MONTY-HALL THEOREM
:
BAYES-PRICE RULE (BAYES THEOREM) FOR A THREE PARAMETER EVENT SPACE

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ABSTRACT

This research report presents the statement of the Monty-Hall Theorem and provides a constructive proof by solving the classical Monty-Hall Problem. It establishes the fact that the probability of winning the prize is indeed unaffected by a switched choice – very much unlike the most prevalent and widely accepted position held by the Leading Subject-Matter-Experts.

Keywords: Monty-Hall Theorem; Bayes-Price Rule; Bayes Theorem

AMS MSC Mathematics Subject Classification: 60A99; 60C99; 62A99; 62C99.

1. INTRODUCTION

The classical “Monty-Hall Problem”, also referred to as the “Three-Door Problem” is based on a game show “Let’s Make a Deal” wherein the host reveals a losing choice to the guest, who had earlier made an initial choice, and in turn offers the guest an enticing option to switch from the initial choice to a second available choice with an aim to enhance the chances of winning the prize. The most prevalent & widely accepted position, as reported in literature, among the leading Subject-Matter-Experts, mathematicians, statisticians, logicians, and rational intellectuals, is that an appropriate detailed study & analysis of the scenario using the well accepted standard approach of Probability & Statistics, would lead to a recommendation to the guest to switch to the second available choice based on the knowledge obtained from the host revealing a losing choice.

We present the statement of our newly formulated Monty-Hall Theorem, and provide a constructive proof by solving the classical Monty-Hall Problem. It establishes that the probability of the guest winning the prize is indeed 1/2 irrespective of whether the guest stays with the initial choice or goes for a switched choice after gathering the information from the host who reveals a losing choice.

2. PROBLEM DESCRIPTION - INPUT DATA

Let us consider the so-called classical Monty-Hall Problem as reported widely in the literature - with a prize hidden behind one of the three doors; a guest making an initial choice of a door to claim the prize; the host who knows the location of the prize as well as the initial choice made by the guest, now *reveals a distinctly different and yet a losing choice*, by opening a third door. Then the host also offers the guest, an option to switch from the initial choice to the now available second choice, anticipating an enhanced chance of winning the prize, based on the knowledge obtained about a losing choice.

Let us represent the events/actions associated with the three doors:
(1) let xr $\in$ {1,2,3} be the door r behind which the prize x is hidden;
(2) let yp $\in$ {1,2,3} be the initial choice of the door p chosen by the guest y;
(3) let zq $\in$ {1,2,3} be the door q opened by the host z to reveal a losing choice.

Let the symbol ‘ai’ denote the event/action [E{(a=i)}] for any ‘agent’ a $\in$ {x,y,z} and ‘door’ i$\in${r,p,q}={1,2,3}.

It is essential to note here that xr and yp are mutually independent of each other as well as independent of zq; whereas zq itself is dependent on both xr and yp, as per the rules of the game. Also, note that the focus must be on the decision-making process & the action to be taken by the guest. So, the *problem formulation* (*modelling*) must necessarily be from the view-point of the guest.

3. ASSUMPTIONS

It is assumed that the prize is hidden randomly behind one of the three doors, each of the events [xr$\in${1,2,3}] being considered equiprobable. Also, the initial choice of the door [yp$\in${1,2,3}] chosen by the guest is also a random (blind) choice.

The host knows the door behind which the prize is hidden and also the door that is the initial choice of the guest. Therefore, the event/action of the host z opening door q, zq$\in${1,2,3} to show a losing choice, is dependent on both yp and xr, as per the rules of the game show. That is, (zq ≠ yp) & (zq ≠ xr). This dependency of zq on yp and xr does indeed limit the available options. It turns out that when yp ≠ xr the host doesn’t have any option except to turn to the one and only one remaining door zq ≠ (yp ≠ xr); whereas when yp = xr the host has the option of choosing between the two doors, that is, zq ≠ (yp = xr). Because the host has this option, at least in a restricted sense, of choosing which of the two doors to open, it introduces an uncertainty for the guest to predict/expect/anticipate the host's decision/action in this regard. Here, it is *assumed* that whenever zq ≠ (yp = xr) the host's choice between the two available options is indeed *equiprobable*.

Table-1 lists the 12 mutually-exclusive together-exhaustive possible alternatives for the combined-triple-event space along with the relevant apriori probabilities.

