Additively-Separable Sum Linear Programming Games

by

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Abstract

We introduce the class of additively-separable linear-programming games. Such games are two-person/two-player games, where one player's "strategy set" is a "closed convex polytope" and the other player's "set of pure strategies" is a non-empty finite set. The objective of the first player is to maximize a real-valued linear payoff function on the closed convex polytope, where the payoff consists partly on a transfer from the second player and depends on the strategies chosen by both players and the rest is independent of the strategy chosen by the second player while being linearly dependent on the strategy chosen by the first player. We allow the second player to choose randomized or mixed strategies and the objective of the second player is to maximize expected pay-off. The expected payoff of the second player is a linear function of the mixed strategy chosen by the second player which is not influenced by the strategy chosen by the first player minus what the second player transfers to the first player. Our main result says that under the assumption that the closed convex polytope is non-empty and bounded, the set of all equilibria of an additively-separable linear programming game is "equivalent" to the solution set of *two different* "primal-dual pairs" of linear programming problems. Further, the optimal value of the second primal-dual pair of linear programming problems defines the value of the additively-separable linear programming game.

Keywords: two-person, game, additively separable sum, equilibrium, linear programming, value

AMS Subject Classifications: 90C05, 90C46, 91A05, 91A10

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1. Introduction:

Classical (or Newtonian) mechanics originated during the seventeenth century, much before cars or automobiles (https://en.wikipedia.org/wiki/Car) were invented in the eighteenth century. Apparently, one of the motivators behind the development of classical mechanics was an "apple" that fell from a tree. Classical mechanics has not only contributed immensely to the development of technology for terrestrial purposes. It continues to be the backbone of the theoretical physics behind "space exploration" as pointed out in Li (2025).

Linear programming which originated in the seminal work of Leonid Kantorovich in 1939, was first used in game theory to provide a simple proof of the minimax theorem for matrix games due to John von Neumann, which was stated and proved in a different way by the latter in 1928. The earliest proof of the minimax theorem using linear programming is due to Dantzig (1951). Numerous other variations and amendments of the proof due to Dantzig (1951) are recorded in the scholarly work of von Stengel (2023), the latter itself providing an amendment of the proof due to Dantzig as well as extending the results in Dantzig's paper. However, von Stengel (2023) and citations therein remain confined to matrix games. To the best of our knowledge, the first equivalence results between the set of equilibria for a more general type of game - bi-matrix game referred to as two-person additively-separable sum game- and a primal-dual pair of linear programming problems, is available in Lahiri (2025a). Further, to the best of our knowledge, the first discussion about the value of this type of a game is available in Lahiri(2025b).

The purpose of this paper is to further extend the results in Lahiri (2025a, 2025b) to a class of games that we refer to as additively-separable linear-programming games. Such games are also two-person/two-player games, where one player's "strategy set" consists of the set of all non-negative solutions of a finite system of linear equations in a finite number of unknowns, i.e., a "closed convex polytope" and the other player's "set of pure strategies" is a non-empty finite set. The objective of the first player is to maximize a real-valued linear payoff function on the closed convex polytope, where the payoff consists partly on a transfer from the second player and depends on the strategies chosen by both players and the rest is independent of the strategy chosen by the second player while being linearly dependent on the strategy chosen by the first player. We allow the second player to choose randomized or mixed strategies and the objective of the second player is to maximize expected pay-off. The expected payoff of the second player is a linear function of the mixed strategy chosen by the second player which is not influenced by the strategy chosen by the first player minus what the second player transfers to the first player. This framework generalizes the framework of a two-person additively-separable sum game, since in the latter case it is assumed that the first player also chooses from the set of mixed strategies over a finite set of pure strategies and the set of all mixed strategies over a finite set of pure strategies is a special type of closed convex polytope that is commonly known as a "simplex". We make the important assumption that the closed convex polytope from which the first player chooses a pay-off maximizing strategy is "non-empty and bounded". This allows us to apply the well-known result that the set of extreme points of a non-empty bounded closed convex polytope is non-empty, finite and further that the polytope itself is the convex hull of its extreme points. This result is included among the four observations that we initially prove in the paper.

Our main result says that the set of all equilibria of an additively-separable linear programming game is "equivalent" to the solution set of *two different* "primal-dual pairs" of linear programming problems. Further, the optimal value of the second primal-dual pair of linear programming problems defines the value of the additively-separable linear programming game. An example of an additively-separable linear programming game is a version of a "minimax stochastic linear programming problem" that is similar to the one defined (by (11) and (12)) in Dupacova (undated) and yet retains some of the dynamic aspects of the "stochastic linear programming game" discussed in Lahiri (2025c). It should be noted that the approach to stochastic programming in Dupacova (undated) and citations therein, is very different from the framework of analysis in Lahiri (2025c), with the latter being a dynamic two-period model relevant in situations where the probability distribution chosen by nature is a "non-behavioral" response to the control variable vector chosen by the decision-maker in the initial period, as for instance in the case of global warming and climate change.

This game theoretic "incremental" extension of the boundary of linear programming modelling begs the question: Can this side of the boundary of linear programming be pushed further?

2. Notations:

For natural numbers m, n, let $\mathbb{R}^{m\times n}$ be the set of real-valued m×n matrices. $\mathbb{R}^{m\times 1}$ denotes the set of all real-valued m-dimensional column vectors and $\mathbb{R}^{1\times n}$ denotes the set of all real-valued n-dimensional row vectors.

The transpose of any finite dimensional real-valued vector x is denoted by x^T.

Given $B\in\mathbb{R}^{m\times n}$, $i\in\{1, \ldots, m\}$ and $j\in\{1, \ldots, n\}$, let $B_i \in \mathbb{R}^{1\times n}$ denote the i^{th} row of B and let $B^j \in\mathbb{R}^{m\times 1}$ denote the j^{th} column of B.

Unless otherwise stated a vector will be assumed to be a column vector.

3. The Framework of Analysis:

Given two natural numbers m, n and $(A, b)\in\mathbb{R}^{m\times n}\times\mathbb{R}^{m\times 1}$, let $X(A, b) = \{x\in\mathbb{R}^n_+ | Ax = b\}$. X(A, b) is said to be a **closed convex polytope** determined by the pair (A, b).

