

LAY-UP OPTIMALITY CONDITIONS FOR STIFFNESS MAXIMIZATION OF ANISOTROPIC COMPOSITE PLATES IN POSTBUCKLING

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Abstract

The present paper deals with the optimization of post-buckled composite plates. The plates have a symmetric lay-up. The layer orientation angles vary in the point-wise or in the plate-wise ways. The von Karman theory is employed. The boundary conditions are the simple support ones or the clamped ones. The structural potential energy is treated as a measure of structural stiffness. For the plate stiffness maximization problem, the first-order necessary conditions of the local optimality are derived. The mathematical treatment of the conditions is performed. The conditions contain two terms. One of them corresponds to the mid-plane strains; another one corresponds to the plate curvatures. The optimality conditions may lead to a co-axiality of some structural tensors. An illustration of the optimality conditions is presented.

Keywords: composite plate, postbuckling, lay-up, optimality, co-axiality.

1. INTRODUCTION

Postbuckling of thin composite plates attracts the attention of numerous researchers, working in various modern application areas. Among these applications, there are the aerospace, the automotive, the marine, and the civil ones. According to design practice in composite structures, the account of the load-carrying capability “above buckling” gives an opportunity of extra weight saving, as compared to existing traditional design.

Composite plates are a part of a considerable number of structures in the above applications. The loading applied to the plates is a combination of compression and shear. The compression dominates for the structures similar to the upper panels of an airplane wing structure. As the examples of known structures designed for pure shear, one may indicate airplane wing ribs and spars, as well as some fuselage panels. As a rule, the shear-loaded plates have the so-called angle-ply symmetric lay-up. For example, buckling is not allowed in airplane structures in the case of loading up to the so-called limit loads (LL) and may be allowed for the loading level between the limit loads and the so-called ultimate loads (UL). The latter loads are 1.5 times higher than the limit loads.

When optimizing the composite plate, an engineer, in fact, does his best to save the structural weight, the material cost, and the production time. Due to that, the optimization problems for the composite plates attract considerable attention at the design phase. The lay-up optimization problem is one of them requiring a solution at the beginning of the design process.

Since the last decade of the previous century, numerous papers dealt with the lay-up optimization of composite plates. The non-exhaustive list [1]-[22] contains the important publications devoted to the postbuckling analysis and optimization. The foundations of the postbuckling theory are described in [23].

There are many reviews and textbooks, describing various aspects of the subject (see Turvey and Marshall [2], Kassapoglou [24], Weaver [25], Gürdal et al. [26], Bruyneel [27], Ghiasi et al. [28, 29]).

Among journal papers, it seems that one of the latest is the paper of Raju et al. [30].

Other interesting publications may be found in the references of the above-indicated papers.

All published papers in the area are devoted to numerical optimization studies. Various optimization criteria are used, in particular: end-shortening, maximal compression strain, maximal deflection, estimated stiffness, etc.

Among papers considering a description of the postbuckling stiffness, the paper [6] should be indicated. In the paper, the compressed post-buckled orthotropic laminated plates are considered. The plate behavior is described by two equations: the von Karman equation, and the strain compatibility equation. The equations are the ones for the deflections and for the Airy stress function. The solutions of the equations are determined numerically. For the numerical process, the deflections are described by a sum of several trigonometric terms. The latter terms satisfy the boundary conditions. The estimated value of the postbuckling stiffness in each direction is equal to the derivative of the corresponding load with respect to the compression strain. The maximization of the initial postbuckling stiffness is based on the estimation and is performed numerically.

“Quick answer” approximations are the result of the closed-form postbuckling solutions. As references, the papers [3], [11], [12] may be indicated. In the papers, the shear-loaded structures are considered.

The numerical perturbation approach to postbuckling analysis should be mentioned as rather popular. For references, one may refer to [3], [8], and [15].

Several papers are devoted to approximate solutions. The ones devoted to the finite strip approach [9] may be indicated in this regard.

Several papers, in particular, [1] and [10], describe software projects. The projects deal with postbuckling analysis and optimization of aerospace structures.

The dissertation [19] considers, in particular, aspects of plate postbuckling. The plates under discussion are made of the specially orthotropic balanced material, with a) maximum (compression) strain as the failure criterion, b) blending constraints (not more than four neighboring plies of the same angle), c) discrete set of ply angles. It is noticed, that failure-optimized and buckling-optimized lay-ups may be considerably different in some cases of loading. Wing-box optimization is also considered.

The paper [20] minimizes the postbuckling dynamic response and maximizes the buckling load, using a special convolution function for two criteria. Numerical examples are considered. In general, the particular objectives conflict with each other necessitating a multi-objective formulation. The latter formulation includes the postbuckling energy and the critical buckling load.

The paper [21] considers a discrete version of the firefly algorithm (DFA). The algorithm is adapted to lay-up optimization of post-buckled composite structures.

In [22], the authors present a novel method of design optimization. The method is a gradient-based one and is intended for optimization of postbuckling performance. The considered structures are made of composite material. The postbuckling analysis of the structures is based on Koiter's asymptotic method. The corresponding design sensitivities of the Koiter's factors are derived, for further usage in the gradient-based optimization. Subsequently, new design optimization formulations are presented; they are based on the sensitivities. The proposed optimization formulations are intended for optimization of

postbuckling stability. In conclusion, the square plate and the curved panel illustrate the concept.

