A Two-Period Stochastic Linear Programming Game

by

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Abstract

We formulate and discuss a model of two-period stochastic optimization which allows for the possibility of the probability distribution over a non-empty finite set of states of nature-exactly one member of which will be realized in the future period- to depend on the control variable chosen in the initial period. In the second and terminal period, the decision maker chooses the value of a control variable vector which along with the control variable vector chosen in the initial period, solve a linear programming problem. The parameters of the linear programming are determined by an uncertain state of nature that is realized at the beginning of the terminal period. In order to obtain specific results, we assume that in the initial period, the decision maker can choose any one alternative from a finite set of alternatives, each such alternative being "a system of linear inequality constraints." It is further assumed that solution sets of any two different systems are mutually disjoint and the probability distribution of the uncertain components of the state variables that is realized in the second period is determined by the system of linear inequalities chosen in the initial period. We refer to the overall problem as a "two-period stochastic linear programming game". Our first result for a two-period stochastic linear programming game is that the optimal value for such a game as well as the optimal solutions for each of the finite number of stochastic linear programs can be found by solving a linear programming problem and our second result in this context says that the (optimal) solution for the dual of this linear programming contains a component that determines "at least one" stochastic linear programming problem any solution of which when substituted in its objective function, yields the optimal value of the two-period stochastic linear programming game. There is no need to solve any other subproblem or the primal problem to derive the optimal value of the game.

Keywords: two-period linear optimal control, stochastic linear programming, probability distribution, game

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1. Introduction: In this note, we discuss a model of two-period stochastic linear programming which allows for the possibility of the probability distribution over a non-empty finite set of states of nature-exactly one member of which will be realized in the future period- to depend on the control variable chosen in the initial period. While stochastic linear programming originates in the work of Dantzig (1955), Dantzig and Madansky (1960) and Walkup and Wets (1967) an extremely lucid presentation of the core idea of two-period stochastic linear programming as pursued in the existing literature can be found in Sen and Higle (1999). A comprehensive survey of the same is available in Shapiro and Philpot (2007). The voluminous works by Kall and Wallace (1994), Prekopa (1995) and Shapiro, Dentcheva and Ruszczynski (2009), takes the reader to the mathematical and methodological frontiers of the existing literature on stochastic programming. However, none of the works mentioned so far allow the probability distribution over the set of future states of nature to depend on the choice made in the initial period.

We propose a model here where in the second and terminal period, the decision maker chooses the value of a control variable vector which along with the control variable chosen in the initial period, solve a linear programming problem. The parameters of the linear programming problem are determined by an uncertain state of nature that is realized at the beginning of the terminal period. We assume that the uncertain component of the state of nature realized at the beginning of the second period belongs to a non-empty finite set of possible states of nature. The probability distribution of the uncertain state of nature may depend on the control variable chosen by the decision maker in the initial period. This makes the problem faced by the decision maker an "interactive decision-making problem", i.e., a "game" (as in "game theory"). The control variable chosen in the initial period is assumed to belong to an arbitrary non-empty set and the control variable chosen in the period also appears in the linear constraints of the second period for the overall linear programming problem. We call this very general framework a "two-period linear optimal control problem". At this level of generality, what continues to remain valid is "Bellman's equation of dynamic programming". We however wish to be more specific about the structure of our model so that it can comfortably fit into the framework of linear programming.

Our main model assumes that in the initial period, the decision maker can choose any one alternative from a finite set of alternatives, each such alternative being "a system of linear inequality constraints." It is further assumed that solution sets of any two different systems are mutually disjoint and the probability distribution of the uncertain components of the state variables that is realized in the second period is determined by the system of linear inequalities chosen in the initial period. Having chosen a linear inequality system, the decision maker is confronted with a stochastic linear programming problem that is very similar to the ones discussed in Sen and Higle (1999), i.e. choose control variable vectors for the two periods to maximize the expected value of a linear objective function. The problem faced by the decision maker involves choosing a linear programming problem in the initial period and this makes the overall problem faced by the decision maker a "possibly non-linear" optimization problem. We refer to the overall problem as a "two-period stochastic linear programming game".

Our first result for a two-period stochastic linear programming game is that the optimal value for such a game as well as the optimal solutions for each of the finite number of stochastic linear programs can be found by solving a linear programming problem and our second result in this context says that the (optimal) solution for the dual of this linear programming contains a component that identifies "at least one" stochastic linear programming subproblem, any solution of which when substituted in its objective function, yields the optimal value of the two-period stochastic linear programming game. There is no need to solve any other subproblem or the primal problem to derive the optimal value of the game. The practical implication (value) of these two results can be realized if in the initial stage there are a large number of systems of linear inequality constraints from which the decision maker has to choose one.

While to the best of our knowledge our model and analysis is original, the mathematical technology we use here is similar to the kind used in Sen and Higle (1999). The other works cited here, including Shapiro and Philpott (2007) use more complex mathematical technology than what is used in our analysis. Our notations and framework of analysis may appear more forbidding than the simplicity of Sen and Higle (1999), since the latter is concerned mainly with numerical examples of stochastic linear programming, whereas we are concerned with developments from a theoretical standpoint. That apart, our entire analysis falls squarely within "finite mathematics" in the tradition of John G. Kemeny, Gerald L. Thompson, and J. Laurie Snell.

2. Notations: For natural numbers m, n, let $\mathbb{R}^{m\times n}$ be the set of real-valued m×n matrices. $\mathbb{R}^{m\times 1}$ denotes the set of all real-valued m-dimensional column vectors and $\mathbb{R}^{1\times n}$ denotes the set of all real-valued n-dimensional row vectors.

The transpose of any finite dimensional real-valued vector x is denoted by x^T.

3. Two-Period Stochastic Dynamic Programming: Given natural numbers n_0, n_1, an **initial control-vector** is a column vector in $\mathbb{R}_+^{n_0}$ and a **terminal control-vector** is a column vector in $\mathbb{R}_+^{n_1}$.

In general, we allow for $n_0 \neq n_1$, since (as in Lahiri (2025)), the initial control-vector may include a deterministic terminal state-variable vector as a sub-vector, and this sub-vector may be the only component of the initial control-vector that can influence the choice of the terminal control-vector. Under such situations we may require $n_0 > n_1$.

Let $\mathcal{U}^{(0)}$ be a subset of $\mathbb{R}_+^{n_0}$. $\mathcal{U}^{(0)}$ is the **initial feasible set**.