Note 1: The usual way of defining a closed convex polytope (as in Henk, Richter-Gebert and Ziegler (2017)) is $\{x\in\mathbb{R}^n | Ax \leq b\}$. However, using standard methods in linear programming it can be shown that the "inequality form" is equivalent to the "equality form with non-negativity constraints", although the number of unknown variables may be different for the two forms.

A point x in a closed convex polytope X(A, b) is said to be an **extreme point** of the polytope if the columns in the array $\langle A^j | x_j > 0\rangle$ are linearly independent.

Clearly, a closed convex polytope can have at most a finite number of extreme points.

Let $X^*(A, b)$ denote the set of extreme points of X(A, b).

Let there be two players one of whom chooses a point from X(A, b) and the other chooses a point from a *non-empty finite set of alternatives* Ω, a generic element of which is denoted by θ. Ω is the **set of pure strategies for the second player**.

Let $<p(\theta)|\theta\epsilon\Omega>$ be a finite array of n-dimensional real valued column vectors, π an n-dimensional column vector. Let $\rho:\Omega\rightarrow\mathbb{R}$ be a function.

If the first player chooses $x \epsilon X(A, b)$ and the second player chooses $\theta\epsilon\Omega$, then the payoff to the first player is $p(\theta)^T x + \pi^T x$ and the payoff to the second player is $-p(\theta)^T x + \rho(\theta)$.

We assume that if the second player chooses $\theta\epsilon\Omega$, the first player will respond by choosing a solution of the linear programming problem: Maximize $p(\theta)^T x + \pi^T x$, subject to $x\epsilon X(A, b)$.

We will refer to the four-tuple $((A, b), <p(\theta)|\theta\epsilon\Omega>, \pi, \rho)$ as an **additively-separable sum linear programming (ASS-LP) game**.

Note 2: If m = 1, all entries of the 1×n matrix A are 1 and b = 1, then $((A, b), <p(\theta)|\theta\epsilon\Omega>, \pi, \rho)$ reduces to what is defined in Lahiri (2025a) as a **two-person additively-separable sum (TPASS) game**. The reason for this is that the first player may now be viewed as having 'n' pure strategies to choose from and receives a pay-off equal to $p_j(\theta) + \pi_j$ if the first player chooses the pure strategy 'j' where $j\epsilon\{1, \ldots, n\}$ and the second player chooses $\theta\epsilon\Omega$. Likewise, the second player receives a payoff equal to $- p_j(\theta) + \rho(\theta)$ under the same situation. Here for $j\epsilon\{1, \ldots, n\}$ and $\theta\epsilon\Omega$, $p_j(\theta)$ denotes the j^{th} coordinate of $p(\theta)$.

We allow the second player to choose a randomization over its set of pure strategies.

Let $\Delta(\Omega) = \{q|q:\Omega\rightarrow[0, 1], \sum_{\theta\epsilon\Omega} q(\theta)\}$.

$q\epsilon\Delta(\Omega)$ is said to be a **randomized (or mixed) strategy** for the second player.

A **strategy profile** is a pair $(x, q)\epsilon X(A, b)\times \Delta(\Omega)$.

At strategy profile (x, q) the pay-off to the first player is $\sum_{\theta\epsilon\Omega} q(\theta)p(\theta)^T x + \pi^T x$ and the payoff to the second player is $- \sum_{\theta\epsilon\Omega} q(\theta)[p(\theta)^T x - \rho(\theta)]$.

A strategy profile (x^*, q^*) is said to be an **equilibrium** for the ASS-LP game $((A, b), <p(\theta)|\theta\epsilon\Omega>, \pi, \rho)$ if for all $x \epsilon X(A, b)$ and $q \epsilon \Delta(\Omega)$: $\sum_{\theta\epsilon\Omega} q^*(\theta)p(\theta)^T x + \pi^T x \leq \sum_{\theta\epsilon\Omega} q^*(\theta)p(\theta)^T x^* + \pi^T x^*$ and $- \sum_{\theta\epsilon\Omega} q(\theta)[p(\theta)^T x^* - \rho(\theta)] \leq - \sum_{\theta\epsilon\Omega} q^*(\theta)[p(\theta)^T x^* - \rho(\theta)]$.

For the discussion that follows we shall be assuming the following.

Assumption: X(A, b) is ***non-empty*** **and** ***bounded***.

4. Some Important and Useful Observations about Closed and Convex Polytopes:

We provide below four observations (with proofs) which are both important and useful for the proof of our main result. All these observations are standard results in linear programming as well as the theory of convex polytopes.

Observation 1: An extreme point of X(A, b) *cannot* be expressed as a "strict convex combination" of two distinct points in X(A, b).

For suppose x is an extreme point of X(A, b) and towards a contradiction suppose there exists y, z ϵ X(A, b) with $x \neq y \neq z \neq x$ and $\alpha \epsilon$ (0, 1) such that $x = \alpha y + (1-\alpha)z$. Thus, $x_j = 0$ implies $y_j = 0$ and $z_j = 0$. Thus, $\sum_{\{j|x_j>0\}} A^j y_j$ = b = $\sum_{\{j|x_j>0\}} A^j z_j$ with $y_j \neq z_j$ for some j satisfying $x_j > 0$.

Thus, $\sum_{\{j|x_j>0\}} A^j (y_j - z_j) = 0$ with $y_j - z_j \neq 0$ for some j satisfying $x_j > 0$.

This contradicts the linear independence of the columns in the array $<A^j| x_j > 0>$.

Thus, x cannot be expressed as a "strict convex combination" of two distinct points in X(A, b).

The following observation follows immediately from proposition 4 in topic 2 and Farka lemma in topic 3 of Lahiri (2020).

Observation 2: X(A, b) is the "convex hull" of X^*(A, b).

