Plithogenic Fuzzy Expert System, Neuro-Plithogenic Fuzzy System, and Plithogenic Fuzzy Cognitive Maps with Upside-down Logic

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Abstract

Neuro–Fuzzy Systems combine fuzzy rule–based inference with neural learning, adapting memberships and rule parameters from data for interpretable modeling. Fuzzy Expert Systems map crisp inputs to outputs via fuzzification, T–norm/implication inference, aggregation, and defuzzification. Fuzzy Cognitive Maps capture causal relations on weighted directed graphs with iterative fuzzy updates. We extend these to the plithogenic setting: the Plithogenic Fuzzy Expert System (PFES), the Neuro–Plithogenic–Fuzzy System (NPFS), and Plithogenic Fuzzy Cognitive Maps (PFCM). We formalize a contradiction–aware operator and an Upside–Down transform with contradiction reset that flips memberships/edge–contributions above a threshold and neutralizes the anchor contradiction. We prove PFES, NPFS, and PFCM strictly generalize their classical counterparts and induce valid plithogenic fuzzy structures. Numerical examples illustrate context–dependent reasoning, learning consistency, and robustness under conflicting attributes.

Keywords: Neuro–Fuzzy Systems, Fuzzy Expert System, Neuro–Plithogenic–Fuzzy System, Plithogenic Fuzzy Expert System, Fuzzy Cognitive Maps, Plithogenic Fuzzy Cognitive Maps, Fuzzy Set

Contents in this paper

The remainder of this paper is organized as follows.

1 Preliminaries

This section provides a structured overview of the key concepts and definitions necessary for understanding the main results of this paper. The graphs considered in this paper are assumed to be finite.

1.1 Neuro–Fuzzy System

A fuzzy set generalizes classical sets, allowing each element to have graded membership between 0 and 1 [1, 2]. Neural learning is the process where artificial neural networks adjust connection weights to recognize patterns, optimize tasks, and improve performance [3, 4]. Neuro–Fuzzy Systems combine fuzzy rule-based inference with neural learning, tuning membership and rule parameters from data for adaptive, interpretable modeling (cf. [5–11]). As related concepts to Neuro–Fuzzy Systems, Neuro–Intuitionistic Fuzzy Systems [12, 13] and Neuro–Neutrosophic Systems [14] are well known.

Definition 1.1 (Neuro–Fuzzy System)**.** (cf. [5–8]) Let $X \subseteq \mathbb{R}^n$ be an input space and $Y \subseteq \mathbb{R}^m$ an output space. A *Neuro–Fuzzy System (NFS)* is a parameterized fuzzy inference system

$$\mathcal{N}_\theta = \big(X, Y, \mathcal{R}_\theta, \{\mu_A(\,\cdot\,;\theta)\}, T_\theta, S_\theta, I_\theta, \mathcal{A}_\theta, \mathcal{D}_\theta\big),$$

where:

- $\mathcal{R}_\theta = \{r_j\}_{j=1}^M$ is a data–tunable rule base. Each rule has antecedents $A_{j1}, \ldots, A_{jn}$ with membership functions $\mu_{A_{jk}}(\,\cdot\,;\theta)$ and either
  - (a) **Sugeno/TSK consequent:** r_j : IF x_k is A_{jk} $(k = 1, \ldots, n)$ THEN $y = g_j(x; \varphi_j)$, or
  - (b) **Mamdani consequent:** r_j : IF x_k is A_{jk} THEN y is B_j with membership $\mu_{B_j}(\,\cdot\,;\theta)$.
- T_θ (T–norm), S_θ (S–norm), I_θ (implication) are (possibly parametric) fuzzy connectives; $\mathcal{A}_\theta$ aggregates rule outputs; $\mathcal{D}_\theta$ is a defuzzifier (e.g., centroid).

Given $x = (x_1, \ldots, x_n) \in X$, the jth rule firing strength is

$$w_j(x;\theta) \;=\; T_\theta\big(\mu_{A_{j1}}(x_1;\theta), \ldots, \mu_{A_{jn}}(x_n;\theta)\big), \qquad \bar{w}_j(x;\theta) = \frac{w_j(x;\theta)}{\sum_{i=1}^M w_i(x;\theta)}.$$

The system output $f_\theta(x) \in Y$ is

$$\text{(Sugeno)} \quad f_\theta(x) = \sum_{j=1}^M \bar{w}_j(x;\theta)\, g_j(x;\varphi_j),$$

$$\text{(Mamdani)} \quad f_\theta(x) = \mathcal{D}_\theta\Big(\mathcal{A}_\theta\big(\{\, w_j(x;\theta)\, I_\theta\, \mu_{B_j}(\cdot;\theta) \,\}_{j=1}^M\big)\Big).$$

Learning. Given data $\mathcal{S} = \{(x^{(p)}, y^{(p)})\}_{p=1}^N \subset X \times Y$ and a loss ℓ, the parameters θ (including premise and consequent parameters) are estimated by empirical risk minimization:

$$\theta^\star \in \arg\min_\theta \sum_{p=1}^N \ell\Big(f_\theta(x^{(p)}),\, y^{(p)}\Big),$$

typically via gradient–based backpropagation; in Sugeno models, hybrid schemes combine least–squares for consequents with gradient descent for premises.

Example 1.2 (ANFIS: First–Order Sugeno)**.** Let M rules of the form

$$\text{IF } x_k \text{ is } A_{jk} \; (k = 1, \ldots, n) \text{ THEN } g_j(x;\varphi_j) = a_{j0} + \sum_{k=1}^n a_{jk} x_k.$$

With product T–norm, $w_j(x) = \prod_{k=1}^n \mu_{A_{jk}}(x_k;\theta)$ and $\bar{w}_j = w_j / \sum_i w_i$, the output is

$$f_\theta(x) \;=\; \sum_{j=1}^M \bar{w}_j(x;\theta) \Big(a_{j0} + \sum_{k=1}^n a_{jk} x_k\Big).$$

When premise (membership) parameters are fixed, $f_\theta(x)$ is linear in the consequent coefficients $\{a_{jk}\}$, enabling a forward least–squares step for consequents and a backward gradient step for premises (hybrid learning).

1.2 Fuzzy Expert System (FES)

A Fuzzy Expert System maps crisp inputs to outputs using fuzzy rules: fuzzification, inference via T-norm/implication, aggregation, and defuzzification [15–19]. As related concepts to the Fuzzy Expert System, the Soft Expert System [20,21], Intuitionistic Fuzzy Expert System [22–24], and Neutrosophic Expert System [25–27] are well known.

Definition 1.3 (Fuzzy Expert System (FES))**.** [15–19] Let $X = X_1 \times \cdots \times X_n$ be the input space and Y the output space. A fuzzy expert system is a tuple

$$\mathcal{F} = (\Phi,\ \mathcal{K},\ \mathcal{I},\ \Delta),$$

where:

- **Knowledge base** $\mathcal{K} = (\mathcal{L}, \mathcal{R})$ comprises linguistic term definitions $\mathcal{L}$ (membership functions) and a rule base $\mathcal{R} = \{R_j\}_{j=1}^m$. Each rule has the form

  $$R_j: \text{ IF } x_1 \text{ is } A_{j1}\ \wedge\ \cdots\ \wedge\ x_n \text{ is } A_{jn}\ \text{ THEN } y \text{ is } B_j,$$

  where $A_{jk} \subseteq X_k,\ B_j \subseteq Y$ are fuzzy sets with membership functions $\mu_{A_{jk}} : X_k \to [0,1]$, $\mu_{B_j} : Y \to [0,1]$.

- **Fuzzifier** Φ maps a crisp input $x = (x_1, \dots, x_n) \in X$ to degrees $\{\mu_{A_{jk}}(x_k)\}$. In particular, the singleton fuzzifier is

  $$\Phi(x_k)(\xi) = \begin{cases} 1, & \xi = x_k, \\ 0, & \text{otherwise.} \end{cases}$$

- **Inference mechanism** $\mathcal{I}$ fixes: (i) a T-norm $\otimes$ to combine antecedents, giving the firing strength

  $$w_j(x) \ = \ \bigotimes_{k=1}^{n} \mu_{A_{jk}}(x_k),$$

  (ii) an implication/activation operator $\Rightarrow$, and (iii) an aggregation operator $\oplus$ across rules. The output fuzzy set (conditional on x) has membership

  $$\mu_{\text{out}}(y \mid x) \ = \ \bigoplus_{j=1}^{m} \big(w_j(x) \Rightarrow \mu_{B_j}(y) \big).$$

  Two common specializations are:

  $$\text{(Mamdani type)} \quad \Rightarrow = \otimes, \quad \oplus = \text{an S-norm (e.g., max)},$$
  $$\mu_{\text{out}}(y \mid x) = \bigoplus_{j=1}^{m} \big(w_j(x) \otimes \mu_{B_j}(y)\big);$$
  $$\text{(Logical type)} \quad \Rightarrow = \text{an S-implication}, \quad \oplus = \text{a T-norm}.$$

- **Defuzzifier** Δ maps $\mu_{\text{out}}(\cdot \mid x)$ to a crisp output $y^* \in Y$. For the centroid (center-of-area) method,

  $$y^* \ = \ \frac{\displaystyle\int_Y y\, \mu_{\text{out}}(y \mid x)\, dy}{\displaystyle\int_Y \mu_{\text{out}}(y \mid x)\, dy}.$$

The overall input–output mapping is $f : X \to Y,\ f(x) = \Delta\big(\mathcal{I}(\Phi(x), \mathcal{R})\big)$.

Example 1.4 (Mamdani–type Fuzzy Expert System for heater control: a full numeric walk–through). **Variables.** Let $X = X_1 \times X_2$ with inputs $e \in X_1$ (temperature error, °C) and $\dot{e} \in X_2$ (error change, °C/min). Let the output universe be $Y = [0, 100]$ (heater power %).

Linguistic terms (triangular memberships). For $a < b < c$ define

$$\mathrm{tri}(t; a, b, c) \;:=\; \begin{cases} 0, & t \le a \text{ or } t \ge c, \\ \dfrac{t-a}{b-a}, & a < t \le b, \\ \dfrac{c-t}{c-b}, & b < t < c. \end{cases}$$

Inputs:

$$\mu_N(e) = \mathrm{tri}(e; -4, -2, 0), \quad \mu_Z(e) = \mathrm{tri}(e; -2, 0, 2), \quad \mu_P(e) = \mathrm{tri}(e; 0, 2, 4),$$
$$\mu_C(\dot{e}) = \mathrm{tri}(\dot{e}; -2, -1, 0), \quad \mu_S(\dot{e}) = \mathrm{tri}(\dot{e}; -0.5, 0, 0.5), \quad \mu_H(\dot{e}) = \mathrm{tri}(\dot{e}; 0, 1, 2).$$

Output (three triangular terms on Y):

$$\mu_L(y) = \mathrm{tri}(y; 0, 25, 50), \quad \mu_M(y) = \mathrm{tri}(y; 25, 50, 75), \quad \mu_H(y) = \mathrm{tri}(y; 50, 75, 100).$$

Rule base $\mathcal{R}$ (three rules).

$$R_1 : \textsf{IF } e \text{ is } N \;\wedge\; \dot{e} \text{ is } H \;\; \textsf{THEN } y \text{ is } L,$$
$$R_2 : \textsf{IF } e \text{ is } Z \;\wedge\; \dot{e} \text{ is } C \;\; \textsf{THEN } y \text{ is } M,$$
$$R_3 : \textsf{IF } e \text{ is } P \;\wedge\; \dot{e} \text{ is } C \;\; \textsf{THEN } y \text{ is } H.$$

Inference mechanism (Mamdani type). Use the product T-norm for antecedent conjunction and implication, and the max S-norm for aggregation:

$$w_j(x) \;=\; \prod (\text{antecedent degrees}), \qquad \mu_j(y \mid x) \;=\; w_j(x) \cdot \mu_{B_j}(y), \qquad \mu_{\mathrm{out}}(y \mid x) \;=\; \max_j \mu_j(y \mid x).$$

Defuzzify by the centroid:

$$y^* \;=\; \frac{\displaystyle\int_0^{100} y\, \mu_{\mathrm{out}}(y \mid x)\, dy}{\displaystyle\int_0^{100} \mu_{\mathrm{out}}(y \mid x)\, dy}.$$

Crisp input and fuzzification. Take $e = 1.0$ and $\dot{e} = -0.5$. Then

$$\mu_N(1.0) = 0, \quad \mu_Z(1.0) = \frac{2-1}{2-0} = 0.5, \quad \mu_P(1.0) = \frac{1-0}{2-0} = 0.5,$$

$$\mu_C(-0.5) = \frac{0-(-0.5)}{0-(-1)} = 0.5, \quad \mu_S(-0.5) = 0, \quad \mu_H(-0.5) = 0.$$

Rule firing strengths (product T-norm).

$$w_1 = \mu_N(1.0) \cdot \mu_H(-0.5) = 0,$$

$$w_2 = \mu_Z(1.0) \cdot \mu_C(-0.5) = 0.5 \cdot 0.5 = 0.25,$$

$$w_3 = \mu_P(1.0) \cdot \mu_C(-0.5) = 0.5 \cdot 0.5 = 0.25.$$

Implication (product) and aggregation (max). Only R_2 and R_3 are active (height 0.25):

$$\mu_2(y \mid x) = 0.25\,\mu_M(y),$$

$$\mu_3(y \mid x) = 0.25\,\mu_H(y),$$

$$\mu_{\text{out}}(y \mid x) = \max\big\{0.25\,\mu_M(y),\ 0.25\,\mu_H(y)\big\}.$$

Defuzzification (centroid). The two scaled triangles are identical in height and placed symmetrically about $y = 62.5$ (centers at 50 and 75, same bases of width 50). Hence the aggregated shape $\mu_{\text{out}}(\cdot \mid x)$ is symmetric about $y = 62.5$, so its centroid is exactly

$$y^* = 62.5\,.$$

For $(e, \dot{e}) = (1.0, -0.5)$ the Mamdani FES outputs a heater command of 62.5%, balancing *Medium* and *High* recommendations generated by R_2 and R_3.

1.3 Fuzzy Cognitive Maps

Fuzzy Cognitive Maps model causal relationships between concepts using weighted directed graphs and iterative updates with fuzzy logic functions [28–30]. Related concepts include Intuitionistic Fuzzy Cognitive Maps [31–33], Neutrosophic Cognitive Maps [34–37], and Cognitive Hypermaps [38].