Let q be an n_0 dimensional real column vector in $\mathbb{R}_+^{n_0}$. The vector q comprises the coefficients of the linear function of the initial period control variable vector in the objective function of the optimization problem faced by the decision maker.

Let Ω be a non-empty finite set of "a non-deterministic component" of states of nature. A state of nature in Ω is realized at the beginning of the terminal period. Let θ denote a generic element in Ω.

Given a natural number n_1, for each θϵΩ, let $(p(\theta), A^{(1)}(\theta), B^{(1)}(\theta), b^{(1)}(\theta)) \epsilon \mathbb{R}^{n_1 \times 1} \times \mathbb{R}^{m_1 \times n_0} \times \mathbb{R}^{m_1 \times n_1} \times \mathbb{R}^{m_1 \times 1}$.

For each (u, θ) ϵ $\mathcal{U}^{(0)} \times \Omega$ the **terminal feasible set from** (u, θ) denoted by $\mathcal{F}^{(1)}(u, \theta)$ is defined as $\mathcal{F}^{(1)}(u, \theta) = \{v \epsilon \mathbb{R}_+^{n_1} | A^{(1)}(\theta)u + B^{(1)}(\theta)v \leq b^{(1)}(\theta)\}$.

For each $u \epsilon \mathcal{U}^{(0)}$, let $\pi(.|u):\Omega \to [0, 1]$ be a function satisfying $\sum_{\theta \epsilon \Omega} \pi(\theta|u) = 1$. For each (u, θ)ϵ $\mathcal{U}^{(0)} \times \Omega$, π(θ|u) denotes the probability with which the "non-deterministic component" θ of future state of nature is realized, conditional on u being the value of the control variable chosen in the initial period.

Having chosen $u \epsilon \mathcal{U}^{(0)}$ in the initial period, the problem faced by the decision maker in the terminal period if θϵΩ is realized, is to Maximize $p(\theta)^T u(1)$, subject to $A^{(1)}(\theta)u + B^{(1)}(\theta)u(1) \leq b^{(1)}(\theta)$, $u(1) \epsilon \mathbb{R}_+^{n_1}$.

For (u, θ) ϵ $\mathcal{U}^{(0)} \times \Omega$, let $V(u, \theta) = p(\theta)^T u(1, (u, \theta))$, where $u(1, (u, \theta)) \epsilon \underset{u(1) \epsilon \mathcal{F}^{(1)}(u,\theta)}{\operatorname{argmax}} p(\theta)^T u(1)$.

It is easy to verify that if $\mathcal{U}^{(0)}$ is a non-empty convex set, then for all θϵΩ, the function $V(., \theta)$: $\mathcal{U}^{(0)} \to \mathbb{R}$ as implicitly defined above, is a concave function.

The dynamic programming problem faced by the decision maker is to Maximize $q^T u + \sum_{\theta \epsilon \Omega} \pi(\theta|u) V(u, \theta)$ subject to $u \epsilon \mathcal{U}^{(0)}$.

Consider the following optimization problem which we shall refer to as **two-period linear optimal control (TPLOC) problem**:

Maximize $q^T u(0) + \sum_{\theta \epsilon \Omega} \pi(\theta|u(0)) p(\theta)^T u(1, \theta)$ subject to $u(0) \epsilon \mathcal{U}^{(0)}$ and for all θϵΩ: $A^{(1)}(\theta) u(0) + B^{(1)}(\theta)u(1, \theta) \leq b^{(1)}(\theta)$, $u(1, \theta) \epsilon \mathbb{R}_+^{n_1}$.

If Ω is a singleton, then the TPLOC problem reduces to a **linear programming optimal control (LPOC)** problem.

Theorem 1: $u^*(0)$, $<u^*(1, \theta)|\theta \epsilon \Omega>$ solve TPLOC problem <u>if and only if</u> $[u^*(0) \epsilon \underset{u \epsilon \mathcal{U}^{(0)}}{\operatorname{argmax}} \{q^T u + \sum_{\theta \epsilon \Omega} \pi(\theta|u) V(u, \theta)\}$ & for all θϵΩ, $u^*(1, \theta) \epsilon \underset{u(1) \epsilon \mathcal{F}^{(1)}(u^*(0),\theta)}{\operatorname{argmax}} p^T u(1)]$.

Proof: Part 1: Suppose $u^*(0)$, $<u^*(1, \theta)|\theta \epsilon \Omega>$ solve TPLOC problem. Thus, for all θϵΩ, $u^*(1, \theta) \epsilon \mathcal{F}^{(1)}(u^*(0), \theta)$.

Towards a contradiction suppose, $u^*(0) \notin \underset{u \epsilon \mathcal{U}^{(0)}}{\operatorname{argmax}} \{q^T u + \sum_{\theta \epsilon \Omega} \pi(\theta|u) V(u, \theta)\}$.

Thus, there exists $u(0) \epsilon \mathcal{U}^{(0)}$ such that $q^T u(0) + \sum_{\theta \epsilon \Omega} \pi\big(\theta \big| u(0)\big) V(u(0), \theta) > q^T u^*(0) + \sum_{\theta \epsilon \Omega} \pi\big(\theta \big| u^*(0)\big) V(u^*(0), \theta)$.

Since for all $\theta \epsilon \Omega$, $u^*(1, \theta) \epsilon \mathcal{F}^{(1)}(u^*(0), \theta)$, it must be the case that $V(u^*(0), \theta) \geq p(\theta)^T u^*(1, \theta)$.

Since, $V(u(0), \theta) = p^T u(1, (u(0), \theta))$, where $u(1, (u(0), \theta)) \epsilon \underset{u(1) \epsilon \mathcal{F}^{(1)}(u(0), \theta)}{\operatorname{argmax}} p(\theta)^T u(1)$ it follows that $q^T u(0) + \sum_{\theta \epsilon \Omega} \pi\big(\theta \big| u(0)\big) p(\theta)^T u(1, (u(0), \theta)) > q^T u^*(0) + \sum_{\theta \epsilon \Omega} \pi\big(\theta \big| u(0)\big) p(\theta)^T V(u^*(0), \theta) \geq q^T u^*(0) + \sum_{\theta \epsilon \Omega} \pi\big(\theta \big| u(0)\big) p(\theta)^T u^*(1, \theta)$

Since, $u(0) \epsilon \mathcal{U}^{(0)}$, $A^{(1)}(\theta) u(0) + B^{(1)}(\theta) u(1, u(0)) \leq b^{(1)}(\theta)$, $u(1, u(0)) \epsilon \mathbb{R}_+^{n_1}$ the assumption that $u^*(0)$, $<u^*(1, \theta) | \theta \epsilon \Omega>$ solve the TPLOC problem is contradicted.

