A COMPLEMENTARY ENERGY THEOREM FOR COMPOSITE PLATES IN POSTBUCKLING

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Abstract

A composite von Karman plate in postbuckling is considered. Using the first Piola stress tensor and the displacement gradient tensor, a complementary energy variational theorem is proven. The proof is given in the case of symmetric lay-up. According to the theorem, at the actual stress state of the plate the complementary energy (as a function of the internal forces and of the moments) reaches its stationary value. The stationary feature of the actual state is valid as compared to other states satisfying the static equilibrium and the static boundary conditions. It is shown how the theorem may be generalized to the case of a non-symmetric lay-up.

Keywords: composite plate, postbuckling, complementary energy, first Piola stress tensor.

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1. INTRODUCTION

The behavior of the composite plates is widely explored since the ending decades of the XX century. The considerable attention is paid, in particular, to the postbuckling of the plates, as the phenomenon corresponds to the highest load-carrying capacity.

The reviews of Turvey and Marshall (1995), Falzon and Aliabadi (2008) describe the state-of-the-art in the field.

The majority of papers deals with the various numerical approaches. All the approaches are based on the kinematic description of the structural behavior in the form of equations or the kinematic (or mixed) variational principles. In particular, the paper of Raju *et al*. (2015) is devoted to a special mixed variational approach for the so-called Variable Angle Tow (VAT) plates. The purely static (complementary energy) principle for composite plates in postbuckling is not used yet.

It is known that the complementary energy approach is successful for the nonlinear tasks where it is possible to derive the complementary energy functional of internal forces. For doing that (see Washizu 1975) it is necessary to express the stresses as functions of the nonlinear strains. Such a possibility does not exist in general for the structures. But in several particular cases the inversion of the stress-strain relations is possible. For example, in the paper of Stumpf (1975) devoted to the case of the isotropic nonlinear von Karman plates, the inversion was made and a complementary theorem was proven.

In the present paper we derive a variational theorem for the complementary energy of the composite plates in postbuckling. The derivation is based on the use of the first Piola stress tensor together with the displacement gradient tensor. The plates obey the von Karman law. Due to postbuckling consideration the key attention is paid to the plates with the symmetric lay-up (in the case of the non-symmetric lay-up the buckling level becomes considerably lower). The way of the theorem generalization to the non-symmetric lay-up case is indicated.

The paper content is the following.

Section 2 describes the assumptions and the definitions.

Section 3 is devoted to the description of the potential energy theorem.

In Section 4, the derivation of a complementary energy theorem is presented.

In Section 5, the conclusions are indicated.

2. ASSUMPTIONS AND DEFINITIONS

We consider a flat composite plate made of stacked orthotropic layers. The Cartesian coordinate system XYZ is used. The undeformed (prior to the deformation) plate mid-plane is located in the X-Y plane. The Z axis is orthogonal to the X-Y plane. The plate mid-plane undeformed surface is denoted *S*. The total undeformed plate volume is *V*. The plate is loaded at the part C_1 of the boundary contour *C* by the in-plane loads. The contour is supposed to be smooth. The boundary loads are assumed to be piecewise-smooth. At the remaining C_2 part of the boundary contour the displacements are prescribed.

In postbuckling, the plate is deformed in accordance with the von Karman theory (Washizu 1975).

The plate deflection is assumed to be non-negligible as compared to the plate thickness, but, at the same time, it is very small as compared to the other plate dimensions. Denote the displacements along the X, Y, Z axes at the arbitrary point within the plate v_x, v_y, v_z, respectively. The X, Y, Z displacements at the mid-surface are denoted *u, v, w*, respectively. Neglecting the terms of the lower order of the magnitude, the displacements at the arbitrary point within the plate are written as follows:

$$v_x = u - zw_{,x} \ ; \ v_{,y} = v - zw_{,y} \ ; \ v_z = w \tag{1}$$

where (and everywhere further) the subscript after comma means the differentiation w.r.t. the corresponding coordinate. The non-zero components of the Green strain tensor are:

$$e_{xx} = v_{x,x} + \frac{1}{2} v_{z,x}^2 \;;\; e_{yy} = v_{y,y} + \frac{1}{2} v_{z,y}^2 \;;\; e_{xy} = \frac{1}{2}\left(v_{x,y} + v_{y,x} + v_{z,x} v_{z,y}\right) \tag{2}$$