A considerable number of works is devoted to the VAT (Variable Angle Tow) composites. The studies were initiated in the last decade of the previous century in Virginia Tech, Delft University, Bristol University, and other locations. For references, one may use the most recent papers like [16-18, 25, 30]. In such papers, the advantages of VAT composites for postbuckling load-carrying capacity are explored numerically and compared to the so-called “quasi-isotropic” or “straight fiber” solutions. As a rule, the ply orientation angles to be considered are linear functions of a coordinate.

In [16], the balanced VAT laminates are numerically studied, using the Tsai-Wu failure criterion and controlling end-shortening and maximal deflection.

In [17], the difference in the postbuckling behavior in case of positive and negative shear loading of VAT plates is demonstrated and discussed.

In [18], the optimization of the fiber paths is driven by two distinct requirements, namely local and global stiffness tailoring that influence the buckling performance and static strength, respectively. Finally, the initial postbuckling behavior of the optimized designs is investigated using Koiter's perturbation approach, which reveals that postbuckling stability should be considered when optimizing the VAT panels.

In [30], postbuckling analysis of a variable-angle-tow composite plate is performed using the perturbation-based asymptotic numerical method, which transforms the nonlinear problem into a set of well-posed recursive linear problems. These linear problems are solved using a novel generalized differential-integral quadrature method, and the postbuckling solutions are sought over a finite load step size around the critical buckling point using asymptotic expansions.

Also we should mention as important ones the papers [35]-[38], considering, in parallel with the optimization, the manufacturing aspects for the VAT laminates. The structures are numerically optimized in the cases of the strength, the stiffness, and the buckling requirements.

The today's opinion of the composite optimization community based on numerical and approximate studies is: the surface plies dominate the buckling events whilst the inner plies dominate the postbuckling (see [25]).

Potentially, having more degrees of freedom, the VAT solutions are more promising, as compared to the conventional "straight fiber" solutions. On the other hand, the drawback of the VAT approach is that there are no known theoretical results for verifying, validating, and explaining the solutions obtained. Also, the approach is more expensive, as compared to the conventional solutions. In our opinion, the potential of the latter solutions is not exhausted yet.

As it is said, in available publications, there are no purely theoretical studies of the plate lay-up optimization problems in the postbuckling. The numerical experience until now covers a considerable number of various examples and requires creating a generalized understanding.

The main goal of the present paper is theoretical; the task is to reveal the features of the optimal lay-ups (both point-wise and plate-wise) for the composite plates in the postbuckling.

In the present paper, the first order necessary optimality conditions are derived for a simplified stiffness optimization formulation, with the objective function being the structural potential energy (see, e.g., the paper of Gierlinski and Mroz [35]). It seems that this formulation is the only one leading to the observable results. After that, an illustration of the optimality conditions is presented.

Note that the optimality conditions of other types (Legendre, Weierstrass, Jacobi, etc.) are not considered in the paper.

Section 1 contains an introduction.

Sections 2-4 deal with the consideration of optimal lay-up of a composite plate in the postbuckling under single in-plane loading. The first order necessary optimality conditions for the structural stiffness maximization problem are derived.

In Section 4, the results of Sections 3-4 are analyzed and discussed.

In Section 5, an illustration of the optimality conditions is presented.

Section 6 contains the conclusions of the paper.

2. ENERGY FORMULATION

The consideration of the paper deals with a post-buckled thin, flat composite plate. The Classical Laminated Plate Theory (CLPT) [36] is employed. The plate is of symmetric lay-up and contains $2N$ tape layers. For simplicity, the laminates with the odd number of layers are outside the discussion; the generalization to the case is straightforward. The boundary condition types are simple support or clamped. The contour of the plate is a piecewise smooth one. The Green strain tensor contains both linear and nonlinear terms (w.r.t. deflections). The plate coordinate system is (x, y) ; the loads are the in-plane loads applied at the contour.

The kinematic variational principle (the structural potential energy stationary principle), together with the strain compatibility condition, describes the plate postbuckling nonlinear behavior (see [24], and Section 8 of [37]).

The variational principle for the composite plates under the above assumptions is formulated in the following way. The quantities (u, v, w) are the corresponding plate displacements in the Cartesian coordinates (x, y, z) , respectively; w is the out-of-plane deflection. The corresponding components of the Green strain tensor in the plate mid-plane, with the account of nonlinear terms, are:

$$e_{xx0} = \frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 ; e_{yy0} = \frac{\partial v}{\partial y} + \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2 ; e_{xy0} = \frac{1}{2} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right) \quad (1)$$

The components of the tensor of the plate curvatures are:

$$k_x = -\frac{\partial^2 w}{\partial x^2} ; k_y = -\frac{\partial^2 w}{\partial y^2} ; k_{xy} = -\frac{\partial^2 w}{\partial x \partial y} \quad (2)$$

The quantities in (1), (2) create the following vectors (below we follow the engineering approach of [36]):

$$\overline{\varepsilon^0}^T = (e_{xx0}, e_{yy0}, 2e_{xy0}) ; \overline{k}^T = (k_x, k_y, 2k_{xy}) \quad (3)$$

where the superscript T means a transposition.

In the case of moderate deflections (see [37], Section 8), described by the von Karman theory, the strain potential energy is:

$$\Pi = \int dS \left(\frac{1}{2} \overline{\varepsilon^0}^T A \overline{\varepsilon^0} + \frac{1}{2} \overline{k}^T D \overline{k} \right) \quad (4)$$

where A and D are the (3*3) matrices describing the 2D-mid-plane behavior and the bending-twisting of the plate [36]; the vectors $\overline{\varepsilon^0}, \overline{k}$ are determined in (3). The integration in (4) is performed over the plate mid-plane surface.

