Analytical Pole Extraction for Surface Wave Modes in Microstrip Structures: A Complete Residue Calculus Treatment

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September 18, 2025

Abstract

This paper presents a rigorous analytical treatment of the electric field dyadic Green's function for a canonical microstrip geometry using pole extraction and residue calculus. The formulation provides explicit closed-form expressions for surface wave contributions and asymptotic radiation fields, enabling efficient computation and direct physical insight into electromagnetic wave propagation in layered media. The approach offers computational advantages over traditional numerical integration methods whilst maintaining mathematical rigour.

1 Introduction

The analysis of electromagnetic fields in layered media has been a subject of intensive research since Sommerfeld's pioneering work on wave propagation over a conducting half-space [1]. For microstrip structures, early analytical approaches relied upon quasi-static approximations [2, 3] that, whilst computationally tractable, neglected the finite propagation time and associated surface wave phenomena.

Full-wave analysis of microstrip structures requires the evaluation of Sommerfeld integrals [4], which arise naturally from the inverse Fourier transformation of spectral domain Green's functions. These integrals have traditionally been computed using numerical methods [5–7], as they exhibit highly oscillatory integrands with slowly decaying behaviour, particularly at higher frequencies.

Significant effort has been devoted to developing closed-form spatial domain representations of these Green's functions. Various approaches have been proposed, including the discrete complex image method [8], robust derivation techniques [9], and extensions for all ranges and materials [10]. However, these methods typically rely upon fitting procedures or approximations that, whilst computationally efficient, do not provide exact analytical expressions for the surface wave contributions.

The physical significance of surface waves in microstrip structures has been well established [11, 12]. Surface wave excitation leads to radiation efficiency reduction, unwanted coupling between circuit elements, and spurious radiation patterns. Understanding and quantifying these effects requires accurate knowledge of the surface wave pole locations and their associated residues [13–15].

Recent advances in numerical techniques for evaluating Sommerfeld integrals [16, 17] have improved computational efficiency, yet the fundamental challenge remains: extracting physical insight from numerical results and enabling rapid design optimisation requires analytical expressions that directly relate substrate parameters to surface wave behaviour.

This work addresses these limitations by developing a complete analytical framework for pole extraction in microstrip Green's functions. The approach provides explicit closed-form expressions for surface wave residues, enabling direct calculation of surface wave excitation coefficients from substrate parameters without numerical approximation. The methodology builds upon established spectral domain formulations [6,18] whilst extending the analysis to provide exact residue expressions that directly relate substrate parameters to surface wave behaviour.

2 Problem Formulation and Geometry

Configuration: We consider the canonical layered microstrip geometry:

- **Region 1 (Air):** $z > 0$, permittivity ε_0, permeability μ_0
- **Region 2 (Substrate):** $-d < z < 0$, permittivity $\varepsilon_1 = \varepsilon_r \varepsilon_0$, permeability μ_0
- **Ground Plane:** Perfect electric conductor at $z = -d$

Source: A horizontal electric dipole (HED) with moment $\boldsymbol{p} = p_0 \hat{\boldsymbol{x}} \delta(\boldsymbol{r} - \boldsymbol{r}')$ located at $\boldsymbol{r}' = (0, 0, 0)$ on the interface.

Objective: Derive the electric field dyadic Green's function component $G^E_{xx}(\rho)$ where $\rho = \sqrt{x^2 + y^2}$.

3 Spectral Domain Formulation

The electric field satisfies the vector Helmholtz equation:

$$\nabla \times \nabla \times \boldsymbol{E} - k_i^2 \boldsymbol{E} = j\omega\mu_0 \boldsymbol{J} \tag{1}$$

where $k_i^2 = \omega^2 \mu_0 \varepsilon_i$ in region i.

3.1 Fourier Transform Representation

Following the established spectral domain approach [7], the two-dimensional Fourier transform pair is:

$$\tilde{\boldsymbol{E}}(k_x, k_y, z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \boldsymbol{E}(x, y, z) e^{-j(k_x x + k_y y)} dx\, dy \tag{2}$$

$$\boldsymbol{E}(x, y, z) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{\boldsymbol{E}}(k_x, k_y, z) e^{j(k_x x + k_y y)} dk_x\, dk_y \tag{3}$$

3.2 Spectral Parameters

Define the spectral domain variables:

$$k_\rho = \sqrt{k_x^2 + k_y^2} \quad \text{(radial spectral variable)} \tag{4}$$

$$k_{zi} = \sqrt{k_i^2 - k_\rho^2} \quad \text{for } i = 0, 1 \quad \text{(vertical wavenumbers)} \tag{5}$$

Branch Cut Convention: For all square roots, we choose the branch such that $\text{Im}(k_{zi}) \geq 0$, ensuring outgoing wave behaviour at infinity and proper radiation conditions [4].

4 Modal Analysis and Characteristic Equations

The spectral domain Green's function exhibits poles corresponding to guided wave modes [5]. These occur when the modal determinants vanish.