Definition 1.5 (Fuzzy Cognitive Map (FCM))**.** [28–30] Let $n \in \mathbb{N}$. An FCM is a quadruple

$$\mathcal{F} = \big(C, W, f, \mathbf{a}^{(0)}\big),$$

where:

- $C = \{C_1, \ldots, C_n\}$ is a finite set of *concepts*.
- $W = (w_{ij}) \in [-1, 1]^{n\times n}$ is a weighted adjacency matrix of a directed graph on C, where w_{ij} is the causal influence from C_i to C_j (support if $w_{ij} > 0$, inhibition if $w_{ij} < 0$). Self-loops are usually set to $w_{ii} = 0$.
- $f : \mathbb{R} \to [0, 1]$ is a nondecreasing *transfer (squashing) function* applied componentwise; typical choices include the logistic sigmoid, a clipped linear map $f(x) = \min\{1, \max\{0, x\}\}$, or tanh rescaled to $[0, 1]$.
- $\mathbf{a}^{(0)} \in [0, 1]^n$ is the *initial activation* (state) vector, where $a_j^{(t)}$ is the degree to which concept C_j is active/true at iteration t.

The (synchronous) update rule is

$$\mathbf{a}^{(t+1)} \;=\; f\big(\mathbf{a}^{(t)}W\big), \qquad t = 0, 1, 2, \ldots,$$

where f acts componentwise on the row vector $\mathbf{a}^{(t)}W$. A vector $\mathbf{a}^* \in [0, 1]^n$ is a *fixed point* if $\mathbf{a}^* = f(\mathbf{a}^*W)$. A periodic orbit $(\mathbf{a}^{(t)})$ is a *limit cycle of length L* if $\mathbf{a}^{(t+L)} = \mathbf{a}^{(t)}$ for all large t.

Remark 1.6 (Well-posedness)**.** Since f maps $\mathbb{R}$ to $[0,1]$ componentwise, every iterate $\mathbf{a}^{(t)}$ lies in $[0,1]^n$. Variants allow clamping a subset $U \subseteq \{1,\ldots,n\}$ to prescribed values each iteration.

Example 1.7 (A 3-concept FCM with step-by-step computation)**.** Consider three everyday concepts: C_1 = Marketing, C_2 = Sales, C_3 = Customer Satisfaction. Let the causal weights be

$$W = \begin{pmatrix} 0 & 0.7 & 0.2 \\ -0.2 & 0 & 0.6 \\ 0.1 & 0.3 & 0 \end{pmatrix},$$

so, e.g., Marketing → Sales has weight 0.7 (support), Sales → Marketing has −0.2 (crowding out), and Satisfaction → Sales has 0.3 (positive feedback). Choose the clipped linear transfer $f(x) = \min\{1, \max\{0, x\}\}$ and the initial state

$$\mathbf{a}^{(0)} = (0.60,\ 0.40,\ 0.50).$$

Iteration $t = 0 \to 1$**.** Compute $\mathbf{a}^{(0)}W$ coordinatewise (row × matrix):

$$\begin{aligned}
(\mathbf{a}^{(0)}W)_1 &= 0.60 \cdot 0 + 0.40 \cdot (-0.2) + 0.50 \cdot 0.1 = -0.08 + 0.05 = \ -0.03 \ \xrightarrow{f}\ 0, \\
(\mathbf{a}^{(0)}W)_2 &= 0.60 \cdot 0.7 + 0.40 \cdot 0 + 0.50 \cdot 0.3 = 0.42 + 0 + 0.15 = \ 0.57, \\
(\mathbf{a}^{(0)}W)_3 &= 0.60 \cdot 0.2 + 0.40 \cdot 0.6 + 0.50 \cdot 0 = 0.12 + 0.24 + 0 = \ 0.36.
\end{aligned}$$

Hence $\mathbf{a}^{(1)} = f(\mathbf{a}^{(0)}W) = (0,\ 0.57,\ 0.36)$.

Iteration $t = 1 \to 2$**.** With $\mathbf{a}^{(1)} = (0, 0.57, 0.36)$:

$$\begin{aligned}
(\mathbf{a}^{(1)}W)_1 &= 0 \cdot 0 + 0.57 \cdot (-0.2) + 0.36 \cdot 0.1 = -0.114 + 0.036 = \ -0.078 \ \xrightarrow{f}\ 0, \\
(\mathbf{a}^{(1)}W)_2 &= 0 \cdot 0.7 + 0.57 \cdot 0 + 0.36 \cdot 0.3 = 0.108, \\
(\mathbf{a}^{(1)}W)_3 &= 0 \cdot 0.2 + 0.57 \cdot 0.6 + 0.36 \cdot 0 = 0.342,
\end{aligned}$$

so $\mathbf{a}^{(2)} = (0,\ 0.108,\ 0.342)$.

Iteration $t = 2 \to 3$**.** With $\mathbf{a}^{(2)} = (0, 0.108, 0.342)$:

$$\begin{aligned}
(\mathbf{a}^{(2)}W)_1 &= 0 \cdot 0 + 0.108 \cdot (-0.2) + 0.342 \cdot 0.1 = -0.0216 + 0.0342 = \ 0.0126, \\
(\mathbf{a}^{(2)}W)_2 &= 0 \cdot 0.7 + 0.108 \cdot 0 + 0.342 \cdot 0.3 = 0.1026, \\
(\mathbf{a}^{(2)}W)_3 &= 0 \cdot 0.2 + 0.108 \cdot 0.6 + 0.342 \cdot 0 = 0.0648,
\end{aligned}$$

thus $\mathbf{a}^{(3)} = f(\cdot) = (0.0126,\ 0.1026,\ 0.0648)$.

Iteration $t = 3 \to 4$**.** With $\mathbf{a}^{(3)} = (0.0126, 0.1026, 0.0648)$:

$$\begin{aligned}
(\mathbf{a}^{(3)}W)_1 &= 0.0126 \cdot 0 + 0.1026 \cdot (-0.2) + 0.0648 \cdot 0.1 = -0.02052 + 0.00648 = \ -0.01404 \ \xrightarrow{f}\ 0, \\
(\mathbf{a}^{(3)}W)_2 &= 0.0126 \cdot 0.7 + 0.1026 \cdot 0 + 0.0648 \cdot 0.3 = 0.00882 + 0.01944 = \ 0.02826, \\
(\mathbf{a}^{(3)}W)_3 &= 0.0126 \cdot 0.2 + 0.1026 \cdot 0.6 + 0.0648 \cdot 0 = 0.00252 + 0.06156 = \ 0.06408,
\end{aligned}$$

so $\mathbf{a}^{(4)} = (0,\ 0.02826,\ 0.06408)$.

These iterations illustrate how positive and negative causal links drive the concepts toward a (near) equilibrium under the chosen f. Different f, weights, or clamped inputs can produce distinct fixed points or limit cycles.

1.4 Upside-Down Logic

This subsection presents the mathematical definition of Upside-Down Logic. In brief, Upside-Down Logic flips the truth and falsity of lemmas under contextual transformations, thereby formalizing ambiguity and reversals in reasoning systems [39–44]. The related definitions and notations are given below.

Definition 1.8 (Context)**.** [40, 41] A *context* $\mathcal{C}$ is a collection of parameters or conditions under which lemmas are evaluated. These may include spatial, temporal, semantic, or interpretive settings.

Definition 1.9 (Logical System)**.** (cf. [45]) A *logical system* $\mathcal{M}$ is a mathematical structure that formalizes reasoning. It consists of

$$\mathcal{M} = (\mathcal{P}, \mathcal{V}, v),$$

where:

- $\mathcal{P}$ is the set of lemmas (statements) in the formal language $\mathcal{L}$;
- $\mathcal{V}$ is the set of truth values (e.g., $\{\text{True}, \text{False}\}$ in classical logic);
- $v : \mathcal{P} \to \mathcal{V}$ is a *valuation* (interpretation) that assigns a truth value to each lemma in $\mathcal{P}$.

A logical system may additionally include:

- a set of *axioms* $\mathcal{A} \subseteq \mathcal{P}$ assumed true within the system;
- a set of *inference rules* $\mathcal{I}$ specifying valid transformations used to derive new truths.

Notation 1.10. *Let $\mathcal{P}$ be a set of lemmas and $\mathcal{C}$ a set of contexts. Define the truth-valuation*

$$T : \mathcal{P} \times \mathcal{C} \longrightarrow \{\textit{True}, \textit{False}, \textit{Indeterminate}\},$$

which assigns a truth value to each lemma–context pair.

Notation 1.11. *Let $\mathcal{L}$ be a formal language, and let $\mathcal{M}$ be a logical system with lemma set $\mathcal{P}$, truth-value set $\mathcal{V}$, and valuation $v : \mathcal{P} \to \mathcal{V}$.*

Definition 1.12 (Upside-Down Logic)**.** [40, 41] An *Upside-Down Logic* is a logical system $\mathcal{M}'$ obtained from $\mathcal{M}$ by introducing a transformation U on lemmas and/or contexts such that:

1. For any lemma $A \in \mathcal{P}$ with truth value $v(A)$ in context $\mathcal{C}$, there exists a transformed lemma $U(A)$ and/or a transformed context $U(\mathcal{C})$ for which:
   - *Falsification of the Truth*: If $v(A) = \text{True}$ in $\mathcal{C}$, then $v\big(U(A)\big) = \text{False}$ in $U(\mathcal{C})$.
   - *Truthification of the False*: If $v(A) = \text{False}$ in $\mathcal{C}$, then $v\big(U(A)\big) = \text{True}$ in $U(\mathcal{C})$.
2. The transformation U is well defined and consistent within the resulting system $\mathcal{M}'$.

Example 1.13 (Upside–Down Logic: Context–Switching Policy)**.** Let $\mathcal{M} = (\mathcal{P}, \mathcal{V}, v)$ with $\mathcal{V} = \{\text{True}, \text{False}\}$ and a single lemma

$$A := \text{“The city park closes at 20:00.”}$$

Consider two contexts:

$$\mathcal{C}_{\text{wk}} := \{\text{day=weekday, policy=close at 20:00}\}, \qquad \mathcal{C}_{\text{fest}} := \{\text{day=festival, policy=open until 22:00}\}.$$

Define the valuation

$$v(A \mid \mathcal{C}_{\text{wk}}) = \text{True}, \qquad v(A \mid \mathcal{C}_{\text{fest}}) = \text{False}.$$

Let the Upside–Down transform U act by switching contexts and fixing the lemma:

$$U(\mathcal{C}_{\text{wk}}) = \mathcal{C}_{\text{fest}}, \quad U(\mathcal{C}_{\text{fest}}) = \mathcal{C}_{\text{wk}}, \quad U(A) = A.$$

Then (Falsification of the Truth) $v(A \mid \mathcal{C}_{\text{wk}}) = \text{True}$ implies $v\big(U(A) \mid U(\mathcal{C}_{\text{wk}})\big) = v(A \mid \mathcal{C}_{\text{fest}}) = \text{False}$; and (Truthification of the False) $v(A \mid \mathcal{C}_{\text{fest}}) = \text{False}$ implies $v\big(U(A) \mid U(\mathcal{C}_{\text{fest}})\big) = v(A \mid \mathcal{C}_{\text{wk}}) = \text{True}$. Hence U yields an Upside–Down Logic $\mathcal{M}'$ as in the definition.

Example 1.14 (Street Parking Rule with Day-Context Switch)**.** Let the lemma be

$$A := \text{“Parking on Main St. is permitted at 08:30.”}$$

Define two contexts:

$$\mathcal{C}_{\text{wk}} := \{\text{day=weekday, rule: no parking 07:00–09:00}\},$$

$$\mathcal{C}_{\text{sun}} := \{\text{day=Sunday, rule: parking allowed 00:00–24:00}\}.$$

Valuations at the same clock time $t = 08{:}30$ are

$$v(A \mid \mathcal{C}_{\text{wk}}) = \text{False} \quad (\,08{:}30 \in [07{:}00, 09{:}00)\,), \qquad v(A \mid \mathcal{C}_{\text{sun}}) = \text{True}.$$

Define the Upside–Down transform U by switching the context and fixing the lemma:

$$U(A) = A, \qquad U(\mathcal{C}_{\text{wk}}) = \mathcal{C}_{\text{sun}}, \quad U(\mathcal{C}_{\text{sun}}) = \mathcal{C}_{\text{wk}}.$$

Then we have *Falsification of the Truth* and *Truthification of the False*:

$$v(A \mid \mathcal{C}_{\text{sun}}) = \text{True} \;\Rightarrow\; v\big(U(A) \mid U(\mathcal{C}_{\text{sun}})\big) = v(A \mid \mathcal{C}_{\text{wk}}) = \text{False},$$

$$v(A \mid \mathcal{C}_{\text{wk}}) = \text{False} \;\Rightarrow\; v\big(U(A) \mid U(\mathcal{C}_{\text{wk}})\big) = v(A \mid \mathcal{C}_{\text{sun}}) = \text{True}.$$

Thus U realizes Upside–Down Logic for this real–life parking policy.

Example 1.15 (Time–of–Day Pricing: Off–Peak vs Peak Tariff)**.** Let the lemma be

$$A := \text{“Running the dishwasher now is billed at the \textit{off–peak} rate.”}$$

Define contexts using standard electricity time–of–use windows:

$$\mathcal{C}_{\text{off}} := \{\text{time} = 22{:}30,\ \text{tariff: off–peak 22:00–06:00}\},$$

$$\mathcal{C}_{\text{peak}} := \{\text{time} = 18{:}30,\ \text{tariff: peak 17:00–21:00}\}.$$

Valuations are

$$v(A \mid \mathcal{C}_{\text{off}}) = \text{True} \quad (22{:}30 \in [22{:}00, 06{:}00)), \qquad v(A \mid \mathcal{C}_{\text{peak}}) = \text{False}.$$

Define U by swapping the time–context and fixing the lemma:

$$U(A) = A, \qquad U(\mathcal{C}_{\text{off}}) = \mathcal{C}_{\text{peak}}, \quad U(\mathcal{C}_{\text{peak}}) = \mathcal{C}_{\text{off}}.$$

Then Upside–Down Logic holds:

$$v(A \mid \mathcal{C}_{\text{off}}) = \text{True} \;\Rightarrow\; v\big(U(A) \mid U(\mathcal{C}_{\text{off}})\big) = v(A \mid \mathcal{C}_{\text{peak}}) = \text{False},$$

$$v(A \mid \mathcal{C}_{\text{peak}}) = \text{False} \;\Rightarrow\; v\big(U(A) \mid U(\mathcal{C}_{\text{peak}})\big) = v(A \mid \mathcal{C}_{\text{off}}) = \text{True}.$$

Therefore the transform flips the truth value of A when moving between peak and off–peak contexts, exemplifying Upside–Down Logic in everyday billing.