3. THE POTENTIAL ENERGY THEOREM FOR COMPOSITE PLATES IN POSTBUCKLING

The total potential energy variational principle for the considered configuration is written as follows (see Washizu 1975, Ashton and Whitney 1970):

$$I = \int_S \Pi dS - \int_{C_1} \left(\overline{N}_{x\nu} u + \overline{N}_{y\nu} v\right) dC_1 \tag{3}$$

where $\overline{N}_{x\nu}, \overline{N}_{y\nu}$ are the given forces per unit contour length at the undeformed boundary C_1 (the given values of the displacements at the remaining part C_2 of the contour will be marked further by a top bar also) , Π is the strain potential energy per unit undeformed plate area. The latter energy is derived from the ply strain energy density integrating through the plate thickness.

The ply strains of the laminate are $\varepsilon_x, \varepsilon_y, \varepsilon_{xy}$, respectively. The ply strain energy density π_{ply} in the undeformed volume is written (Gibson 1994):

$$\pi_{ply} = \frac{1}{2}\overline{Q}_{11}\varepsilon_x^2 + \frac{1}{2}\overline{Q}_{22}\varepsilon_y^2 + 2\overline{Q}_{66}\varepsilon_{xy}^2 + \overline{Q}_{12}\varepsilon_x\varepsilon_y + 2\overline{Q}_{16}\varepsilon_x\varepsilon_{xy} + 2\overline{Q}_{26}\varepsilon_y\varepsilon_{xy} \tag{4}$$

where the ply stresses $\sigma_x, \sigma_y, \sigma_{xy}$ and strains $\varepsilon_x, \varepsilon_y, \varepsilon_{xy}$ are related to each other as follows:

$$\begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{pmatrix} = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix} \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ 2\varepsilon_{xy} \end{pmatrix} \tag{5}$$

and $\overline{Q}_{ij}$, i,j=1,2,6, is the matrix of the ply elastic constants.

The first Piola stress tensor components $\tilde{\sigma}_{ij}$ and the Kirchhoff stress tensor components σ_{ij} are given by the relations:

$$\tilde{\sigma}_{ij} = \frac{\partial \pi(v_{i,j})}{\partial v_{j,i}} \tag{6}$$

$$\sigma_{ij} = \frac{\partial \pi(e_{ij})}{\partial e_{ij}} \tag{7}$$

where π is the strain energy density in the undeformed volume. In general, the tensor $\tilde{\sigma}_{ij}$ is non-symmetric. Calculating the components of $\tilde{\sigma}_{ij}$ for a ply, we obtain:

$$\tilde{\sigma}_{xx} = \overline{Q}_{11}\left(v_{x,x} + \frac{1}{2}v_{z,x}^2\right) + \overline{Q}_{12}\left(v_{y,y} + \frac{1}{2}v_{z,y}^2\right) + \overline{Q}_{16}\left(v_{x,y} + v_{y,x} + v_{z,x}v_{z,y}\right) \tag{8}$$

$$\tilde{\sigma}_{xy} = \overline{Q}_{12}\left(v_{x,x} + \frac{1}{2}v_{z,x}^2\right) + \overline{Q}_{22}\left(v_{y,y} + \frac{1}{2}v_{z,y}^2\right) + \overline{Q}_{26}\left(v_{x,y} + v_{y,x} + v_{z,x}v_{z,y}\right) \tag{9}$$

$$\tilde{\sigma}_{xy} = \tilde{\sigma}_{yx} = \overline{Q}_{16}\left(v_{x,x} + \frac{1}{2}v_{z,x}^2\right) + \overline{Q}_{26}\left(v_{y,y} + \frac{1}{2}v_{z,y}^2\right) + \overline{Q}_{66}\left(v_{x,y} + v_{y,x} + v_{z,x}v_{z,y}\right) \tag{10}$$

Direct differentiation of (4) according to (7) gives the *xx, yy, xy, yx* components of the Kirchhoff tensor equal to (8)-(10), respectively.

Using the known relations (see Washizu 1975 and the above note on equality of the in-plane components of the Kirchhoff and the first Piola stress tensors)

$$\tilde{\sigma}_{xz} = \tilde{\sigma}_{xx}v_{z,x} + \tilde{\sigma}_{xy}v_{z,y} \tag{11}$$

$$\tilde{\sigma}_{yz} = \tilde{\sigma}_{yx}v_{z,x} + \tilde{\sigma}_{yy}v_{z,y} \tag{12}$$

we obtain the remaining non-zero components of the $\tilde{\sigma}_{ij}$ tensor. The components of the σ_{ij} tensor differing from *xx, yy, xy, yx* are equal to zero.