Thus, it must be that $u^*(0) \epsilon \underset{u \epsilon \mathcal{U}^{(0)}}{\operatorname{argmax}} \{q^T u + \sum_{\theta \epsilon \Omega} \pi(\theta | u) V(u, \theta)\}$.

Suppose that for some $\theta \epsilon \Omega$, $u^*(1, \theta) \notin \underset{u(1) \epsilon \mathcal{F}^{(1)}(u^*(0), \theta)}{\operatorname{argmax}} p^T u(1)$. Then, there exists $u(1, (u^*(0), \theta)) \epsilon \mathcal{F}^{(1)}(u^*(0), \theta)$ satisfying $p(\theta)^T u^*(1, (u^*(0), \theta)) > p(\theta)^T u(1, u^*(0))$.

Thus, $q^T u^*(0) + p(\theta)^T u^*(1, (u^*(0), \theta)) > q^T u^*(0) + p(\theta)^T u(1, u^*(0))$.

Since, $u^*(0) \epsilon \mathcal{U}^{(0)}$, $A^{(1)}(\theta) u^*(0) + B^{(1)}(\theta) u(1, (u^*(0), \theta)) \leq b^{(1)}(\theta)$, $u(1, u^*(0)) \epsilon \mathbb{R}_+^{n_1}$ the assumption that $u^*(0)$, $<u^*(1, \theta) | \theta \epsilon \Omega>$ solve the TPLOC problem is once again contradicted.

Thus, it must be the case that for all $\theta \epsilon \Omega$, $u^*(1,\theta) \epsilon \underset{u(1) \epsilon \mathcal{F}^{(1)}(u^*(0), \theta)}{\operatorname{argmax}} p(\theta)^T u(1)$.

Part 2: Now suppose that $u^*(0) \epsilon \underset{u \epsilon \mathcal{U}^{(0)}}{\operatorname{argmax}} \{q^T u + \sum_{\theta \epsilon \Omega} \pi(\theta | u) V(u, \theta)\}$ & for all $\theta \epsilon \Omega$, $u^*(1, \theta) \epsilon \underset{u(1) \epsilon \mathcal{F}^{(1)}(u^*(0), \theta)}{\operatorname{argmax}} p^T u(1)$.

Towards a contradiction suppose that $u^*(0)$, $<u^*(1, \theta) | \theta \epsilon \Omega>$ does not solve the TPLOC problem.

Thus, there exists $u(0)$, $<u(1, \theta) | \theta \epsilon \Omega>$ such that $u(0) \epsilon \mathcal{U}^{(0)}$ and for all $\theta \epsilon \Omega$: $A^{(1)}(\theta) u(0) + B^{(1)}(\theta) u(1) \leq b^{(1)}(\theta)$, $u(1, \theta) \epsilon \mathbb{R}_+^{n_1}$ <u>and</u> $q^T u(0) + \sum_{\theta \epsilon \Omega} \pi(\theta | u) p(\theta)^T u(1, \theta) > q^T u^*(0) + \sum_{\theta \epsilon \Omega} \pi(\theta | u) p(\theta)^T u^*(1, \theta)$.

Since for all $\theta \epsilon \Omega$: $A^{(1)}(\theta) u(0) + B^{(1)}(\theta) u(1) \leq b^{(1)}(\theta)$, $u(1, \theta) \epsilon \mathbb{R}_+^{n_1}$ it must be the case that for all $\theta \epsilon \Omega$, $V(u(0), \theta) \geq p(\theta)^T u(1,\theta)$.

Thus, $q^T u(0) + \sum_{\theta \epsilon \Omega} \pi(\theta | u) V(u, \theta) > q^T u^*(0) + \sum_{\theta \epsilon \Omega} \pi(\theta | u) p(\theta)^T u^*(1, \theta)$.

Since for all $\theta \epsilon \Omega$, $u^*(1, \theta) \epsilon \underset{u(1) \epsilon \mathcal{F}^{(1)}(u^*(0), \theta)}{\operatorname{argmax}} p(\theta)^T u(1)$, it must be that for all $\theta \epsilon \Omega$, $p^T u^*(1, \theta) = V(u^*(0), \theta)$.

Thus, $q^T u(0) + \sum_{\theta\epsilon\Omega}\pi(\theta|u)\, V(u,\theta) > q^T u^*(0) + \sum_{\theta\epsilon\Omega}\pi(\theta|u) p(\theta)^T\, u^*(1,\theta)$, contradicting our assumption that $u^*(0) \,\epsilon\, \underset{u\epsilon\mathcal{U}^{(0)}}{\operatorname{argmax}}\{q^T u + \sum_{\theta\epsilon\Omega}\pi(\theta|u)\, V(u,\theta)\}$.

Thus, it must be the case that $u^*(0)$, $<u^*(1, \theta)|\theta\epsilon\Omega>$ solve the TPLOC problem. Q.E.D.

4. A Two-Period Stochastic Linear Programming Game: At the level of generality at which TPLOC has been discussed so far, it is difficult to relate either the dynamic programming problem or its equivalent TPLOC problem to a linear programming problem. Thus, we will be a little more specific about the feasible set of control variables in the initial period, so that the problem can be formulated "almost like" a two-period linear programming problem.

Suppose that for some natural numbers m_0 and K, $<(A^{(0,k)}, b^{(0,k)})|\, k = 1, \ldots, K>$ is an array in $\mathbb{R}^{m_0\times n_0}\times\mathbb{R}^{m_0\times 1}$.

For $k\epsilon\{1, \ldots, K\}$, let $\mathcal{F}^{(0)}(k) = \{u\epsilon\mathbb{R}_+^{n_0}|\, A^{(0,k)}u \leq b^{(0,k)}\}$.

Initially Mutually Disjoint Feasible Sets: For what follows, we assume that sets in the collection $\{\mathcal{F}^{(0)}(k)|k\epsilon\{1, \ldots, K\}\}$ are "mutually disjoint".

Let $\mathcal{U}^{(0)} = \bigcup_{k=1}^{K}\mathcal{F}^{(0)}(k)$ and for each $k\epsilon\{1, \ldots, K\}$, let $\pi^{(k)}:\Omega \to [0, 1]$ be a function such that for all $(u, \theta)\epsilon\, \mathcal{F}^{(0)}(k)\times\Omega$, $\pi(\theta|u) = \pi^{(k)}(\theta)$.

