Two-Person Zero Sum Games with Random Rewards: A Linear Programming Approach

By

Somdeb Lahiri (Email: somdeb.lahiri@gmail.com)

ORCID: https://orcid.org/0000-0002-5247-3497

(Formerly with) PD Energy University, Gandhinagar (EU-G), India.

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Abstract

We consider bi-matrix games between a row player and a column player. The row player's pay-off depends on a part that is determined by the strategy choices of both players plus a part that depends solely on what is chosen by the column player. Similarly, the column player's pay-off depends on a part that is determined by the strategy choices of both players plus a part that depends solely on what is chosen by the row player. The sum of the parts of the payoffs that depend on the strategies of both players is equal to zero. We show that an equilibrium exists and a strategy profile is an equilibrium for such a game if and only if it is an equilibrium for the two-person zero-sum game determined by the interdependent parts of the pay-off matrices. Our proof relies almost entirely on the well-known result considering existence of equilibrium for two-person zero-sum games and the complete characterization of the set of such equilibria by the solution of a pair of linear programming problems that are dual to each other.

Keywords: bi-matrix games, two-person zero-sum, random rewards, additively separable sum

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1. Notations: For positive integers r, s, let $\mathbb{R}^{r\times s}$ denote the set of all r×s real valued matrices and $\mathbb{R}^r$ be the set of all r-dimensional real valued column vectors. Given any real-valued r×s matrix P, and (i, j)∈{1, …, r}×{1, …, s}, we will denote the $(i, j)^{th}$ entry of P by p_{ij}, and we will denote the transpose of P by P^T.

For (i, j)∈{1, …, r}×{1, …, s}, let P_i denote the i^{th} row of P and let P^j denote the j^{th} column of P.

For any positive integer r, let $\Delta^{r-1} = \{x\in\mathbb{R}^r_+ | \sum_{i=1}^{r} x_i = 1\}$.

For any positive integer r, let $E^{(r)}$ denote the r-dimensional column vector all whose coordinates are equal to 1.

2. Bi-matrix games and their equilibria: Given positive integers m, n, a m×n **bi-matrix game** (i.e., a two-player interactive decision-making problem where one player is called the "row player" and the other player is called the "column player") is an ordered pair of m×n real valued matrices (C, D) such that the row player can choose any row i∈{1, …, m} and the column player can choose any column $j \in \{1, \ldots, n\}$ and having done so, receive a payoff of c_{ij} and d_{ij} respectively.

A bi-matrix game (C, D) is said to be a **two-person zero-sum (TPZS) game** if D = - C.

A **strategy for the row player** is a column vector $x \in \Delta^{m-1}$. A **strategy for the column player** is a column vector $y \in \Delta^{n-1}$.

A pair $(x, y) \in \Delta^{m-1} \times \Delta^{n-1}$ is said to be a **strategy profile**.

A strategy profile (x^*, y^*) is said to be an **equilibrium** for the bi-matrix game (C, D) if for all $i \in \{1, \ldots, m\}$, $C_i y^* \leq x^{*T} C y^*$ and for all $j \in \{1, \ldots, n\}$, $x^{*T} D^j \leq x^{*T} D y^*$.

4. TPZS-RR Games: In Lahiri (2025) there is the definition and analysis of two-person additively-separable sum (TPASS) games.

An m×n bi-matrix game (C, D) is said to be an m×n **two-person additively-separable sum (TPASS) game** if there exists and ordered triplet $(A, \pi, \rho) \in \mathbb{R}^{m \times n} \times \mathbb{R}^m \times \mathbb{R}^n$, such that for all $(i, j) \in \{1, \ldots, m\} \times \{1, \ldots, n\}$, $c_{ij} = a_{ij} + \pi_i$ and $d_{ij} = -a_{ij} + \rho_j$.

If this be the case then the ordered triplet (A, π, ρ) is said to be an m×n TPASS game

An m×n bi-matrix game (C, D) is said to be an m×n **Two-Person Zero Sum game with Random Rewards (TPZS-RR game)** if there exists and ordered triplet $(A, \pi, \rho) \in \mathbb{R}^{m \times n} \times \mathbb{R}^m \times \mathbb{R}^n$, such that for all $(i, j) \in \{1, \ldots, m\} \times \{1, \ldots, n\}$, $c_{ij} = a_{ij} + \rho_j$ and $d_{ij} = -a_{ij} + \pi_i$.