Integrating over z through the plate thickness, we introduce the stress (force) and the moment resultants N_{ij} and M_{ij}:

$$N_{xx} = \int_{-h/2}^{+h/2} \tilde{\sigma}_{xx} dz \tag{13}$$

$$N_{yy} = \int_{-h/2}^{+h/2} \tilde{\sigma}_{yy} dz \tag{14}$$

$$N_{xy} = N_{yx} = \int_{-h/2}^{+h/2} \tilde{\sigma}_{xy} dz \tag{15}$$

$$N_{xz} = \int_{-h/2}^{+h/2} \tilde{\sigma}_{xz} dz \tag{16}$$

$$N_{yz} = \int_{-h/2}^{+h/2} \tilde{\sigma}_{yz} dz \tag{17}$$

$$M_{xx} = \int_{-h/2}^{+h/2} z\tilde{\sigma}_{xx} dz \tag{18}$$

$$M_{yy} = \int_{-h/2}^{+h/2} z\tilde{\sigma}_{yy} dz \tag{19}$$

$$M_{xy} = M_{yx} = \int_{-h/2}^{+h/2} z\tilde{\sigma}_{xy} dz \tag{20}$$

Substituting (8)-(12) into (13)-(20) and remembering the definition of *A, B, D* stiffness matrices for laminates via $\overline{Q}_{ij}$, *i,j*=1,2,6 (Gibson 1994, Section 7.3), we obtain these forces and moments.

Rewrite the formula for the total potential energy as

$$I = \int_S \left(\frac{1}{2} \vec{e}_0^{\,T} A \vec{e}_0 + \vec{e}_0^{\,T} B \vec{\omega} + \frac{1}{2} \vec{\omega}^T D \vec{\omega} \right) dS - \int_{C_1} \left(\overline{N}_{xv} u + \overline{N}_{yv} v \right) dC_1 \tag{21}$$

where *T* means the transposition, the vectors $\vec{e}_0, \vec{\omega}$ are the mid-surface 2D-stains and the mid-surface curvatures:

$$\vec{e}_0 = \left. \begin{pmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{pmatrix} \right|_{z=0} \quad ; \quad \vec{\omega} = \left. \begin{pmatrix} -\partial^2 w / \partial x^2 \\ -\partial^2 w / \partial y^2 \\ -2\,\partial^2 w / \partial x \partial y \end{pmatrix} \right|_{z=0} \tag{22}$$

The first integral in (21) is obtained from (4) (with account of (1), (2)) after integration through the plate thickness.

The variation of the total potential energy (3, 21) is written as follows:

$$\delta I = \int_S \delta\left(\frac{1}{2}\vec{\varepsilon}_0^{\,T} A \vec{\varepsilon}_0 + \vec{\varepsilon}_0^{\,T} B \vec{\omega} + \frac{1}{2}\vec{\omega}^{\,T} D \vec{\omega}\right) dS - \int_{C_1} \left(\overline{N}_{x\nu}\delta u + \overline{N}_{y\nu}\delta v\right) dC_1 \tag{23}$$

Using the known relation (Washizu 1975)

$$\delta\pi = \frac{\partial \pi}{\partial v_{j,i}}\delta v_{j,i} = \widetilde{\sigma}_{i,j}\delta v_{j,i} \tag{24}$$

(hereinafter the repetitive indices mean the summation on them) and substituting (1) into it, after integration over the plate thickness, we obtain the variation of the strain energy *Π* per unit undeformed area:

$$\delta\Pi = N_{xx}\delta u_{,x} + N_{yy}\delta u_{,y} + N_{xy}\left(\delta u_{,y} + \delta v_{,x}\right) + N_{xz}\delta w_{,x} + N_{yz}\delta w_{,y} - M_{xx}\delta w_{,xx} - M_{yy}\delta w_{,yy} - 2M_{xy}\delta w_{,xy} \tag{25}$$