1.5 Plithogenic Set

A Plithogenic Set [46–48] models elements with attribute-based membership and contradiction functions, extending fuzzy [1,49], intuitionistic [50,51], and neutrosophic sets [52,53].

Definition 1.16 (Plithogenic Set)**.** [46,54] Let S be a universal set and $P \subseteq S$ a nonempty subset. A *Plithogenic Set* is a quintuple

$$PS = (P,\ v,\ Pv,\ pdf,\ pCF),$$

where

- v is an attribute,
- Pv is the set of possible values of the attribute v,
- $pdf : P \times Pv \to [0,1]^s$ is the *Degree of Appurtenance Function (DAF)*,[1]
- $pCF : Pv \times Pv \to [0,1]^t$ is the *Degree of Contradiction Function (DCF)*.

The DCF satisfies, for all $a, b \in Pv$,

$$\text{Reflexivity: } pCF(a,a) = 0, \qquad \text{Symmetry: } pCF(a,b) = pCF(b,a).$$

Here $s \in \mathbb{N}$ is the appurtenance dimension and $t \in \mathbb{N}$ the contradiction dimension.

Example 1.17 (Plithogenic Set: Laptops with Two–Dimensional Appurtenance)**.** Let the universe S be a laptop catalogue and $P = \{\ell_1, \ell_2, \ell_3\} \subset S$ the models under evaluation. Fix the attribute $v =$"design aspect" with value set $Pv = \{\text{Light}, \text{Power}, \text{Cheap}\}$. Define a degree of appurtenance function $pdf : P \times Pv \to [0,1]^2$ whose components encode (utility, reliability):

$$\begin{aligned}
&pdf(\ell_1, \text{Light}) = (0.90, 0.60), \quad && pdf(\ell_1, \text{Power}) = (0.40, 0.70), \quad && pdf(\ell_1, \text{Cheap}) = (0.55, 0.65),\\
&pdf(\ell_2, \text{Light}) = (0.70, 0.80), && pdf(\ell_2, \text{Power}) = (0.85, 0.75), && pdf(\ell_2, \text{Cheap}) = (0.30, 0.60),\\
&pdf(\ell_3, \text{Light}) = (0.50, 0.55), && pdf(\ell_3, \text{Power}) = (0.60, 0.50), && pdf(\ell_3, \text{Cheap}) = (0.85, 0.70).
\end{aligned}$$

Let the contradiction function $pCF : Pv \times Pv \to [0,1]$ be symmetric with

$$pCF(\text{Light}, \text{Light}) = pCF(\text{Power}, \text{Power}) = pCF(\text{Cheap}, \text{Cheap}) = 0,$$

$$\begin{aligned}
pCF(\text{Light}, \text{Power}) &= pCF(\text{Power}, \text{Light}) = 0.7,\\
pCF(\text{Light}, \text{Cheap}) &= pCF(\text{Cheap}, \text{Light}) = 0.3,\\
pCF(\text{Power}, \text{Cheap}) &= pCF(\text{Cheap}, \text{Power}) = 0.6.
\end{aligned}$$

Then $PS = (P, v, Pv, pdf, pCF)$ satisfies the plithogenic set axioms (reflexive and symmetric pCF; vector–valued pdf), hence is a valid Plithogenic Set.

Definition 1.18 (Plithogenic Fuzzy Set ($s = 1,\ t = 1$))**.** [46,56,57] A *Plithogenic Fuzzy Set* is a Plithogenic Set $PS = (P, v, Pv, pdf, pCF)$ with

$$pdf : P \times Pv \longrightarrow [0,1], \qquad pCF : Pv \times Pv \longrightarrow [0,1].$$

For $x \in P$ and attribute value $a \in Pv$, write

$$\mu_P(x \mid a) \ := \ pdf(x, a) \in [0,1],$$

the (single) fuzzy membership degree of x w.r.t. a. The contradiction between two attribute values is the scalar $c(a,b) := pCF(a,b) \in [0,1]$ with $c(a,a) = 0$ and $c(a,b) = c(b,a)$.

[1] In the literature, DAF is defined in slightly different ways: some variants use powerset–valued constructions, others the simple cube $[0,1]^s$. We adopt the latter (classical) form here; cf. [55].

Example 1.19 (Plithogenic Fuzzy Set: Houses with Scalar Appurtenance and Contradiction)**.** Let $P = \{h_1, h_2, h_3\}$ be houses and take the attribute value set $Pv = \{\text{Expensive, Cheap, Modern, Traditional}\}$. Define the scalar membership $pdf : P \times Pv \to [0,1]$ by

$$pdf(h_1, \text{Expensive}) = 0.9, \quad pdf(h_1, \text{Cheap}) = 0.1, \quad pdf(h_1, \text{Modern}) = 0.8, \quad pdf(h_1, \text{Traditional}) = 0.2,$$
$$pdf(h_2, \text{Expensive}) = 0.4, \quad pdf(h_2, \text{Cheap}) = 0.6, \quad pdf(h_2, \text{Modern}) = 0.5, \quad pdf(h_2, \text{Traditional}) = 0.5,$$
$$pdf(h_3, \text{Expensive}) = 0.2, \quad pdf(h_3, \text{Cheap}) = 0.8, \quad pdf(h_3, \text{Modern}) = 0.3, \quad pdf(h_3, \text{Traditional}) = 0.7.$$

Let the contradiction map $pCF : Pv \times Pv \to [0,1]$ be symmetric with $pCF(a,a) = 0$ and, for $a \neq b$,

$$pCF(\text{Expensive, Cheap}) = 1.0, \quad pCF(\text{Modern, Traditional}) = 0.9,$$
$$pCF(\text{Expensive, Modern}) = 0.3, \quad pCF(\text{Expensive, Traditional}) = 0.6,$$
$$pCF(\text{Cheap, Modern}) = 0.6, \quad pCF(\text{Cheap, Traditional}) = 0.3.$$

Then $PS = (P, v, Pv, pdf, pCF)$ is a Plithogenic Fuzzy Set (the case $s = t = 1$ in the definition), with scalar appurtenance $\mu_P(h_i \mid a) = pdf(h_i, a)$ and admissible symmetric contradiction degrees.

1.6 Upside-Down Logic in Plithogenic Fuzzy Set with contradiction reset

Upside-Down Logic in a Plithogenic Fuzzy Set with contradiction reset flips fuzzy memberships when contradictions exceed a threshold, then neutralizes contradictions, ensuring consistency in contexts where values previously conflicted (cf. [58]).

Definition 1.20 (Upside-Down Logic in Plithogenic Fuzzy Set with contradiction reset)**.** (cf. [58]) Let $PS = (P, v, Pv, pdf, pCF)$ be a Plithogenic Fuzzy Set with

$$\mu_P(x \mid a) := pdf(x, a) \in [0,1], \qquad c(a,b) := pCF(a,b) \in [0,1],$$

and $c(a,a) = 0,\ c(a,b) = c(b,a)$. Fix a reference (anchor) attribute $b \in Pv$ and a threshold $\tau \in [0,1]$. Define the *activation set*

$$A_\tau(b) \ := \ \{\, a \in Pv \ : \ c(a,b) \geq \tau \,\}.$$

The *Upside-Down transform with contradiction reset* produces a new Plithogenic Fuzzy Set

$$PS^{U_{b,\tau}} \ = \ (P,\ v,\ Pv,\ pdf^{U_{b,\tau}},\ pCF^{U_{b,\tau}})$$

by

$$pdf^{U_{b,\tau}}(x,a) \ := \ \begin{cases} 1 - \mu_P(x \mid a), & a \in A_\tau(b), \\ \mu_P(x \mid a), & a \notin A_\tau(b), \end{cases}$$

and the updated contradiction map $pCF^{U_{b,\tau}}$ defined for all $u, v \in Pv$ by

$$pCF^{U_{b,\tau}}(u,v) \ :=$$

$$\begin{cases} 0, & \{u,v\} = \{a,b\} \text{ for some } a \in A_\tau(b), \\ pCF(u,v), & \text{otherwise.} \end{cases}$$

That is, whenever the flip is triggered for the pair (a,b) (i.e. $a \in A_\tau(b)$), the post-transform contradiction between a and b is *reset to zero.*

Example 1.21 (Concrete instance of the Upside–Down transform with contradiction reset)**.** Let $P = \{h_1, h_2, h_3\}$ be houses and $Pv = \{\text{Expensive, Cheap, Modern}\}$. Define scalar memberships $\mu_P(x \mid a) = pdf(x, a) \in [0,1]$ by

	Expensive	Cheap	Modern
h_1	0.80	0.20	0.60
h_2	0.30	0.70	0.40
h_3	0.50	0.50	0.20

and a symmetric contradiction map $c(a,b) = pCF(a,b)$:

$c(a,b)$	Expensive	Cheap	Modern
Expensive	0	0.90	0.75
Cheap	0.90	0	0.40
Modern	0.75	0.40	0

Choose the anchor attribute $b =$ Expensive and threshold $\tau = 0.70$. Then the activation set is

$$A_\tau(b) = \{\, a \in Pv : \; c(a, \text{Expensive}) \geq 0.70 \,\} = \{\text{Cheap}, \text{Modern}\}.$$

Applying the Upside–Down transform with contradiction reset, for $a \in A_\tau(b)$ we flip memberships $pdf^{U_{b,\tau}}(x,a) = 1 - pdf(x,a)$; otherwise we keep them unchanged. The transformed memberships are

	Expensive	Cheap	Modern
h_1	0.80	$1 - 0.20 = 0.80$	$1 - 0.60 = 0.40$
h_2	0.30	$1 - 0.70 = 0.30$	$1 - 0.40 = 0.60$
h_3	0.50	$1 - 0.50 = 0.50$	$1 - 0.20 = 0.80$

The contradiction map is reset to 0 between the flipped attributes and the anchor:

$c^{U_{b,\tau}}(a,b)$	Expensive	Cheap	Modern
Expensive	0	0	0
Cheap	0	0	0.40
Modern	0	0.40	0

i.e.,

$$c^{U_{b,\tau}}(\text{Expensive}, \text{Cheap}) = c^{U_{b,\tau}}(\text{Cheap}, \text{Expensive}) = 0$$

and

$$c^{U_{b,\tau}}(\text{Expensive}, \text{Modern}) = c^{U_{b,\tau}}(\text{Modern}, \text{Expensive}) = 0$$

, while the non–anchor pair remains unchanged

$$c^{U_{b,\tau}}(\text{Cheap}, \text{Modern}) = c(\text{Cheap}, \text{Modern}) = 0.40$$

. This realizes the definition: all attributes sufficiently contradictory to the anchor are flipped, and their post–transform contradiction with the anchor is reset to zero.

Example 1.22 (Daily Commuting Choice under a Safety–First Context)**. Universe and attributes.** Let $P = \{\text{Car}, \ \text{Bicycle}, \ \text{Subway}\}$ and $Pv = \{\text{Safety}, \text{Speed}, \text{Cost}\}$. The plithogenic memberships $pdf(x,a) \in [0,1]$ are

	Safety	Speed	Cost
Car	0.80	0.60	0.20
Bicycle	0.40	0.90	0.80
Subway	0.90	0.70	0.60

The symmetric contradiction map $c(a,b)$ is

$c(a,b)$	Safety	Speed	Cost
Safety	0	0.80	0.30
Speed	0.80	0	0.50
Cost	0.30	0.50	0

Anchor and threshold. Take the anchor $b =$ Safety with threshold $\tau = 0.70$. Then the activation set is

$$A_\tau(b) = \{\, a \in Pv : \; c(a, \text{Safety}) \geq 0.70 \,\} = \{\text{Speed}\}.$$

Upside–Down transform with contradiction reset. For $a \in A_\tau(b)$ we flip pdf and reset the corresponding contradiction to 0:

$$pdf^{U_{b,\tau}}(x, \text{Speed}) = 1 - pdf(x, \text{Speed}),$$

$$c^{U_{b,\tau}}(\text{Speed}, \text{Safety}) = c^{U_{b,\tau}}(\text{Safety}, \text{Speed}) = 0.$$

All other entries remain unchanged. Hence

	Safety	Speed (flipped)	Cost
Car	0.80	$1-0.60=0.40$	0.20
Bicycle	0.40	$1-0.90=0.10$	0.80
Subway	0.90	$1-0.70=0.30$	0.60

and the updated contradiction matrix entries are

$$c^{U_{b,\tau}}(\text{Safety}, \text{Speed}) = c^{U_{b,\tau}}(\text{Speed}, \text{Safety}) = 0,$$

$$c^{U_{b,\tau}}(\text{Safety}, \text{Cost}) = 0.30,$$

$$c^{U_{b,\tau}}(\text{Speed}, \text{Cost}) = 0.50.$$

Consider the lemma A: "Bicycle is fast" with truth proxy $pdf(\text{Bicycle}, \text{Speed})$. Before UD, $pdf = 0.90 > 0.5$ (*True*); after UD, $pdf^{U_{b,\tau}} = 0.10 < 0.5$ (*False*), while the contradiction with the anchor is reset to 0 to avoid double penalization.

