphi(e) \;=\; \max\Big\{\, \rho(L_c(e), \tau_c(e)),\; \rho(L_s(e), \tau_s(e)) \Big\} \in [0,1],$$

where $L_\bullet(e)$ is the (core/support) common-overlap length and $\tau_\bullet(e)$ is the minimum (core/support) tolerance across e. This is exactly the fuzzy tolerance *hyper*edge construction on an ordinary vertex set; $\eta(v,e) = \mu(e)$ for $v \in e$ and 0 otherwise gives the standard admissible incidence. Hence $\mathfrak{S}^{(0)}_{\mathrm{FT}}$ is a fuzzy tolerance hypergraph.

(b) FTSHG ⇒ Tolerance SuperHypergraph (crisp, unit-height, 1-cut). Assume each $\mathcal{I}_v$ and $\mathcal{T}_v$ is a crisp interval, and $h(\mathcal{I}_v) = h(\mathcal{T}_v) = 1$, so $\sigma \equiv 1$. Then for any $e \subseteq V$,

$$\rho(x,a) = \min\{1, x/a\} \in \{0,1\} \quad \text{with } a > 0,$$

so

$$\varphi(e) = 1 \iff L_c(e) \geq \tau_c(e) \;\text{ or }\; L_s(e) \geq \tau_s(e).$$

Because $\sigma \equiv 1$, we have $\mu(e) = \min\{\varphi(e), 1\} = \varphi(e)$, hence

$$\mu(e) \geq 1 \iff \varphi(e) = 1 \iff \ell\Big(\bigcap_{v \in e} I_v\Big) \;\geq\; \min_{v \in e} t_v,$$

where $I_v = c(\mathcal{I}_v) = s(\mathcal{I}_v)$ and $t_v = \ell\big(c(\mathcal{T}_v)\big) = \ell\big(s(\mathcal{T}_v)\big)$. Therefore the 1-cut $\mathcal{E}^{[1]} = \{e : \mu(e) \geq 1\}$ is precisely the tolerance-superedge set, yielding the (crisp) Tolerance SuperHyperGraph $(V, \mathcal{E}^{[1]})$.

(c) FTSHG ⇒ Fuzzy Tolerance Graph (2-section via μ_2). Define a fuzzy graph on V by $\mu_2(u,v) := \mu(\{u,v\})$ for $u \neq v$. Then

$$\mu_2(u,v) = \min\Big\{ \max\big\{\rho\big(\ell(c(\mathcal{I}_u) \cap c(\mathcal{I}_v)),$$

$$\min\{\ell(c(\mathcal{T}_u)), \ell(c(\mathcal{T}_v))\}\big),\; \rho\big(\ell(s(\mathcal{I}_u) \cap s(\mathcal{I}_v)),\; \min\{\ell(s(\mathcal{T}_u)), \ell(s(\mathcal{T}_v))\}\big)\big\},\; \sigma(u), \sigma(v) \Big\},$$

which is the standard fuzzy tolerance edge-membership: a normalized overlap (core/support channels) bounded by vertex heights. Hence the 2-section is a Fuzzy Tolerance Graph; in the crisp, unit-height setting of (b), its 1-cut is the classical tolerance graph.

(d) FTSHG ⇒ n-SuperHyperGraph (edge t-cuts). For any fixed $t \in (0,1]$, define

$$\mathcal{E}^{[t]} := \{\, e \in \mathcal{E} : \; \mu(e) \geq t \,\}.$$

Then $\mathcal{E}^{[t]} \subseteq \mathcal{P}^*(V)$ and $V \subseteq \mathcal{P}^n(V_0)$ by hypothesis, so $(V, \mathcal{E}^{[t]})$ is an n-SuperHyperGraph (finite, loopless, with nonempty superedges). This shows every FTSHG yields an n-SuperHyperGraph at each level-cut.

Collecting (a)–(d) gives the claimed boxed implications. □

In the future, it is expected that further investigations will be conducted on extended systems employing Intuitionistic Fuzzy Sets [43, 44], Hyper Fuzzy Sets [45, 46], Hesitant Fuzzy Sets [47, 48], Neutrosophic Sets [49, 50], and Plithogenic Sets [51, 52].

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Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Use of Generative AI and AI-Assisted Tools

I use generative AI and AI-assisted tools for tasks such as English grammar checking, and I do not employ them in any way that violates ethical standards.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

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