The structural potential energy is:

$$U = \int dS \left(\frac{1}{2} \overline{\varepsilon^0}^T A \overline{\varepsilon^0} + \frac{1}{2} \overline{k}^T D \overline{k} \right) - W \quad (5)$$

where W is the potential of external “dead” forces. The potential depends on the forces, on the displacements at the loaded part of the external boundary, on the external pressure (if any), and on the corresponding deflections of the plate. In the present paper, the dependence on contour in-plane forces p_x, p_y is considered only (the plate is supposed to be located in x - y plane):

$$W = p_x u + p_y v \quad (6)$$

where p_x, p_y are the given x, y components of the contour forces, respectively. The kinematic variational principle states that in equilibrium the structure satisfies

$$\delta U = \delta \Pi - \delta W \quad (7)$$

for all kinematically admissible displacements (u, v, w) .

The principle (7) leads to the equilibrium equation for the deflections w and for the stress flows N_x, N_y, N_{xy} within the plate (see [37] for the usual sign convention). The plate deflections and the mid-plane strains must also satisfy the compatibility equation:

$$\frac{\partial^2 e_{xx0}}{\partial y^2} + \frac{\partial^2 e_{yy0}}{\partial x^2} - 2 \frac{\partial^2 e_{xy0}}{\partial x \partial y} = \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 - \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} \quad (8)$$

3. POINT-WISE OPTIMALITY CONDITIONS

For the orthotropic lamina, we follow the approach and the notations of [36], Section 2.6.

The stress tensor components within the lamina in the coordinates (x, y) are:

$$\begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ 2\varepsilon_{xy} \end{pmatrix} \quad (9)$$

where $\bar{Q}$ is the lamina stiffness matrix, and $\varepsilon_x, \varepsilon_y, \varepsilon_{xy}$ are the lamina infinitesimal strain tensor components. The components of the lamina stiffness matrix are the known algebraic functions of four layer elastic constants U_1, U_2, U_3, U_4 and of trigonometric functions of the angle θ_m (where m is the ply number). The angle is the angle between the fiber direction and the x -axis (measured in the direction of counter-clockwise rotation from x to y).

According to [36], the derivatives of the lamina stiffness matrix $\bar{Q}_{ij}$ w.r.t. the orientation angle θ_m are:

$$\begin{aligned}\frac{d\bar{Q}_{11}}{d\theta_m} &= -2U_2 \sin 2\theta_m - 4U_3 \sin 4\theta_m = -4\bar{Q}_{16} \\ \frac{d\bar{Q}_{22}}{d\theta_m} &= 2U_2 \sin 2\theta_m - 4U_3 \sin 4\theta_m = 4\bar{Q}_{26} \\ \frac{d\bar{Q}_{12}}{d\theta_m} &= 4U_3 \sin 4\theta_m = 2(\bar{Q}_{16} - \bar{Q}_{26})\end{aligned}\quad (10)$$

Determination of A_{ij} and D_{ij} via the lamina parameters is performed according to the general formulas [36], $ij=11, 12, 22, 16, 26, 66$:

$$A_{ij} = \sum_{m=1}^N (\bar{Q}_{ij})_m (z_m - z_{m-1}) \quad (11)$$

$$D_{ij} = \frac{1}{3} \sum_{m=1}^N (\bar{Q}_{ij})_m (z_m^3 - z_{m-1}^3) \quad (12)$$

where z_m, z_{m-1} are the top and the bottom z -coordinates of the m -th layer ($z=0$ corresponds to the mid-plane).

The derivatives of the stiffness matrices A, D w.r.t. the lamina angles θ_m are, $ij=11, 12, 22; m=1, \dots, N$:

$$\begin{aligned}\frac{dA_{ij}}{d\theta_m} &= (z_m - z_{m-1}) \frac{d(\bar{Q}_{ij})_m}{d\theta_m} \\ \frac{dD_{ij}}{d\theta_m} &= \frac{1}{3} (z_m^3 - z_{m-1}^3) \frac{d(\bar{Q}_{ij})_m}{d\theta_m}\end{aligned}\quad (13)$$

Below we consider the structural potential energy U from (5) as a measure of the structural stiffness. Such an approach for the stiffness of nonlinear plates was used in many papers, for example, in the paper of Gierlinski and Mroz [35].

The goal of lay-up optimization is to maximize the postbuckling stiffness value. The ply orientation angles are varied in the point-wise or in the plate-wise way.

In this sub-Section, we consider the point-wise case and assume, that the lamina orientation angles θ_m are smooth functions of the mid-plane location (x, y) , $m=1, \dots, N$.