4.1 Transverse Magnetic (TM) Modes

Characteristic Equation: $D_{\text{TM}}(k_\rho) = 0$ where

$$D_{\text{TM}}(k_\rho) = \varepsilon_r k_{z0} + k_{z1} \cot(k_{z1} d) \tag{6}$$

4.2 Transverse Electric (TE) Modes

Characteristic Equation: $D_{\text{TE}}(k_\rho) = 0$ where

$$D_{\text{TE}}(k_\rho) = k_{z0} + k_{z1} \cot(k_{z1} d) \tag{7}$$

4.3 Spectral Domain Green's Function

For a horizontal electric dipole at the interface, the xx-component of the spectral Green's function is [6]:

$$\tilde{G}_{xx}^E(k_\rho) = \frac{j\omega\mu_0}{4\pi k_{z0}} \left[\frac{k_0^2}{D_{\text{TM}}(k_\rho)} - \frac{k_\rho^2}{D_{\text{TE}}(k_\rho)} \right] \tag{8}$$

5 Spatial Green's Function via Sommerfeld Integration

5.1 Hankel Transform

Due to cylindrical symmetry in the transverse plane:

$$G_{xx}^E(\rho) = \int_0^\infty \tilde{G}_{xx}^E(k_\rho) J_0(k_\rho \rho) k_\rho \, dk_\rho \tag{9}$$

This integral is the **Sommerfeld integral** [19], which has no closed-form solution in general and requires complex analysis for evaluation.

6 Complex Plane Analysis and Pole Extraction

6.1 Integration Path Deformation

We deform the integration path in the complex k_ρ-plane to systematically capture:

1. **Surface wave poles** (discrete spectrum)

2. **Branch point contributions** (continuous spectrum/radiation)

This approach follows established principles of complex analysis applied to layered media problems [4].

6.2 Surface Wave Pole Locations

The poles $\{k_p^{(n)}\}$ are the solutions to:

- $D_{\mathrm{TM}}(k_p^{(n)}) = 0$ (TM surface waves)
- $D_{\mathrm{TE}}(k_p^{(n)}) = 0$ (TE surface waves)

Physical Constraints: For bound surface waves [11]:

- $|k_p^{(n)}| > k_0$ (phase velocity less than speed of light)
- $\mathrm{Im}(k_p^{(n)}) < 0$ (exponential decay with distance)

7 Residue Theorem Application

7.1 Green's Function Decomposition

Applying Cauchy's residue theorem [20]:

$$G_{xx}^E(\rho) = \sum_{\text{surface waves}} G_{xx,\mathrm{sw}}^E(\rho) + G_{xx,\mathrm{rad}}^E(\rho) \tag{10}$$

7.2 Residue Calculations

For a simple pole at $k_\rho = k_p$:

$$\text{Residue} = \lim_{k_\rho \to k_p} (k_\rho - k_p)\tilde{G}_{xx}^E(k_\rho) \tag{11}$$

7.3 TM Surface Wave Analysis

Residue Expression:

$$\mathrm{Res}_{\mathrm{TM}}(k_p^{\mathrm{TM}}) = \frac{j\omega\mu_0 k_0^2}{4\pi k_{z0}(k_p^{\mathrm{TM}})} \cdot \frac{1}{D'_{\mathrm{TM}}(k_p^{\mathrm{TM}})} \tag{12}$$

Derivative Calculation: Using the chain rule and standard differentiation formulae [21]:

$$D'_{\mathrm{TM}}(k_\rho) = -\varepsilon_r \frac{k_\rho}{k_{z0}} + \frac{\partial}{\partial k_\rho}\left[k_{z1}\cot(k_{z1}d)\right] \tag{13}$$

where:

$$\frac{\partial}{\partial k_\rho}\left[k_{z1}\cot(k_{z1}d)\right] = -\frac{k_\rho}{k_{z1}}\left[\cot(k_{z1}d) + k_{z1}d\csc^2(k_{z1}d)\right] \tag{14}$$

Therefore:

$$D'_{\mathrm{TM}}(k_\rho) = -\frac{k_\rho}{k_{z0}}\varepsilon_r - \frac{k_\rho}{k_{z1}}\left[\cot(k_{z1}d) + k_{z1}d\csc^2(k_{z1}d)\right] \tag{15}$$

TM Surface Wave Contribution:

$$G^E_{xx,\mathrm{TM}}(\rho) = -2\pi j \cdot \mathrm{Res}_{\mathrm{TM}}(k_p^{\mathrm{TM}}) \cdot J_0(k_p^{\mathrm{TM}}\rho)k_p^{\mathrm{TM}} \tag{16}$$

$$= \frac{\omega\mu_0 k_0^2 (k_p^{\mathrm{TM}})^2}{2k_{z0}(k_p^{\mathrm{TM}})D'_{\mathrm{TM}}(k_p^{\mathrm{TM}})} J_0(k_p^{\mathrm{TM}}\rho) \tag{17}$$