In such a situation the optimization problem faced by the decision maker is the following: Choose $(u^*(0), <u^*(1, \theta)|\, |\theta\epsilon\Omega>)$ and $<(u^{(k)*}(0), <u^{(k)*}(1, \theta)|\theta\epsilon\Omega>)|k = 1, \ldots, K>$ such that $(u^*(0), <u^*(1, \theta)|\, |\theta\epsilon\Omega>) \,\epsilon\, \underset{<(u^{(k)},<u^{(k)}(1,\theta)|\theta\epsilon\Omega>|k\epsilon\{1,\ldots,K\}>}{\operatorname{argmax}} \{q^T u^{(k)*}(0) + \sum_{\theta\epsilon\Omega}\pi^{(k)}(\theta)p(\theta)^T u^{(k)*}(1,\theta)\}$, and for each $k\epsilon\{1, \ldots, K\}$, $(u^{(k)*}(0), <u^{(k)*}(1, \theta)|\theta\epsilon\Omega>)$ solves:

Maximize $q^T u^{(k)}(0) + \sum_{\theta\epsilon\Omega}\pi^{(k)}(\theta)p(\theta)^T u^{(k)}(1,\theta)$ subject to $A^{(0,k)}u^{(k)}(0) \leq b^{(0,k)}$, $u^{(k)}(0)\epsilon\mathbb{R}_+^{n_0}$, $A^{(1)}(\theta)\, u^{(k)}(0) + B^{(1)}(\theta)u^{(k)}(1, \theta) \leq b^{(1)}(\theta)$, $u^{(k)}(1, \theta)\epsilon\mathbb{R}_+^{n_1}$.

TPLOC now amounts to choosing "a best" from K different linear programming problems. TPLOC may now be referred to as **"two-period stochastic linear programming (TPSLP) game**".

The TPSLP game is equivalent to the following optimization problem: Maximize $\sum_{k=1}^{K}[q^T u^{(k)}(0) + \sum_{\theta\epsilon\Omega}\pi^{(k)}(\theta)p(\theta)^T u^{(k)}(1,\theta)]$ subject to $A^{(0,k)}u^{(k)}(0) \leq b^{(0,k)}$, $u^{(k)}(0)\epsilon\mathbb{R}_+^{n_0}$, $A^{(1)}(\theta)\, u^{(k)}(0) + B^{(1)}(\theta)u^{(k)}(1, \theta) \leq b^{(1)}(\theta)$, $u^{(k)}(1, \theta)\epsilon\mathbb{R}_+^{n_1}$ for all $\theta\epsilon\Omega$ and $k\epsilon\{1, \ldots, K\}$, $(u(0), <u(1, \theta)|\, |\theta\epsilon\Omega>)\epsilon \underset{<(u^{(k)},<u^{(k)}(1,\theta)|\theta\epsilon\Omega>|k\epsilon\{1,\ldots,K\}>}{\operatorname{argmax}} \{q^T u^{(k)}(0) + \sum_{\theta\epsilon\Omega}\pi^{(k)}(\theta)p(\theta)^T u^{(k)}(1,\theta)\}$.

The following result is an immediate consequence of the duality theorems and complementary slackness conditions of linear programming.

Theorem 2: ($u^*(0)$, $<u^*(1, \theta)|\theta\epsilon\Omega>$), $<(u^{(k)*}(0), <u^{(k)*}(1, \theta)|\theta\epsilon\Omega>)|k = 1, \ldots, K>$ solve the TPSLP game <u>if and only if</u>:

(1) ($u^*(0)$, $<u^*(1, \theta)|\theta\epsilon\Omega>$)$\epsilon \underset{<(u^{(k)*}, <u^{(k)*}(1,\theta)|\theta\epsilon\Omega>|k\epsilon\{1,\ldots,K\}>}{\text{argmax}} \{q^T u^{(k)*}(0) + \sum_{\theta\epsilon\Omega}\pi^{(k)}(\theta)p(\theta)^T u^{(k)*}(1,\theta)\}$; &

(2) for each $k\epsilon\{1, \ldots, K\}$ there exists ($y^{(k)}(0)$, $<y^{(k)}(\theta)|\theta\epsilon\Omega>$) that along with ($u^{(k)*}(0)$, $<u^{(k)*}(1, \theta)|\theta\epsilon\Omega>$) satisfy the following conditions:

(i) $A^{(0,k)}u^{(k)*}(0) \leq b^{(0,k)}$, $A^{(1)}(\theta)u^{(k)*}(0) + B^{(1)}(\theta)u^{(k)*}(1, \theta) \leq b^{(1)}(\theta)$, $u^{(k)*}(0)\epsilon\mathbb{R}_+^{n_0}$ and for all $u^{(k)}(1, \theta)\epsilon\mathbb{R}_+^{n_1}$ for all $\theta\epsilon\Omega$.

(ii) $[y^{(k)}(0)]^T A^{(0,k)} + [y^{(k)}(\theta)]^T A^{(1)}(\theta) \geq q^T$, $[y^{(k)}(\theta)]^T A^{(1)}(\theta) \geq \pi^{(k)}(\theta)p(\theta)^T$, $y^{(k)}(\theta) \epsilon\mathbb{R}_+^{m_1}$ for all $\theta\epsilon\Omega$ and $y^{(k)}(0)\epsilon\mathbb{R}_+^{m_0}$.

(iii) $[y^{(k)}(0)]^T(A^{(0,k)}u^{(k)*}(0) - b^{(0,k)}) = 0$, $[y^{(k)}(\theta)]^T(A^{(1)}(\theta)u^{(k)*}(0) + B^{(1)}(\theta)u^{(k)*}(1, \theta) - b^{(1)}(\theta)) = 0$ for all $\theta\epsilon\Omega$.

(iv) $([y^{(k)}(0)]^T A^{(0,k)} + [y^{(k)}(\theta)]^T A^{(1)}(\theta) - q^T)\, u^{(k)*}(0) = 0$, $([y^{(k)}(\theta)]^T A^{(1)}(\theta) - \pi^{(k)}(\theta)p(\theta)^T)\, u^{(k)*}(1, \theta) = 0$ for all $\theta\epsilon\Omega$.

5. The Optimal Value: While the TPSLP game is not a "totally" linear programming formulation of the problem faced by the decision maker, it is possible to find the optimal value for the two-period stochastic linear programming game by solving a single linear programming problem. The relevant result is the following.