Example 1.23 (Hiring under Budget Pressure with Experience as Anchor)**. Universe and attributes.** Let $P = \{\text{Alice, Bob, Carol}\}$ and $Pv = \{\text{Experience}, \text{SalaryLow}, \text{CultureFit}\}$. Memberships $pdf(x, a)$ are

	Experience	SalaryLow	CultureFit
Alice	0.90	0.20	0.70
Bob	0.60	0.80	0.50
Carol	0.40	0.90	0.60

The symmetric contradiction map $c(a, b)$ is

$c(a,b)$	Experience	SalaryLow	CultureFit
Experience	0	0.85	0.20
SalaryLow	0.85	0	0.40
CultureFit	0.20	0.40	0

Anchor and threshold. Choose $b = \text{Experience}$ and $\tau = 0.80$. Then

$$A_\tau(b) = \{\, a : \; c(a, \text{Experience}) \geq 0.80 \,\} = \{\text{SalaryLow}\}.$$

Upside–Down transform with contradiction reset. Flip the activated attribute and reset its contradiction with the anchor:

$$pdf^{U_{b,\tau}}(x, \text{SalaryLow}) = 1 - pdf(x, \text{SalaryLow}),$$

$$c^{U_{b,\tau}}(\text{SalaryLow}, \text{Experience}) = c^{U_{b,\tau}}(\text{Experience}, \text{SalaryLow}) = 0,$$

others unchanged. Thus

	Experience	SalaryLow (flipped)	CultureFit
Alice	0.90	$1-0.20=0.80$	0.70
Bob	0.60	$1-0.80=0.20$	0.50
Carol	0.40	$1-0.90=0.10$	0.60

and the updated contradiction entries are

$$c^{U_{b,\tau}}(\text{Experience}, \text{SalaryLow}) = 0,$$

$$c^{U_{b,\tau}}(\text{Experience}, \text{CultureFit}) = 0.20,$$

$$c^{U_{b,\tau}}(\text{SalaryLow}, \text{CultureFit}) = 0.40.$$

Upside–Down Logic manifestation. Lemma A: "Bob has low salary demand" uses $pdf(\text{Bob}, \text{SalaryLow})$. Initially $0.80 > 0.5$ (*True*); after UD, $1 - 0.80 = 0.20 < 0.5$ (*False*). The contradiction between *Experience* (anchor) and *SalaryLow* is reset to 0, so subsequent reasoning treats the flipped attribute as non–contradictory to the anchor.

2 Main Results

The results of this paper are presented below.

2.1 Neuro–Plithogenic–Fuzzy System

The Neuro–Plithogenic–Fuzzy System with Upside–Down Transform resets contradictions by flipping memberships, enhancing learning consistency and robust reasoning under conflicting attributes.

Definition 2.1 (Plithogenic Activation/Combination Operator)**.** Let $T : [0,1]^2 \to [0,1]$ be a T–norm, $S : [0,1]^2 \to [0,1]$ an S–norm, and $w : [0,1] \to [0,1]$ a nondecreasing function with $w(0) = 0$ and $w(1) = 1$. For $\alpha, \beta, c \in [0,1]$, define the *plithogenic combination*

$$\alpha \odot_c \beta := \big(1 - w(c)\big)\, T(\alpha, \beta) + w(c)\, S(\alpha, \beta).$$

Then $\odot_c$ is a convex interpolation between T and S controlled by the *contradiction degree* c; in particular, $\alpha \odot_0 \beta = T(\alpha, \beta)$ and $\alpha \odot_1 \beta = S(\alpha, \beta)$.

Example 2.2 (Numerical instance of the Plithogenic Activation/Combination Operator)**.** Let the T–norm and S–norm be

$$T(\alpha, \beta) = \alpha\beta, \qquad S(\alpha, \beta) = \max\{\alpha, \beta\},$$

and choose the interpolation weight $w(c) = c$ (linear in the contradiction degree $c \in [0,1]$). For $\alpha = 0.6$ and $\beta = 0.3$ one has $T(\alpha, \beta) = 0.18$ and $S(\alpha, \beta) = 0.6$. Therefore,

$$\alpha \odot_c \beta = (1 - c)\, 0.18 + c\, 0.6.$$

Concrete values:

$$\alpha \odot_0 \beta = 0.18 \quad (\text{pure } T), \qquad \alpha \odot_{0.25} \beta = 0.75 \cdot 0.18 + 0.25 \cdot 0.6 = 0.285,$$

$$\alpha \odot_{0.75} \beta = 0.25 \cdot 0.18 + 0.75 \cdot 0.6 = 0.495, \qquad \alpha \odot_1 \beta = 0.6 \quad (\text{pure } S).$$

Thus $\odot_c$ continuously interpolates between conjunctive/product behavior and disjunctive/max behavior as c increases.

Definition 2.3 (Neuro–Plithogenic–Fuzzy System (NPFS))**.** Let $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$. Let Pv be a finite set of attribute values and let $c : Pv \times Pv \to [0,1]$ be a *contradiction map* satisfying $c(a, a) = 0$ and $c(a, b) = c(b, a)$ for all $a, b \in Pv$. Fix a T–norm T, an S–norm S, an implication/activation $\Rightarrow$, an aggregation $\oplus$, a defuzzifier Δ, and a monotone w as in Definition 2.1. An *NPFS* is a tuple

$$\mathcal{NP}_\theta = \big(X, Y, Pv, c, \mathcal{R}_\theta, \{\mu_{A_{jk}}(\,\cdot\,;\theta)\}, \{\mu_{B_j}(\,\cdot\,;\theta)\}, T, S, \Rightarrow, \oplus, \odot_\cdot, \Delta\big),$$

where the (trainable) rule base $\mathcal{R}_\theta = \{R_j\}_{j=1}^{M}$ consists of rules

$$R_j : \ \mathsf{IF}\ x_1 \text{ is } A_{j1} \wedge \cdots \wedge x_n \text{ is } A_{jn}\ \mathsf{THEN}\ \big(y \text{ is } B_j\big) \ \text{ with attribute } a(j) \in Pv,$$

the A_{jk} and B_j being fuzzy sets with memberships $\mu_{A_{jk}} : X_k \to [0,1]$ and $\mu_{B_j} : Y \to [0,1]$, and where $a(j)$ tags the consequent of rule j with an attribute value.

Inference for a given context/anchor $b \in Pv$. Given a crisp input $x = (x_1, \ldots, x_n) \in X$ and a chosen *anchor attribute* $b \in Pv$ (representing the desired/prevailing attribute in the context), define the rule firing strengths

$$w_j(x;\theta) := T\big(\mu_{A_{j1}}(x_1;\theta), \ldots, \mu_{A_{jn}}(x_n;\theta)\big),$$

$$\bar{w}_j(x;\theta) := \frac{w_j(x;\theta)}{\sum_{i=1}^{M} w_i(x;\theta)}.$$

The *plithogenic activation* of rule j on each $y \in Y$ is

$$\mu_j(y \mid x, b) \ := \ \big(w_j(x;\theta)\big) \ \odot_{c(a(j),\,b)} \ \mu_{B_j}(y;\theta).$$

The aggregated output fuzzy set (conditional on (x, b)) is

$$\mu_{\text{out}}(y \mid x, b) \ := \ \bigoplus_{j=1}^{M} \mu_j(y \mid x, b),$$

and the crisp output is $f_\theta(x; b) := \Delta\big(\mu_{\text{out}}(\cdot \mid x, b)\big) \in Y$.

Plithogenic appurtenance induced by the NPFS. Let $\rho_{j,a} \in [0, 1]$ be rule–to–attribute weights with $\sum_{a\in Pv} \rho_{j,a} = 1$. Define, for each $x \in X$ and $a \in Pv$, the *NPFS–induced degree of appurtenance*

$$pdf_\theta(x, a) \ := \ \sum_{j=1}^{M} \rho_{j,a} \ \bar{w}_j(x;\theta) \ \in [0, 1]. \tag{1}$$

(The map $x \mapsto pdf_\theta(x, a)$ will provide the plithogenic fuzzy structure in Theorem 2.26.)

Learning. Given data $\mathcal{S} = \{(x^{(p)}, y^{(p)}, b^{(p)})\}_{p=1}^{N}$ and a loss ℓ, estimate θ (and optionally $\rho_{j,a}$, c, w) by

$$\theta^\star \ \in \ \arg\min_\theta \ \sum_{p=1}^{N} \ell\big(f_\theta(x^{(p)}; b^{(p)}),\ y^{(p)}\big).$$

Example 2.4 (A minimal Neuro–Plithogenic–Fuzzy System (NPFS) with two rules)**.** Let $n = 1$, $X \subset \mathbb{R}$, $Y = \{0, 1\}$ (“reject”, “accept”). Take $Pv = \{\text{Eco}, \text{Speed}\}$ with anchor $b = \text{Eco}$ and contradiction map

$$c(\text{Eco}, \text{Eco}) = 0, \qquad c(\text{Speed}, \text{Eco}) = c(\text{Eco}, \text{Speed}) = 0.8.$$

Use $T(\alpha, \beta) = \alpha\beta$, $S(\alpha, \beta) = \max\{\alpha, \beta\}$, $w(c) = c$, aggregation $\oplus = \max$, and discrete centroid defuzzifier $y^* = \dfrac{\sum_{y\in\{0,1\}} y\, \mu_{\text{out}}(y)}{\sum_{y\in\{0,1\}} \mu_{\text{out}}(y)}$. Consider two rules:

$$R_1 : \mathsf{IF}\ x \text{ is } A_{11}\ \ \mathsf{THEN}\ y \text{ is } B_1 \quad (a(1) = \text{Eco}),$$
$$R_2 : \mathsf{IF}\ x \text{ is } A_{21}\ \ \mathsf{THEN}\ y \text{ is } B_2 \quad (a(2) = \text{Speed}).$$

At a given input x, suppose $\mu_{A_{11}}(x) = 0.6$, $\mu_{A_{21}}(x) = 0.5$; hence $w_1 = 0.6$, $w_2 = 0.5$. Let the consequents be fuzzy sets on Y with memberships

$$\mu_{B_1}(0) = 0.2,\ \mu_{B_1}(1) = 0.9, \qquad \mu_{B_2}(0) = 0.7,\ \mu_{B_2}(1) = 0.3.$$

Plithogenic activation (Definition 2.1) yields, for $y \in \{0, 1\}$,

$$\mu_j(y \mid x, b) = \big(1 - c(a(j), b)\big)\, w_j\, \mu_{B_j}(y) + c(a(j), b)\, \max\{w_j, \mu_{B_j}(y)\}.$$

Compute for $y = 0$ (reject):

$$\mu_1(0 \mid x, b) = (1 - 0) \cdot (0.6 \cdot 0.2) = 0.12,$$

$$\mu_2(0 \mid x, b) = (1 - 0.8) \cdot (0.5 \cdot 0.7) + 0.8 \cdot \max\{0.5, 0.7\} = 0.07 + 0.56 = 0.63,$$

so $\mu_{\text{out}}(0 \mid x, b) = \max\{0.12, 0.63\} = 0.63$. For $y = 1$ (accept):

$$\mu_1(1 \mid x, b) = (1 - 0) \cdot (0.6 \cdot 0.9) = 0.54,$$

$$\mu_2(1 \mid x, b) = (1 - 0.8) \cdot (0.5 \cdot 0.3) + 0.8 \cdot \max\{0.5, 0.3\} = 0.03 + 0.40 = 0.43,$$

hence $\mu_{\text{out}}(1 \mid x, b) = \max\{0.54, 0.43\} = 0.54$. Defuzzifying,

$$y^* \ = \ \frac{0 \cdot 0.63 + 1 \cdot 0.54}{0.63 + 0.54} \ = \ \frac{0.54}{1.17} \ \approx \ 0.462.$$

Thus the NPFS produces a context–aware output that balances rule evidence via the contradiction–controlled operator $\odot_c$.

Remark 2.5 (Well–definedness and bounds)**.** By convexity in Definition 2.1 and the facts $T, S \in [0,1]$, one has $0 \le \mu_j(y \mid x, b) \le 1$ and thus $0 \le \mu_{\text{out}}(y \mid x, b) \le 1$. Moreover, $pdf_\theta(x, a) \in [0,1]$ and $\sum_{a \in Pv} pdf_\theta(x, a) = 1$ by (1).

Theorem 2.6 (NPFS generalizes the Neuro–Fuzzy System)**.** *Consider the NPFS of Definition 2.3 with the following specialization:*

$$c(a,b) \equiv 0 \quad \textit{for all } a, b \in Pv, \qquad w \equiv \mathrm{id}, \qquad \odot_0 = T.$$

Then for every input x and anchor b,

$$\mu_j(y \mid x, b) = T\big(w_j(x;\theta), \mu_{B_j}(y;\theta)\big), \qquad \mu_{\text{out}}(y \mid x, b) = \bigoplus_{j=1}^{M} T\big(w_j, \mu_{B_j}\big),$$

which coincides with the standard Mamdani–type Neuro–Fuzzy inference. Consequently, $f_\theta(x; b)$ equals the Neuro–Fuzzy output with the same $(T, \oplus, \Delta)$; hence NPFS strictly generalizes the Neuro–Fuzzy System.

Proof. If $c \equiv 0$, then by Definition 2.1, $\alpha \odot_0 \beta = (1 - w(0))T(\alpha, \beta) + w(0)S(\alpha, \beta) = T(\alpha, \beta)$ since $w(0) = 0$. Substituting into the NPFS equations yields the classical Mamdani pipeline; the defuzzified output therefore matches exactly the Neuro–Fuzzy mapping with the same $(T, \oplus, \Delta)$. □

Theorem 2.7 (NPFS carries a Plithogenic–Fuzzy Set structure)**.** *Fix θ, the attribute set Pv, and the contradiction map c from Definition 2.3. Let $P := X$ and let v denote the (categorical) attribute whose values are Pv. Define $PS_\theta := (P, v, Pv, pdf_\theta, c)$ where pdf_θ is given by (1). Then PS_θ is a Plithogenic–Fuzzy Set, i.e.,*

$$pdf_\theta : \ P \times Pv \to [0,1], \qquad c : \ Pv \times Pv \to [0,1],$$

with $c(a,a) = 0$ and $c(a,b) = c(b,a)$ for all $a, b \in Pv$.

Proof. By construction, $pdf_\theta(x, a) = \sum_j \rho_{j,a} \bar{w}_j(x;\theta)$ is a convex combination of numbers in $[0,1]$; hence $pdf_\theta(x, a) \in [0,1]$. Moreover $\sum_{a \in Pv} pdf_\theta(x, a) = \sum_j \bar{w}_j(x;\theta) \sum_a \rho_{j,a} = 1$, so each $x \in P$ has a well–defined distribution of appurtenance degrees across Pv. The map c is assumed reflexive on the diagonal and symmetric by Definition 2.3. Therefore $(P, v, Pv, pdf_\theta, c)$ satisfies the axioms of a Plithogenic–Fuzzy Set. □

2.2 Upside–Down Transform with Contradiction Reset in NPFS

The Upside–Down Transform with Contradiction Reset in NPFS flips fuzzy memberships and resets contradictions, enabling context-aware adaptive reasoning.

