Linear Partially Zero-Sum Bi-matrix Games

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Abstract

We consider bi-matrix games between a row player and a column player. The row player's pay-off depends on a part that is determined by the strategy choices of both players plus a part that depends solely on what is chosen by the column player plus another part that depends solely on what is chosen by the row player. Similarly, the column player's pay-off depends on a part that is determined by the strategy choices of both players plus a part that depends solely on what is chosen by the row player plus another part that depends solely on what is chosen by the column player. The sum of the parts of the payoffs that simultaneously depend on the strategies of both players is equal to zero. We call such games, linear partially zero-sum bi-matrix games. We show that a strategy profile is an equilibrium for such a game if and only if it is an equilibrium for the two-person additively-separable game obtained by ignoring the part of the pay-offs determined by the strategy chosen by the opponent. Thus, an equilibrium for the kind of game we introduce here exists and the set of equilibria of any such game is equal to the projection of the set of solutions of a corresponding linear programming problem into the set of all strategy profiles. In the special case, where the pay-offs to a player in a linear partially zero-sum bi-matrix game do not include a part that solely depends on the player's own strategy, the set of equilibria for such games coincide with the set of equilibria for the matrix (zero-sum) game determined entirely by interdependent pay-offs.

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1. Notations: For positive integers r, s, let $\mathbb{R}^{r\times s}$ denote the set of all r×s real valued matrices and $\mathbb{R}^r$ be the set of all r-dimensional real valued column vectors. Given any real-valued r×s matrix P, and (i, j)$\in$\{1, …, r\}×\{1, …, s\}, we will denote the (i, j)th entry of P by p_{ij}, and we will denote the transpose of P by P^T.

For (i, j)$\in$\{1, …, r\}×\{1, …, s\}, let P_i denote the ith row of P and let P^j denote the jth column of P.

For any positive integer r, let $\Delta^{r-1} = \{x \epsilon \mathbb{R}^r_+ | \sum_{i=1}^{r} x_i = 1\}$.

For any positive integer r, let $E^{(r)}$ denote the r-dimensional column vector all whose coordinates are equal to 1.

2. Bi-matrix games and their equilibria: Given positive integers m, n, a m×n **bi-matrix game** (i.e., a two-player interactive decision-making problem where one player is called the "row player" and the other player is called the "column player") is an ordered pair of m×n real valued matrices (C, D) such that the row player can choose any row $i \epsilon \{1, \ldots, m\}$ and the column player can choose any column $j \epsilon \{1, \ldots, n\}$ and having done so, receive a payoff of c_{ij} and d_{ij} respectively.

A bi-matrix game (C, D) is said to be a **two-person zero-sum (TPZS) game** if D = - C.

A **strategy for the row player** is a column vector $x \epsilon \Delta^{m-1}$. A **strategy for the column player** is a column vector $y \epsilon \Delta^{n-1}$.

A pair $(x, y) \epsilon \Delta^{m-1} \times \Delta^{n-1}$ is said to be a **strategy profile**.

A strategy profile (x^*, y^*) is said to be an **equilibrium** for the bi-matrix game (C, D) if for all $i \epsilon \{1, \ldots, m\}$, $C_i y^* \leq x^{*T} C y^*$ and for all $j \epsilon \{1, \ldots, n\}$, $x^{*T} D^j \leq x^{*T} D y^*$.

4. Linear Partially Zero-Sum Bi-matrix Games: In Lahiri (2025) there is the definition and analysis of two-person additively-separable sum (TPASS) games.

An m×n bi-matrix game (C, D) is said to be an m×n **two-person additively-separable sum (TPASS) game** if there exists and ordered triplet $(A, \pi, \rho) \epsilon \mathbb{R}^{m \times n} \times \mathbb{R}^m \times \mathbb{R}^n$, such that for all $(i, j) \epsilon \{1, \ldots, m\} \times \{1, \ldots, n\}$, $c_{ij} = a_{ij} + \pi_i$ and $d_{ij} = -a_{ij} + \rho_j$.