For if x ϵ X(A, b) and x does not belong to the convex hull of X^*(A, b) it follows from a direct application of the Farka lemma, that there exists an n-dimensional real valued column vector ξ such that $\xi^T x^* < \xi^T x$ for all $x^* \epsilon X^*$(A, b). This is contrary to a well-known result in linear programming (proposition 4 in topic 2 of Lahiri (2020)), which says that since $\{\xi^T y$: y ϵ X(A, b)$\}$ is bounded above (which is implied by our assumption that X(A, b) is bounded), the linear programming problem [Maximize $\xi^T y$, subject to yϵ X(A, b)] has a solution in X^*(A, b). Thus, X(A, b) must be a subset of the convex hull of X^*(A, b). Since $X^*(A, b) \subset X(A, b)$ and X(A, b) is a convex set, it follows that X(A, b) is the "convex hull" of X^*(A, b).

Observation 3: A strategy profile (x^*, q^*) is an equilibrium for the ASS-LP game ((A, b), $<p(\theta)|\theta\epsilon\Omega>$, π, ρ) <u>if and only if</u> for all x ϵ X^*(A, b) and $\theta \epsilon \Omega$: $\sum_{\theta\epsilon\Omega} q^*(\theta) p(\theta)^T x + \pi^T x \leq$ $\sum_{\theta\epsilon\Omega} q^*(\theta) p(\theta)^T x^* + \pi^T x^*$ and $-p(\theta)^T x^* + \rho(\theta) \leq - \sum_{\theta\epsilon\Omega} q^*(\theta)[p(\theta)^T x^* - \rho(\theta)]$.

Observation 3 is an immediate consequence of observation 2 and the definition of equilibrium.

Observation 4: (x^*, q^*) is an equilibrium for the ASS-LP game ((A, b), $<p(\theta)|\theta\epsilon\Omega>$, π, ρ) <u>if and only if</u> for all x ϵ X^*(A, b) and $\theta \epsilon \Omega$: $\sum_{\theta\epsilon\Omega} q^*(\theta)[(p(\theta)^T + \pi^T)x - \rho(\theta)] \leq$ $\sum_{\theta\epsilon\Omega} q^*(\theta)[(p(\theta)^T + \pi^T)x^* - \rho(\theta)]$ and $- [(p(\theta)^T + \pi^T)x^* - \rho(\theta)] \leq -$ $\sum_{\theta\epsilon\Omega} q^*(\theta)[(p(\theta)^T + \pi^T)x^* - \rho(\theta)]$.

Observation 4 follows immediately from the fact that for x, $x^0 \epsilon X^*$(A, b) and $\theta, \theta^0 \epsilon \Omega$: $[p(\theta)^T x + \pi^T x \geq p(\theta)^T x^0 + \pi^T x^0$ <u>if and only if</u> $(p(\theta)^T + \pi^T)x - \rho(\theta) \geq (p(\theta)^T + \pi^T)x^0 - \rho(\theta)]$ & $[-p(\theta)^T x + \rho(\theta) \geq -p(\theta^0)^T x + \rho(\theta^0)$ <u>if and only if</u> $-(p(\theta)^T + \pi^T)x + \rho(\theta) \geq -(p(\theta^0)^T + \pi^T)x + \rho(\theta^0)]$.

5. The Main Result:

In this section we provide the main result. The proof of this result is very similar to the proofs leading to theorem 1 in Lahiri (2025a). However, the theorem we prove here is valid in a framework, a special case of which is the framework in which the entire analysis in Lahiri (2025a) is carried out. This is explained in note 2 in section 3.

Theorem 1: Let ((A, b), <p(θ)|θ∈Ω>, π, ρ) be an ASS-LP game.

(i) ((A, b), <p(θ)|θ∈Ω>, π, ρ) has an equilibrium.

(ii) (x^*, q^*) is an equilibrium for the ASS-LP game ((A, b), <p(θ)|θ∈Ω>, π, ρ) <u>if and only if</u> there exist real numbers α^*, β^* and a function λ^*: $X^*(A, b)\rightarrow[0,1]$ satisfying $\sum_{x\in X^*(A,b)} \lambda^*(x) = 1$ as well as $\sum_{x\in X^*(A,b)} \lambda^*(x)x = x^*$ such that:

q^*, α^* solve the linear programming problem **LP-1**: [Maximize $\sum_{\theta\in\Omega} q(\theta)\rho(\theta) - \alpha$, subject to $\sum_{\theta\in\Omega} q(\theta)p(\theta)^T x - \alpha \leq -\pi^T x$ for all $x \in X^*(A, b)$, $\sum_{\theta\in\Omega} q(\theta) = 1$, $q(\theta) \geq 0$ for all $\theta\in\Omega$, $\alpha\in\mathbb{R}$]

&

λ^*, β^* solve its dual, i.e., the linear programming problem **Dual LP-1**: [Minimize $-\sum_{x\in X^*(A,b)} \lambda(x)\pi^T x + \beta$, subject to $\sum_{x\in X^*(A,b)} \lambda(x)p(\theta)^T x + \beta \geq \rho(\theta)$ for all $\theta\in\Omega$, $\sum_{x\in X^*(A,b)} \lambda(x) = 1$, $\lambda(x) \geq 0$ for all $x \in X^*(A, b)$, $\beta\in\mathbb{R}$].

Further, λ^*, q^*, α^*, β^* satisfies $\alpha^* = \sum_{\theta\in\Omega} q^*(\theta)p(\theta)^T x^* + \pi^T x^*$ and $\beta^* = -\sum_{\theta\in\Omega} q^*(\theta)p(\theta)^T x^* + \sum_{\theta\in\Omega} q^*(\theta)\rho(\theta)$.