Definition 2.8 (Upside–Down Transform with Contradiction Reset in NPFS)**.** Let

$$\mathcal{NP}_\theta = (X, Y, Pv, c, \mathcal{R}_\theta, \{\mu_{A_{jk}}\}, \{\mu_{B_j}\}, T, S, \Rightarrow, \oplus, \odot_\cdot, \Delta)$$

be a Neuro–Plithogenic–Fuzzy System as in Definition 2.3, where each rule

$$R_j : \ \textsf{IF } x_1 \text{ is } A_{j1} \ \wedge \cdots \wedge x_n \text{ is } A_{jn} \ \ \textsf{THEN } y \text{ is } B_j \quad \text{is tagged by } a(j) \in Pv.$$

Fix an *anchor attribute* $b \in Pv$ and a threshold $\tau \in [0,1]$. Define the activation set

$$A_\tau(b) \ := \ \{\, a \in Pv \ : \ c(a,b) \ge \tau \,\}.$$

The *Upside–Down (UD) transform with contradiction reset* produces a new NPFS

$$\mathcal{NP}_\theta^{U_{b,\tau}} = (X, Y, Pv, c^{U_{b,\tau}}, \mathcal{R}_\theta, \{\mu_{A_{jk}}\}, \{\mu_{B_j}^{U_{b,\tau}}\}, T, S, \Rightarrow, \oplus, \odot_\cdot^{U_{b,\tau}}, \Delta)$$

by the following operations:

(a) **Consequent flip (Upside–Down):**

$$\mu_{B_j}^{U_{b,\tau}}(y) \;:=\; \begin{cases} 1-\mu_{B_j}(y), & \text{if } a(j)\in A_\tau(b),\\ \mu_{B_j}(y), & \text{otherwise,}\end{cases} \qquad \forall\, y\in Y.$$

(b) **Contradiction reset:** for all $u,v\in Pv$,

$$c^{U_{b,\tau}}(u,v) \;:=\; \begin{cases} 0, & \text{if } \{u,v\}=\{a,b\} \text{ for some } a\in A_\tau(b),\\ c(u,v), & \text{otherwise.}\end{cases}$$

(c) **Plithogenic activation update:** the plithogenic combination used in the inference step is reparameterized by the updated contradiction: for any $\alpha,\beta\in[0,1]$ and any pair $(a,b)\in Pv\times Pv$,

$$\alpha \odot^{U_{b,\tau}}_{c^{U_{b,\tau}}(a,b)} \beta \;:=\; \big(1-w(c^{U_{b,\tau}}(a,b))\big)\,T(\alpha,\beta) \;+\; w(c^{U_{b,\tau}}(a,b))\,S(\alpha,\beta),$$

where $w:[0,1]\to[0,1]$ is a fixed nondecreasing function with $w(0)=0$, $w(1)=1$.

Given a crisp input $x=(x_1,\dots,x_n)\in X$, define the (unchanged) rule firing strengths

$$w_j(x;\theta)=T\big(\mu_{A_{j1}}(x_1),\dots,\mu_{A_{jn}}(x_n)\big), \qquad \bar{w}_j(x;\theta)=\frac{w_j(x;\theta)}{\sum_{i=1}^{M} w_i(x;\theta)}.$$

Then the transformed rule activations and output membership are

$$\mu_j^{U_{b,\tau}}(y\mid x,b)=\; w_j(x;\theta)\;\odot^{U_{b,\tau}}_{c^{U_{b,\tau}}(a(j),\,b)}\;\mu_{B_j}^{U_{b,\tau}}(y), \qquad \mu_{\mathrm{out}}^{U_{b,\tau}}(y\mid x,b)=\bigoplus_{j=1}^{M}\mu_j^{U_{b,\tau}}(y\mid x,b),$$

and the crisp output is $f_\theta^{U_{b,\tau}}(x;b)=\Delta\big(\mu_{\mathrm{out}}^{U_{b,\tau}}(\cdot\mid x,b)\big)$.

Remark 2.9. The transform flips only those consequents whose attributes are sufficiently contradictory to the anchor ($a(j)\in A_\tau(b)$), and *then* neutralizes the corresponding contradiction by setting $c^{U_{b,\tau}}(a(j),b)=0$. Hence, after the flip, those rules interact conjunctively (via T) with the context, eliminating double penalization.

Example 2.10 (Two–Rule Numerical Illustration with Discrete Defuzzification)**.** Let $n=1$, $X\subseteq\mathbb{R}$, and take a discrete output universe $Y=\{L,H\}$ identified with $\{0,1\}$. Choose $T(\alpha,\beta)=\alpha\beta$, $S(\alpha,\beta)=\max\{\alpha,\beta\}$, and $w(c)=c$. Let $Pv=\{\mathrm{Cost},\mathrm{Speed}\}$ and anchor $b=\mathrm{Cost}$. Assume the contradiction map

$$c(\mathrm{Cost},\mathrm{Cost})=0, \qquad c(\mathrm{Cost},\mathrm{Speed})=c(\mathrm{Speed},\mathrm{Cost})=0.8.$$

Threshold $\tau=0.7$ gives $A_\tau(b)=\{\mathrm{Speed}\}$.

Rules and memberships. There are two rules ($M=2$):

$$\begin{aligned} R_1 &: \textsf{IF } x \text{ is } A_{11} \;\; \textsf{THEN } y \text{ is } B_1 \quad (a(1)=\mathrm{Cost}),\\ R_2 &: \textsf{IF } x \text{ is } A_{21} \;\; \textsf{THEN } y \text{ is } B_2 \quad (a(2)=\mathrm{Speed}).\end{aligned}$$

For an input x, suppose

$$\mu_{A_{11}}(x)=0.6, \qquad \mu_{A_{21}}(x)=0.5 \;\;\Longrightarrow\;\; w_1=0.6,\; w_2=0.5.$$

Let the consequent memberships on Y be

$$\mu_{B_1}(L)=0.2,\; \mu_{B_1}(H)=0.9, \qquad \mu_{B_2}(L)=0.8,\; \mu_{B_2}(H)=0.3.$$

Aggregation is $\oplus = \max$, and defuzzification uses the discrete centroid

$$y^* \;=\; \frac{\sum_{y\in Y} y\,\mu_{\text{out}}(y)}{\sum_{y\in Y}\mu_{\text{out}}(y)} \;\in [0,1].$$

Before UD transform. Plithogenic activation uses $\odot_c(\alpha,\beta) = (1-c)\alpha\beta + c\max\{\alpha,\beta\}$.

$$\begin{aligned}
\mu_1(L \mid x,b) &= (1-0)\cdot(0.6\cdot 0.2) + 0\cdot\max\{0.6,0.2\} = 0.12,\\
\mu_2(L \mid x,b) &= (1-0.8)\cdot(0.5\cdot 0.8) + 0.8\cdot\max\{0.5,0.8\}\\
&= 0.2\cdot 0.4 + 0.8\cdot 0.8 = 0.08 + 0.64 = 0.72,\\
\Rightarrow\;\; \mu_{\text{out}}(L \mid x,b) &= \max\{0.12,\,0.72\} = 0.72.
\end{aligned}$$

Similarly for H:

$$\begin{aligned}
\mu_1(H \mid x,b) &= (1-0)\cdot(0.6\cdot 0.9) = 0.54,\\
\mu_2(H \mid x,b) &= (1-0.8)\cdot(0.5\cdot 0.3) + 0.8\cdot\max\{0.5,0.3\}\\
&= 0.2\cdot 0.15 + 0.8\cdot 0.5 = 0.03 + 0.40 = 0.43,\\
\Rightarrow\;\; \mu_{\text{out}}(H \mid x,b) &= \max\{0.54,\,0.43\} = 0.54.
\end{aligned}$$

Discrete centroid:

$$y^*_{\text{pre}} = \frac{0\cdot 0.72 + 1\cdot 0.54}{0.72+0.54} = \frac{0.54}{1.26} \approx 0.4286.$$

After UD transform with contradiction reset. Since $a(2) = \text{Speed} \in A_\tau(b)$, flip B_2 and reset $c(\text{Speed},\text{Cost})$ to 0:

$$\mu^U_{B_2}(L) = 1-0.8 = 0.2,\quad \mu^U_{B_2}(H) = 1-0.3 = 0.7,\qquad c^U(\text{Speed},\text{Cost}) = 0.$$

Rule 1 is unchanged. Using $\odot^U_0(\alpha,\beta) = \alpha\beta$ for the pair $(\text{Speed},\text{Cost})$:

$$\begin{aligned}
\mu^U_1(L \mid x,b) &= 0.6\cdot 0.2 = 0.12, &\mu^U_2(L \mid x,b) &= 0.5\cdot 0.2 = 0.10,\\
\mu^U_1(H \mid x,b) &= 0.6\cdot 0.9 = 0.54, &\mu^U_2(H \mid x,b) &= 0.5\cdot 0.7 = 0.35.
\end{aligned}$$

Aggregate:

$$\mu^U_{\text{out}}(L \mid x,b) = \max\{0.12,0.10\} = 0.12,\qquad \mu^U_{\text{out}}(H \mid x,b) = \max\{0.54,0.35\} = 0.54.$$

Discrete centroid:

$$y^*_{\text{post}} = \frac{0\cdot 0.12 + 1\cdot 0.54}{0.12+0.54} = \frac{0.54}{0.66} \approx 0.8182.$$

Effect. The UD transform reduces the influence at L (after flipping the contradictory rule) and neutralizes the contradiction with the anchor, yielding a much larger crisp output ($0.4286 \to 0.8182$) under the same input x.

2.3 Plithogenic Fuzzy Expert System

A Plithogenic Fuzzy Expert System integrates fuzzy inference with plithogenic attributes, modeling contradiction degrees for nuanced expert reasoning and decision-making.

Definition 2.11 (Plithogenic Activation/Combination Operator)**.** Let $T : [0,1]^2 \to [0,1]$ be a T-norm, $S : [0,1]^2 \to [0,1]$ an S-norm, and $w : [0,1] \to [0,1]$ a nondecreasing function with $w(0) = 0$ and $w(1) = 1$. For $\alpha,\beta,c \in [0,1]$, define the *plithogenic combination*

$$\alpha \;\odot_c\; \beta \;:=\; \big(1-w(c)\big)\,T(\alpha,\beta) \;+\; w(c)\,S(\alpha,\beta).$$

Then $\odot_c$ continuously interpolates between T and S using the *contradiction degree* c: in particular, $\alpha \odot_0 \beta = T(\alpha,\beta)$ and $\alpha \odot_1 \beta = S(\alpha,\beta)$.

Example 2.12 (Numerical illustration of the Plithogenic Activation/Combination Operator)**.** Let the T-norm and S-norm be

$$T(\alpha,\beta)=\alpha\beta, \qquad S(\alpha,\beta)=\max\{\alpha,\beta\},$$

and choose the interpolation $w(c)=c$ for $c\in[0,1]$. For $\alpha=0.6$ and $\beta=0.3$ we have $T(\alpha,\beta)=0.18$ and $S(\alpha,\beta)=0.6$. Hence, for any contradiction degree c,

$$\alpha\odot_c\beta=(1-c)\,T(\alpha,\beta)+c\,S(\alpha,\beta)=(1-c)\cdot 0.18+c\cdot 0.6.$$

Concrete values:

$$\alpha\odot_0\beta=0.18 \quad (\text{pure } T), \qquad \alpha\odot_{0.25}\beta=0.75\cdot 0.18+0.25\cdot 0.6=0.285,$$

$$\alpha\odot_{0.75}\beta=0.25\cdot 0.18+0.75\cdot 0.6=0.495, \qquad \alpha\odot_1\beta=0.6 \quad (\text{pure } S).$$

Thus $\odot_c$ continuously interpolates between conjunctive (product) and disjunctive (max) behavior as c increases.

Definition 2.13 (Plithogenic Fuzzy Expert System (PFES))**.** Let $X=X_1\times\cdots\times X_n$ be the input space and Y the output universe. Let Pv be a finite set of attribute values and $c: Pv\times Pv\to[0,1]$ a *contradiction map* satisfying

$$c(a,a)=0, \qquad c(a,b)=c(b,a) \quad \text{for all } a,b\in Pv.$$

Fix a fuzzifier Φ, a T-norm T, an S-norm S, an implication/activation $\Rightarrow$, an aggregation operator $\oplus$, a plithogenic operator $\odot_{(\cdot)}$ from Definition 2.11, and a defuzzifier Δ. A *PFES* is a tuple

$$\mathcal{PF}=\big(\Phi,\ \mathcal{K},\ \mathcal{I},\ \Delta,\ Pv,\ c,\ \odot_{(\cdot)}\big),$$

where the knowledge base $\mathcal{K}=(\mathcal{L},\mathcal{R})$ consists of linguistic terms $\mathcal{L}$ (membership functions) and a rule base $\mathcal{R}=\{R_j\}_{j=1}^M$ with rules of the form

$$R_j:\ \textsf{IF}\ x_1 \text{ is } A_{j1}\ \wedge\cdots\wedge\ x_n \text{ is } A_{jn}\ \ \textsf{THEN}\ y \text{ is } B_j \quad \text{tagged with } a(j)\in Pv,$$

where $A_{jk}\subseteq X_k$ and $B_j\subseteq Y$ are fuzzy sets with memberships $\mu_{A_{jk}}: X_k\to[0,1]$ and $\mu_{B_j}: Y\to[0,1]$.

Inference under an anchor attribute. Given a crisp input $x=(x_1,\ldots,x_n)\in X$ and a chosen *anchor* (context) $b\in Pv$, the fuzzifier Φ yields degrees $\{\mu_{A_{jk}}(x_k)\}$ and the rule firing strengths

$$w_j(x)=T\big(\mu_{A_{j1}}(x_1),\ldots,\mu_{A_{jn}}(x_n)\big), \qquad \bar{w}_j(x)=\frac{w_j(x)}{\sum_{i=1}^M w_i(x)}.$$

The plithogenic activation of rule j on $y\in Y$ is

$$\mu_j(y\mid x,b)\ :=\ w_j(x)\ \odot_{c(a(j),\,b)}\ \mu_{B_j}(y),$$

and the aggregated output fuzzy set is

$$\mu_{\text{out}}(y\mid x,b)\ :=\ \bigoplus_{j=1}^{M}\mu_j(y\mid x,b).$$

The crisp output is $f(x;b):=\Delta\big(\mu_{\text{out}}(\cdot\mid x,b)\big)\in Y$.

