Two-Person Zero Sum Games with Random Rewards

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Abstract

We consider bi-matrix games between a row player and a column player. The row player's pay-off depends on a part that is determined by the strategy choices of both players plus a part that depends solely on what is chosen by the column player. Similarly, the column player's pay-off depends on a part that is determined by the strategy choices of both players plus a part that depends solely on what is chosen by the row player. The sum of the parts of the payoffs that depend on the strategies of both players is equal to zero. We show that a strategy profile is an equilibrium for such a game if and only if it is an equilibrium for the two-person zero-sum game determined by the interdependent parts of the pay-off matrices. Thus, an equilibrium for the kind of game we introduce here exists and the set of equilibria of any such game is equal to the projection of the set of solutions of a corresponding linear programming problem into the set of all strategy profiles.

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1. Notations: For positive integers r, s, let $\mathbb{R}^{r\times s}$ denote the set of all r×s real valued matrices and $\mathbb{R}^r$ be the set of all r-dimensional real valued column vectors. Given any real-valued r×s matrix P, and $(i, j)\in\{1, \ldots, r\}\times\{1, \ldots, s\}$, we will denote the $(i, j)^{th}$ entry of P by p_{ij}, and we will denote the transpose of P by P^T.

For $(i, j)\in\{1, \ldots, r\}\times\{1, \ldots, s\}$, let P_i denote the i^{th} row of P and let P^j denote the j^{th} column of P.

For any positive integer r, let $\Delta^{r-1} = \{x\in\mathbb{R}^r_+ \mid \sum_{i=1}^{r} x_i = 1\}$.

For any positive integer r, let $E^{(r)}$ denote the r-dimensional column vector all whose coordinates are equal to 1.

2. Bi-matrix games and their equilibria: Given positive integers m, n, a m×n **bi-matrix game** (i.e., a two-player interactive decision-making problem where one player is called the "row player" and the other player is called the "column player") is an ordered pair of m×n real valued matrices (C, D) such that the row player can choose any row $i \in \{1, \ldots, m\}$ and the column player can choose any column $j \in \{1, \ldots, n\}$ and having done so, receive a payoff of c_{ij} and d_{ij} respectively.

A bi-matrix game (C, D) is said to be a **two-person zero-sum (TPZS) game** if $D = -C$.

A **strategy for the row player** is a column vector $x \in \Delta^{m-1}$. A **strategy for the column player** is a column vector $y \in \Delta^{n-1}$.

A pair $(x, y) \in \Delta^{m-1} \times \Delta^{n-1}$ is said to be a **strategy profile**.

A strategy profile (x^*, y^*) is said to be an **equilibrium** for the bi-matrix game (C, D) if for all $i \in \{1, \ldots, m\}$, $C_i y^* \leq x^{*T} C y^*$ and for all $j \in \{1, \ldots, n\}$, $x^{*T} D^j \leq x^{*T} D y^*$.

4. TPZS-RR Games: In Lahiri (2025) there is the definition and analysis of two-person additively-separable sum (TPASS) games.

An m×n bi-matrix game (C, D) is said to be an m×n **two-person additively-separable sum (TPASS) game** if there exists and ordered triplet $(A, \pi, \rho) \in \mathbb{R}^{m \times n} \times \mathbb{R}^m \times \mathbb{R}^n$, such that for all $(i, j) \in \{1, \ldots, m\} \times \{1, \ldots, n\}$, $c_{ij} = a_{ij} + \pi_i$ and $d_{ij} = -a_{ij} + \rho_j$.

If this be the case then the ordered triplet (A, π, ρ) is said to be an m×n TPASS game

An m×n bi-matrix game (C, D) is said to be an m×n **Two-Person Zero Sum game with Random Rewards (TPZS-RR game)** if there exists and ordered triplet $(A, \pi, \rho) \in \mathbb{R}^{m \times n} \times \mathbb{R}^m \times \mathbb{R}^n$, such that for all $(i, j) \in \{1, \ldots, m\} \times \{1, \ldots, n\}$, $c_{ij} = a_{ij} + \rho_j$ and $d_{ij} = -a_{ij} + \pi_i$.

If this be the case then the ordered triplet (A, π, ρ) is said to be an m×n TPZS-RR game.

