The Value of Two-Person Additively-Separable Sum Games

By

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Abstract

In this paper, introduce the concept of the value of a two-person additively-separable sum game and note that the set of equilibria of a two-person additively-separable sum game is equal to the set of equilibria of a closely related two-person zero-sum game. Results concerning the value of a two-person additively-separable sum game are very similar to the results for the value of a two-person zero-sum game.

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1. In Lahiri (2025) a sub-class of bi-matrix games called two-person (hereafter referred to as two-player) additively-separable sum (TPASS) games is introduced. In such games, the sum of the pay-offs of the two players is additively separable. The row player's pay-off at each pair of pure strategies, is the sum of two numbers, the first of which may be dependent on the pure strategy chosen by the column player and the second being independent of the pure strategy chosen by the column player. The column player's pay-off at each pair of pure strategies, is also the sum of two numbers, the first of which may be dependent on the pure strategy chosen by the row player and the second being independent of the pure strategy chosen by the row player. The sum of the inter-dependent components of the pay-offs of the two players is assumed to be zero. In the same paper, there is a proof of the existence of equilibrium for such games and a proof of the result that the set of equilibria for such games is the projection on the set of strategy pairs of the solutions of a pair of linear programming problems that are dual to each other. This result is a generalization of the corresponding and well-known result for two-person zero-sum games. It is also shown there that a (randomized or mixed) strategy pair is an equilibrium of the game if and only if there exist two other real numbers such that the three together solve a certain linear programming problem. In order to prove this result, we need to appeal to the existence of an equilibrium for the TPASS game.

In this paper, introduce the concept of the value of a TPASS sum game and note that the set of equilibria of a TPASS game is equal to the set of equilibria of a closely related two-person zero-sum game. Results concerning the value of TPASS games are very similar to the results for the value of two-person zero-sum games.

2. Consider an ordered triplet (A, π, ρ) where for some positive integers m and n, A is an m×n real valued matrix, π is an m-dimensional real-valued column vector and ρ is an n-dimensional real-valued column vector. For $(i, j) \in \{1, \dots, m\} \times \{1, \dots, n\}$, let π_i denote the i^{th} coordinate of π, ρ_j denote the j^{th} coordinate of ρ, and let a_{ij} denote the entry at the intersection of the i^{th} row and j^{th} column of A.

There are two players in this game (i.e., interactive decision-making problem)- the row player and the column player.

For $(i, j) \in \{1, \dots, m\} \times \{1, \dots, n\}$, if the row player chooses the i^{th} row and the column player chooses column j, then the payoff to the row player is $a_{ij} + \pi_i$ and the payoff to the column player is $- a_{ij} + \rho_j$.

Note that the sum of the pay-offs to the row player and the column player if the row player chooses the i^{th} row and the column player chooses j^{th} column is $\pi_i + \rho_j$.

We refer to the triplet (A, π, ρ) as a **two-person additively-separable sum** (TPASS) game.

A TPASS game (A, π, ρ) is said to be a **two-person zero-sum (TPZS) game** if $\pi = \rho = 0$.

A TPZS game (A, 0, 0) is simply represented by the matrix A.

A timelessly reliable source of information on TPZS games is Ferguson (undated).

We allow for randomized (mixed) strategies for the row and column players.

For any non-negative integer ℓ, let $\Delta^{\ell} = \{x \in \mathbb{R}_{+}^{\ell+1} | \sum_{k=1}^{\ell+1} x_k = 1\}$. We will interpret points in Δ^{ℓ} as ℓ-dimensional column vectors.

The (randomized or mixed) strategy set for the row player is Δ^{m-1} and the (randomized or mixed) strategy set for the column player is Δ^{n-1}.

A pair $(p, q) \in \Delta^{m-1} \times \Delta^{n-1}$ is a (**randomized or mixed**) **strategy pair**.

The pay-off function for the row-player is the function $f^R: \Delta^{m-1} \times \Delta^{n-1} \rightarrow \mathbb{R}$ such that for all $(p, q) \in \Delta^{m-1} \times \Delta^{n-1}$, $f^R(p, q) = p^T A q + p^T \pi$.

The pay-off function for the column-player is the function $f^C: \Delta^{m-1} \times \Delta^{n-1} \rightarrow \mathbb{R}$ such that for all $(p, q) \in \Delta^{m-1} \times \Delta^{n-1}$, $f^C(p, q) = -p^T A q + \rho^T q$.

The following concept is available in Nash (1951).

$(p^*, q^*) \in \Delta^{m-1} \times \Delta^{n-1}$ is said to be **an equilibrium** of the TPASS game (A, π) if for all $(p, q) \in \Delta^{m-1} \times \Delta^{n-1}$: $f^R(p^*, q^*) \geq f^R(p, q^*)$ and $f^C(p^*, q^*) \geq f^C(p^*, q)$.

Let $R(\pi)$ be the $m\times n$ real matrix, such that for all $i\in\{1, \ldots, m\}$, every entry in the i^{th} row of $R(\pi)$ is π_i.