Following the usual variational procedure [38] and taking into account the variational principle (7), we obtain the first order necessary conditions of local optimality in the case. The maximization of the structural stiffness means that the first variation of U with respect to the orientation angles is equal to zero.

$$\delta U = \delta U \Big|_{\substack{A-\text{non-varied} \\ D-\text{non-varied}}} + \delta U \Big|_{(u,v,w)-\text{non-varied}} = 0 \quad (14)$$

The first item in (14) is equal to zero due to (7). In the second item, the matrices A , D are functions of the lamina orientation angles only. From the second item, we obtain a relation for the orientation angles of i -th layer at a current point, $i=1, \dots, N$:

$$\delta U \Big|_{(u,v,w)-\text{non-varied}} = \sum_{i=1}^N \int dS \left(\frac{1}{2} \overline{\varepsilon^0}^T \frac{dA}{d\theta_i} \overline{\varepsilon^0} + \frac{1}{2} \overline{k}^T \frac{dD}{d\theta_i} \overline{k} \right) \delta \theta_i = 0 \quad (15)$$

and, keeping in mind the independence of the θ_i components on each other, $i=1, \dots, N$, and the basic Lemma of calculus of variations [38], we obtain the first order necessary optimality conditions for the post-buckled plate stiffness maximization:

$$\frac{1}{2} \overline{\varepsilon^0}^T \frac{dA}{d\theta_i} \overline{\varepsilon^0} + \frac{1}{2} \overline{k}^T \frac{dD}{d\theta_i} \overline{k} = 0, \quad i=1, \dots, N \quad (16)$$

Further, we calculate the first item in (16) (the item corresponds to the mid-plane 2D-strains), in the axes of the principal 2D mid-plane strains $\varepsilon_1, \varepsilon_2$ (the whole tensor is determined in (1)), where $\varepsilon_1 \geq \varepsilon_2$. The second item in (16) (the item corresponds to the plate bending and twisting), we calculate in the axes of the principal plate curvatures k_1, k_2 , where $k_1 \geq k_2$. Making necessary transformations in (16) and remembering that the absolute value of

the Jacobian of a transition from the principal mid-plane strain coordinates to the principal curvature coordinates is equal to 1.0 (the transition is a simple rotation), we obtain for $i=1, \dots, N$:

$$\frac{1}{2} \left\{ (\varepsilon_1, \varepsilon_2) \begin{bmatrix} \frac{dA_{11}^{pr.str.}}{d\theta_i} & \frac{dA_{12}^{pr.str.}}{d\theta_i} \\ \frac{dA_{12}^{pr.str.}}{d\theta_i} & \frac{dA_{22}^{pr.str.}}{d\theta_i} \end{bmatrix} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix} \right\} + \frac{1}{2} \left\{ (k_1, k_2) \begin{bmatrix} \frac{dD_{11}^{pr.cur.}}{d\theta_i} & \frac{dD_{12}^{pr.cur.}}{d\theta_i} \\ \frac{dD_{12}^{pr.cur.}}{d\theta_i} & \frac{dD_{22}^{pr.cur.}}{d\theta_i} \end{bmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \right\} = 0 \quad (17)$$

where *pr.str.*, *pr.cur.* correspond to the principal mid-plane strain axes and to the principal curvature axes.

Analyzing (17), we consider at the beginning the first item there.

Substituting in (17) the relations (10)-(13), we obtain for every i -th layer, $i=1, \dots, N$:

$$\left\{ \varepsilon_1^2 [-2U_2 \sin 2(\theta_i - \varphi) - 4U_3 \sin 4(\theta_i - \varphi)] + \right. \\ \left. + \varepsilon_2^2 [2U_2 \sin 2(\theta_i - \varphi) - 4U_3 \sin 4(\theta_i - \varphi)] + 2\varepsilon_1 \varepsilon_2 4U_3 \sin 4(\theta_i - \varphi) \right\} \frac{1}{2} (z_i - z_{i-1}) \quad (18)$$

where φ is the angle between the global x -axis and the ε_1 axis.

Analogously the second item gives for every i -th layer, $i=1, \dots, N$:

$$\left\{ k_1^2 [-2U_2 \sin 2(\theta_i - \psi) - 4U_3 \sin 4(\theta_i - \psi)] + \right. \\ \left. + k_2^2 [2U_2 \sin 2(\theta_i - \psi) - 4U_3 \sin 4(\theta_i - \psi)] + 2k_1 k_2 4U_3 \sin 4(\theta_i - \psi) \right\} \frac{1}{6} (z_i^3 - z_{i-1}^3) \quad (19)$$

where ψ is the angle between the global x -axis and the k_1 axis.

The sum of (18) and (19), equated to zero, is the first order necessary optimality condition for every layer, $i=1, \dots, N$

$$\left\{ \varepsilon_1^2 [-2U_2 \sin 2(\theta_i - \varphi) - 4U_3 \sin 4(\theta_i - \varphi)] + \right. \\ \left. + \varepsilon_2^2 [2U_2 \sin 2(\theta_i - \varphi) - 4U_3 \sin 4(\theta_i - \varphi)] + 2\varepsilon_1 \varepsilon_2 4U_3 \sin 4(\theta_i - \varphi) \right\} \frac{1}{2} (z_i - z_{i-1}) + \\ + \left\{ k_1^2 [-2U_2 \sin 2(\theta_i - \psi) - 4U_3 \sin 4(\theta_i - \psi)] + \right. \\ \left. + k_2^2 [2U_2 \sin 2(\theta_i - \psi) - 4U_3 \sin 4(\theta_i - \psi)] + 2k_1 k_2 4U_3 \sin 4(\theta_i - \psi) \right\} \frac{1}{6} (z_i^3 - z_{i-1}^3) = 0 \quad (20)$$