(iii) (x^*, q^*) is an equilibrium for the ASS-LP game ((A, b), <p(θ)|θ∈Ω>, π, ρ) <u>if and only if</u> there exist real numbers u^*, v^* and a function λ^*: $X^*(A, b)\rightarrow[0,1]$ satisfying $\sum_{x\in X^*(A,b)} \lambda^*(x) = 1$ as well as $\sum_{x\in X^*(A,b)} \lambda^*(x)x = x^*$ such that:

q^*, u^* solve the linear programming problem **LP-2**: [Maximize $-u$, subject to $\sum_{\theta\in\Omega} q(\theta)[(p(\theta)^T + \pi^T)x - \rho(\theta)] - u \leq 0$ for all $x \in X^*(A, b)$, $\sum_{\theta\in\Omega} q(\theta) = 1$, $q(\theta) \geq 0$ for all $\theta\in\Omega$, $u\in\mathbb{R}$]

&

λ^*, v^* solve its dual, i.e., the linear programming problem **Dual LP-2**: [Minimize v, subject to $\sum_{x\in X^*(A,b)} \lambda(x)[(p(\theta)^T + \pi^T)x - \rho(\theta)] + v \geq 0$ for all $\theta\in\Omega$, $\sum_{x\in X^*(A,b)} \lambda(x) = 1$, $\lambda(x) \geq 0$ for all $x \in X^*(A, b)$, $v\in\mathbb{R}$].

Further, λ^*, q^*, u^*, v^* satisfies $u^* = \sum_{\theta\epsilon\Omega} q^*(\theta)[(p(\theta)^T + \pi^T)x^* - \rho(\theta)] = -v^*$ and we call u^* the **value** of the ASS-LP game ((A, b), <p(θ)|θ∊Ω>, π, ρ).

Proof: In what follows we shall be frequently using the facts:

If 'f' is a real-valued function on X(A,b), then for all $x \epsilon X(A, b)$ and $q \epsilon \Delta(\Omega)$, $\sum_{\theta\epsilon\Omega} q(\theta) f(x) = f(x)$, &

If 'g' is a real-valued function on $\Delta(\Omega)$, then for all $\lambda: X^*(A, b) \rightarrow [0,1]$ satisfying $\sum_{x\epsilon X^*(A,b)} \lambda(x) = 1$ and $q \epsilon \Delta(\Omega)$, $\sum_{x\epsilon X^*(A,b)} \lambda(x)\, g(q) = g(q)$.

(i) Consider the linear programming problem LP-1:

Maximize $\sum_{\theta\epsilon\Omega} q(\theta)\rho(\theta) - \alpha$, subject to $\sum_{\theta\epsilon\Omega} q(\theta) p(\theta)^T x - \alpha \leq -\pi^T x$ for all $x \epsilon X^*(A, b)$, $\sum_{\theta\epsilon\Omega} q(\theta) = 1$, $q(\theta) \geq 0$ for all $\theta \epsilon \Omega$, $\alpha \in \mathbb{R}$.

Clearly q, α satisfying the constraints must satisfy $\alpha \geq \max\{\sum_{\theta\epsilon\Omega} q(\theta)(p(\theta)^T + \pi^T)x \mid x \epsilon X^*(A, b)\}$.

For $q \epsilon \Delta(\Omega)$ and $x \epsilon X^*(A, b)$, $\sum_{\theta\epsilon\Omega} q(\theta)(p(\theta)^T + \pi^T)x \geq \min\{(p(\theta)^T + \pi^T)x \mid \theta \epsilon \Omega\}$.

Thus, q, α satisfying the constraints must satisfy $\alpha \geq \max\{\min\{(p(\theta)^T + \pi^T)x \mid \theta\epsilon\Omega\} \mid x \epsilon X^*(A, b)\}$
$= \max_{x\epsilon X^*(A,b)} \min_{\theta\epsilon\Omega} (p(\theta)^T + \pi^T)x$.

The set $\{\sum_{\theta\epsilon\Omega} q(\theta)\rho(\theta) \mid q \epsilon \Delta(\Omega)\} \leq \max\{\rho(\theta) \mid \theta\epsilon\Omega\}$.

Thus, q, α satisfying the constraints must satisfy $\sum_{\theta\epsilon\Omega} q(\theta)\rho(\theta) - \alpha \leq \max\{\rho(\theta) \mid \theta\epsilon\Omega\} - \max_{x\epsilon X^*(A,b)} \min_{\theta\epsilon\Omega} (p(\theta)^T + \pi^T)x < +\infty$.

Thus, the objective function of LP1 is bounded above on the set of pairs (q, α) satisfying the constraints and hence LP-1 has a solution. Let q^*, α^* solve LP-1.

By the weak duality theorem of linear programming, we know that the dual of LP1 has a solution.

The dual of LP-1 is the linear programming problem Dual LP-1:

Minimize $-\sum_{x\epsilon X^*(A,b)} \lambda(x)\pi^T x + \beta$, subject to $\sum_{x\epsilon X^*(A,b)} \lambda(x) p(\theta)^T x + \beta \geq \rho(\theta)$ for all $\theta\epsilon\Omega$, $\sum_{x\epsilon X^*(A,b)} \lambda(x) = 1$, $\lambda(x) \geq 0$ for all $x \epsilon X^*(A, b)$, $\beta \in \mathbb{R}$.

Let λ^*, β^* solve Dual LP-1 and let $x^* = \sum_{x\epsilon X^*(A,b)} \lambda^*(x)x$.

Thus, q^*, α^*, λ^*, β^* must satisfy:

(1) $\sum_{\theta\epsilon\Omega} q^*(\theta)p(\theta)^T x - \alpha^* \leq -\pi^T x$ for all $x \epsilon X^*(A, b)$, $\sum_{\theta\epsilon\Omega} q^*(\theta) = 1$, $q^*(\theta) \geq 0$ for all $\theta\epsilon\Omega$, $\alpha^* \in \mathbb{R}$.

(2) $\sum_{x\epsilon X^*(A,b)} \lambda^*(x)p(\theta)^T x + \beta^* \geq \rho(\theta)$ for all $\theta\epsilon\Omega$, $\sum_{x\epsilon X^*(A,b)} \lambda^*(x) = 1$, $\lambda^*(x) \geq 0$ for all $x \epsilon X^*(A, b)$, $\beta^* \in \mathbb{R}$.

In addition, the following complementary slackness conditions must also be satisfied.

(3) $(\sum_{\theta\epsilon\Omega} q^*(\theta)p(\theta)^T x - \alpha^* + \pi^T x)\, \lambda^*(x) = 0$ for all $x \epsilon X^*(A, b)$.