Sl.No.	[xr]	[yp]	[xr&yp]	[zq]	[xr&yp&zq]		P[xr]	P[yp]	P[zq | (xr & yp)]	P[xr&yp&zq]
01	1	1	11	2	112		1/3	1/3	1/2	1/18
02	1	1	11	3	113		1/3	1/3	1/2	1/18
03	1	2	12	3	123		1/3	1/3	1	1/9
04	1	3	13	2	132		1/3	1/3	1	1/9
05	2	1	21	3	213		1/3	1/3	1	1/9
06	2	2	22	1	221		1/3	1/3	1/2	1/18
07	2	2	22	3	223		1/3	1/3	1/2	1/18
08	2	3	23	1	231		1/3	1/3	1	1/9
09	3	1	31	2	312		1/3	1/3	1	1/9
10	3	2	32	1	321		1/3	1/3	1	1/9
11	3	3	33	1	331		1/3	1/3	1/2	1/18
12	3	3	33	2	332		1/3	1/3	1/2	1/18

Table-1: Twelve *combined-triplet-event* possibilities along with its *joint-probabilities*.
[xr]: prize x behind door r; [yp]: guest y choses door p; [zq]: host z reveals door q
Twelve Mutually-Exclusive Together-Exhaustive Alternative-Possibilities

4. MONTY-HALL THEOREM

Given that the initial choice of the guest is, say door-1 (event [y1]); and that the host opens the door, say door-3 (event [z3]) to reveal a losing choice, that is different from the door behind which the prize is hidden, and also different from the initial choice of the guest; then the probability of the guest winning the prize is given by the *aposteriori* (conditional to [z3]) *probability* of the prize being hidden behind the door-1 (event [x1]); that is, P[x1 **|** z3]. In the case of the classical Monty-Hall Problem, this value may be computed by the application of the *Bayes-Price Rule* (*Bayes Theorem*) for the case of *three parameter event(sample)space*; and it is equal to 0.50 - therefore the option of the switched choice doesn't yield any enhancement in the chances of winning the prize.

PROOF

The proof is simply by solving the problem, following the below enumerated steps.

For each required value, a general expression is given first; followed by the classical case.

(1) INPUT DATA

P[x1]; P[x2]; P[x3]; P[y1]; P[y2]; P[y3];
P[z3 | x1y1]; P[z3 | x1y2]; P[z3 | x2y1]; P[z3 | x2y2];
P[z2 | x1y1]; P[z2 | x1y3]; P[z2 | x3y1]; P[z2 | x3y3];
P[z1 | x2y2]; P[z1 | x2y3]; P[z1 | x3y2]; P[z1 | x3y3];

(2) JOINT PROBABILITIES FOR INDEPENDENT EVENTS [xr & yp]

P[x1y1] = P[x1]*P[y1]; P[x1y2] = P[x1]*P[y2]; P[x1y3] = P[x1]*P[y3];
P[x2y1] = P[x2]*P[y1]; P[x2y2] = P[x2]*P[y2]; P[x2y3] = P[x2]*P[y3];
P[x3y1] = P[x3]*P[y1]; P[x3y2] = P[x3]*P[y2]; P[x3y3] = P[x3]*P[y3];

(3) VALIDITY CHECK FOR NON-ZERO APRIORI PROBABILITIES

Check and confirm the *validity of input data values* for application of Bayes-Price Rule (Bayes Theorem). The presence of *zero-value* for any of the *apriori probabilities* leading to the intended conditional used to derive the required *aposteriori* (*conditional*) *probabilities*, can result in *spurious results*. Appropriate alternative approach may be needed in such cases. For the classical Monty-Hall Problem, the *joint probabilities* listed above leading to the required conditionality of the host opening a door (say z3) will be used in the below calculations.