As we see, the conditions for every layer are highly nonlinear ones. Transforming (20), we obtain:

$$\begin{aligned}
& 2 \sin 2(\theta_i - \varphi) \left\{ \varepsilon_1^2 [-U_2 - 4U_3 \cos 2(\theta_i - \varphi)] + \right. \\
& \left. + \varepsilon_2^2 [U_2 - 4U_3 \cos 2(\theta_i - \varphi)] + 8\varepsilon_1 \varepsilon_2 U_3 \cos 2(\theta_i - \varphi) \right\} \frac{1}{2} (z_i - z_{i-1}) + \\
& + 2 \sin 2(\theta_i - \psi) \left\{ k_1^2 [-U_2 - 4U_3 \cos 2(\theta_i - \psi)] + \right. \\
& \left. + k_2^2 [U_2 - 4U_3 \cos 2(\theta_i - \psi)] + 8k_1 k_2 U_3 \cos 2(\theta_i - \psi) \right\} \frac{1}{6} (z_i^3 - z_{i-1}^3) = 0
\end{aligned} \tag{21}$$

and, finally, dividing by $-(z_i - z_{i-1})4U_3$:

$$\begin{aligned}
& \sin 2(\theta_i - \varphi) \left[\frac{U_2}{4U_3} (\varepsilon_1^2 - \varepsilon_2^2) + (\varepsilon_1 - \varepsilon_2)^2 \cos 2(\theta_i - \varphi) \right] + \\
& + \sin 2(\theta_i - \psi) \left[\frac{U_2}{4U_3} (k_1^2 - k_2^2) + (k_1 - k_2)^2 \cos 2(\theta_i - \psi) \right] \frac{1}{3} (z_i^2 + z_i z_{i-1} + z_{i-1}^2) = 0
\end{aligned} \tag{22}$$

As we see, the dependence on material comes through the parameter $\frac{U_2}{4U_3}$. It is interesting to

note that all multipliers of the trigonometric functions are non-negative. The explicit dependence on the layer z location comes through the last multiplier of the second item. The coupling between the layers occurs due to the principal 2D-strain lines, the principal curvature lines, the values of the principal 2D-strains and of the principal curvatures. In Section 5 we discuss the ply locations with dominating one or both items of the optimality conditions.

Analyzing (22), one may say also that the innermost layers and the outermost layers play different roles there. In particular, it follows from the first item in (22) that the innermost layers are most sensitive to the principal 2D-strains and their principal directions. As to the outermost layers, they are most sensitive (due to z^2 multiplier of the second item) to the principal plate curvatures and their principal directions.

If we make a sum of (21) for all k and keep in mind (10)-(16), we obtain the following united condition

$$\left[A_{16}^{pr.str.} \varepsilon_1^2 - A_{26}^{pr.str.} \varepsilon_2^2 - (A_{16}^{pr.str.} - A_{26}^{pr.str.}) \varepsilon_1 \varepsilon_2 \right] + \left[D_{16}^{pr.cur.} k_1^2 - D_{26}^{pr.cur.} k_2^2 - (D_{16}^{pr.cur.} - D_{26}^{pr.cur.}) k_1 k_2 \right] = 0 \tag{23}$$

or, after using the definitions of the stiffness matrices A and D , the relation with the clear physical meaning

$$N_{xy}^{pr.str.}(\varepsilon_1 - \varepsilon_2) + M_{xy}^{pr.cur.}(k_1 - k_2) = 0 \quad (24)$$

The relation (24) is a linear combination of the optimality conditions (21) or (22) and, hence, also it is the optimality condition uniting the ones for every layer. The first item in (24), if considered separately as equal to zero, means that the nonlinear mid-plane strain tensor is co-axial with the stress flow tensor (or, for $\varepsilon_1 \neq \varepsilon_2$, the shear flow in the principal strain axes is equal to zero). The second item in (24), if considered separately as equal to zero, means that the tensor of the curvatures is co-axial with the moment tensor (or, for $k_1 \neq k_2$, the twisting moment in the principal curvature axes is equal to zero).

It should be noted that the result, in the case of an orthotropic plate with a point-wise orthotropic orientation, follows from (24) with the number of layers $N=1$.

The united condition (24), in fact, follows from (15) in case of the local ply orientation angle variations, having the same value. It means that the ply variations correspond to an infinitesimal point-wise ply material rotation.

4. PLATE-WISE OPTIMALITY CONDITIONS

In the case of the plate-wise variation of the layer fiber angles, the first order necessary conditions of local optimality for every layer are different from (22), as the basic Lemma of calculus of variations [38] is not applicable in the case. The conditions in the case mean that the integral of (21) or (22) over the plate is equal to zero.

A generalization of (22) to the plate-wise case is written as follows, $i=1, \dots, N$:

$$\int_S dx dy \left[\sin 2(\theta_i - \varphi) \left[\frac{U_2}{4U_3} (\varepsilon_1^2 - \varepsilon_2^2) + (\varepsilon_1 - \varepsilon_2)^2 \cos 2(\theta_i - \varphi) \right] + \right. \\ \left. + \sin 2(\theta_i - \psi) \left[\frac{U_2}{4U_3} (k_1^2 - k_2^2) + (k_1 - k_2)^2 \cos 2(\theta_i - \psi) \right] \frac{1}{3} (z_i^2 + z_i z_{i-1} + z_{i-1}^2) \right] = 0 \quad (25)$$

where S is the plate mid-plane surface.

A generalization of (24) for the case is

$$\int_S dx dy [N_{xy}^{pr.str.} (\varepsilon_1 - \varepsilon_2) + M_{xy}^{pr.cur.} (k_1 - k_2)] = 0 \quad (26)$$

5. DISCUSSION

Analysis of the optimality conditions (22)-(26) leads to the following conclusions.