Further we denote n_x, n_y the components of the normal vector to the boundary contour. Using them, we introduce the *x, y, z* stress resultants $N_{x\nu}, N_{y\nu}, N_{z\nu}$ at the boundary contour, and the moment results there also:

$$\begin{cases} N_{x\nu} = n_x N_{xx} + n_y N_{yx} \\ N_{y\nu} = n_x N_{xy} + n_y N_{yy} \\ N_{z\nu} = n_x N_{xz} + n_y N_{yz} \end{cases} \qquad \begin{cases} M_{x\nu} = n_x M_{xx} + n_y M_{yx} \\ M_{y\nu} = n_x M_{xy} + n_y M_{yy} \\ M_{\nu} = n_x M_{x\nu} + n_y M_{y\nu} \end{cases} \tag{26}$$

In the further description we use the *x* and *y* components of the normal n_x, n_y to the undeformed boundary contour in the relations (where *s* is the direction along the contour)

$$\begin{aligned} \frac{\partial}{\partial x} &= n_x \frac{\partial}{\partial n} - n_y \frac{\partial}{\partial s} \\ \frac{\partial}{\partial y} &= n_y \frac{\partial}{\partial n} + n_x \frac{\partial}{\partial s} \end{aligned} \tag{27}$$

where *n, s* mean the normal and the tangential direction to the contour. Integating (23) by parts and using the Gauss divergence theorem, we obtain the variation of the total potential energy functional:

$$\begin{aligned}\delta I = &-\int_S \left[\left(N_{xx,x} + N_{xy,y}\right)\delta u + \left(N_{xy,x} + N_{yy,y}\right)\delta v + \left(M_{xx,xx} + 2M_{xy,xy} + M_{yy,yy} + N_{xz,x} + N_{yz,y}\right)\delta w\right] ds + \\ &+ \int_{C_1} \left[\left(N_{x\nu} - \overline{N}_{x\nu}\right)\delta u + \left(N_{y\nu} - \overline{N}_{y\nu}\right)\delta v + N_{z\nu}\delta w - M_\nu \delta w_{,n}\right] dC_1 + \\ &+ \int_{C_2} \left[\left(N_{x\nu}\delta u + N_{y\nu}\delta v + N_{z\nu}\delta w - M_\nu \delta w_{,n}\right)\right] dC_2\end{aligned} \tag{28}$$

On the part C_2 of the boundary contour we have:

$$\delta u = \delta v = \delta w = M_\nu \delta w_{,n} = 0 \tag{29}$$

where the latter term (the product of the normal moment and the normal derivative) covers both the simple support and the clamped conditions. Then the stationarity of *I* w.r.t. the displacements leads to the equilibrium equations at the plate (in *S*):

$$\begin{aligned} N_{xx,x} + N_{xy,y} &= 0 \\ N_{xy,x} + N_{yy,y} &= 0 \\ M_{xx,xx} + 2M_{xy,xy} + M_{yy,yy} + N_{xz,x} + N_{yz,y} &= 0 \end{aligned} \tag{30}$$

where, according to (11), (12), (16), (17)

$$\begin{aligned} N_{xz,x} &= N_{xx} w_{,x} + N_{xy} w_{,y} \\ N_{yz,x} &= N_{yx} w_{,x} + N_{yy} w_{,y} \end{aligned} \tag{31}$$

The third equation of (30), after substituting of (31), leads to the known *z*-equilibrium equation of the von Karman plate:

$$\frac{\partial^2 M_{xx}}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_{yy}}{\partial y^2} + \frac{\partial}{\partial x}\left(N_{xx}\frac{\partial w}{\partial x} + N_{xy}\frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial y}\left(N_{xy}\frac{\partial w}{\partial x} + N_{yy}\frac{\partial w}{\partial y}\right) = 0 \tag{32}$$

The linear boundary conditions on C_1 are:

$$
\begin{aligned}
&N_{x\nu} - \overline{N}_{x\nu} = 0 \\
&N_{y\nu} - \overline{N}_{y\nu} = 0 \\
&N_{z\nu} = 0 \\
&M_{\nu} = 0
\end{aligned}
\tag{33}
$$

4. A COMPLEMENTARY ENERGY THEOREM

Now, after indicating the total potential energy theorem, we proceed with the complementary energy theorem derivation.