(4) APRIORI PROBABILITY FOR [z3] AS PER THE RULES OF THE GAME

P[z3] = P[z3 | x1y1]*P[x1y1] + P[z3 | x2y1]*P[x2y1] + P[z3 | x1y2]*P[x1y2] + P[z3 | x2y2]*P[x2y2];
= P[x1y1z3] + P[x1y2z3] + P[x2y1z3] + P[x2y2z3];
= 1/18 + 1/9 + 1/9 + 1/18;
= 1/3;

(5) APRIORI CONDITIONAL (w.r.t. x1) MARGINAL (w.r.t. yp) PROBABILITY FOR z3

P[z3 | x1] = (P[z3 | x1y1] * P[x1y1] + P[z3 | x1y2] * P[x1y2]) / (P[x1]);
= (P[z3x1y1] + P[z3x1y2]) / (P[x1]);
= (1/18 + 1/9) / (1/3);
= 1/2;

(6) APOSTERIORI CONDITIONAL (w.r.t. z3) MARGINAL (w.r.t. yp) PROBABILITY FOR x1

P[x1 | z3] = (P[z3 | x1] * P[x1]) / (P[z3]);
= (1/2 * 1/3) / (1/3);
= 1/2;

END OF PROOF

5. DISCUSSION

It is to be noted here that the theorem and the proof uses some specific labels for the doors, just for convenience; namely, door-3 [z3] for the door opened by the host to reveal a losing choice, and door-1 [y1] for the guest's initial choice, thus leading to a decision making problem for the guest that requires the computation of the value P[x1 | z3] and its complementary value P[x2 | z3]. However, the result is neither restricted by nor dependent on these specific labels. This is evident from the symmetry in the data entries in Table-1, which shows that xr and yp are interchangeable for any given zq, and that the entries are identical for the three subsets corresponding to each of the values for zq $\in$ {1,2,3}. Or, if one wishes, one can re-write the theorem in three parts, one corresponding to each of the cases with the host opening a door zq $\in$ {1,2,3}.

Also, one can always consider a scenario wherein the three doors are exactly identical from the viewpoint of the guest, and that the initial choice of the guest is then labelled as door-1, and that the door that is opened by the host is then labelled as door-3, thus leaving the remaining door to be labelled as door-2. Therefore, it gets established that *irrespective of whichever be the door opened by the host, each of the remaining two doors have equal probability of having the prize hidden behind it*.

6. EARLIER ERRONEOUS RESULT

The Monty-Hall Theorem reaffirms common-sense based rational & intellectual reasoning, confirmed by the results obtained through the computations shown in the proof. Note that the Monty-Hall Problem is not a problem with possibly multiple correct solutions. Therefore, the above theorem indirectly points out the erroneous result that has been the widely accepted position by the Leading Subject Matter Experts who claim that a switched choice has a clear advantage, with the chances of winning the prize being 2/3 as against only 1/3 for staying with the initial choice.

There seems to be various approaches adopted by the Leading Subject Matter Experts, to derive the very same erroneous result. Almost all of them are centered around the use (rather the erroneous use) of the four apriori probabilities: (1) P[z3x1y1]; (2) P[z3x2y1]; (3) P[z3x1y2]; (4) P[z3x2y2]; leading to the intended conditional [z3] that is supposed to be used appropriately to derive the required *aposteriori* (*conditional*) *probabilities*: P[x1 | z3] to be compared with P[x2 | z3] in the decision-making problem faced by the guest.

Some consider only the two apriori terms (1) & (2) while leaving out (*error of omission*) the other two terms (3) & (4) mentioned above; as-if fixing [z3y1] as the conditionality rather than [z3]; and derive the aposteriori probabilities: P[x1 | z3y1] to be compared with P[x2 | z3y1] - only to recommend a switched choice from [y1] to [y2] – which in itself is indeed a serious *Logical Fallacy*. This is exactly similar to the physical analogy of chasing the proverbial mirage-waters, wherein that perception itself vanishes, since the very conditions that caused such a perception are violated (no more valid) by the very action of moving towards it.

Some others seem to go wrong in their application of the Bayes-Price Rule (Bayes Theorem) - *error of commission* - in a situation with zero value associated with apriori probabilities P[y2] & P[y3] - as-if fixing [y1] as a pre-condition – an issue of concern that has been clearly mentioned in the above proof while *insisting on a check for the validity of input data* before further processing to derive aposteriori (conditional to [z3]) probabilities.

One of the most striking errors is the claim that the chances of winning by staying with the initial choice is given by (P[x1y1z2]+P[x1y1z3]) whereas the chances of winning by a switched choice is given by (P[x1y2z3]+P[x1y3z2]) as-if fixing [x1] as a pre-condition! Similarly, another equally intriguing approach adopted by some others is to compare (P[x1y1z3]+P[x2y2z3]) with (P[x1y2z3]+P[x2y1z3]) while correctly considering [z3] as the aposteriori condition! We are amazed as to how these approaches can be justified by either any rational intellectual reasoning or any theory based on the fundamentals of Probability & Statistics.