If this be the case then the ordered triplet (A, π, ρ) is said to be an m×n TPZS-RR game.

For what follows we will require the following well known result available in Appendix 5 of Luce and Raiffa (1957).

Theorem 1: (i) (x^*, y^*) is an equilibrium for the TPZS game (A, -A) <u>if and only if</u> there exists real numbers U^*, V^* such that x^*, V^* solve the linear programming problem [Minimize V subject to $-x^T A - VE^{(n)} \leq 0$, j = 1, …, n, $\sum_{i=1}^{m} x_i = 1$, $x_i \geq 0$, i = 1, …, m, $V \in \mathbb{R}$] and y^*, U^* solve the linear programming problem [Maximize -U, subject to $Ay - UE^{(m)} \leq 0$, $\sum_{j=1}^{n} y_j = 1$, $y_j \geq 0$, j = 1, …, n, $U \in \mathbb{R}$]. The two linear programming problems are dual to each other and hence $U^* + V^* = 0$.

(ii) (A, -A) has an equilibrium.

The proof of theorem 1 is a simple application of the "duality theorem of linear programming" as is the case for the proof of the main result in Lahiri (2025), the latter being a generalization of theorem 1.

5. The Linear Programming Problem: While the major stepping stone to our main result is an immediate consequence of the "Equivalence Theorem" in Mangasarian and Stone (1964), the proof that we provide relies on theorem 1 and little else. In addition to proving the existence of equilibrium for TPZS-RR games, the main result in this paper is a significant and non-trivial complete characterization of the set of equilibria for TPZS-RR games in terms of the set of equilibria of the TPZS game determined by the interdependent parts of the pay-offs.

Consider the linear programming LP-1:

Maximize $\pi^T x + \rho^T y - u - v$, subject to $(A_i + \rho^T)y - u \leq 0$, $\sum_{j=1}^{n} y_j = 1$, $x^T(-A_j + \pi) - v \leq 0$, $\sum_{i=1}^{m} x_i = 1$, $x_i \geq 0$, $i = 1, \ldots, m$, $y_j \geq 0$, $j = 1, \ldots, n$, $u, v \in \mathbb{R}$.

Proposition 1: Let (A, π, ρ) be a $m \times n$ TPZS-RR game. Then, the following statements are equivalent.

(i) x^*, y^*, u^*, v^* solve LP-1.

(ii) y^*, u^* solve the linear programming problem LP-2: [Maximize $\rho^T y - u$, subject to $(A_i + \rho^T)y - u \leq 0$, $i = 1, \ldots, m$, $\sum_{j=1}^{n} y_j = 1$, $y_j \geq 0$, $j = 1, \ldots, n$, $u \in \mathbb{R}$]

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x^*, v^* solve the linear programming problem LP-3: [Maximize $\pi^T x - v$, subject to $x^T(-A_j + \pi)x - v \leq 0$, $j = 1, \ldots, n$, $\sum_{i=1}^{m} x_i = 1$, $x_i \geq 0$, $i = 1, \ldots, m$, $v \in \mathbb{R}$]

(iii) y^*, $U^* = u^* - \rho^T y^*$ solve the linear programming problem LP-4: [Maximize $-U$, subject to $A_i y - U \leq 0$, $i = 1, \ldots, m$, $\sum_{j=1}^{n} y_j = 1$, $y_j \geq 0$, $j = 1, \ldots, n$, $U \in \mathbb{R}$],

&

x^*, $V^* = v^* - \pi^T x^*$ solve the linear programming problem LP-5: [Maximize $-V$, subject to $-x^T A^j - V \leq 0$, $j = 1, \ldots, n$, $\sum_{i=1}^{m} x_i = 1$, $x_i \geq 0$, $i = 1, \ldots, m$, $V \in \mathbb{R}$].

Proof: (i) <u>if and only if</u> (ii): Suppose x^*, y^*, u^*, v^* solve LP-1.

Towards a contradiction suppose that for some y, u satisfying $(A_i + \rho^T)y - u \leq 0$, $i = 1, \ldots, m$, $\sum_{j=1}^{n} y_j = 1$, $y_j \geq 0$, $j = 1, \ldots, n$, $u \in \mathbb{R}$, it is the case that $\rho^T y - u > \rho^T y^* - u^*$.