(4) $(\sum_{x\epsilon X^*(A,b)} \lambda^*(x)p(\theta)^T x + \beta^* - \rho(\theta))q^*(\theta) = 0$ for all $\theta\epsilon\Omega$.

Taking the sum over all $x \epsilon X^*(A, b)$ in (3) and using $x^* = \sum_{x\epsilon X^*(A,b)} \lambda^*(x)x$ we get, $\sum_{\theta\epsilon\Omega} q^*(\theta)p(\theta)^T x^* + \pi^T x^* = \alpha^*$.

Taking the sum over all $\theta\epsilon\Omega$ in (4) and using $x^* = \sum_{x\epsilon X^*(A,b)} \lambda^*(x)x$ we get,

$- \sum_{\theta\epsilon\Omega} q^*(\theta)p(\theta)^T x^* + \sum_{\theta\epsilon\Omega} q^*(\theta)\rho(\theta) = \beta^*$.

Let $x \epsilon X^*(A, b)$ and $\theta \epsilon \Delta(\Omega)$.

From (1) we get $\sum_{\theta\epsilon\Omega} q^*(\theta)p(\theta)^T x + \pi^T x \leq \alpha^* = \sum_{\theta\epsilon\Omega} q^*(\theta)p(\theta)^T x^* + \pi^T x^*$ and from (2) we get $- \sum_{x\epsilon X^*(A,b)} \lambda^*(x)p(\theta)^T x + \rho(\theta) \leq \beta^* = - \sum_{\theta\epsilon\Omega} q^*(\theta)p(\theta)^T x^* + \sum_{\theta\epsilon\Omega} q^*(\theta)\rho(\theta)$.

Thus, from observation 3 it follows that (x^*, q^*) is an equilibrium for the ASS-LP game ((A, b), <p(θ)|θϵΩ>, π, ρ).

Note that as mentioned after the statements of (3) and (4), $\alpha^* = \sum_{\theta\epsilon\Omega} q^*(\theta)p(\theta)^T x^* + \pi^T x^*$ and $\beta^* = - \sum_{\theta\epsilon\Omega} q^*(\theta)p(\theta)^T x^* + \sum_{\theta\epsilon\Omega} q^*(\theta)\rho(\theta)$.

(ii) The proof that <u>if</u> there exist real numbers α^*, β^* and a function $\lambda^*: X^*(A, b) \rightarrow [0,1]$ satisfying $\sum_{x\epsilon X^*(A,b)} \lambda^*(x) = 1$ as well as $\sum_{x\epsilon X^*(A,b)} \lambda^*(x)x = x^*$ such that q^*, α^* solve LP-1and λ^*, β^* solve Dual LP-1, <u>then</u> (x^*, q^*) is an equilibrium for the ASS-LP game, and further $\alpha^* = \sum_{\theta\epsilon\Omega} q^*(\theta)p(\theta)^T x^* + \pi^T x^*$ and $\beta^* = - \sum_{\theta\epsilon\Omega} q^*(\theta)p(\theta)^T x^* + \sum_{\theta\epsilon\Omega} q^*(\theta)\rho(\theta)$, is exactly the same as the proof of (i).

Hence, let us suppose that (x^*, q^*) is an equilibrium for the ASS-LP game ((A, b), <p(θ)|θϵΩ>, π, ρ).

Thus, by observation (3), for all $x \in X^*(A, b)$ and $\theta \in \Omega$: $\sum_{\theta\epsilon\Omega} q^*(\theta)p(\theta)^T x + \pi^T x \leq \sum_{\theta\epsilon\Omega} q^*(\theta)p(\theta)^T x^* + \pi^T x^*$ and $-p(\theta)^T x^* + \rho(\theta) \leq - \sum_{\theta\epsilon\Omega} q^*(\theta)[p(\theta)^T x^* - \rho(\theta)]$.

Let $\alpha^* = \sum_{\theta\epsilon\Omega} q^*(\theta)p(\theta)^T x^* + \pi^T x^*$ and $\beta^* = - \sum_{\theta\epsilon\Omega} q^*(\theta)[p(\theta)^T x^* - \rho(\theta)]$.

Since $x^* \epsilon X(A, b)$, by observation 2, there exists a function $\lambda^*: X^*(A, b) \rightarrow [0,1]$ (not necessarily unique) satisfying $\sum_{x\epsilon X^*(A,b)} \lambda^*(x) = 1$ as well as $\sum_{x\epsilon X^*(A,b)} \lambda^*(x)x = x^*$.

Choose any such λ^*.

Suppose λ, β satisfy $\sum_{x\epsilon X^*(A,b)} \lambda(x)p(\theta)^T x + \beta \geq \rho(\theta)$ for all $\theta\epsilon\Omega$, $\sum_{x\epsilon X^*(A,b)} \lambda(x) = 1$, $\lambda(x) \geq 0$ for all $x \in X^*(A, b)$, $\beta \in \mathbb{R}$.

Then, $-\sum_{x\epsilon X^*(A,b)} \lambda(x)\pi^T x + \beta \geq -\sum_{x\epsilon X^*(A,b)} \lambda(x)\pi^T x + \rho(\theta) - \sum_{x\epsilon X^*(A,b)} \lambda(x)p(\theta)^T x$ for all $\theta\epsilon\Omega$.

Multiplying both sides of the inequality by $q^*(\theta)$ and summing over all $\theta\epsilon\Omega$ we get $-\sum_{x\epsilon X^*(A,b)} \lambda(x)\pi^T x + \beta \geq -\pi^T \sum_{x\epsilon X^*(A,b)} \lambda(x)x + \sum_{\theta\epsilon\Omega} q^*(\theta)\rho(\theta) - \sum_{\theta\epsilon\Omega} q^*(\theta)p(\theta)^T [\sum_{x\epsilon X^*(A,b)} \lambda(x)x] \geq -\pi^T x^* + \sum_{\theta\epsilon\Omega} q^*(\theta)\rho(\theta) - \sum_{\theta\epsilon\Omega} q^*(\theta)p(\theta)^T x^*$, since (x^*, q^*) is an equilibrium implies $\pi^T x^* + \sum_{\theta\epsilon\Omega} q^*(\theta)p(\theta)^T x^* \geq \pi^T \sum_{x\epsilon X^*(A,b)} \lambda(x)x + \sum_{\theta\epsilon\Omega} q^*(\theta)p(\theta)^T [\sum_{x\epsilon X^*(A,b)} \lambda(x)x]$.