Theorem 3: ($u^*(0)$, $<u^*(1, \theta)|\theta\epsilon\Omega>$), $<(u^{(k)*}(0), <u^{(k)*}(1, \theta)|\theta\epsilon\Omega>)|k = 1, \ldots, K>$ solve the TPSLP game <u>if and only if</u> $<(u^{(k)*}(0), <u^{(k)*}(1, \theta)|\theta\epsilon\Omega>)|k = 1, \ldots, K>$, $M^* = q^T u^*(0) + \sum_{\theta\epsilon\Omega}\pi^{(\hat{k})}(\theta)p(\theta)^T u^*(1,\theta)$ with ($u^*(0)$, $<u^*(1, \theta)|\theta\epsilon\Omega>$) = ($u^{(\hat{k})*}$, $<u^{(\hat{k})*}(1, \theta)|\theta\epsilon\Omega>$)$\epsilon \underset{<(u^{(k)*}, <u^{(k)*}(1,\theta)|\theta\epsilon\Omega>|k\epsilon\{1,\ldots,K\}>}{\text{argmax}} \{q^T u^{(k)*}(0) + \sum_{\theta\epsilon\Omega}\pi^{(k)}(\theta)p(\theta)^T u^{(k)*}(1,\theta)\}$ solve the following linear programming problem denoted **LP-1**:

Maximize $\sum_{k=1}^{K} 2[q^T u^{(k)}(0) + \sum_{\theta\epsilon\Omega}\pi^{(k)}(\theta)p(\theta)^T u^{(k)}(1,\theta)] - M$, subject to $q^T u^{(k)}(0) + \sum_{\theta\epsilon\Omega}\pi^{(k)}(\theta)p(\theta)^T u^{(k)}(1,\theta) - M \leq 0$, $A^{(0,k)}u^{(k)}(0) \leq b^{(0,k)}$, $u^{(k)}(0)\epsilon\mathbb{R}_+^{n_0}$, $A^{(1)}(\theta)\, u^{(k)}(0) + B^{(1)}(\theta)u^{(k)}(1, \theta) \leq b^{(1)}(\theta)$, $u^{(k)}(1, \theta)\epsilon\mathbb{R}_+^{n_1}$ for all $\theta\epsilon\Omega$ and $k\epsilon\{1, \ldots, K\}$, $M\epsilon\mathbb{R}$.

M^* is said to be the optimal value of the TPSLP game.

Proof: Part 1: Suppose, ($u^*(0)$, $<u^*(1, \theta)|\theta\epsilon\Omega>$), $<(u^{(k)*}(0), <u^{(k)*}(1, \theta)|\theta\epsilon\Omega>)|k = 1, \ldots, K>$ solve the TPSLP game.

Thus, there exists ($u^{(\hat{k})^*}$, <$u^{(\hat{k})^*}$(1, θ)|θ∊Ω>)∊ $\underset{<(u^{(k)*},<u^{(k)*}(1,\theta)|\theta\epsilon\Omega>|k\epsilon\{1,\ldots,K\}>}{\text{argmax}}$ $\{q^T u^{(k)*}(0) + \sum_{\theta\epsilon\Omega}\pi^{(k)}(\theta)p(\theta)^T u^{(k)*}(1,\theta)\}$ and for each k∊{1, …, K}, ($u^{(k)*}(0)$, <$u^{(k)*}$(1, θ)|θ∊Ω>) solves Maximize $q^T u^{(k)}(0) + \sum_{\theta\epsilon\Omega}\pi^{(k)}(\theta)p(\theta)^T u^{(k)}(1,\theta)$ subject to $A^{(0,k)}u^{(k)}(0) \leq b^{(0,k)}$, $u^{(k)}(0)\epsilon\mathbb{R}_+^{n_0}$, $A^{(1)}(\theta)\, u^{(k)}(0) + B^{(1)}(\theta)u^{(k)}(1, \theta) \leq b^{(1)}(\theta)$, $u^{(k)}(1, \theta)\epsilon\mathbb{R}_+^{n_1}$.

Let $M^* = q^T u^{(\hat{k})*}(0) + \sum_{\theta\epsilon\Omega}\pi^{(\hat{k})}(\theta)p(\theta)^T u^{(\hat{k})*}(1,\theta)$.

Thus, $\sum_{k=1}^{K} 2[q^T u^{(k)*}(0) + \sum_{\theta\epsilon\Omega}\pi^{(k)*}(\theta)p(\theta)^T u^{(k)*}(1,\theta)] - M^* = 2\sum_{k\neq\hat{k}}[q^T u^{(k)*}(0) + \sum_{\theta\epsilon\Omega}\pi^{(k)*}(\theta)p(\theta)^T u^{(k)*}(1,\theta)] + q^T u^{(\hat{k})*}(0) + \sum_{\theta\epsilon\Omega}\pi^{(\hat{k})}(\theta)p(\theta)^T u^{(\hat{k})*}(1,\theta)$.

Towards a contradiction suppose, there exists <($u^{(k)}(0)$, <$u^{(k)}$(1, θ)|θ∊Ω>)|k = 1, …, K>, M such that $q^T u^{(k)}(0) + \sum_{\theta\epsilon\Omega}\pi^{(k)}(\theta)p(\theta)^T u^{(k)}(1,\theta) - M \leq 0$, $A^{(0,k)}u^{(k)}(0) \leq b^{(0,k)}$, $u^{(k)}(0)\epsilon\mathbb{R}_+^{n_0}$, $A^{(1)}(\theta)\, u^{(k)}(0) + B^{(1)}(\theta)u^{(k)}(1, \theta) \leq b^{(1)}(\theta)$, $u^{(k)}(1, \theta)\epsilon\mathbb{R}_+^{n_1}$ for all θ∊Ω and k∊{1, …, K}, M∊ℝ and further $\sum_{k=1}^{K} 2[q^T u^{(k)}(0) + \sum_{\theta\epsilon\Omega}\pi^{(k)}(\theta)p(\theta)^T u^{(k)}(1,\theta)] - M > 2\sum_{k\neq\hat{k}}[q^T u^{(k)*}(0) + \sum_{\theta\epsilon\Omega}\pi^{(k)*}(\theta)p(\theta)^T u^{(k)*}(1,\theta)] + q^T u^{(\hat{k})*}(0) + \sum_{\theta\epsilon\Omega}\pi^{(\hat{k})}(\theta)p(\theta)^T u^{(\hat{k})*}(1,\theta)$.