Example 2.14 (A concrete Plithogenic Fuzzy Expert System with step–by–step inference)**.** Let $X=X_1$ with a single crisp input $x\in X_1$, and $Y=\{0,1\}$ (“reject”, “accept”). Consider the attribute value set $Pv=\{\text{Safety},\text{Cost}\}$ and choose the anchor (context) $b=\text{Safety}$. Define a symmetric contradiction map $c: Pv\times Pv\to[0,1]$ by

$$c(\text{Safety},\text{Safety})=0, \qquad c(\text{Safety},\text{Cost})=c(\text{Cost},\text{Safety})=0.7.$$

Set the operators and modules as follows:

$$T(\alpha,\beta)=\alpha\beta,\quad S(\alpha,\beta)=\max\{\alpha,\beta\},\quad w(c)=c,\quad \oplus=\max,\quad \Phi=\text{singleton fuzzifier}.$$

Use the discrete centroid defuzzifier

$$\Delta(\mu_{\text{out}})=\frac{\sum_{y\in\{0,1\}} y\,\mu_{\text{out}}(y)}{\sum_{y\in\{0,1\}} \mu_{\text{out}}(y)}.$$

The rule base $\mathcal{R}=\{R_1,R_2\}$ is:

$$\begin{aligned}
R_1 &: \mathsf{IF}\ x \text{ is } A_{11}\ \mathsf{THEN}\ y \text{ is } B_1 \quad (\text{tag } a(1)=\text{Safety}),\\
R_2 &: \mathsf{IF}\ x \text{ is } A_{21}\ \mathsf{THEN}\ y \text{ is } B_2 \quad (\text{tag } a(2)=\text{Cost}).
\end{aligned}$$

At the given input x, suppose

$$\mu_{A_{11}}(x)=0.7,\qquad \mu_{A_{21}}(x)=0.6,$$

so the rule firing strengths (with T as product) are

$$w_1=T(\mu_{A_{11}}(x))=0.7,\qquad w_2=T(\mu_{A_{21}}(x))=0.6.$$

Let the consequent fuzzy sets on Y be

$$\mu_{B_1}(0)=0.2,\ \mu_{B_1}(1)=0.8,\qquad \mu_{B_2}(0)=0.6,\ \mu_{B_2}(1)=0.3.$$

Plithogenic activation (Definition 2.11) uses the contradiction between the rule tag and the anchor, i.e. $c(a(1),b)=0$ and $c(a(2),b)=0.7$:

$$\mu_j(y\mid x,b)\ =\ w_j\ \odot_{c(a(j),\,b)}\ \mu_{B_j}(y)\ =\ (1-c)\,T(w_j,\mu_{B_j}(y))\ +\ c\,S(w_j,\mu_{B_j}(y)).$$

For $y=1$ (accept):

$$\mu_1(1\mid x,b)=(1-0)\,T(0.7,0.8)+0\cdot S(0.7,0.8)=0.7\cdot 0.8=0.56,$$

$$\mu_2(1\mid x,b)=(1-0.7)\,T(0.6,0.3)+0.7\,S(0.6,0.3)=0.3\cdot 0.18+0.7\cdot 0.6=0.054+0.42=0.474.$$

Aggregate by $\oplus=\max$:

$$\mu_{\text{out}}(1\mid x,b)=\max\{0.56,\ 0.474\}=0.56.$$

For $y=0$ (reject):

$$\mu_1(0\mid x,b)=(1-0)\,T(0.7,0.2)=0.14,$$

$$\mu_2(0\mid x,b)=(1-0.7)\,T(0.6,0.6)+0.7\,S(0.6,0.6)=0.3\cdot 0.36+0.7\cdot 0.6=0.108+0.42=0.528,$$

hence

$$\mu_{\text{out}}(0\mid x,b)=\max\{0.14,\ 0.528\}=0.528.$$

Finally, defuzzify:

$$f(x;b)=\Delta(\mu_{\text{out}})=\frac{0\cdot 0.528+1\cdot 0.56}{0.528+0.56}=\frac{0.56}{1.088}\approx 0.5147.$$

Thus, the PFES outputs a context–aware score balancing rule evidence via the contradiction–controlled plithogenic operator $\odot_c$.

Remark 2.15 (Well-posedness)**.** Since $T,S\in[0,1]$ and $0\le w(c)\le 1$, one has $0\le \mu_j(y\mid x,b)\le 1$ and $0\le \mu_{\text{out}}(y\mid x,b)\le 1$, so Δ is applicable (e.g., centroid).

Theorem 2.16 (PFES generalizes the classical Fuzzy Expert System)**.** *Let $\mathcal{PF}$ be a PFES as in Definition 2.13 and specialize to*

$$c(a,b)\equiv 0 \textit{ for all } a,b\in Pv,\qquad w(0)=0,$$

so that $\odot_0=T$ by Definition 2.11. Then, for every input x and anchor b,

$$\mu_j(y\mid x,b)\ =\ T\big(w_j(x),\mu_{B_j}(y)\big),\qquad \mu_{\text{out}}(y\mid x,b)=\bigoplus_{j=1}^{M} T\big(w_j(x),\mu_{B_j}(y)\big),$$

which coincides with the standard Mamdani-type fuzzy expert inference. Hence $f(x;b)$ equals the output of a Fuzzy Expert System with the same $(\Phi,T,\Rightarrow,\oplus,\Delta)$; therefore PFES strictly generalizes FES.

Proof. With $c \equiv 0$ and $w(0) = 0$, Definition 2.11 yields

$$\alpha \odot_0 \beta = (1-0)\,T(\alpha,\beta) + 0 \cdot S(\alpha,\beta) = T(\alpha,\beta).$$

Substituting $\alpha = w_j(x)$ and $\beta = \mu_{B_j}(y)$ gives $\mu_j(y \mid x, b) = T(w_j(x), \mu_{B_j}(y))$ and thus the classical Mamdani aggregation. Defuzzification is identical, so the overall mapping matches the usual FES. □

Definition 2.17 (PFES-induced Plithogenic Fuzzy Membership)**.** Let $P := X$ and let v denote the categorical attribute with value set Pv. Define, for $x \in P$ and $a \in Pv$, the *PFES-induced degree of appurtenance*

$$pdf(x,a) \;:=\; \sum_{j:\ a(j)=a} \bar{w}_j(x) \;\in [0,1], \qquad \text{with} \quad \bar{w}_j(x) = \frac{w_j(x)}{\sum_{i=1}^{M} w_i(x)}.$$

Theorem 2.18 (PFES carries a Plithogenic-Fuzzy Set structure)**.** *Fix a PFES $\mathcal{PF}$ with Pv and contradiction map c. Let $PS = (P, v, Pv, pdf, c)$ where $P = X$ and pdf is given by Definition 2.17. Then PS is a Plithogenic-Fuzzy Set, i.e.,*

$$pdf: \; P \times Pv \to [0,1], \qquad c: \; Pv \times Pv \to [0,1],$$

with $c(a,a) = 0$ and $c(a,b) = c(b,a)$ for all $a, b \in Pv$.

Proof. (Range) For each fixed x, $0 \le \bar{w}_j(x) \le 1$ and $\sum_{j=1}^{M} \bar{w}_j(x) = 1$. Hence $0 \le pdf(x,a) \le 1$ for every a as a sum of a subfamily of a probability vector. *(Normalization)* Summing over all $a \in Pv$,

$$\sum_{a \in Pv} pdf(x,a) \;=\; \sum_{a \in Pv} \sum_{j:\ a(j)=a} \bar{w}_j(x) \;=\; \sum_{j=1}^{M} \bar{w}_j(x) \;=\; 1,$$

so the appurtenance degrees across Pv form a proper distribution for each x. *(Contradiction)* By assumption on c, one has $c(a,a) = 0$ and $c(a,b) = c(b,a)$, meeting the usual plithogenic-fuzzy axioms. Therefore (P, v, Pv, pdf, c) is a Plithogenic-Fuzzy Set. □

2.4 Upside–Down Transform with Contradiction Reset in Plithogenic Fuzzy Expert System

Upside–Down Transform with Contradiction Reset in a Plithogenic Fuzzy Expert System flips contradictory fuzzy rule outputs and neutralizes contradictions, ensuring consistent reasoning outcomes.

Definition 2.19 (Upside–Down Transform with Contradiction Reset in PFES)**.** Let a Plithogenic Fuzzy Expert System (PFES) be

$$\mathcal{PF} = \big(\Phi,\ \mathcal{K},\ \mathcal{I},\ \Delta,\ Pv,\ c,\ \odot_{(\cdot)}\big),$$

where $\mathcal{K} = (\mathcal{L}, \mathcal{R})$ and $\mathcal{R} = \{R_j\}_{j=1}^{M}$ with rules

$$R_j: \ \mathsf{IF}\ x_1 \text{ is } A_{j1} \ \wedge \cdots \wedge \ x_n \text{ is } A_{jn} \ \ \mathsf{THEN}\ y \text{ is } B_j,$$

each tagged by an attribute value $a(j) \in Pv$. Here $c : Pv \times Pv \to [0,1]$ is a contradiction map with $c(a,a) = 0$ and $c(a,b) = c(b,a)$, and the plithogenic activation (for $\alpha, \beta, c \in [0,1]$) is

$$\alpha \ \odot_c \ \beta \ := \ \big(1 - w(c)\big)\,T(\alpha,\beta) \ + \ w(c)\,S(\alpha,\beta),$$

for fixed T (T–norm), S (S–norm), and a nondecreasing $w : [0,1] \to [0,1]$ with $w(0) = 0$, $w(1) = 1$.

Fix an *anchor attribute* $b \in Pv$ and a threshold $\tau \in [0,1]$. Define the activation set

$$A_\tau(b) \ := \ \{\, a \in Pv : \ c(a,b) \ge \tau \,\}.$$

The *Upside–Down (UD) transform with contradiction reset* produces a new PFES

$$\mathcal{PF}^{U_{b,\tau}} = \big(\Phi,\ \mathcal{K}^{U_{b,\tau}},\ \mathcal{I}^{U_{b,\tau}},\ \Delta,\ Pv,\ c^{U_{b,\tau}},\ \odot_{(\cdot)}^{U_{b,\tau}}\big)$$

by:

(a) **Consequent flip (Upside–Down):** for each rule R_j,

$$\mu_{B_j}^{U_{b,\tau}}(y) \;:=\; \begin{cases} 1-\mu_{B_j}(y), & \text{if } a(j) \in A_\tau(b), \\ \mu_{B_j}(y), & \text{otherwise}, \end{cases} \qquad \forall\, y \in Y,$$

and set $\mathcal{K}^{U_{b,\tau}} = (\mathcal{L}, \{R_j^{U_{b,\tau}}\}_{j=1}^{M})$ with consequents $B_j^{U_{b,\tau}}$.

(b) **Contradiction reset:** for all $u, v \in Pv$,

$$c^{U_{b,\tau}}(u,v) \;:=\; \begin{cases} 0, & \text{if } \{u,v\} = \{a,b\} \text{ for some } a \in A_\tau(b), \\ c(u,v), & \text{otherwise}. \end{cases}$$

(c) **Plithogenic activation update:** for any $\alpha, \beta \in [0,1]$ and $(a,b) \in Pv \times Pv$,

$$\alpha \;\odot^{U_{b,\tau}}_{c^{U_{b,\tau}}(a,b)}\; \beta \;:=\; \big(1 - w(c^{U_{b,\tau}}(a,b))\big)\, T(\alpha,\beta) \;+\; w(c^{U_{b,\tau}}(a,b))\, S(\alpha,\beta).$$

Inference (unchanged premises). Given $x = (x_1, \ldots, x_n)$, the fuzzifier Φ yields degrees $\mu_{A_{jk}}(x_k)$ and rule firing strengths

$$w_j(x) = T\big(\mu_{A_{j1}}(x_1), \ldots, \mu_{A_{jn}}(x_n)\big), \qquad \bar{w}_j(x) = \frac{w_j(x)}{\sum_{i=1}^{M} w_i(x)}.$$

The transformed rule activations and aggregated output are

$$\mu_j^{U_{b,\tau}}(y \mid x, b) = w_j(x) \;\odot^{U_{b,\tau}}_{c^{U_{b,\tau}}(a(j),\,b)}\; \mu_{B_j}^{U_{b,\tau}}(y), \qquad \mu_{\text{out}}^{U_{b,\tau}}(y \mid x, b) = \bigoplus_{j=1}^{M} \mu_j^{U_{b,\tau}}(y \mid x, b),$$

and the crisp output is $f^{U_{b,\tau}}(x; b) = \Delta\big(\mu_{\text{out}}^{U_{b,\tau}}(\cdot \mid x, b)\big)$.

Remark 2.20. Only consequents whose attributes are sufficiently contradictory to the anchor ($a(j) \in A_\tau(b)$) are flipped. After the flip, the reset $c^{U_{b,\tau}}(a(j), b) = 0$ prevents double penalization and makes their activation conjunctive ($w(c^U) = 0$).

Example 2.21 (Two–Rule Numeric PFES with Discrete Defuzzification)**.** Let $n = 1$, $X \subset \mathbb{R}$, and $Y = \{L, H\} \equiv \{0, 1\}$. Choose $T(\alpha,\beta) = \alpha\beta$, $S(\alpha,\beta) = \max\{\alpha,\beta\}$, $w(c) = c$, and $\oplus = \max$. Let $Pv = \{\text{Cost}, \text{Speed}\}$ with anchor $b = \text{Cost}$ and

$$c(\text{Cost}, \text{Cost}) = 0, \qquad c(\text{Cost}, \text{Speed}) = c(\text{Speed}, \text{Cost}) = 0.8.$$

Set threshold $\tau = 0.7$, hence $A_\tau(b) = \{\text{Speed}\}$.