7.4 TE Surface Wave Analysis

TE Derivative:

$$D'_{\mathrm{TE}}(k_\rho) = -\frac{k_\rho}{k_{z0}} - \frac{k_\rho}{k_{z1}}\left[\cot(k_{z1}d) + k_{z1}d\csc^2(k_{z1}d)\right] \tag{18}$$

TE Surface Wave Contribution:

$$G^E_{xx,\mathrm{TE}}(\rho) = -\frac{\omega\mu_0 (k_p^{\mathrm{TE}})^4}{2k_{z0}(k_p^{\mathrm{TE}})D'_{\mathrm{TE}}(k_p^{\mathrm{TE}})} J_0(k_p^{\mathrm{TE}}\rho) \tag{19}$$

8 Asymptotic Radiation Field

8.1 Steepest Descent Method

For large ρ, the radiation integral is evaluated using the method of steepest descent [21]. The saddle point occurs at:

$$k_\rho^{\mathrm{saddle}} = k_0 \sin\theta \tag{20}$$

where θ is the observation angle from the vertical axis.

8.2 Asymptotic Form

The far-field radiation contribution has the asymptotic behaviour:

$$G^E_{xx,\mathrm{rad}}(\rho) \sim \sqrt{\frac{2}{\pi k_0 \rho}} e^{-jk_0\rho} F_{xx}(\theta) \quad \text{as } \rho \to \infty \tag{21}$$

where $F_{xx}(\theta)$ is the radiation pattern factor obtained by evaluating the spectral Green's function at the saddle point.

9 Complete Analytical Solution

9.1 Final Expression

The complete electric field Green's function is given by:

$$\boxed{G^E_{xx}(\rho) = \sum_n \frac{\omega\mu_0 A_n (k_p^{(n)})^2}{2k_{z0}(k_p^{(n)})D'(k_p^{(n)})} J_0(k_p^{(n)}\rho) + G^E_{xx,\mathrm{rad}}(\rho)} \tag{22}$$

where:

- $A_n = k_0^2$ for TM modes
- $A_n = -(k_p^{(n)})^2$ for TE modes
- $D'(k_p^{(n)})$ is D'_{TM} or D'_{TE} evaluated at the respective poles
- The summation includes all bound surface wave modes

9.2 Convergence Properties

Mathematical Convergence: The series converges absolutely due to the exponential decay factor $\text{Im}(k_p^{(n)}) < 0$ in the pole locations, ensuring $|J_0(k_p^{(n)}\rho)| \to 0$ as $\rho \to \infty$.

Physical Interpretation: Each term represents the excitation and propagation of a distinct surface wave mode, with the Bessel function $J_0(k_p^{(n)}\rho)$ describing the transverse field distribution.

10 Discussion and Applications

10.1 Theoretical Contributions

The present formulation provides several advances over existing approaches:

1. **Explicit Residue Expressions:** Unlike previous methods that rely upon numerical evaluation [16, 19], the present approach yields closed-form expressions for surface wave residues.
2. **Physical Decomposition:** The clear separation of electromagnetic field into discrete surface wave modes and continuous radiation spectrum facilitates physical understanding of wave propagation mechanisms.
3. **Computational Efficiency:** Analytical evaluation replaces computationally expensive numerical integration of Sommerfeld integrals, potentially offering substantial computational savings for parametric studies.

10.2 Engineering Applications

The analytical framework enables direct application to practical microwave engineering problems:

1. **Antenna Arrays:** Direct calculation of mutual coupling coefficients for design optimization
2. **Circuit Analysis:** Prediction of spurious radiation and surface wave losses from material parameters
3. **Substrate Selection:** Analytical guidance for minimizing surface wave excitation in specific applications

10.3 Comparison with Existing Methods

Compared to established techniques such as the discrete complex image method [8, 10], the present approach provides exact rather than approximate representations of surface wave contributions. While numerical methods [16] can achieve high accuracy, they require iterative evaluation for each parameter set, whereas the analytical expressions herein enable direct computation across parameter ranges.

10.4 Limitations and Extensions

The formulation presented is restricted to the canonical two-layer microstrip geometry. Extension to multilayer structures [22] would require modification of the characteristic equations whilst maintaining the fundamental pole extraction methodology. Additionally, the treatment of lossy substrates would introduce complex-valued material parameters, affecting the pole locations and residue calculations.

11 Conclusion

This work presents a complete analytical framework for the electric field Green's function in microstrip geometries based upon rigorous pole extraction and residue calculus. The approach provides explicit closed-form expressions for surface wave contributions, enabling direct analytical computation of electromagnetic coupling and radiation in microstrip structures.

The key contribution lies in the derivation of exact residue expressions for surface wave poles, complementing existing numerical approaches [16, 19] with analytical tools that enable direct parameter-to-performance relationships. The methodology demonstrates how classical complex analysis techniques can be systematically applied to layered media problems, providing a foundation for further analytical developments.

Future work will focus on experimental validation of the theoretical predictions and extension of the framework to more complex geometries, including multilayer substrates and finite ground plane effects. The analytical expressions developed herein provide a robust foundation for such extensions whilst maintaining the computational advantages of closed-form evaluation.

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