Let $C(\rho)$ be the $m\times n$ real matrix, such that for all $j\in\{1, \ldots, n\}$, every entry in the j^{th} column of $C(\rho)$ is ρ_j.

Thus, for all $(p, q)\in \Delta^{m-1}\times\Delta^{n-1}$, $p^T\pi = p^TR(\pi)q$ and $\rho^Tq = p^TC(\rho)q$.

For any positive integer ℓ, let $e^{(\ell)}$ denote the ℓ-dimensional sum column vector, i.e., the ℓ-dimensional vector, all coordinates of which are 1.

3. The main result in Lahiri (2025) is the following.

Theorem 1: Let (A, π, ρ) be a TPASS game.

(i) (A, π, ρ) has an equilibrium.

(ii) (p^*, q^*) is an equilibrium for the TPASS game (A, π, ρ) if and only if there exist real numbers α^*, β^* such that q^*, α^* solve [Maximize $\rho^Tq - \alpha$, subject to $Aq - \alpha e^{(m)} \leq -\pi$, $e^{(n)T}q = 1$, $q\in \mathbb{R}^n_+$, $\alpha\in \mathbb{R}$] and p^*, β^* solve its dual [Minimize $-\pi^Tp + \beta$, subject to $p^TA + \beta e^{(n)T} \geq \rho^T$, $-p^Te^{(m)} = -1$, $p\in \mathbb{R}^m_+$, $\beta\in \mathbb{R}$]. Further, p^*, q^*, α^*, β^* satisfies $\alpha^* = p^{*T}Aq^* + p^{*T}\pi$ and $\beta^* = - p^{*T}Aq^* + \rho^Tq^*$.

Based on theorem 1, we can now introduce the concept of a value for a TPASS game.

The **value** of a TPASS game (A, π, ρ), denoted by $V(A, \pi, \rho)$ is equal to the optimal value of the linear programming problem [Minimize $\alpha - \rho^Tq$, subject to $Aq - \alpha e^{(m)} \leq -\pi$, $e^{(n)T}q = 1$, $q\in \mathbb{R}^n_+$, $\alpha\in \mathbb{R}$], i.e., $V(A, \pi, \rho) = \alpha^* - \rho^Tq^*$, where q^*, α^* solve [Maximize $\rho^Tq - \alpha$, subject to $Aq - \alpha e^{(m)} \leq -\pi$, $e^{(n)T}q = 1$, $q\in \mathbb{R}^n_+$, $\alpha\in \mathbb{R}$].

For a TPZS game A, instead of denoting the value as $V(A, 0, 0)$, it is customary to denote it by $V(A)$.

It is easy to see that (p^*, q^*) is an equilibrium for the TPASS game (A, π, ρ) if and only if (p^*, q^*) is an equilibrium for the TPZS game $A + R(\pi) - C(\rho)$. Further, $V(A, \pi, \rho)$ is equal to "the value of the TPZS game $A + R(\pi) - C(\rho)$", i.e., $V(A, \pi, \rho) = V(A + R(\pi) - C(\rho))$.

A good source of information about the value of a TPZS game is chapter 3 (and in particular theorem 3.1) of Ferguson (undated).

Hence, $V(A, \pi, \rho) = \max_{p\epsilon\Delta^{m-1}} \min_{q\epsilon\Delta^{n-1}} (p^TAq + p^T\pi - \rho^Tq) = \min_{q\epsilon\Delta^{n-1}} \max_{p\epsilon\Delta^{m-1}} (p^TAq + p^T\pi - \rho^Tq)$.

Thus, we have the following result.

Theorem 2: Let (A, π, ρ) be a TPASS game.

(i) (A, π, ρ) has an equilibrium.

(ii) (p^*, q^*) is an equilibrium for the TPASS game (A, π, ρ) if and only if (p^*, q^*) is an equilibrium for the TPZS game $A + R(\pi) - C(\rho)$.

(iii) Both $\max_{p\in\Delta^{m-1}} \min_{q\epsilon\Delta^{n-1}} (p^T Aq + p^T\pi - \rho^T q)$ and $\min_{q\in\Delta^{n-1}} \max_{p\epsilon\Delta^{m-1}} (p^T Aq + p^T\pi - \rho^T q)$ exist and are equal to each other, their common value being denoted by $V(A, \pi, \rho)$.

(iv) If (p^*, q^*) is an equilibrium for the TPASS game (A, π, ρ), then $p^{*T}Aq^* + p^{*T}\pi - \rho^T q^* = V(A, \pi, \rho)$.

Note: The converse of (iv) in theorem 2 is not necessarily true. Consider the 2×2 TPZS game $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Clearly, the value of the game is zero, with $\left(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\right)$ being an equilibrium. However, $(p, q) = \begin{pmatrix}\frac{1}{2}\\ \frac{1}{2}\end{pmatrix}, \begin{pmatrix}\frac{1}{2}\\ \frac{1}{2}\end{pmatrix}$ is a strategy pair with $p^TAq = 0$, and yet $\begin{pmatrix}1\\0\end{pmatrix}^T Aq = \frac{1}{2} > 0 = p^TAq$.

Thus, (p, q) is not an equilibrium.

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