Then for x^*, y, u, v^* it is the case that $(A_i + \rho^T)y - u \leq 0$, $\sum_{j=1}^{n} y_j = 1$, $x^{*T}(-A_j + \pi) - v^* \leq 0$, $\sum_{i=1}^{m} x_i^* = 1$, $x_i^* \geq 0$, $i = 1, \ldots, m$, $y_j \geq 0$, $j = 1, \ldots, n$, $u, v^* \in \mathbb{R}$ and $\pi^T x^* + \rho^T y - u - v^* > \pi^T x^* + \rho^T y^* - u^* - v^*$, contradicting x^*, y^*, u^*, v^* solve LP-1.

Thus, y^*, u^* solve LP-2.

A similar argument shows that x^*, v^* solve LP-3.

Now suppose y^*, u^* solve LP-2 and x^*, v^* solve LP-3.

Towards a contradiction there exist x, y, u, v satisfying $(A_i + \rho^T)y - u \leq 0$, $\sum_{j=1}^{n} y_j = 1$, $x^T(-A_j + \pi) - v \leq 0$, $\sum_{i=1}^{m} x_i = 1$, $x_i \geq 0$, $i = 1, \ldots, m$, $y_j \geq 0$, $j = 1, \ldots, n$, $u, v \in \mathbb{R}$ and $\pi^T x + \rho^T y - u - v > \pi^T x^* + \rho^T y^* - u^* - v^*$.

Thus, either $\rho^T y - u > \rho^T y^* - u^*$ or $\pi^T x - v > \pi^T x^* - v^*$.

In the first case $(A_i + \rho^T)y - u \leq 0$, i = 1, …, m, $\sum_{j=1}^{n} y_j = 1$, $y_j \geq 0$, j = 1, …, n, $u \in \mathbb{R}$ and $\rho^T y - u > \rho^T y^* - u^*$ contradicts y^*, u^* solve LP-2.

In the second situation $x^T(-A_j + \pi) - v \leq 0$, j = 1, …, n, $\sum_{i=1}^{m} x_i = 1$, $x_i \geq 0$, $v \in \mathbb{R}$ and $\pi^T x - v > \pi^T x^* - v^*$ contradicts x^*, v^* solve LP-3.

Thus, x^*, y^*, u^*, v^* solve LP-1.

This proves (i) is equivalent to (ii).

(ii) <u>if and only if</u> (iii): Suppose y^*, u^* solve LP-2 and x^*, v^* solve LP-3 and let $U^* = u^* - \rho^T y^*$, $V^* = v^* - \pi^T x^*$.

Towards a contradiction suppose there exist y, U such that $A_i y - U \leq 0$, i = 1, …, m, $\sum_{j=1}^{n} y_j = 1$, $y_j \geq 0$, j = 1, …, n, $U \in \mathbb{R}$ and $U < U^*$. Let $u = U + \rho^T y$.

Thus, $(A_i + \rho^T)y - u \leq 0$, $\sum_{j=1}^{n} y_j = 1$, $y_j \geq 0$, j = 1, …, n, $u \in \mathbb{R}$ and $u - \rho^T y < u^* - \rho^T y^*$ (i.e., $\rho^T y^* - u^* < \rho^T y - u$) thereby contradicting y^*, u^* solve LP-2.

Thus, y^*, U^* solve LP-4. Similarly, it follows that x^*, V^* solve LP-5

Now suppose that y^*, U^* solve LP-4 and x^*, V^* solve LP-5.

Towards a contradiction there exists y, u satisfying $(A_i + \rho^T)y - u \leq 0$, i = 1, …, m, $\sum_{j=1}^{n} y_j = 1$, $y_j \geq 0$, j = 1, …, n, $u \in \mathbb{R}$ and $\rho^T y - u > \rho^T y^* - u^* = -U^*$.

Let $U = u - \rho^T y$. Thus $A_i y - U \leq 0$, i = 1, …, m, $\sum_{j=1}^{n} y_j = 1$, $y_j \geq 0$, j = 1, …, n, $U \in \mathbb{R}$ and $-U < -U^*$, contradicting y^*, U^* solve LP-4.

Thus, y^*, u^* solve LP-2.

It follows via a similar argument that x^*, v^* solve LP-3.