Rules and memberships.

$$R_1 : \textsf{IF } x \text{ is } A_{11} \;\; \textsf{THEN } y \text{ is } B_1 \quad (a(1) = \text{Cost}),$$
$$R_2 : \textsf{IF } x \text{ is } A_{21} \;\; \textsf{THEN } y \text{ is } B_2 \quad (a(2) = \text{Speed}).$$

For the given x, suppose $\mu_{A_{11}}(x) = 0.6$, $\mu_{A_{21}}(x) = 0.5$, so $w_1 = 0.6$, $w_2 = 0.5$. Let

$$\mu_{B_1}(L) = 0.2,\; \mu_{B_1}(H) = 0.9, \qquad \mu_{B_2}(L) = 0.8,\; \mu_{B_2}(H) = 0.3.$$

Defuzzification uses the discrete centroid

$$y^* \;=\; \frac{\sum_{y\in Y} y\, \mu_{\text{out}}(y)}{\sum_{y\in Y} \mu_{\text{out}}(y)} \;\in [0,1].$$

Before UD transform. Plithogenic activation $\odot_c(\alpha,\beta) = (1-c)\alpha\beta + c\max\{\alpha,\beta\}$ gives

$$\begin{aligned}
&\mu_1(L \mid x,b) = (1-0)\cdot(0.6\cdot 0.2) = 0.12,\\
&\mu_2(L \mid x,b) = (1-0.8)\cdot(0.5\cdot 0.8) + 0.8\cdot\max\{0.5,0.8\} = 0.08+0.64 = 0.72,\\
\Rightarrow\ &\mu_{\text{out}}(L \mid x,b) = \max\{0.12,0.72\} = 0.72;\\
&\mu_1(H \mid x,b) = (1-0)\cdot(0.6\cdot 0.9) = 0.54,\\
&\mu_2(H \mid x,b) = (1-0.8)\cdot(0.5\cdot 0.3) + 0.8\cdot\max\{0.5,0.3\} = 0.03+0.40 = 0.43,\\
\Rightarrow\ &\mu_{\text{out}}(H \mid x,b) = \max\{0.54,0.43\} = 0.54.
\end{aligned}$$

Thus

$$y^*_{\text{pre}} = \frac{0\cdot 0.72 + 1\cdot 0.54}{0.72+0.54} = \frac{0.54}{1.26} \approx 0.4286.$$

After UD transform with contradiction reset. Since $a(2) = \text{Speed} \in A_\tau(b)$, flip B_2 and reset $c(\text{Speed},\text{Cost})$ to 0:

$$\mu^U_{B_2}(L) = 1-0.8 = 0.2, \quad \mu^U_{B_2}(H) = 1-0.3 = 0.7, \qquad c^U(\text{Speed},\text{Cost}) = 0.$$

Then with $\odot^U_0(\alpha,\beta) = \alpha\beta$,

$$\begin{aligned}
\mu^U_1(L \mid x,b) &= 0.6\cdot 0.2 = 0.12, \quad \mu^U_2(L \mid x,b) = 0.5\cdot 0.2 = 0.10,\\
\mu^U_1(H \mid x,b) &= 0.6\cdot 0.9 = 0.54, \quad \mu^U_2(H \mid x,b) = 0.5\cdot 0.7 = 0.35,
\end{aligned}$$

so

$$\mu^U_{\text{out}}(L \mid x,b) = \max\{0.12,0.10\} = 0.12, \qquad \mu^U_{\text{out}}(H \mid x,b) = \max\{0.54,0.35\} = 0.54,$$

and

$$y^*_{\text{post}} = \frac{0\cdot 0.12 + 1\cdot 0.54}{0.12+0.54} = \frac{0.54}{0.66} \approx 0.8182.$$

Effect. Flipping the contradictory rule and neutralizing its contradiction with the anchor shifts the decision markedly toward H $(0.4286 \to 0.8182)$ under the same input.

2.5 Plithogenic Fuzzy Cognitive Maps

Plithogenic Fuzzy Cognitive Maps integrate attribute-specific causal matrices using contradiction-aware weights, yielding context-dependent concept dynamics and iterative updates convergence properties (cf. [59–62]).

Notation 2.22. *We adopt the standard plithogenic ingredients (Pv, c):*

- *Pv is a finite set of* attribute values *(design aspects, criteria, contexts, ...).*
- *$c : Pv \times Pv \to [0,1]$ is a* degree of contradiction function (DCF) *satisfying*

$$c(a,a) = 0, \qquad c(a,b) = c(b,a) \quad \text{for all } a,b \in Pv.$$

Let $\varphi : [0,1] \to (0,\infty)$ be a fixed monotone nonincreasing *attenuation with $\varphi(0) = 1$ (e.g. $\varphi(u) = 1-u$). For an* anchor *(context) $b \in Pv$, define normalized* plithogenic context weights

$$\gamma(a \mid b) \ := \ \frac{\varphi\big(c(a,b)\big)}{\sum_{u\in Pv} \varphi\big(c(u,b)\big)} \in (0,1], \qquad \sum_{a\in Pv} \gamma(a \mid b) = 1. \tag{2}$$

Definition 2.23 (Plithogenic Fuzzy Cognitive Map (PFCM))**.** Fix $n \in \mathbb{N}$, an attribute value set Pv, a DCF c, and an attenuation φ as above. For each $a \in Pv$, let $W^{(a)} = (w_{ij}^{(a)}) \in [-1, 1]^{n\times n}$ be the attribute–specific influence matrix. Let $f : \mathbb{R} \to [0, 1]$ be a nondecreasing transfer function and $\mathbf{a}^{(0)} \in [0, 1]^n$ an initial state.

For an anchor (context) $b \in Pv$, the *effective (contextual) influence matrix* is the convex combination

$$\widehat{W}(b) \;:=\; \sum_{a\in Pv} \gamma(a \mid b)\, W^{(a)}, \qquad \gamma(a \mid b) \text{ as in (7).} \tag{3}$$

The *Plithogenic Fuzzy Cognitive Map* is the system

$$\mathcal{PFCM} = \Big(C, \{W^{(a)}\}_{a\in Pv}, Pv, c, \varphi, f, \mathbf{a}^{(0)}\Big),$$

whose synchronous update under anchor b is

$$\mathbf{a}^{(t+1)} \;=\; f\big(\mathbf{a}^{(t)}\, \widehat{W}(b)\big), \qquad t = 0, 1, 2, \ldots, \tag{4}$$

with f acting componentwise.

Theorem 2.24 (PFCM generalizes classical FCM)**.** *Assume either $|Pv| = 1$ or $c(a, b) \equiv 0$ for all $a, b \in Pv$ and $\varphi \equiv 1$. Then $\gamma(a \mid b) = 1$ for the unique a, so $\widehat{W}(b) = W^{(a)}$ and the PFCM update (9) reduces to the classical FCM update with $W = W^{(a)}$. Hence every FCM is a special case of a PFCM.*

Proof. If $|Pv| = 1$ with value a, then the denominator in (7) equals $\varphi(c(a, b)) = \varphi(0) = 1$ and $\gamma(a \mid b) = 1$, giving $\widehat{W}(b) = W^{(a)}$. If $c \equiv 0$ and $\varphi \equiv 1$, then $\gamma(a \mid b) = 1/|Pv|$ and $W^{(a)}$ can be chosen so that their average equals the single W of the classical FCM (or simply take $|Pv| = 1$). In all cases (9) coincides with the FCM iteration. □

Definition 2.25 (Attribute–weighted row strengths and plithogenic membership)**.** For each concept C_i and attribute $a \in Pv$ define the (nonnegative) absolute row strength

$$S_i(a) \;:=\; \sum_{j=1}^{n} \big|\, w_{ij}^{(a)} \,\big|.$$

Fix an anchor $b \in Pv$. Set

$$pdf(i, a \mid b) \;:=\; \frac{\varphi\big(c(a, b)\big)\, S_i(a)}{\displaystyle\sum_{u\in Pv} \varphi\big(c(u, b)\big)\, S_i(u)} \;\in [0, 1], \tag{5}$$

whenever the denominator is nonzero; if it is zero (all row strengths vanish), set $pdf(i, a \mid b) = \frac{1}{|Pv|}$ for all a.

Theorem 2.26 (PFCM induces a Plithogenic–Fuzzy Set)**.** *Let $P := C$ and v be the categorical attribute with value set Pv. For any anchor $b \in Pv$, the pair*

$$PS_b = \big(P,\ v,\ Pv,\ pdf(\,\cdot\,, \cdot \mid b),\ c\big)$$

is a Plithogenic–Fuzzy Set: $pdf(\cdot, \cdot \mid b) : P \times Pv \to [0, 1]$ satisfies $\sum_{a\in Pv} pdf(i, a \mid b) = 1$ for every i, and c is reflexive and symmetric.

Proof. By construction $S_i(a) \geq 0$ and $\varphi\big(c(a, b)\big) > 0$, so the numerator and denominator of (5) are nonnegative; if the denominator is positive then $pdf(i, a \mid b) \in [0, 1]$ and

$$\sum_{a\in Pv} pdf(i, a \mid b) = \frac{\sum_a \varphi(c(a, b))S_i(a)}{\sum_u \varphi(c(u, b))S_i(u)} = 1.$$

If the denominator is zero we defined a uniform distribution. The DCF c has $c(a, a) = 0$ and $c(a, b) = c(b, a)$ by assumption. Thus the quintuple satisfies the axioms of a plithogenic–fuzzy set. □

Example 2.27. Let $n = 3$ with concepts $C_1 = \text{Marketing}$, $C_2 = \text{Sales}$, $C_3 = \text{Satisfaction}$. Take two attribute values

$$Pv = \{\text{Cost, Speed}\}, \qquad c(\text{Cost}, \text{Cost}) = c(\text{Speed}, \text{Speed}) = 0, \quad c(\text{Cost}, \text{Speed}) = 0.8,$$

and $\varphi(u) = 1 - u$. Choose anchor $b = \text{Cost}$; then

$$\varphi\big(c(\text{Cost}, \text{Cost})\big) = 1, \qquad \varphi\big(c(\text{Speed}, \text{Cost})\big) = 1 - 0.8 = 0.2.$$

Hence, using (7) with denominator $1 + 0.2 = 1.2 = 6/5$,

$$\gamma(\text{Cost} \mid b) = \frac{1}{1.2} = \frac{5}{6}, \qquad \gamma(\text{Speed} \mid b) = \frac{0.2}{1.2} = \frac{1}{6}.$$

Attribute–specific influence matrices:

$$W^{(\text{Cost})} = \begin{pmatrix} 0 & 0.5 & 0.1 \\ -0.1 & 0 & 0.4 \\ 0.05 & 0.2 & 0 \end{pmatrix}, \qquad W^{(\text{Speed})} = \begin{pmatrix} 0 & 0.9 & 0.3 \\ -0.3 & 0 & 0.6 \\ 0.1 & 0.4 & 0 \end{pmatrix}.$$

By (8),

$$\widehat{W}(b) = \frac{5}{6} W^{(\text{Cost})} + \frac{1}{6} W^{(\text{Speed})} = \begin{pmatrix} 0 & \frac{17}{30} & \frac{2}{15} \\ -\frac{2}{15} & 0 & \frac{13}{30} \\ \frac{7}{120} & \frac{7}{30} & 0 \end{pmatrix},$$

where each entry is computed exactly, e.g.

$$\widehat{w}_{12} = \tfrac{5}{6} \cdot 0.5 + \tfrac{1}{6} \cdot 0.9 = \tfrac{5}{12} + \tfrac{9}{60} = \tfrac{25+9}{60} = \tfrac{34}{60} = \tfrac{17}{30}.$$

Choose the clipped linear transfer $f(x) = \min\{1, \max\{0, x\}\}$ and initial state

$$\mathbf{a}^{(0)} = (0.6,\ 0.4,\ 0.5) = \left(\tfrac{3}{5},\ \tfrac{2}{5},\ \tfrac{1}{2}\right).$$

Compute $\mathbf{a}^{(1)} = f\big(\mathbf{a}^{(0)}\,\widehat{W}(b)\big)$ coordinatewise:

$$(\mathbf{a}^{(0)}\widehat{W})_1 = 0.6 \cdot 0 + 0.4 \cdot \left(-\tfrac{2}{15}\right) + 0.5 \cdot \left(\tfrac{7}{120}\right) = -\tfrac{4}{75} + \tfrac{7}{240} = -\tfrac{64}{1200} + \tfrac{35}{1200} = -\tfrac{29}{1200} \approx -0.02417 \xrightarrow{f} 0,$$

$$(\mathbf{a}^{(0)}\widehat{W})_2 = 0.6 \cdot \tfrac{17}{30} + 0.4 \cdot 0 + 0.5 \cdot \tfrac{7}{30} = \tfrac{17}{50} + \tfrac{7}{60} = \tfrac{102+35}{300} = \tfrac{137}{300} \approx 0.45667,$$

$$(\mathbf{a}^{(0)}\widehat{W})_3 = 0.6 \cdot \tfrac{2}{15} + 0.4 \cdot \tfrac{13}{30} + 0.5 \cdot 0 = \tfrac{2}{25} + \tfrac{13}{75} = \tfrac{6+13}{75} = \tfrac{19}{75} \approx 0.25333.$$

Thus

$$\mathbf{a}^{(1)} = \left(0,\ \tfrac{137}{300},\ \tfrac{19}{75}\right) \approx (0,\ 0.4567,\ 0.2533)\ .$$

Finally, extract the plithogenic–fuzzy membership for node C_1 under anchor $b = \text{Cost}$ via (5). Absolute row strengths for $i = 1$:

$$S_1(\text{Cost}) = |0| + |0.5| + |0.1| = 0.6, \qquad S_1(\text{Speed}) = |0| + |0.9| + |0.3| = 1.2.$$

Weight by φ:

$$\varphi(c(\text{Cost}, b))S_1(\text{Cost}) = 1 \cdot 0.6 = 0.6, \quad \varphi(c(\text{Speed}, b))S_1(\text{Speed}) = 0.2 \cdot 1.2 = 0.24,$$

denominator $= 0.84 = \frac{21}{25}$. Hence

$$pdf(1, \text{Cost} \mid b) = \frac{0.6}{0.84} = \frac{60}{84} = \frac{5}{7} \approx 0.714285\ , \qquad pdf(1, \text{Speed} \mid b) = \frac{0.24}{0.84} = \frac{24}{84} = \frac{2}{7} \approx 0.285714\ .$$

One checks $pdf(1, \text{Cost} \mid b) + pdf(1, \text{Speed} \mid b) = 1$, as guaranteed by Theorem 2.26.

2.6 Upside–Down Transform with Contradiction Reset in Plithogenic Fuzzy Cognitive Maps

The transform flips highly contradictory attribute matrices, resets anchor contradictions to zero, and recomputes context-weighted influences preserving overall update stability.

Notation 2.28 (Notation (Recall))**.** *Let $n \in \mathbb{N}$ and $C = \{C_1, \dots, C_n\}$ be the concept set. Fix a finite set of* attribute values *Pv and a* degree of contradiction function *$c : Pv \times Pv \to [0, 1]$ with*

$$c(a, a) = 0, \qquad c(a, b) = c(b, a) \quad \text{for all } a, b \in Pv. \tag{6}$$

For each $a \in Pv$ let $W^{(a)} = (w_{ij}^{(a)}) \in [-1, 1]^{n\times n}$ be the attribute–specific influence matrix. Let $f : \mathbb{R} \to [0, 1]$ be a nondecreasing transfer (squashing) function applied componentwise and let $\mathbf{a}^{(0)} \in [0, 1]^n$ be an initial state.