7. A CHALLENGE TO THE LEADING SUBJECT MATTER EXPERTS

Let us rephrase the Monty-Hall Problem, now adorned with a *jewel-on-the-crown* as below:

(1.1) The prize is hidden behind one of the three doors.
(1.2) I the guest make an initial choice of which door it could be, say door-1, to claim my prize.
(1.3) Then Monty the host opens a different door, say door-3, revealing a losing choice.
(2.1) I am given an option to withdraw/cancel the earlier choice of door-1 and switch to door-2.
(2.2) I appreciate the knowledge of a losing choice and also Monty's offer of the option to switch.
(3.1) I grab Monty's offer, withdraw/cancel my earlier choice of door-1.
(3.2) Then I re-evaluate the two choices available for me now, namely door-1 or door-2.
(3.3) I find that the chances of winning are exactly the same between the two available choices;
(4.1) Now that YOU enter the Hall, I seek YOUR recommendation. What is YOUR recommendation?
(4.2) TO SWITCH OR NOT TO SWITCH : THAT IS THE QUESTION!

Note that your answer must necessarily be independent of my initial-choice; although Monty's choice of opening a door to reveal a losing choice was dependent on my initial choice which he had to avoid as per the rules of the game. Hope your expert advice is *NEITHER* an exemplification of a well-known proverb "*the grass is always greener on the other side*" *NOR* any enticement to chase the proverbial mirage-waters wherein that perception itself vanishes, since the very conditions that caused such a perception are violated by the very action of moving towards it.

8. CONCLUSION

The Monty-Hall Theorem establishes the correct approach in solving the classical Monty-Hall Problem. It establishes the fact that the probability of winning the prize is indeed unaffected by a switched choice – very much unlike the most prevalent and widely accepted position held by the Leading Subject-Matter-Experts. A brief mention is made about the error of commission

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10. ACKNOWLEDGEMENT

Let us acknowledge that, looking back, it seems as if Marilyn cast an enchantingly deep spell over the Frequentists who in turn pushed the Probabilists to mistake Bayes-Price, paying the price through errors of commission and/or errors of omission, riddled with some Logical Fallacy as well. We need to come out of that long drawn intellectual hibernation of over six decades, and wake up to reality.

I must necessarily confess here that the *core idea behind this analysis is so stunningly & elusively simple*, that one may simply be taken aback in a profound wonder-struck jaw-drop-silence, maybe with an after-thought: "*oh my goodness, how could it be that it never flashed on me any time earlier*"! as was also the case in an earlier research work reported in [13] by this author.

On this most auspicious vidyaa(vijaya)daSami day [2025OCT02] I was blessed with the vision to formulate the *Monty-Hall Theorem* and its *proof* - as a *concise* & *precise* approach - the preferred style of presentation for the target audience consisting of Mathematicians, Statisticians, Logicians, eminent scientists etc.

11. DEDICATION

To my ಅಜ್ಜ(ajja) Karinja Halemane Keshava Bhat & ಅಜ್ಜಿ(ajji) Thirumaleshwari, ಅಪ್ಪ(appa) Shama Bhat & ಅಮ್ಮ(amma) Thirumaleshwari, for their *teachings through love*, that *quality matters more than quantity*; to my wife Vijayalakshmi for her *ever consistent love & support*; to my daughter Sriwidya.Bharati and my twin sons Sriwidya.Ramana & Sriwidya.Prawina for their *love & affection*.

Whereas *this Original Author-Creator* holds the (PIPR:©:) Perpetual Intellectual Property Rights, his legal heirs (three children mentioned above) may avail the same for perpetuity.

To all the *cool-headed brave-hearts*, eagerly awaited but probably yet to be visible among the world professionals, especially the *Subject-Matter-Experts*, who would be attracted to and certainly capable of effectively understanding without any prejudice and appreciating the deeper insights enshrined in this short research report, who may opt for innate rational-&-intellectual common-sense and simple creativity over any sophisticated and/or complex theory in problem-solving to resolve any seemingly paradoxical scenarios possibly arising therefrom.