This proves (ii) is equivalent to (iii) and the proposition. Q.E.D.

Using proposition 1 and theorem 1 we can now prove the following.

Proposition 2: LP-1 has a solution and if x^*, y^*, u^*, v^* solve LP-1, then $\pi^T x^* + \rho^T y^* - u^* - v^* = 0$.

Proof: By part (ii) of theorem 1, the TPZS game (A, -A) has a solution (x^*, y^*).

By part (i) of theorem 1, there exists U^*, V^* such that y^*, U^* solve [Minimize U, subject to $A_i y - U \leq 0$, i = 1, …, m, $\sum_{j=1}^{n} y_j = 1$, $y_j \geq 0$, j = 1, …, n, $U \in \mathbb{R}$] and x^*, V^* solve [Minimize V, subject to $-x^T A^j - V \leq 0$, j = 1, …, n, $\sum_{i=1}^{m} x_i = 1$, $x_i \geq 0$, i = 1, …, m, $V \in \mathbb{R}$]. Further, $U^* + V^* = 0$.

y^*, U^* solve [Minimize U, subject to $A_i y - U \leq 0$, i = 1, …, m, $\sum_{j=1}^{n} y_j = 1$, $y_j \geq 0$, j = 1, …, n, U $\epsilon\mathbb{R}$] if and only if y^*, U^* solve LP-4 and x^*, V^* solve [Minimize V, subject to $-x^T A^j - V \leq 0$, j = 1, …, n, $\sum_{i=1}^{m} x_i = 1$, $x_i \geq 0$, i = 1, …, m, V $\epsilon\mathbb{R}$] if and only if x^*, V^* solve LP-5.

Let $u^* = U^* + \rho^T y^*$ and $v^* = V^* + \pi^T x^*$.

By the equivalence of parts (i) and (iii) of proposition 2, it follows that x^*, y^*, u^*, v^* solve LP-1.

Thus, LP-1 has a solution.

Further, $U^* + V^* = 0$ implies $u^* + v^* = \pi^T x^* + \rho^T y^*$, so that $\pi^T x^* + \rho^T y^* – u^* - v^* = 0$. Q.E.D.

With proposition 2 in hand, we can state and prove the "major stepping stone" to our main result.

Proposition 3: (x^*, y^*) is an equilibrium for the m×n TPZS-RR game (A, π, ρ) if and only if there exists real numbers u^*, v^* such that x^*, y^*, u^*, v^* solves LP-1.

Proof: Suppose (x^*, y^*) is an equilibrium for the m×n TPZS-RR game (A, π, ρ). Let $u^* = x^{*T}Ay^* + \rho^T y^*$ and $v^* = -x^{*T}Ay^* + \pi^T x^*$.

Then, $(A_i + \rho^T)y^* – u^* \leq 0$, $\sum_{j=1}^{n} y_j^* = 1$, $(-A_j^T + \pi^T)x^* – v^* \leq 0$, $\sum_{i=1}^{m} x_i^* = 1$, $x_i^* \geq 0$, i = 1, …, m, $y_j^* \geq 0$, j = 1, …, n, u^*, $v^* \epsilon\mathbb{R}$.

Further, $\pi^T x^* + \rho^T y^* - u^* - v^* = \pi^T x^* + \rho^T y^* - (u^* + v^*) = 0$.

By proposition 2, it follows that x^*, y^*, u^*, v^* solve LP-1.

Now suppose, x^*, y^*, u^*, v^* solve LP-1. By proposition 2, it must be the case that $\pi^T x^* + \rho^T y^* - u^* - v^* = 0$.

Further, $(A_i + \rho^T)y^* – u^* \leq 0$ for all i = 1, …, m and $(-A_j^T + \pi^T)x^* – v^* \leq 0$ for all j = 1, …, n.

Thus, $x^{*T}Ay^* + \rho^T y^* – u^* \leq 0$ and $-x^{*T}Ay^* + \pi^T x^* – v^* \leq 0$.

Summing the two we get $\pi^T x^* + \rho^T y^* - u^* - v^* \leq 0$.

Since, $\pi^T x^* + \rho^T y^* - u^* - v^* = 0$, it must be the case that $x^{*T}Ay^* + \rho^T y^* – u^* = 0$ and $-x^{*T}Ay^* + \pi^T x^* – v^* = 0$, i.e., $x^{*T}Ay^* + \rho^T y^* = u^*$ and $-x^{*T}Ay^* + \pi^T x^* = v^*$.