Note that (x^*, y^*) is an equilibrium for the m×n TPZS-RR game (A, π, ρ) if and only if [for all $i \in \{1, \ldots, m\}$, $A_i y^* + \rho^T y^* \leq x^{*T} A y^* + \rho^T y^*$ & for all $j \in \{1, \ldots, n\}$, $-x^{*T} A^j + \pi^T x^* \leq -x^{*T} A y^* + \pi^T x^*$].

5. Equivalence Theorem for the set of equilibria of TPZS-RR game:

We begin this section with a general lemma about bi-matrix games.

Lemma 1: Let (A, B) and (C, D) be two m×n bi-matrix games such that for some $\pi \in \mathbb{R}^m$ and $\rho \in \mathbb{R}^n$, the following holds: for all $(i, j) \in \{1, \ldots, m\} \times \{1, \ldots, n\}$, $c_{ij} = a_{ij} + \rho_j$ and $d_{ij} = b_{ij} + \pi_i$. Then, (x^*, y^*) is an equilibrium for (A, B) if and only if (x^*, y^*) is an equilibrium for (C, D).

Proof: (x^*, y^*) is an equilibrium for (C, D) if and only if [for all $(i, j) \in \{1, \ldots, m\} \times \{1, \ldots, n\}$: $C_i y^* \leq x^{*T} C y^*$ and $x^{*T} D^j \leq x^* D y^*$].

[for all $(i, j) \in \{1, \ldots, m\} \times \{1, \ldots, n\}$: $C_i y^* \leq x^{*T} C y^*$ and $x^{*T} D^j \leq x^* D y^*$] if and only if [for all $(i, j) \in \{1, \ldots, m\} \times \{1, \ldots, n\}$: $A_i y^* + \rho^T y^* \leq x^{*T} A y^* + \rho^T y^*$ and $x^{*T} B^j + \pi^T x^* \leq x^* B y^* + \pi^T x^*$].

The latter statement is equivalent to [for all (i, j)ϵ\{1, …, m\}×\{1, …, n\}: $A_i y^* \leq x^{*T}Ay^*$ and $x^{*T}B^j \leq x^*By^*$], i.e., (x^*, y^*) is an equilibrium for (A, B). Q. E. D.

An immediate consequence of lemma 1 is the following result.

Proposition 1: (x^*, y^*) is an equilibrium for the m×n TPZS-RR game (A, π, ρ) <u>if and only if</u> (x^*, y^*) is an equilibrium for the m×n TPZS game (A, -A).

The following well known result is available in Appendix 5 of Luce and Raiffa (1957).

Theorem 1: (i) (x^*, y^*) is an equilibrium for the TPZS game (A, -A) <u>if and only if</u> there exists real numbers U^*, V^* such that x^*, V^* solve the linear programming problem [Minimize V subject to $-x^TA - VE^{(n)} \leq 0$, j = 1, …, n, $\sum_{i=1}^{m} x_i = 1$, $x_i \geq 0$, i = 1, …, m, $V \epsilon \mathbb{R}$] and y^*, U^* solve the linear programming problem [Maximize -U, subject to $Ay - UE^{(m)} \leq 0$, $\sum_{j=1}^{n} y_j = 1$, $y_j \geq 0$, j = 1, …, n, $U \epsilon \mathbb{R}$]. The two linear programming problems are dual to each other and hence $U^* + V^* = 0$.

(ii) (A, -A) has an equilibrium.

The proof of theorem 1 is a simple application of the "duality theorem of linear programming" as is the case for the proof of the main result in Lahiri (2025), the latter being a generalization of theorem 1.

As a consequence of proposition 1 and theorem 1, we get the following result.

Proposition 2: (i) (x^*, y^*) is an equilibrium for the m×n TPZS-RR game (A, π, ρ) <u>if and only if</u> there exists real numbers U^*, V^* such that x^*, V^* solve the linear programming problem [Minimize V subject to $-x^TA - VE^{(n)} \leq 0$, j = 1, …, n, $\sum_{i=1}^{m} x_i = 1$, $x_i \geq 0$, i = 1, …, m, $V \epsilon \mathbb{R}$] and y^*, U^* solve the linear programming problem [Maximize -U, subject to $Ay - UE^{(m)} \leq 0$, $\sum_{j=1}^{n} y_j = 1$, $y_j \geq 0$, j = 1, …, n, $U \epsilon \mathbb{R}$]. The two linear programming problems are dual to each other and hence $U^* + V^* = 0$.

(ii) (A, π, ρ) has an equilibrium.

References

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