The first conclusion corresponds to the case, when the load value is slightly higher than the buckling level. In the case, the second item (related to the curvatures) of the optimality conditions plays the key role. The reason is that the order of magnitude of the first (2D-strain) item at (24), (26) is the same as one of the product of the second (bending-twisting) item and the quantity $\frac{\bar{w}^2}{t^2}$. In the latter ratio, the quantities $t, \bar{w}$ are the plate thickness and the

maximal deflection, with the following inequality being valid: $\frac{\bar{w}^2}{t^2} \ll 1$ (see also [37],

Section 8.5). Indeed, the relations estimating the order of magnitude of the items in (24), (26) are written as follows:

$$u_{,x} \sim u_{,y} \sim v_{,x} \sim v_{,y} \sim (w_{,x})^2 \sim (w_{,y})^2 \quad (27)$$

$$D \sim t^2 A \quad (28)$$

where x and y after the comma mean differentiation w.r.t. the corresponding coordinate, and (27) is taken from [37]. The order of magnitude of the items in the lhs of the optimality condition (24) may be written as (omitting the mutual multiplier the order of magnitude A):

$$\left(\frac{\bar{w}^2}{a^2}\right)^2 + t^2 \left(\frac{\bar{w}}{a^2}\right)^2 \quad (29)$$

where a is the characteristic in-plane plate dimension. Analyzing the latter relation, one comes to the above first conclusion.

The second conclusion corresponds to the case, when the load value is higher (but not slightly and not much) than the buckling level. In the case, the order of magnitude of the ratio $\bar{w}^2/t^2$ becomes about 1., and the both items in (29) are of the same order of magnitude. This means that both items in (24) (curvature-driven and 2D-strain-driven) play the comparable roles in the plate behavior.

The third conclusion corresponds to the case, when the load value is considerably higher than the buckling level. In the case, the ratio $\bar{w}^2/t^2$ becomes much higher than 1., and the first item dominates in (29) and in (24) (the item corresponds to 2D mid-plane strains). On the other hand, in the case, we are very close to the border of applicability of the von Karman approach. Due to that, we propose in the case the following conjecture: the role of the first item (related to the 2D-strains) of the optimality conditions becomes stronger than the role of the second item.

In particular, the so-called diagonal tension behavior (see [24]) of the shear loaded post-buckled structure, in our opinion, is a confirmation of the conjecture (to some extent).

The optimality condition (26) is analyzed in a similar way.

Similar to (24), the united condition (26), in fact, corresponds to the plate-wise ply orientation angle variations, having the same value. It means that the ply variations correspond to an infinitesimal plate-wise material rotation.

Now, similar to the above analysis, we analyze (22) from the point of view of the driving items of the optimality conditions for the plies at a particular plate point.

Analogously to (29), the order of magnitude of the items in the lhs of the optimality condition (22) may be written as:

$$\left(\frac{\bar{w}^2}{a^2}\right)^2 + z^2 \left(\frac{\bar{w}}{a^2}\right)^2 \quad (30)$$

Analyzing the relation, one comes to the following conclusions (keeping in mind that $3 \cdot 3 \sim 10$). If for some coordinate z_0 :

$$\frac{\bar{w}}{z_0} > 3 \quad (31)$$

then the 2D-strain item dominates for plies below z_0 (and above the mid-plane). If

$$\frac{\bar{w}}{z_0} < 1/3 \quad (32)$$

then the bending item dominates for plies above z_0 up to the external surface.

In between of regions described by the inequalities (31), (32), the both items of the optimality condition (22) play the roles of the same order of the magnitude.

Thus, it follows from the above-obtained optimality conditions that the key drivers of the lay-up optimality for the innermost plies are the mid-plane strains; the key drivers for the outermost plies are the mid-surface curvatures.

It is interesting to mention that (see [24, 28, 29, 39]) the lay-up of the maximal upper estimate of the first buckling value corresponds to the second item in (22)-(26). In the case of pre-buckling, the lay-up solution with the maximal stiffness value corresponds to the first item in (22)-(26).

Concluding this Section, we note that optimality conditions of other types are not under discussion here, as well as the conditions at the contour non-smoothness points. All the conditions are beyond the scope of the present paper.

6. AN ILLUSTRATION

In this Section, we illustrate the obtained optimality conditions by the plate shown in Fig. 1. It is demonstrated that there are some regions on the plate where the lay-up optimality conditions are valid. For determining the lay-up distribution of the whole plate, the proper numerical approaches should be employed.

Fig. 1 shows is a flat square composite plate with the dimensions $a \times a$. The plate is loaded by the 1-D in-plane uniform compression and has the simple support boundary conditions along all edges. The origin of the X - Y coordinate system is located at the plate center. The dashed lines connect the midpoints of the plate opposite sides. The lines coincide the coordinate system axes. It is assumed that the plate is made of a specially orthotropic material [36] composed of symmetrically stacked 0° and 90° tape layers, where the 0° direction is the bending-strongest direction coinciding the X -axis. The stacking is $[90^\circ, 0^\circ]_{NS}$, where $2 \times N$ is the total number of plies and S means symmetric.

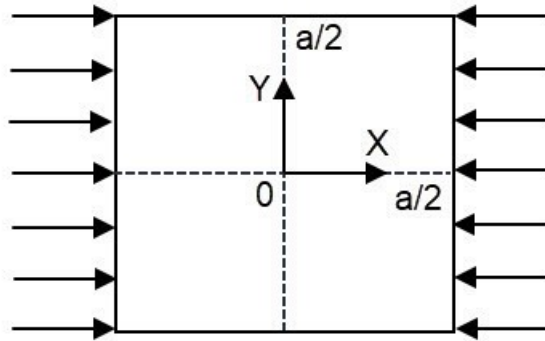


Fig. 1. Square plate loaded by the 1-D uniform in-plane compression

The buckling solution for the orthotropic plate is known [36]. The eigenmode deflection w is equal to a product of the corresponding sines.