However, $- \sum_{\theta\epsilon\Omega} q^*(\theta)[p(\theta)^T x^* - \rho(\theta)] = \beta^*$ implies $-\pi^T x^* + \sum_{\theta\epsilon\Omega} q^*(\theta)\rho(\theta) - \sum_{\theta\epsilon\Omega} q^*(\theta)p(\theta)^T x^* = -\pi^T x^* + \beta^* = - \sum_{x\epsilon X^*(A,b)} \lambda^*(x)\pi^T x + \beta^*$

Thus, $-\sum_{x\epsilon X^*(A,b)} \lambda(x)\pi^T x + \beta \geq - \sum_{x\epsilon X^*(A,b)} \lambda^*(x)\pi^T x + \beta^*$.

Since for all $\theta \in \Omega$, $-p(\theta)^T x^* + \rho(\theta) \leq - \sum_{\theta\epsilon\Omega} q^*(\theta)[p(\theta)^T x^* - \rho(\theta)] = \beta^*$, it follows that $p(\theta)^T x^* + \beta^* \geq \rho(\theta)$ for all $\theta \in \Omega$ and hence λ^*, β^* satisfies the constraints of Dual LP-1.

Thus, λ^*, β^* solve Dual LP-1.

Now suppose q, α satisfy $\sum_{\theta\epsilon\Omega} q(\theta)p(\theta)^T x - \alpha \leq -\pi^T x$ for all $x \in X^*(A, b)$, $\sum_{\theta\epsilon\Omega} q(\theta) = 1$, $q(\theta) \geq 0$ for all $\theta\epsilon\Omega$, $\alpha \in \mathbb{R}$.

Therefore, $\sum_{\theta\epsilon\Omega} q(\theta)\rho(\theta) - \alpha \leq \sum_{\theta\epsilon\Omega} q(\theta)\rho(\theta) - \sum_{\theta\epsilon\Omega} q(\theta)p(\theta)^T x - \pi^T x$ for all $x \in X^*(A, b)$.

Multiplying both sides of the inequality by $\lambda^*(x)$ and summing over all $x \in X^*(A, b)$ we get $\sum_{\theta\epsilon\Omega} q(\theta)\rho(\theta) - \alpha \le \sum_{\theta\epsilon\Omega} q(\theta)\rho(\theta) - \sum_{\theta\epsilon\Omega} q(\theta)p(\theta)^T x^* - \pi^T x^* \le \sum_{\theta\epsilon\Omega} q^*(\theta)\rho(\theta) - \sum_{\theta\epsilon\Omega} q^*(\theta)p(\theta)^T x^* - \pi^T x^*$, since (x^*, q^*) is an equilibrium.

Since $\sum_{\theta\epsilon\Omega} q^*(\theta)p(\theta)^T x^* + \pi^T x^* = \alpha^*$, it must be that $\sum_{\theta\epsilon\Omega} q^*(\theta)\rho(\theta) - \sum_{\theta\epsilon\Omega} q^*(\theta)p(\theta)^T x^* - \pi^T x^* = \sum_{\theta\epsilon\Omega} q^*(\theta)\rho(\theta) - \alpha^*$.

Thus, $\sum_{\theta\epsilon\Omega} q(\theta)\rho(\theta) - \alpha \le \sum_{\theta\epsilon\Omega} q^*(\theta)\rho(\theta) - \alpha^*$.

Since for all $x \in X^*(A, b)$, $\sum_{\theta\epsilon\Omega} q^*(\theta)p(\theta)^T x + \pi^T x \le \sum_{\theta\epsilon\Omega} q^*(\theta)p(\theta)^T x^* + \pi^T x^* = \alpha^*$, it follows that $\sum_{\theta\epsilon\Omega} q^*(\theta)p(\theta)^T x - \alpha^* \le \pi^T x$ for all $x \in X^*(A, b)$.

Thus, q^*, α^* satisfy all the constraints of LP-1.

Hence q^*, α^* solve LP-1.

Note that we have defined $\alpha^* = \sum_{\theta\epsilon\Omega} q^*(\theta)p(\theta)^T x^* + \pi^T x^*$ and $\beta^* = - \sum_{\theta\epsilon\Omega} q^*(\theta)p(\theta)^T x^* + \sum_{\theta\epsilon\Omega} q^*(\theta)\rho(\theta)$.

This proves (ii).

(iii) **Part 1:** There exists α^*, u^* such that q^*, α^* solve LP-1 if and only if q^*, u^* solve LP-2.

Suppose, q^*, α^* solve LP-1.

q^*, α^* solve LP-1

<u>if and only if</u>

q^*, α^* satisfy the constraints of LP-1 and $\sum_{\theta\epsilon\Omega} q^*(\theta)\rho(\theta) - \alpha^* \ge \sum_{\theta\epsilon\Omega} q(\theta)\rho(\theta) - \alpha$ for all q, α satisfying the constraints of LP-1.

q, α satisfy the constraints of LP -1 <u>if and only if</u> $\sum_{\theta\epsilon\Omega} q(\theta)p(\theta)^T x - \alpha \le -\pi^T x$ for all $x \in X^*(A, b)$, $\sum_{\theta\epsilon\Omega} q(\theta) = 1$, $q(\theta) \ge 0$ for all $\theta\epsilon\Omega$, $\alpha \in \mathbb{R}$.

$\sum_{\theta\epsilon\Omega} q(\theta)p(\theta)^T x - \alpha \le -\pi^T x$ for all $x \in X^*(A, b)$, $\sum_{\theta\epsilon\Omega} q(\theta) = 1$, $q(\theta) \ge 0$ for all $\theta\epsilon\Omega$, $\alpha \in \mathbb{R}$ <u>if and only if</u> $\sum_{\theta\epsilon\Omega} q(\theta)[(p(\theta)^T + \pi^T)x - \rho(\theta)] + \sum_{\theta\epsilon\Omega} q(\theta)\rho(\theta) - \alpha \le 0$ for all $x \in X^*(A, b)$, $\sum_{\theta\epsilon\Omega} q(\theta) = 1$, $q(\theta) \ge 0$ for all $\theta\epsilon\Omega$, $\alpha \in \mathbb{R}$.