Fix a monotone nonincreasing *attenuation $\varphi : [0, 1] \to (0, \infty)$ with $\varphi(0) = 1$ (e.g. $\varphi(u) = 1 - u$). For an* anchor (context) *$b \in Pv$ define normalized* context weights

$$\gamma(a \mid b) \;:=\; \frac{\varphi\big(c(a, b)\big)}{\sum\limits_{u\in Pv} \varphi\big(c(u, b)\big)} \in (0, 1], \qquad \sum_{a\in Pv} \gamma(a \mid b) = 1. \tag{7}$$

The effective influence matrix *in context b is*

$$\widehat{W}(b) \;:=\; \sum_{a\in Pv} \gamma(a \mid b)\, W^{(a)} \;\in [-1, 1]^{n\times n}, \tag{8}$$

and the synchronous update is

$$\mathbf{a}^{(t+1)} \;=\; f\big(\mathbf{a}^{(t)}\, \widehat{W}(b)\big), \qquad t = 0, 1, 2, \dots . \tag{9}$$

We call the tuple $\mathcal{PFCM} = (C, \{W^{(a)}\}_{a\in Pv}, Pv, c, \varphi, f, \mathbf{a}^{(0)})$ a Plithogenic Fuzzy Cognitive Map *(PFCM).*

Definition 2.29 (UD + CR on a PFCM)**.** Let $\mathcal{PFCM}$ be as above, fix an anchor $b \in Pv$ and a threshold $\tau \in [0, 1]$. Define the *activation set*

$$A_\tau(b) \;:=\; \{\, a \in Pv \;:\; c(a, b) \geq \tau \,\}.$$

The *Upside–Down transform with contradiction reset $U_{b,\tau}$* produces the transformed system

$$\mathcal{PFCM}^{U_{b,\tau}} = (C, \{W^{(a),U}\}_{a\in Pv}, Pv, c^U, \varphi, f, \mathbf{a}^{(0)})$$

by

$$\text{(flip of contradictory attributes)} \quad W^{(a),U} \;:=\; \begin{cases} -\,W^{(a)}, & a \in A_\tau(b), \\ W^{(a)}, & a \notin A_\tau(b), \end{cases} \tag{10}$$

$$\text{(contradiction reset)} \quad c^U(u, v) \;:=\; \begin{cases} 0, & \{u, v\} = \{a, b\} \text{ for some } a \in A_\tau(b), \\ c(u, v), & \text{otherwise.} \end{cases} \tag{11}$$

Using c^U in (7) yields new weights

$$\gamma^U(a \mid b) \;:=\; \frac{\varphi\big(c^U(a, b)\big)}{\sum_{u\in Pv} \varphi\big(c^U(u, b)\big)} \in (0, 1],$$

and the new effective matrix and update law are

$$\widehat{W}^U(b) \;:=\; \sum_{a\in Pv} \gamma^U(a \mid b)\, W^{(a),U}, \qquad \mathbf{a}^{(t+1)} \;=\; f\big(\mathbf{a}^{(t)}\, \widehat{W}^U(b)\big). \tag{12}$$

Remark 2.30 (Well–posedness)**.** Since each $W^{(a),U} \in [-1,1]^{n\times n}$ and $\gamma^U(\cdot \mid b)$ is a convex weight vector, $\widehat{W}^U(b) \in [-1,1]^{n\times n}$ and (12) is well defined. Because f maps $\mathbb{R}$ into $[0,1]$ componentwise, every iterate $\mathbf{a}^{(t)} \in [0,1]^n$ both before and after $U_{b,\tau}$.

Definition 2.31 (Edge–sign lemmas and valuations)**.** Fix $b \in Pv$. For each $a \in Pv$ and ordered pair (i,j) define the lemma

$$L^{(a)}_{ij,+} \;:=\; \text{“the contribution of attribute } a \text{ to the effective edge } i\to j \text{ in context } b \text{ is nonnegative”}.$$

Let v be the valuation

$$v\big(L^{(a)}_{ij,+}\big) \;=\; \begin{cases} \text{True}, & \gamma(a \mid b)\, w^{(a)}_{ij} \geq 0 \;\text{ and } w^{(a)}_{ij} \neq 0,\\ \text{False}, & \gamma(a \mid b)\, w^{(a)}_{ij} < 0,\\ \text{Indeterminate}, & w^{(a)}_{ij} = 0. \end{cases}$$

After applying $U_{b,\tau}$, evaluate the same lemmas in the transformed system using the valuation v^U:

$$v^U\big(L^{(a)}_{ij,+}\big) \;=\; \begin{cases} \text{True}, & \gamma^U(a \mid b)\, w^{(a),U}_{ij} \geq 0 \;\text{ and } w^{(a),U}_{ij} \neq 0,\\ \text{False}, & \gamma^U(a \mid b)\, w^{(a),U}_{ij} < 0,\\ \text{Indeterminate}, & w^{(a),U}_{ij} = 0. \end{cases}$$

Theorem 2.32 (UD + CR endows PFCM with an Upside–Down Logic)**.** *Let $U_{b,\tau}$ be as in Definition 2.29 and consider the logical system $\mathcal{M} = (\mathcal{P}, \{\text{True}, \text{False}, \text{Indeterminate}\}, v)$ with $\mathcal{P} = \{L^{(a)}_{ij,+}\}$. Define the transformed system $\mathcal{M}' = (\mathcal{P}, \{\text{True}, \text{False}, \text{Indeterminate}\}, v^U)$. Then, for every $a \in A_\tau(b)$, $i,j \in \{1,\dots,n\}$ with $w^{(a)}_{ij} \neq 0$, one has*

$$v\big(L^{(a)}_{ij,+}\big) = \text{True} \;\Rightarrow\; v^U\big(L^{(a)}_{ij,+}\big) = \text{False}, \qquad v\big(L^{(a)}_{ij,+}\big) = \text{False} \;\Rightarrow\; v^U\big(L^{(a)}_{ij,+}\big) = \text{True}. \tag{13}$$

In words: within the activated attribute set $A_\tau(b)$, the UD + CR map falsifies truths and truthifies falsities*—hence realizes an Upside–Down Logic on these lemmas.*

Proof. Fix $a \in A_\tau(b)$ and indices i,j with $w^{(a)}_{ij} \neq 0$. By (10), $w^{(a),U}_{ij} = -w^{(a)}_{ij}$. By (11), $c^U(a,b) = 0$, so $\varphi(c^U(a,b)) = \varphi(0) = 1$. Hence $\gamma^U(a \mid b) = \dfrac{1}{\sum_{u\in Pv} \varphi\big(c^U(u,b)\big)} \in (0,1]$, in particular $\gamma^U(a \mid b) > 0$. Likewise $\gamma(a \mid b) > 0$ by (7). Therefore the signs satisfy

$$\operatorname{sgn}\big(\gamma^U(a \mid b)\, w^{(a),U}_{ij}\big) = \operatorname{sgn}\big(\gamma^U(a \mid b)\big)\, \operatorname{sgn}\big(-w^{(a)}_{ij}\big) = -\operatorname{sgn}\big(w^{(a)}_{ij}\big),$$

while

$$\operatorname{sgn}\big(\gamma(a \mid b)\, w^{(a)}_{ij}\big) = \operatorname{sgn}\big(w^{(a)}_{ij}\big),$$

since both $\gamma(\cdot \mid b)$ and $\gamma^U(\cdot \mid b)$ are strictly positive. Consequently,

$$\gamma(a \mid b)\, w^{(a)}_{ij} \geq 0 \iff \gamma^U(a \mid b)\, w^{(a),U}_{ij} \leq 0,$$

with strict inequalities because $w^{(a)}_{ij} \neq 0$. Hence the truth value of $L^{(a)}_{ij,+}$ flips between True and False under $U_{b,\tau}$, establishing (13). If $w^{(a)}_{ij} = 0$, then $w^{(a),U}_{ij} = 0$ and both valuations yield *Indeterminate*, which is consistent with the statement. □

Proposition 2.33 (Effect on the effective matrix entries)**.** *For any $a \in A_\tau(b)$ and any (i,j), if $w^{(a)}_{ij} \neq 0$ then the* attribute–level contribution *to the effective entry flips sign:*

$$\gamma(a \mid b)\, w^{(a)}_{ij} \quad \text{and} \quad \gamma^U(a \mid b)\, w^{(a),U}_{ij} = -\,\gamma^U(a \mid b)\, w^{(a)}_{ij}$$

have opposite signs. Consequently, the transformed effective entry satisfies

$$\widehat{w}^U_{ij}(b) = \sum_{u\notin A_\tau(b)} \gamma^U(u \mid b)\, w^{(u)}_{ij} \;-\; \sum_{u\in A_\tau(b)} \gamma^U(u \mid b)\, w^{(u)}_{ij},$$

which differs from $\widehat{w}_{ij}(b) = \sum_u \gamma(u \mid b)\, w^{(u)}_{ij}$ by the sign inversion of all activated–attribute contributions together with a reweighting caused by the reset c^U.

Proof. The displayed identities follow directly from (10), (11), and the definitions of $\widehat{W}(b)$ and $\widehat{W}^U(b)$. The sign claim is identical to the argument in the proof of Theorem 2.32. ☐

Example 2.34. Let $n = 3$ and $Pv = \{\text{Cost}, \text{Speed}\}$. Take

$$c(\text{Cost}, \text{Cost}) = c(\text{Speed}, \text{Speed}) = 0, \qquad c(\text{Cost}, \text{Speed}) = c(\text{Speed}, \text{Cost}) = 0.8,$$

choose $\varphi(u) = 1 - u$, anchor $b = \text{Cost}$, and threshold $\tau = 0.7$. Then $A_\tau(b) = \{\text{Speed}\}$, and

$$\varphi\big(c(\text{Cost}, b)\big) = 1, \quad \varphi\big(c(\text{Speed}, b)\big) = 1 - 0.8 = 0.2.$$

Hence, before UD,

$$\gamma(\text{Cost} \mid b) = \frac{1}{1 + 0.2} = \frac{5}{6}, \qquad \gamma(\text{Speed} \mid b) = \frac{0.2}{1.2} = \frac{1}{6}.$$

Attribute matrices (all entries exact):

$$W^{(\text{Cost})} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{10} \\ -\frac{1}{10} & 0 & \frac{2}{5} \\ \frac{1}{20} & \frac{1}{5} & 0 \end{pmatrix}, \qquad W^{(\text{Speed})} = \begin{pmatrix} 0 & \frac{9}{10} & \frac{3}{10} \\ -\frac{3}{10} & 0 & \frac{3}{5} \\ \frac{1}{10} & \frac{2}{5} & 0 \end{pmatrix}.$$

Effective matrix before UD:

$$\widehat{W}(b) = \tfrac{5}{6}\, W^{(\text{Cost})} + \tfrac{1}{6}\, W^{(\text{Speed})} = \begin{pmatrix} 0 & \frac{17}{30} & \frac{2}{15} \\ -\frac{2}{15} & 0 & \frac{13}{30} \\ \frac{7}{120} & \frac{7}{30} & 0 \end{pmatrix}.$$

Apply $U_{b,\tau}$. Since $A_\tau(b) = \{\text{Speed}\}$, we flip $W^{(\text{Speed})}$ and reset $c^U(\text{Speed}, b) = 0$. Thus $\varphi\big(c^U(\text{Cost}, b)\big) = \varphi\big(c^U(\text{Speed}, b)\big) = 1$ and

$$\gamma^U(\text{Cost} \mid b) = \gamma^U(\text{Speed} \mid b) = \tfrac{1}{2}, \qquad W^{(\text{Speed}),U} = -\,W^{(\text{Speed})}.$$

Hence the *transformed* effective matrix is

$$\widehat{W}^U(b) = \tfrac{1}{2}\, W^{(\text{Cost})} + \tfrac{1}{2}\,\big(-\,W^{(\text{Speed})}\big).$$

For example, the (1, 2)–entry changes from

$$\widehat{w}_{12}(b) = \tfrac{5}{6} \cdot \tfrac{1}{2} + \tfrac{1}{6} \cdot \tfrac{9}{10} = \tfrac{5}{12} + \tfrac{3}{20} = \tfrac{25+18}{120} = \tfrac{43}{120} \approx 0.3583$$

to

$$\widehat{w}^U_{12}(b) = \tfrac{1}{2} \cdot \tfrac{1}{2} + \tfrac{1}{2} \cdot \big(-\tfrac{9}{10}\big) = \tfrac{1}{4} - \tfrac{9}{20} = -\tfrac{1}{5} = -0.2,$$

i.e. its sign is inverted. This realizes Theorem 2.32 concretely for the lemma $L^{(\text{Speed})}_{12,+}$:

$$v\big(L^{(\text{Speed})}_{12,+}\big) = \text{True} \quad \longrightarrow \quad v^U\big(L^{(\text{Speed})}_{12,+}\big) = \text{False}.$$

(The strict flip holds for any edge where $w^{(\text{Speed})}_{ij} \neq 0$.)

3 Conclusion

In this paper, we examined the extensions of these frameworks into the Plithogenic Set domain, namely the Plithogenic Fuzzy Expert System and the Neuro–Plithogenic–Fuzzy System. In future work, we expect research to progress on further extensions employing Neutrosophic Sets [52, 53], Intuitionistic Fuzzy Sets [50], Graphs [63], HyperGraphs [64, 65], and SuperHyperGraphs [48, 66, 67].

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Data Availability

Since this research is purely theoretical and mathematical, no empirical data or computational analysis was utilized. Researchers are encouraged to expand upon these findings with data-oriented or experimental approaches in future studies.

Ethical Statement

As this study does not involve experiments with human participants or animals, no ethical approval was required.

Conflicts of Interest

The authors declare that they have no conflicts of interest related to the content or publication of this paper.

Code Availability

No code or software was developed for this study.

Clinical Trial

This study did not involve any clinical trials.

Consent to Participate

Not applicable.

Use of Generative AI and AI-Assisted Tools

I use generative AI and AI-assisted tools for tasks such as English grammar checking, and I do not employ them in any way that violates ethical standards.

Disclaimer

This work presents theoretical ideas and frameworks that have not yet been empirically validated. Readers are encouraged to explore practical applications and further refine these concepts. Although care has been taken to ensure accuracy and appropriate citations, any errors or oversights are unintentional. The perspectives and interpretations expressed herein are solely those of the authors and do not necessarily reflect the viewpoints of their affiliated institutions.

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