If this be the case then the ordered triplet (A, π, ρ) is said to be an m×n TPASS game.

For the m×n TPASS game (A, π, ρ) at the strategy profile (x, y), the pay-off to the row player is $x^T Ay + x^T \pi$ and the pay-off to the column player is $-x^T Ay + \rho y$.

For natural numbers m, n, let $\pi^C, \pi^R \epsilon \mathbb{R}^m$ be m-dimensional real-valued column vectors and ρ^C, $\rho^R \epsilon \mathbb{R}^n$ be n-dimensional real-valued row vectors.

Given a m×n real valued matrix A, consider the m×n bi-matrix game $(A + \Pi^R + P^R, -A + \Pi^C + P^C) \epsilon \mathbb{R}^{m \times n} \times \mathbb{R}^{m \times n}$, where Π^R is the m×n real-valued matrix every column of which is π^R, P^R is the m×n real-valued matrix every row of which is ρ^R, Π^C is the m×n real-valued matrix every column of which is π^C and P^C is the m×n real-valued matrix every row of which is ρ^C.

We will refer to $(A + \Pi^R + P^R, -A + \Pi^C + P^C)$ as an m×n, **Linear Partially Zero-Sum Bi-matrix (LPZSBM) game**.

For the m×n LPZSBM game $(A + \Pi^R + P^R, -A + \Pi^C + P^C)$ at the strategy profile (x, y), the pay-off to the row player is $x^T Ay + \rho^R y + x^T \pi^R$ and the pay-off to the column player is $-x^T Ay + \rho^C y + x^T \pi^C$.

5. Equivalence theorem for Linear Partially Zero-Sum Bi-matrix Games: We begin this section with a general lemma about bi-matrix games.

Lemma 1: Let (A, B) and (C, D) be two m×n bi-matrix games such that for some $\pi \epsilon \mathbb{R}^m$ and $\rho \epsilon \mathbb{R}^n$, the following holds: for all (i, j)ϵ\{1, …, m\}×\{1, …, n\}, $c_{ij} = a_{ij} + \rho_j$ and $d_{ij} = b_{ij} + \pi_i$. Then, (x^*, y^*) is an equilibrium for (A, B) if and only if (x^*, y^*) is an equilibrium for (C, D).

Proof: (x^*, y^*) is an equilibrium for (C, D) if and only if [for all (i, j)ϵ\{1, …, m\}×\{1, …, n\}: $C_i y^* \leq x^{*T}Cy^*$ and $x^{*T}D^j \leq x^*Dy^*$].

[for all (i, j)ϵ\{1, …, m\}×\{1, …, n\}: $C_i y^* \leq x^{*T}Cy^*$ and $x^{*T}D^j \leq x^*Dy^*$] if and only if [for all (i, j)ϵ\{1, …, m\}×\{1, …, n\}: $A_i y^* + \rho^T y^* \leq x^{*T}Ay^* + \rho^T y^*$ and $x^{*T}B^j + \pi^T x^* \leq x^*By^* + \pi^T x^*$].

The latter statement is equivalent to [for all (i, j)ϵ\{1, …, m\}×\{1, …, n\}: $A_i y^* \leq x^{*T}Ay^*$ and $x^{*T}B^j \leq x^*By^*$], i.e., (x^*, y^*) is an equilibrium for (A, B). Q. E. D.

From lemma 1, it follows that (x^*, y^*) is an equilibrium for the m×n LPZSBM game $(A+\Pi^R + P^R, -A+\Pi^C + P^C)$ if and only if (x^*, y^*) is an equilibrium for the m×n bi-matrix game $(A+\Pi^R, -A+P^C)$.

However, the m×n bi-matrix game $(A+\Pi^R, -A+P^C)$ is simply the m×n TPASS game (A, π^R, ρ^C).

Thus, in view of lemma 1 above, proposition 1 and its proof in Lahiri (2025) and proposition 3 of Lahiri (2025) we have the following interesting result.