Thus, $(A_i + \rho^T)y^* \leq x^{*T}Ay^* + \rho^T y^*$ for all i = 1, …, m and $- x^{*T}A^j + \pi^T x^* \leq -x^{*T}Ay^* + \pi^T x^*$ for all j = 1, …, n.

Thus, (x^*, y^*) is an equilibrium for the TPZS-RR game (A, π, ρ). Q.E.D.

Note 1: While proposition 3 is a special case of the "Equivalence Theorem" in Mangasarian and Stone (1964), our proof in this special case uses simpler technology than what is required to prove the general "existence of equilibrium" theorem for non-cooperative games due to John F. Nash that Mangasarian and Stone (1964) need to use for the proof of the more general equivalence theorem.

6. Equivalence Theorem for the set of equilibria of TPZS-RR game: An immediate consequence of theorem 1 and propositions 2, 3 is the following result.

Theorem 2: The m×n TPZS-RR game (A, π, ρ) has an equilibrium. (x^*, y^*) is an equilibrium for the TPZS-RR game (A, π, ρ) if and only if (x^*, y^*) is an equilibrium for the m×n TPZS game (A, -A).

Proof: By proposition 2, LP-1 has a solution x^*, y^*, u^*, v^* and by proposition 3 it follows that (x^*, y^*) is an equilibrium for the TPZS-RR game (A, π, ρ).

Thus, the TPZS-RR game (A, π, ρ) has an equilibrium.

By proposition 3, (x^*, y^*) is an equilibrium for the m×n TPZS-RR game (A, π, ρ) if and only if there exists real numbers u^*, v^* such that x^*, y^*, u^*, v^* solve LP-1 and by proposition 2, $\pi^T x^* + \rho^T y^* - u^* - v^* = 0$.

By the equivalence of parts (i) and (iii) of proposition 1, x^*, y^*, u^*, v^* solve LP-1 if and only if y^*, U^* with $U^* = u^* - \rho^T y^*$ solve LP- 4 and x^*, V^* with $V^* = v^* - \pi^T x^*$ solve LP-5.

For U^*, V^* as defined above, $U^* + V^* = 0$ if and only if $\pi^T x^* + \rho^T y^* - u^* - v^* = 0$.

Thus, (x^*, y^*) is an equilibrium for the m×n TPZS-RR game (A, π, ρ) if and only if [there exists real numbers U^*, V^* such that y^*, U^* solve LP- 4, x^*, V^* solve LP-5 and $U^* + V^* = 0$].

y^*, U^* solve LP- 4, x^*, V^* solve LP-5 and $U^* + V^* = 0$ if and only if y^*, U^* solve [Minimize U, subject to $A_i y - U \leq 0$, i = 1, …, m, $\sum_{j=1}^{n} y_j = 1$, $y_j \geq 0$, j = 1, …, n, U $\epsilon\mathbb{R}$] and x^*, V^* solve [Minimize V, subject to $-x^T A^j - V \leq 0$, j = 1, …, n, $\sum_{i=1}^{m} x_i = 1$, $x_i \geq 0$, i = 1, …, m, V $\epsilon\mathbb{R}$] and $U^* + V^* = 0$.

Thus, (x^*, y^*) is an equilibrium for the m×n TPZS-RR game (A, π, ρ) if and only if [there exists real numbers u^*, v^* such that x^*, y^*, U^*, V^* if and only if y^*, U^* solve [Minimize U, subject to $A_i y - U \leq 0$, i = 1, …, m, $\sum_{j=1}^{n} y_j = 1$, $y_j \geq 0$, j = 1, …, n, U $\epsilon\mathbb{R}$] and x^*, V^* solve [Minimize V, subject to $-x^T A^j - V \leq 0$, j = 1, …, n, $\sum_{i=1}^{m} x_i = 1$, $x_i \geq 0$, i = 1, …, m, V $\epsilon\mathbb{R}$] and $U^* + V^* = 0$].

By part (i) of theorem 1, (x^*, y^*) is an equilibrium for the m×n TPZS-RR game (A, π, ρ) if and only if (x^*, y^*) is an equilibrium for the m×n TPZS game (A, -A). Q.E.D.

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