Ler $u^* = \alpha^* - \sum_{\theta\epsilon\Omega} q^*(\theta)\rho(\theta)$ and let q, u satisfy the constraints of LP-2, i.e.

$\sum_{\theta\epsilon\Omega} q(\theta)[(p(\theta)^T + \pi^T)x - \rho(\theta)] - u \leq 0$ for all $x \epsilon X^*(A, b)$, $\sum_{\theta\epsilon\Omega} q(\theta) = 1$, $q(\theta) \geq 0$ for all $\theta\epsilon\Omega$, $u\in \mathbb{R}$.

Let $\alpha = u + \sum_{\theta\epsilon\Omega} q(\theta)\rho(\theta)$.

Thus, $\sum_{\theta\epsilon\Omega} q(\theta)[(p(\theta)^T + \pi^T)x - \rho(\theta)] + \sum_{\theta\epsilon\Omega} q(\theta)\rho(\theta) - \alpha \leq 0$ for all $x \epsilon X^*(A, b)$, $\sum_{\theta\epsilon\Omega} q(\theta) = 1$, $q(\theta) \geq 0$ for all $\theta\epsilon\Omega$, $\alpha\in \mathbb{R}$.

Hence, $\sum_{\theta\epsilon\Omega} q(\theta)p(\theta)^T x - \alpha \leq -\pi^T x$ for all $x \epsilon X^*(A, b)$, $\sum_{\theta\epsilon\Omega} q(\theta) = 1$, $q(\theta) \geq 0$ for all $\theta\epsilon\Omega$, $\alpha\in \mathbb{R}$.

Further, q^*, α^* satisfy the constraints of LP-1 if and only if $\sum_{\theta\epsilon\Omega} q^*(\theta)p(\theta)^T x - \alpha^* \leq -\pi^T x$ for all $x \epsilon X^*(A, b)$, $\sum_{\theta\epsilon\Omega} q^*(\theta) = 1$, $q^*(\theta) \geq 0$ for all $\theta\epsilon\Omega$, $\alpha^*\in \mathbb{R}$.

Therefore, $u^* = \alpha^* - \sum_{\theta\epsilon\Omega} q^*(\theta)\rho(\theta)$ implies $\sum_{\theta\epsilon\Omega} q^*(\theta)p(\theta)^T x - \sum_{\theta\epsilon\Omega} q^*(\theta)\rho(\theta) - u^* \leq -\pi^T x$ for all $x \epsilon X^*(A, b)$, $\sum_{\theta\epsilon\Omega} q^*(\theta) = 1$, $q^*(\theta) \geq 0$ for all $\theta\epsilon\Omega$, $u^*\in \mathbb{R}$.

Thus, $\sum_{\theta\epsilon\Omega} q^*(\theta)(p(\theta)^T + \pi^T)x - \sum_{\theta\epsilon\Omega} q^*(\theta)\rho(\theta) - u^* \leq 0$ for all $x \epsilon X^*(A, b)$, $\sum_{\theta\epsilon\Omega} q^*(\theta) = 1$, $q^*(\theta) \geq 0$ for all $\theta\epsilon\Omega$, $u^*\in \mathbb{R}$.

Hence, $\sum_{\theta\epsilon\Omega} q^*(\theta)[(p(\theta)^T + \pi^T)x - \rho(\theta)] - u^* \leq 0$ for all $x \epsilon X^*(A, b)$, $\sum_{\theta\epsilon\Omega} q^*(\theta) = 1$, $q^*(\theta) \geq 0$ for all $\theta\epsilon\Omega$, $u^*\in \mathbb{R}$.

Thus, q^*, u^* satisfies the constraints of LP-2.

Our assumptions $\sum_{\theta\epsilon\Omega} q^*(\theta)\rho(\theta) - \alpha^* \geq \sum_{\theta\epsilon\Omega} q(\theta)\rho(\theta) - \alpha$, $u^* = \alpha^* - \sum_{\theta\epsilon\Omega} q^*(\theta)\rho(\theta)$ and $u = \alpha - \sum_{\theta\epsilon\Omega} q(\theta)\rho(\theta)$ now imply $-u^* \geq -u$.

Conversely, suppose q^*, u^* solve LP-2.

q^*, u^* solve LP-2

if and only if

q^*, u^* satisfy the constraints of LP-2 and $-u^* \geq -u$ for all q, u satisfying the constraints of LP-2.

q, u satisfy the constraints of LP-2 if and only if $\sum_{\theta\epsilon\Omega} q(\theta)[(p(\theta)^T + \pi^T)x - \rho(\theta)]$ - u ≤ 0 for all x ϵ X^*(A, b), $\sum_{\theta\epsilon\Omega} q(\theta) = 1$, q(θ) ≥ 0 for all θϵΩ, u∈ℝ.

Let $\alpha^* = u^* + \sum_{\theta\epsilon\Omega} q^*(\theta)\rho(\theta)$ and suppose q, α satisfy the constraints of LP-1, i.e., $\sum_{\theta\epsilon\Omega} q(\theta)p(\theta)^T x$ - α ≤ -π^Tx for all x ϵ X^*(A, b), $\sum_{\theta\epsilon\Omega} q(\theta) = 1$, q(θ) ≥ 0 for all θϵΩ, α∈ℝ.

Let u = α - $\sum_{\theta\epsilon\Omega} q(\theta)\rho(\theta)$.

Thus, $\sum_{\theta\epsilon\Omega} q(\theta)[(p(\theta)^T + \pi^T)x - \rho(\theta)]$ - u ≤ 0 for all x ϵ X^*(A, b), $\sum_{\theta\epsilon\Omega} q(\theta) = 1$, q(θ) ≥ 0 for all θϵΩ, u∈ℝ.