The following assumptions are made. The plate in the postbuckling process develops the first buckling mode. The mode has the lowest value of the critical buckling compression force flow and is a single one (not multiple). For the mode the whole plate has one half-wave of the deflections.

Observing the plate behavior under moderate deflections, one may say that the plate (v , w) displacements are symmetric relative to the Y -axis. The out-of-plane displacements w are also symmetric relative to the X -axis. The X -direction displacements u are anti-symmetric relative to the Y -axis. The Y -direction displacements v (parallel to the Y -axis) are anti-symmetric relative to the X -axis. At the dashed line coinciding the X -axis the following equalities are valid

$$v = 0 \quad ; \quad \frac{\partial w}{\partial y} = 0 \quad (33)$$

At the dashed line coinciding the Y -axis we also have

$$u = 0 \quad ; \quad \frac{\partial w}{\partial x} = 0 \quad (34)$$

The indicated observations, after a simple analysis, lead to a conclusion that the dashed lines are the principal curvature lines and the principal mid-plane strain lines. Using the features of the specially orthotropic material, we obtain that at the lines the twisting moment and the shear flow in X - Y coordinates are equal to zero. Hence, at the dashed lines the united point-wise optimality condition (24) is valid.

As the ply orientation directions coincide the X or Y directions, we obtain that at the dashed lines the corresponding sines in (22) are equal to zero. Hence, at the lines the layer first order necessary point-wise optimality conditions (22) are valid.

It should be noted that the lay-up solution with the above features may not be unique. Other lay-ups composed of 0° and 90° may obey the above assumptions regarding the first buckling mode and lead to the same results at the dashed lines. It is explained by the fact that the analysis of this Section is based on the first order necessary conditions of local optimality. The final choice of the lay-up could be made using other optimality conditions (Legendre, Weierstrass, etc.). Another option for the choice is to compare the several solutions following from the first order local optimality conditions.

7. CONCLUSIONS

- The first order necessary optimality conditions for the postbuckling stiffness maximization problem are highly nonlinear ones;
- The optimality conditions contain two items. One of them corresponds to the plate bending-twisting and is mainly governed by the outermost layers; another one corresponds to the mid-plane strains and is mainly governed by the innermost layers; the layers in the middle are governed by both items;
- The principal mid-plane strain lines and the principal curvature lines play an important role in the conditions;
- The united point-wise optimality condition contains two terms. The first term corresponds to the bending-twisting and equals zero when the tensor of the curvatures is co-axial with the moment tensor. The second term corresponds to the mid-plane strains and equals zero when the mid-plane strain tensor is co-axial with the mid-plane stress flow tensor;
- The united optimality conditions of the paper correspond to the optimality conditions for some orthotropic (point-wise or plate-wise) plate;

- The outermost layers mainly influence the postbuckling behavior at the beginning of the postbuckling;
- The innermost layers play a comparable role with the outermost layers in the case of the developed postbuckling.

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REFERENCES

- [1] Bisagni C, Vescovini R. A fast procedure for the design of composite stiffened panel. *Aeron J* 2015, 119 (No. 212): 185-201.
- [2] Turvey GJ, Marshall IH, eds. *Buckling and Postbuckling of Composite Plates*. London: Chapman and Hall, 1995.
- [3] Wu Z, Raju G, White S, Weaver PM. Optimal design of postbuckling behavior of laminated composite plates using lamination parameters. In: *55th AIAA/ ASME/ ASCE/ AHS/ ASC Structures, Structural Dynamics, and Materials Conference*, National Harbor, Maryland, USA, 2014: Paper AIAA 2014-1376.
- [4] Diaconu CG, Weaver PM. Approximate solution and optimum design of compression-loaded, postbuckled laminated composite plates. *AIAA J* 2005, 43 (4): 906-914.
- [5] Shin DS, Gürdal Z, O'Hayden G. Minimum weight design of laminated composite plates for postbuckling performance. *Appl Mech Rev* 1991, 44 (No. 11, part 2): S219-S231.
- [6] Adali S, Verijenko VE, Walker M. Optimal laminate configurations with symmetric lay-ups for maximum post-buckling stiffness. *Compos Eng* 1994, 4 (No. 11): 1119-1127.

- [7] Adali S, Walker M, Verijenko VE. Multiobjective optimization of laminated plates for maximum prebuckling, buckling and postbuckling strength using continuous and discrete ply angles. *Compos Struct* 1996, 35: 117-130.
- [8] Rahman T, Ijsselmuiden ST, Abdalla MM. Postbuckling analysis of variable stiffness composite plates using a finite element-based perturbation method. *Intl J Struct Stab Dynamics* 2011, 11: 735- 753.
- [9] Dawe DJ. Finite strip buckling and postbuckling analysis. In: Turvey GJ, Marshall IH, editors. *Buckling and postbuckling of composite plates*. London: Chapman and Hall, 1995. p. 108-153.
- [10] Degenhardt R, Kling A, Rohwer K, Thomson RS. Design and analysis of stiffened composite panels including post-buckling and collapse. *Comput & Struct* 2008, 86: 919-929.
- [11] Beerhorst M, Seibel M, Mittelstedt C. Fast analytical method describing the postbuckling behavior of long, symmetric, balanced laminated composite plates under biaxial compression and shear. *Compos Struct* 2012, 94: 2001–2009.
- [12] Mittelstedt C, Erdmann H, Schröder K-U. Postbuckling of imperfect rectangular composite plates under inplane shear closed-form approximate solutions. *Arch Appl Mech* 2011, 81: 1409–1426.
- [13] Nemoto K, Kasuya H, Kikugawa H, Asaka T. Postbuckling Behavior of Composite Laminated Plates with Initial Imperfections under Biaxial Compression. *Mater Transactions* 2009, 50: 299-304.
- [14] Kassapoglou C, Dobyns AL. Simultaneous cost and weight minimization of postbuckled composite panels under combined compression and shear. *Struct Multidisc Optim* 2001, 21: 372-382.
- [15] Zhang J-W, Shen H-S. Postbuckling of orthotropic rectangular plates in biaxial compression. *J Eng Mech* 1991, 117: 1158-1170.