We know that $q^T u^{(k)*}(0) + \sum_{\theta\epsilon\Omega}\pi^{(k)}(\theta)p(\theta)^T u^{(k)*}(1,\theta) \geq q^T u^{(k)}(0) + \sum_{\theta\epsilon\Omega}\pi^{(k)}(\theta)p(\theta)^T u^{(k)}(1,\theta)$ for all k∊{1, …, K}.

Thus, it must be the case that $q^T u^{(\hat{k})*}(0) + \sum_{\theta\epsilon\Omega}\pi^{(\hat{k})}(\theta)p(\theta)^T u^{(\hat{k})*}(1,\theta) < 2[q^T u^{(\hat{k})}(0) + \sum_{\theta\epsilon\Omega}\pi^{(\hat{k})}(\theta)p(\theta)^T u^{(\hat{k})}(1,\theta)] - M = q^T u^{(\hat{k})}(0) + \sum_{\theta\epsilon\Omega}\pi^{(\hat{k})}(\theta)p(\theta)^T u^{(\hat{k})}(1,\theta) + [q^T u^{(\hat{k})}(0) + \sum_{\theta\epsilon\Omega}\pi^{(\hat{k})}(\theta)p(\theta)^T u^{(\hat{k})}(1,\theta) - M]$.

Since, $q^T u^{(\hat{k})}(0) + \sum_{\theta\epsilon\Omega}\pi^{(\hat{k})}(\theta)p(\theta)^T u^{(\hat{k})}(1,\theta) - M \leq 0$, we get $2[q^T u^{(\hat{k})}(0) + \sum_{\theta\epsilon\Omega}\pi^{(\hat{k})}(\theta)p(\theta)^T u^{(\hat{k})}(1,\theta)] - M \leq q^T u^{(\hat{k})}(0) + \sum_{\theta\epsilon\Omega}\pi^{(\hat{k})}(\theta)p(\theta)^T u^{(\hat{k})}(1,\theta)$.

Thus, $q^T u^{(\hat{k})*}(0) + \sum_{\theta\epsilon\Omega}\pi^{(\hat{k})}(\theta)p(\theta)^T u^{(\hat{k})*}(1,\theta) < q^T u^{(\hat{k})}(0) + \sum_{\theta\epsilon\Omega}\pi^{(\hat{k})}(\theta)p(\theta)^T u^{(\hat{k})}(1,\theta)$, contradicting ($u^{(\hat{k})^*}$, <$u^{(\hat{k})^*}$(1, θ)|θ∊Ω>) solves the linear programming problem relevant for $\hat{k}$ in the definition of the TPSLP game.

Thus, it must be the case that <($u^{(k)*}(0)$, <$u^{(k)*}$(1, θ)|θ∊Ω>)|k = 1, …, K>, M^* solve LP-1.

Part 2: Suppose <($u^{(k)*}(0)$, <$u^{(k)*}$(1, θ)|θ∊Ω>)|k = 1, …, K>, M^* solve LP-1.

Thus, $M^* = q^T u^*(0) + \sum_{\theta\epsilon\Omega}\pi^{(\hat{k})}(\theta)p(\theta)^T u^*(1,\theta)$ with ($u^*(0)$, <u^*(1, θ)|θ∊Ω>) = ($u^{(\hat{k})^*}$, <$u^{(\hat{k})^*}$(1, θ)|θ∊Ω>)∊ $\underset{<(u^{(k)*},<u^{(k)*}(1,\theta)|\theta\epsilon\Omega>|k\epsilon\{1,\ldots,K\}>}{\text{argmax}}$ $\{q^T u^{(k)*}(0) + \sum_{\theta\epsilon\Omega}\pi^{(k)}(\theta)p(\theta)^T u^{(k)*}(1,\theta)\}$.

Towards a contradiction suppose that ($u^*(0)$, <u^*(1, θ)|θ∊Ω>) along with <($u^{(k)*}(0)$, <$u^{(k)*}$(1, θ)|θ∊Ω>)|k = 1, …, K> does not solve the TPSLP game.

If for all kϵ{1, …, K}, ($u^{(k)^*}$, <$u^{(k)^*}$(1, θ)|θϵΩ>) solves Maximize $q^T u^{(k)}(0) + \sum_{\theta\epsilon\Omega}\pi^{(k)}(\theta)p(\theta)^T u^{(k)}(1,\theta)$ subject to $A^{(0,k)}u^{(k)}(0) \leq b^{(0,k)}$, $u^{(k)}(0)\epsilon\mathbb{R}_+^{n_0}$, $A^{(1)}(\theta)\, u^{(k)}(0) + B^{(1)}(\theta)u^{(k)}(1, \theta) \leq b^{(1)}(\theta)$, $u^{(k)}(1, \theta)\epsilon\mathbb{R}_+^{n_1}$ for all θϵΩ, then ($u^*(0)$, <$u^*(1, \theta)$|θϵΩ>) along with <($u^{(k)*}(0)$, <$u^{(k)*}(1, \theta)$|θϵΩ>)|k = 1, …, K> solve the TPSLP game.

Thus, suppose there exists $k^\#\epsilon${1, …, K} and ($\tilde{u}^{k^\#}(0)$, <$\tilde{u}^{(k^\#)}(1, \theta)$|θ$\epsilon$Ω>) such that $A^{(0,k)}\, \tilde{u}^{(k^\#)}(0) \leq b^{(0,k)}$, $\tilde{u}^{(k^\#)}(0)\epsilon\mathbb{R}_+^{n_0}$, $A^{(1)}(\theta)\, \tilde{u}^{(k^\#)}(0) + B^{(1)}(\theta)\, \tilde{u}^{(k^\#)}(1, \theta) \leq b^{(1)}(\theta)$, $\tilde{u}^{(k^\#)}(1, \theta)\epsilon\mathbb{R}_+^{n_1}$ for all θϵΩ and $q^T\, \tilde{u}^{(k^\#)}(0) + \sum_{\theta\epsilon\Omega}\pi^{(k^\#)}(\theta)p(\theta)^T\tilde{u}^{(k^\#)}(1,\theta) > q^T u^{(k^\#)*}(0) + \sum_{\theta\epsilon\Omega}\pi^{(k^\#)*}(\theta)p(\theta)^T u^{(k^\#)*}(1,\theta)$.