Proposition 1: (i) (x^*, y^*) is an equilibrium for the m×n LPZSBM game $(A+\Pi^R + P^R, -A+\Pi^C + P^C)$ if and only if (x^*, y^*) is an equilibrium for the m×n TPASS game (A, π^R, ρ^C).

(ii) $(A+\Pi^R + P^R, -A+\Pi^C + P^C)$ has an equilibrium.

(iii) (x^*, y^*) is an equilibrium for the LPZSBM game $(A+\Pi^R + P^R, -A+\Pi^C + P^C)$ if and only if there exist real numbers α^*, β^* such that y^*, α^* solves \{Maximize $\rho^C y - \alpha$, subject to $Aq - \alpha E^{(m)} \leq -\pi^R$, $E^{(n)T}y = 1$, $y \in \mathbb{R}^n_+$, $\alpha \in \mathbb{R}$] and x^*, β^* solves its dual [Minimize $-x^T\pi^R + \beta$, subject to $x^TA + \beta e^{(n)T} \geq \rho^C$, $x^TE^{(m)} = 1$, $x \in \mathbb{R}^m_+$, $\beta \in \mathbb{R}$]. Further, the α^*, β^* associated with x^*, y^* satisfies $\alpha^* = x^{*T}Ay^* + x^{*T}\pi^R$ and $\beta^* = -x^{*T}Ay^* + \rho^C q^*$.

Note 1: Propositions 1 and 2 (along with the proof of proposition 1) in Lahiri (2025) generalizes the well-known result in Appendix 5 of Luce and Raiffa (1957). As in the case of the result in Luce and Raiffa (1957), the proofs of the main results in Lahiri (2025) are based on the duality theorem of linear programming.

6. The set of equilibria of TPZS-RR game: In this section we discuss a special case of LPZSBM games.

An m×n bi-matrix game (C, D) is said to be an m×n **Two-Person Zero Sum game with Random Rewards (TPZS-RR game)** if there exists and ordered triplet (A, π, ρ)$\epsilon\mathbb{R}^{m\times n}\times\mathbb{R}^{m}\times\mathbb{R}^{n}$, such that for all (i, j)ϵ{1, …, m}×{1, …, n}, $c_{ij} = a_{ij} + \rho_j$ and $d_{ij} = -a_{ij} + \pi_i$.

If this be the case then the ordered triplet (A, π, ρ) is said to be an m×n TPZS-RR game.

For the m×n TPZS-RR game (A, π, ρ) at the strategy profile (x, y), the pay-off to the row player is $x^TAy+ \rho^Ty$ and the pay-off to the column player is $-x^TAy+ \pi^Tx$.

An immediate consequence of lemma 1 is the following result.

Proposition 2: (x^*, y^*) is an equilibrium for the m×n TPZS-RR game (A, π, ρ) <u>if and only if</u> (x^*, y^*) is an equilibrium for the m×n TPZS game (A, -A).

As a consequence of propositions 1 and 2, we get the following result.

Proposition 3: (i) The m×n TPZS-RR game (A, π, ρ) has an equilibrium.

(ii) (x^*, y^*) is an equilibrium for (A, π, ρ) <u>if and only if</u> there exists real numbers α^*, β^* such that x^*, β^* solve the linear programming problem [Minimize α subject to $-x^TA - \beta E^{(n)} \leq 0$, j = 1, …, n, $\sum_{i=1}^{m} x_i = 1$, $x_i \geq 0$, i = 1, …, m, $\beta\epsilon\mathbb{R}$] and y^*, α^* solve the linear programming problem [Maximize -U, subject to $Ay - \alpha E^{(m)} \leq 0$, $\sum_{j=1}^{n} y_j = 1$, $y_j \geq 0$, j = 1, …, n, $\alpha\epsilon\mathbb{R}$]. The two linear programming problems are dual to each other and hence $\alpha^* + \beta^* = 0$.

References

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