- [16] Wu Z, Weaver PM, Raju G. Postbuckling optimization of variable angle tow composite plates. *Compos Struct* 2013, 103: 34-42.
- [17] Raju G, Wu Z, Weaver PM. Buckling and postbuckling of variable angle tow composite plates under in-plane shear loading. *Intl J Solids Struct*, 2015, 58: 270-287.
- [18] Groh RM, Weaver P. Mass optimization of variable angle tow, variable thickness panels with static failure and buckling constraints. In: [56th AIAA/ASCE/AHS/ASC Structures, Structural Dynamics, and Materials Conference](#), Kissimmee, Florida, USA, 5-9 January 2015.
- [19] Seresta O. Buckling, flutter, and postbuckling optimization of composite structures. PhD dissertation in Aerospace Engineering. Virginia Tech., USA, 2007
- [20] Fares ME, Youssif YG, Elshoraky AE. Non-linear design and control optimization of composite laminated plates with buckling and postbuckling objectives. *Intl J of Non-Lin Mech* 2006; 41 (6–7): 807-824.
- [21] Koide RM, Lima LM, Ferreira APCS, Luersen MA. Postbuckling optimization of laminated composite rectangular plates and stiffened panel under compression using firefly algorithm. In: 24th ABCM International Congress of Mechanical Engineering. December 3-8, 2017, Curitiba, PR, Brazil.
- [22] Henrichsen SR, Weaver PM, Lindgaard E, Lund E. Postbuckling Optimization of Composite Structure Using Koiter's Method. *Intl J of Numer Meth in Eng* 2016, 108 (8): 902–940.
- [23] Hutchinson JW, Koiter WT. Postbuckling theory. *Appl Mech Rev* 1970, 23: 1353-1366.
- [24] Kassapoglou C. *Design and Analysis of Composite Structures: With Applications to Aerospace Structures*. John Wiley & Sons, 2010.
- [25] Weaver P Anisotropic Elastic Tailoring in Laminated Composite Plates and Shells. In: Falzon BG, Aliabadi MH *Buckling and postbuckling structures: experimental, analytical and numerical studies*, Imperial College Press, 2008, pp 177-224.

- [26] Gürdal Z, Haftka R, Hajela P Design and Optimization of Laminated Composite Materials. Wiley, New York, 1999
- [27] Bruyneel M. Optimization of laminated composite structures: problems, solution procedures and applications. In: Durand LP (ed) Composite materials research progress. Nova Science Publ Inc, 2008, pp 51-107
- [28] Ghiasi H, Pasini D, Lessard L, Optimum Stacking Sequence Design of Composite Materials Part I: Constant Stiffness Design. Compos Struct 2009, 90: 1-11
- [29] Ghiasi H, Fayazbakhsh K, Pasini D, Lessard L, Optimum Stacking Sequence Design of Composite Materials Part II: Variable Stiffness Design. Compos Struct 2010, 93: 1-13
- [30] Raju G, Wu Z, White S, Weaver P, Optimal Postbuckling Design of Variable Angle Tow Composite Plates. AIAA J, 2018, 56: 2045-2061
- [31] Peeters DMJ, Hesse S, Abdalla MM. Stacking sequence optimisation of variable stiffness laminates with manufacturing constraints. Compos Struct 2015, 125: 596-604
- [32] Peeters DMJ, Hesse S, Abdalla MM. Multi-objective and multi-constraint optimisation of variable stiffness composite laminates. European Conference on Composite Materials, Sevilla, Spain, June 22-26, 2014.
- [33] D Peeters, Z Hong, M Abdalla. A compliance approximation method applied to variable stiffness composite optimisation. Struct Multidisc Optim 2018, 58 (5): 1981-2001
- [34] Meddaikar YM, Irisarri F-X, Abdalla MM. Laminate optimization of blended composite structures using a modified Shepard's method and stacking sequence tables. Struct Multidisc Optim 2017, 55 (2): 535-546
- [35] Gierlinski J, Mroz Z, Optimal design of elastic plates and beams taking large deflections and shear forces into account. Acta Mech 1981, 39: 77-92
- [36] Gibson RF. Principles of Composite Material Mechanics. New York, McGraw-Hill Inc. 1994

- [37] Washizu K. Variational Methods in Elasticity and Plasticity. 2nd ed. Oxford, New York, Pergamon Press. 1975.
- [38] Gelfand IM, Fomin SV. Calculus of Variations. Dover Publ. 2000.
- [39] Selyugin S. On Choice of Optimal Anisotropy of Composite Plates against Buckling, with Special Attention to Bending-twisting Coupling. Struct Multidisc Optim 2013, 48: 279–294