If $q^T\, \tilde{u}^{(k^\#)}(0) + \sum_{\theta\epsilon\Omega}\pi^{(k^\#)}(\theta)p(\theta)^T\tilde{u}^{(k^\#)}(1,\theta) \leq M^*$, then by letting ($u^{(k)}(0)$, <$u^{(k)}(1, \theta)$|θ$\epsilon$Ω>) = ($u^{(k)*}(0)$, <$u^{(k)*}(1, \theta)$|θ$\epsilon$Ω>) for all $k \neq k^\#$, ($u^{(k^\#)}(0)$, <$u^{(k^\#)}(1, \theta)$|θϵΩ>) = ($\tilde{u}^{k^\#}(0)$, <$\tilde{u}^{(k^\#)}(1, \theta)$|θ$\epsilon$Ω>) and M = M*, we get $\sum_{k=1}^{K} 2[q^T u^{(k)}(0) + \sum_{\theta\epsilon\Omega}\pi^{(k)}(\theta)p(\theta)^T u^{(k)}(1,\theta)] - M^* > \sum_{k=1}^{K} 2[q^T u^{(k)*}(0) + \sum_{\theta\epsilon\Omega}\pi^{(k)}(\theta)p(\theta)^T u^{(k)*}(1,\theta)] - M^*$, contradicting our assumption that <($u^{(k)*}(0)$, <$u^{(k)*}(1, \theta)$|θϵΩ>)|k = 1, …, K>, M* solve LP-1.

Thus, it must be the case that $q^T\, \tilde{u}^{(k^\#)}(0) + \sum_{\theta\epsilon\Omega}\pi^{(k^\#)}(\theta)p(\theta)^T\tilde{u}^{(k^\#)}(1,\theta) > M^*$.

Let <($u^{(k)}(0)$, <$u^{(k)}(1, \theta)$|θϵΩ>)| k = 1, …, K> be defined as in the case of $q^T\, \tilde{u}^{(k^\#)}(0) + \sum_{\theta\epsilon\Omega}\pi^{(k^\#)}(\theta)p(\theta)^T\tilde{u}^{(k^\#)}(1,\theta) \leq M^*$ and let $M = q^T\, \tilde{u}^{(k^\#)}(0) + \sum_{\theta\epsilon\Omega}\pi^{(k^\#)}(\theta)p(\theta)^T\tilde{u}^{(k^\#)}(1,\theta)$.

Then, $\sum_{k=1}^{K} 2[q^T u^{(k)}(0) + \sum_{\theta\epsilon\Omega}\pi^{(k)}(\theta)p(\theta)^T u^{(k)}(1,\theta)] - M = \sum_{k\neq k^\#} 2[q^T u^{(k)}(0) + \sum_{\theta\epsilon\Omega}\pi^{(k)}(\theta)p(\theta)^T u^{(k)}(1,\theta)] + q^T\, \tilde{u}^{(k^\#)}(0) + \sum_{\theta\epsilon\Omega}\pi^{(k^\#)}(\theta)p(\theta)^T\tilde{u}^{(k^\#)}(1,\theta) > \sum_{k\neq k^\#} 2[q^T u^{(k)*}(0) + \sum_{\theta\epsilon\Omega}\pi^{(k)}(\theta)p(\theta)^T u^{(k)*}(1,\theta)] + q^T u^{(k^\#)*}(0) + \sum_{\theta\epsilon\Omega}\pi^{(k^\#)*}(\theta)p(\theta)^T u^{(k^\#)*}(1,\theta) \geq \sum_{k\neq k^\#} 2[q^T u^{(k)*}(0) + \sum_{\theta\epsilon\Omega}\pi^{(k)}(\theta)p(\theta)^T u^{(k)*}(1,\theta)] + q^T u^{(k^\#)*}(0) + \sum_{\theta\epsilon\Omega}\pi^{(k^\#)*}(\theta)p(\theta)^T u^{(k^\#)*}(1,\theta) + [q^T u^{(k^\#)*}(0) + \sum_{\theta\epsilon\Omega}\pi^{(k^\#)*}(\theta)p(\theta)^T u^{(k^\#)*}(1,\theta) - M^*]$, since $q^T u^{(k^\#)*}(0) + \sum_{\theta\epsilon\Omega}\pi^{(k^\#)*}(\theta)p(\theta)^T u^{(k^\#)*}(1,\theta) - M^* \leq 0$.

Thus, $\sum_{k=1}^{K} 2[q^T u^{(k)}(0) + \sum_{\theta\epsilon\Omega}\pi^{(k)}(\theta)p(\theta)^T u^{(k)}(1,\theta)] - M > \sum_{k=1}^{K} 2[q^T u^{(k)*}(0) + \sum_{\theta\epsilon\Omega}\pi^{(k)*}(\theta)p(\theta)^T u^{(k)*}(1,\theta)] - M^*$, contradicting our assumption that <($u^{(k)*}(0)$, <$u^{(k)*}(1, \theta)$|θϵΩ>)|k = 1, …, K>, M* solve LP-1.

Hence, it must be that ($u^*(0)$, <$u^*(1, \theta)$|θϵΩ>) along with <($u^{(k)*}(0)$, <$u^{(k)*}(1, \theta)$|θϵΩ>)|k = 1, …, K> solve the TPSLP game. Q.E.D.

Applying the duality theorem and complementary slackness conditions of linear programming to the linear programming problem we obtain a procedure to obtain at least one (optimal) solution to the TPSLP game.

Theorem 4: $(u^*(0), \langle u^*(1, \theta)|\theta\epsilon\Omega\rangle), \langle(u^{(k)*}(0), \langle u^{(k)*}(1, \theta)|\theta\epsilon\Omega\rangle)|k = 1, \ldots, K\rangle$ solve the TPSLP game <u>if and only if</u>

(1) There exists an array of real numbers $\langle y^{(k)*}| k = 1, \ldots, K\rangle$, with $y^{(\hat{k})*} > 0$ for some $\hat{k}\epsilon\{1, \ldots, K\}$, an array of n_0 dimensional real-valued column vectors $\langle z^{(k)*}| k = 1, \ldots, K\rangle$ and an array of n_1 dimensional real-valued column vectors $\langle w^{(k)*}(\theta)| k = 1, \ldots, K\rangle$ that solve the following linear programming problem denoted **Dual LP-1**:

Minimize $\sum_{k=1}^{K}[z^{(k)T}b^{(0,k)} + \sum_{\theta\epsilon\Omega}w^{(k)}(\theta)^T b^1(\theta)]$, subject to $y^{(k)}q^T + z^{(k)T}A^{(0,k)} + w^{(k)}(\theta)^T A^1(\theta) \geq 2q^T$, $y^{(k)}\pi^{(k)}(\theta)p(\theta)^T + w^{(k)}(\theta)^T B^1(\theta) \geq 2\pi^{(k)}(\theta)p(\theta)^T$, $\sum_{k=1}^{K} y^{(k)} = 1$, $y^{(k)}\epsilon\mathbb{R}_+$, $z^{(k)}\epsilon\mathbb{R}^{n_0\times 1}$, $w^{(k)}(\theta)\epsilon\mathbb{R}^{n_1\times 1}$ for all $\theta\epsilon\Omega$ and $k = 1, \ldots, K$.