The complementary energy density π_c and the strain energy density π_{ply} for a ply satisfy the Legendre transformation (if the displacement gradient tensor $v_{i,j}$ can be expressed through the first Piola stress tensor):

$$
\pi_c = \widetilde{\sigma}_{lk} v_{l,k} - \pi_{ply}\left(v_{i,j}\right) \tag{34}
$$

Using (1), we have:

$$
\widetilde{\sigma}_{lk} v_{l,k} = \widetilde{\sigma}_{xx}\left(u_{,x} - zw_{,xx}\right) + \widetilde{\sigma}_{yy}\left(u_{,y} - zw_{,yy}\right) + \widetilde{\sigma}_{xy}\left(u_{,y} + v_{x} - 2zw_{,xy}\right) + \widetilde{\sigma}_{xz} w_{,x} + \widetilde{\sigma}_{yz} w_{,y} \tag{35}
$$

The relation (4) may be re-written as:

$$
\pi_{ply} = \frac{1}{2}\widetilde{\sigma}_{xx}\left(u_{,x} - zw_{,xx} + \frac{1}{2}w_{,x}^2\right) + \frac{1}{2}\widetilde{\sigma}_{yy}\left(v_{,y} - zw_{,yy} + \frac{1}{2}w_{,y}^2\right) + \frac{1}{2}\widetilde{\sigma}_{xy}\left(u_{,y} + v_{,x} - 2zw_{,xx} + w_{,x}w_{,y}\right) \tag{36}
$$

Subtracting (36) from (35), we have the ply complementary energy density:

$$
\begin{aligned}
\pi_c = {} & \frac{1}{2}\widetilde{\sigma}_{xx}\left(u_{,x} - zw_{,xx} - \frac{1}{2}w_{,x}^2\right) + \frac{1}{2}\widetilde{\sigma}_{yy}\left(v_{,y} - zw_{,yy} - \frac{1}{2}w_{,y}^2\right) + \\
& + \frac{1}{2}\widetilde{\sigma}_{xy}\left(u_{,y} + v_{,x} - 2zw_{,xx} - w_{,x}w_{,y}\right) + \widetilde{\sigma}_{xz} w_{,x} + \widetilde{\sigma}_{yz} w_{,y}
\end{aligned}
\tag{37}
$$

Integrating (37) over z through the plate thickness, we obtain the complementary energy per unit undeformed mid-plane area π_c:

$$\Pi_c = \frac{1}{2}N_{xx}\left(u_{,x} - \frac{1}{2}w_{,x}^2\right) + \frac{1}{2}N_{yy}\left(v_{,y} - \frac{1}{2}w_{,y}^2\right) + + \frac{1}{2}N_{xy}\left(u_{,y} + v_{,x} - w_{,x}w_{,y}\right) + N_{xz}w_{,x} + N_{yz}w_{,y} +$$
$$+ \frac{1}{2}M_{xx}\left(-w_{,xx}\right) + \frac{1}{2}M_{yy}\left(-w_{,yy}\right) + \frac{1}{2}M_{xy}\left(-2w_{,xy}\right)$$

(38)

Integrating (11), (12) over the plate thickness and inverting the relations, we have:

$$w_{,x} = \frac{N_{yy}N_{xz} - N_{xy}N_{yz}}{N_{xx}N_{yy} - N_{xy}^2} = \gamma_1(N_{ij}) \tag{39}$$

$$w_{,y} = \frac{N_{xx}N_{yz} - N_{xy}N_{xz}}{N_{xx}N_{yy} - N_{xy}^2} = \gamma_2(N_{ij}) \tag{40}$$

Now we define the vectors:

$$\vec{N} = \left(N_{xx}, N_{yy}, N_{xy}\right)^T \tag{41}$$

$$\vec{M} = \left(M_{xx}, M_{yy}, M_{xy}\right)^T \tag{42}$$

For simplification of the transformations, we consider further the plates with the symmetric lay-up only. This means that the B stiffness matrix is composed of zeros. Such an approach looks reasonable for postbuckling analysis, as otherwise the buckling level may come down considerably, decreasing the total load-carrying capacity.

The derivation of the theorem for the case $B\neq 0$ is straightforward. It may be done using the obvious equality:

$$\left(A - BD^{-1}B\right)\vec{\varepsilon}_0 = \vec{N} - BD^{-1}\vec{M} \tag{43}$$

and obtaining the Green strains, the displacement gradient components, and the curvatures as the functions of the forces and the moments.