(2) Both $(u^*(0), \langle u^*(1, \theta)|\theta\epsilon\Omega\rangle), (u^{(\hat{k})^*}, \langle u^{(\hat{k})^*}(1, \theta)|\theta\epsilon\Omega\rangle)\epsilon \underset{\langle(u^{(k)*}, \langle u^{(k)*}(1,\theta)|\theta\epsilon\Omega\rangle|k\epsilon\{1,\ldots,K\}\rangle}{\operatorname{argmax}} \{q^T u^{(k)*}(0) + \sum_{\theta\epsilon\Omega}\pi^{(k)}(\theta)p(\theta)^T u^{(k)*}(1,\theta)\}$.

Proof: By theorem 3 we know that $(u^*(0), \langle u^*(1, \theta)|\theta\epsilon\Omega\rangle), \langle(u^{(k)*}(0), \langle u^{(k)*}(1, \theta)|\theta\epsilon\Omega\rangle)|k = 1, \ldots, K\rangle$ solve the TPSLP game <u>if and only if</u> $\langle(u^{(k)*}(0), \langle u^{(k)*}(1, \theta)|\theta\epsilon\Omega\rangle)|k = 1, \ldots, K\rangle$, $M^* = q^T u^*(0) + \sum_{\theta\epsilon\Omega}\pi^{(\hat{k})}(\theta)p(\theta)^T u^*(1,\theta)$ with $(u^*(0), \langle u^*(1, \theta)|\theta\epsilon\Omega\rangle)$ $\epsilon \underset{\langle(u^{(k)*}, \langle u^{(k)*}(1,\theta)|\theta\epsilon\Omega\rangle|k\epsilon\{1,\ldots,K\}\rangle}{\operatorname{argmax}} \{q^T u^{(k)*}(0) + \sum_{\theta\epsilon\Omega}\pi^{(k)}(\theta)p(\theta)^T u^{(k)*}(1,\theta)\}$ solve LP-1 (as defined in the statement of theorem 3).

By the duality theorem of linear programming, we know that $\langle(u^{(k)*}(0), \langle u^{(k)*}(1, \theta)|\theta\epsilon\Omega\rangle)|k = 1, \ldots, K\rangle$, M^* solve LP-1 <u>if and only if</u> there exists an array of real numbers $\langle y^{(k)*}| k = 1, \ldots, K\rangle$, an array of n_0 dimensional real-valued column vectors $\langle z^{(k)*}| k = 1, \ldots, K\rangle$ and an array of n_1 dimensional real-valued column vectors $\langle w^{(k)*}(\theta)| k = 1, \ldots, K\rangle$ that solve Dual LP-1 as defined in the statement of this theorem.

By the complementary slackness conditions, it must be the case that $y^{(k)*}(q^T u^{(k)*}(0) + \sum_{\theta\epsilon\Omega}\pi^{(k)}(\theta)p(\theta)^T u^{(k)*}(1,\theta) - M^*) = 0$ for $k = 1, \ldots, K$.

Since, $\sum_{k=1}^{K} y^{(k)*} = 1$, there exists $\hat{k}\epsilon\{1, \ldots, K\}$ such that $y^{(\hat{k})*} > 0$ and hence $q^T u^{(\hat{k})*}(0) + \sum_{\theta\epsilon\Omega}\pi^{(\hat{k})}(\theta)p(\theta)^T u^{(\hat{k})*}(1,\theta) - M^* = 0$.

Since M^* is the optimal value of the TPSLP game, it must be the case that $(u^{(\hat{k})*}(0), \langle u^{(\hat{k})*}(1,\theta)|\theta\epsilon\Omega\rangle)\epsilon \underset{\langle(u^{(k)*}, \langle u^{(k)*}(1,\theta)|\theta\epsilon\Omega\rangle|k\epsilon\{1,\ldots,K\}\rangle}{\operatorname{argmax}} \{q^T u^{(k)*}(0) + \sum_{\theta\epsilon\Omega}\pi^{(k)}(\theta)p(\theta)^T u^{(k)*}(1,\theta)\}$. Q.E.D.

Note 1: Theorem 4 implies that simply by solving Dual LP-1, one can identify "at least one (stochastic) linear programming subproblem", any (optimal) solution of which when substituted in its objective function, yields the optimal value of the TPSLP game. There is no need to solve any other subproblem or LP-1 to derive the optimal value of the TPSLP game.

Note 2: Since $\underset{<(u^{(k)*}, <u^{(k)*}(1,\theta)|\theta\epsilon\Omega>|k\epsilon\{1,\ldots,K\}>}{\operatorname{argmax}} \{q^T u^{(k)*}(0) + \sum_{\theta\epsilon\Omega} \pi^{(k)}(\theta) p(\theta)^T u^{(k)*}(1,\theta)\}$ may not be a singleton we cannot say that $(u^*(0), <u^*(1, \theta)|\theta\epsilon\Omega>) = (u^{(\hat{k})^*}, <u^{(\hat{k})^*}(1, \theta)|\theta\epsilon\Omega>)$. However, if $(u^*(0), <u^*(1, \theta)|\theta\epsilon\Omega>) = (u^{(\ddot{k})^*}, <u^{(\ddot{k})^*}(1, \theta)|\theta\epsilon\Omega>)$, then it must be the case that $q^T u^{(\ddot{k})*}(0) + \sum_{\theta\epsilon\Omega} \pi^{(\ddot{k})}(\theta) p(\theta)^T u^{(\ddot{k})*}(1,\theta) = q^T u^{(\hat{k})*}(0) + \sum_{\theta\epsilon\Omega} \pi^{(\hat{k})}(\theta) p(\theta)^T u^{(\hat{k})*}(1,\theta)$.

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