Now, substituting (39)-(42) into (38), we obtain the final expression for the complementary energy density per undeformed plate area:

$$\Pi_c = \frac{1}{2}\vec{N}^T A^{-1}\vec{N} + \frac{1}{2}\vec{M}^T D^{-1}\vec{M} + \frac{1}{2}N_{xz}\gamma_1(N_{ij}) + \frac{1}{2}N_{yz}\gamma_2(N_{ij}) \tag{44}$$

The functions γ_1, γ_2 in (38), (39) are incompatible in the case of an arbitrary stress state. They become the corresponding deflection derivatives for the actual state only.

For creating the complementary energy functional I_c we substitute into (3) the relation (34), integrated over the plate thickness. Then, integrating by parts and using the Gauss divergence theorem, we obtain the functional I as follows:

$$
\begin{aligned}
I = &-\int_S \Pi_c dS + \int_{C_2} \left[N_{x\nu}\bar{u} + N_{y\nu}\bar{v} + N_{xz}\bar{w} \right] - \int_S \left[\left(N_{xx,x} + N_{xy,y} \right) u + \left(N_{xy,x} + N_{yy,y} \right) v + \right. \\
&+ \left. \left(M_{xx,xx} + 2M_{xy,xy} + M_{yy,yy} + N_{xz,x} + N_{yz,y} \right) w \right] dS + \\
&+ \int_{C_1} \left[\left(N_{x\nu} - \bar{N}_{x\nu} \right) \delta u + \left(N_{y\nu} - \bar{N}_{y\nu} \right) \delta v + N_{z\nu} \delta w - M_\nu \delta w_{,n} \right] dC_1 + \\
&+ \int_{C_2} \left[\left(N_{x\nu} \delta u + N_{y\nu} \delta v + N_{z\nu} \delta w - M_\nu \delta w_{,n} \right) \right] dC_2
\end{aligned}
\tag{45}
$$

Below the geometric boundary conditions on C_2 are used, namely, the values of the displacements are prescribed there and

$$
u = \bar{u} \;;\; v = \bar{v} \;;\; w = \bar{w} \;;\; M_\nu w_{,n} = 0 \tag{46}
$$

If the equilibrium equations (30), (31) and the static boundary conditions (33) are valid for the variations of the stress state, then the complementary variational functional I_c in (45) is written in the following form (Π_c comes from (44)):

$$
I_c = -\int_S \Pi_c dS + \int_{C_2} \left[N_{x\nu}\bar{u} + N_{y\nu}\bar{v} + N_{z\nu}\bar{w} - M_\nu w_{,n} \right] dS \tag{47}
$$

The variation of Π_c reads:

$$
\begin{aligned}
\delta\Pi_c = &\frac{\partial \Pi_c}{\partial N_{xx}} \delta N_{xx} + \frac{\partial \Pi_c}{\partial N_{yy}} \delta N_{yy} + \frac{\partial \Pi_c}{\partial N_{xy}} \delta N_{xy} + \frac{\partial \Pi_c}{\partial N_{xz}} \delta N_{xz} + \frac{\partial \Pi_c}{\partial N_{yz}} \delta N_{yz} + \\
&+ \frac{\partial \Pi_c}{\partial M_{xx}} \delta M_{xx} + \frac{\partial \Pi_c}{\partial M_{yy}} \delta M_{yy} + \frac{\partial \Pi_c}{\partial M_{xy}} \delta M_{xy}
\end{aligned}
\tag{48}
$$

It is necessary to prove that for the stress states satisfying the static equilibrium conditions (30), (31) and the static boundary conditions (33) the actual stress state corresponds to a stationary point. Observing (45), one may say that the displacements there may be considered as the Lagrange multipliers, when he/she looks for a stationary value of (47).

We augment (47) by the items corresponding to the equilibrium equations and the static boundary conditions with the Lagrange multipliers $\alpha_x, \alpha_y, \alpha_z$. After that, using the Gauss divergence theorem and integrating by parts, we obtain:

$$\delta I_c = -\int_S \begin{bmatrix} \delta \vec{N}^T A^{-1} \vec{N} + \delta \vec{M}^T D^{-1} \vec{M} - \left(\frac{1}{2}\gamma_1^2 + \alpha_{x,x}\right)\delta N_{xx} - \left(\frac{1}{2}\gamma_2^2 + \alpha_{y,y}\right)\delta N_{yy} - \\ -\left(\gamma_1\gamma_2 + \alpha_{x,y} + \alpha_{y,x}\right)\delta N_{xy} + \left(\gamma_1 - \alpha_{z,x}\right)\delta N_{xz} + \left(\gamma_2 - \alpha_{z,y}\right)\delta N_{yz} + \\ \alpha_{z,xx}\delta M_{xx} + \alpha_{z,yy}\delta M_{yy} + \alpha_{z,xy}\delta M_{xy} \end{bmatrix} dS -$$
$$- \int_{C_2} \left[\left(\alpha_x - \bar{u}\right)\delta N_{x\nu} + \left(\alpha_y - \bar{v}\right)\delta N_{y\nu} + \left(\alpha_z - \bar{w}\right)\delta N_{z\nu} - \left(\alpha_{z,n} - w_{,n}\right)\delta M_\nu\right] dC_2 = 0 \tag{49}$$

If we interpret the Lagrange multipliers as the displacements, we obtain the geometric boundary conditions on C_2:

$$\alpha_x = \bar{u} \;;\; \alpha_y = \bar{v} \;;\; \alpha_z = \bar{w} \;;\; M_\nu(\alpha_z) = 0 \text{ or } \alpha_{z,n} = \bar{w}_{z,n} \tag{50}$$

Also, in S we have the compatibility conditions for γ_1, γ_2 as functions of $N_{xx}, N_{yy}, N_{xy}, N_{xz}, N_{yx}$:

$$\gamma_1 = \alpha_{z,x} \;;\; \gamma_2 = \alpha_{z,y} \tag{51}$$

and the force (moment) – displacement (curvature) relations:

$$\begin{cases} A_{11}^{-1}N_{xx} + A_{12}^{-1}N_{yy} + A_{16}^{-1}N_{xy} = \alpha_{x,x} + \frac{1}{2}\gamma_1^2 \\ A_{12}^{-1}N_{xx} + A_{22}^{-1}N_{yy} + A_{26}^{-1}N_{xy} = \alpha_{y,y} + \frac{1}{2}\gamma_2^2 \\ A_{16}^{-1}N_{xx} + A_{26}^{-1}N_{yy} + A_{66}^{-1}N_{xy} = \alpha_{x,y} + \alpha_{y,x} + \gamma_1\gamma_2 \end{cases} \tag{52}$$

$$\begin{cases} D_{11}^{-1}M_{xx} + D_{12}^{-1}M_{yy} + D_{16}^{-1}M_{xy} = -\alpha_{z,xx} \\ D_{12}^{-1}M_{xx} + D_{22}^{-1}M_{yy} + D_{26}^{-1}M_{xy} = -\alpha_{z,yy} \\ D_{16}^{-1}M_{xx} + D_{26}^{-1}M_{yy} + D_{66}^{-1}M_{xy} = -\alpha_{z,xy} \end{cases} \tag{53}$$

where A_{ij}^{-1}, D_{ij}^{-1} mean the elements of the inverse matrices to *A, D*. The rhs of the relations (52) looks similar to the rhs of (2) for *z*=0. The $\alpha_{z,xx}, \alpha_{z,yy}, \alpha_{z,xy}$ may be treated as the plate curvatures. Thus, the relations (51)-(53) are equivalent to the known strain compatibility condition for the von Karman plates:

$$\frac{\partial^2 e_{xx0}}{\partial y^2}+\frac{\partial^2 e_{yy0}}{\partial x^2}-2\frac{\partial^2 e_{xy0}}{\partial x\partial y}=\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2-\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} \tag{54}$$

where $e_{xx0}, e_{yy0}, e_{xy0}$ are the Green strains at z=0. It is supposed in the both sides of (54) that:

$$\begin{cases} u=\alpha_x \\ v=\alpha_y \\ w=\alpha_z \end{cases} \tag{55}$$

We have proven that the stationary value of the complementary energy corresponds to the solution of the composite plate in postbuckling.

5. CONCLUSIONS

- The complementary energy theorem for the composite plates in postbuckling is proven
- According to the theorem, the actual stress state of the plate corresponds to a stationary value of the complementary energy functional in the case that the resultants of the first Piola stress tensor satisfy the static equilibrium and the static boundary conditions.
- The theorem is proven for the symmetric plate lay-ups (having better buckling resistance than other ones). The way of the proof of the theorem for a non-symmetric lay-up is clearly indicated.

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