- u^* ≥ - u implies $\sum_{\theta\epsilon\Omega} q^*(\theta)\rho(\theta)$ - α^* ≥ $\sum_{\theta\epsilon\Omega} q(\theta)\rho(\theta)$ - α.

Further, q^*, u^* satisfy the constraints of LP-2 if and only if $\sum_{\theta\epsilon\Omega} q^*(\theta)(p(\theta)^T + \pi^T)x$ - $\sum_{\theta\epsilon\Omega} q^*(\theta)\rho(\theta)$ – u^* ≤ 0 for all x ϵ X^*(A, b), $\sum_{\theta\epsilon\Omega} q^*(\theta) = 1$, q^*(θ) ≥ 0 for all θϵΩ, u^*∈ℝ.

Therefore, $\alpha^* = u^* + \sum_{\theta\epsilon\Omega} q^*(\theta)\rho(\theta)$ implies $\sum_{\theta\epsilon\Omega} q^*(\theta)(p(\theta)^T + \pi^T)x$ - α^* ≤ 0 for all x ϵ X^*(A, b), $\sum_{\theta\epsilon\Omega} q^*(\theta) = 1$, q^*(θ) ≥ 0 for all θϵΩ, α^*∈ℝ, i.e., $\sum_{\theta\epsilon\Omega} q^*(\theta)p(\theta)^T x$ - α^* ≤ -π^Tx for all x ϵ X^*(A, b), $\sum_{\theta\epsilon\Omega} q^*(\theta) = 1$, q^*(θ) ≥ 0 for all θϵΩ, α^*∈ℝ

Thus, q^*, α^* solve LP-1.

Further, in either case $u^* = \sum_{\theta\epsilon\Omega} q^*(\theta)[p(\theta)^T x^* - \rho(\theta)] + \pi^T x^*$ if and only if $\alpha^* = \sum_{\theta\epsilon\Omega} q^*(\theta)p(\theta)^T x^* + \pi^T x^*$.

Part 2: For λ^*: X^*(A, b)→[0,1] satisfying $\sum_{x\epsilon X^*(A,b)} \lambda^*(x) = 1$ as well as $\sum_{x\epsilon X^*(A,b)} \lambda^*(x)x = x^*$: there exists a real numbers β^* such that λ^*, β^* solve Dual LP-1 if and only if there exists a real number v^* such that λ^*, v^* solve Dual LP-2.

λ^*, β^* solve Dual LP-1

if and only if

λ^*, β^* satisfy the constraints of Dual LP-1 and $-\sum_{x\epsilon X^*(A,b)} \lambda^*(x)\pi^T x + \beta^* \leq -\sum_{x\epsilon X^*(A,b)} \lambda(x)\pi^T x + \beta$ for all λ, β that satisfy the constraints of Dual LP-1.

Similarly, λ^*, v^* solve Dual LP-2

if and only if

λ^*, v^* satisfy the constraints of Dual LP-2 and $v^* \leq v$ for all λ, v satisfying the constraints of Dual LP-2.

Following steps very similar to that used in the proof of part 1, it turns out that the required equivalence is obtained if the relationship between β^* and v^* is given by the equation $\beta^* - v^* = \pi^T x^*$.

Further, $-v^* = \sum_{\theta \epsilon \Omega} q^*(\theta)[p(\theta)^T x^* - \rho(\theta)] + \pi^T x^*$ <u>if and only if</u> $\beta^* = -\sum_{\theta \epsilon \Omega} q^*(\theta)[p(\theta)^T x^* - \rho(\theta)]$. Q.E.D.

6. A Minimax Stochastic Linear Programming Game:

Consider a two-period stochastic programming game in which in the initial period, the first player gets to choose a control variable vector x(0) from a bounded closed polytope and in the second or terminal period the first player chooses a control variable vector x(1) from another bounded closed polytope that depends on the control variable vector chosen by the player in the first period as well as the prevailing state of nature. The second player responds by choosing a mixed strategy $q \epsilon \Delta(\Omega)$. While the payoff to the two players in the initial period depend only on x(0), their payoffs in the second period depend on both x(1) and q. We assume that for all x(0), x(1), q, the sum of the payoffs to the two players is zero.

For natural number m_0, n_0, m_1, n_1 let $A^{(0)} \epsilon \mathbb{R}^{m_0 \times n_0}$, $\pi^{(0)} \epsilon \mathbb{R}^{n_0}$, $b^{(0)} \epsilon \mathbb{R}^{m_0}$ and for each $\theta \epsilon \Omega$, let $A^{(1)}(\theta) \epsilon \mathbb{R}^{m_0 \times n_0}$, $B^{(1)}(\theta) \epsilon \mathbb{R}^{m_1 \times n_1}$, $p^{(1)}(\theta) \epsilon \mathbb{R}^{n_1}$ and $b^{(1)}(\theta) \epsilon \mathbb{R}^{m_1}$.

In a minimax stochastic linear programming game:

The first player's problem is the following:

Given $q \epsilon \Delta(\Omega)$: Choose $(x(0), \langle x(1, \theta) | \theta \epsilon \Omega \rangle)$ so as to Maximize $[\pi^{(0)T} x(0) + \sum_{\theta \epsilon \Omega} q(\theta) p^{(1)}(\theta)^T x(1, \theta)]$ subject to $A^{(0)} x(0) = b^{(0)}$, $x(0) \epsilon \mathbb{R}_+^{n_0}$, $A^{(1)}(\theta)\, x(0) + B^{(1)}(\theta) x(1, \theta) = b^{(1)}(\theta)$, $x(1, \theta) \epsilon \mathbb{R}_+^{n_1}$.

&

The second player's problem is the following:

Given $(x(0), \langle x(1, \theta) | \theta \epsilon \Omega \rangle)$ satisfying $A^{(0)} x(0) = b^{(0)}$, $x(0) \epsilon \mathbb{R}_+^{n_0}$, $A^{(1)}(\theta)\, x(0) + B^{(1)}(\theta) x(1, \theta) = b^{(1)}(\theta)$, $x(1, \theta) \epsilon \mathbb{R}_+^{n_1}$, choose $q \epsilon \Delta(\Omega)$ to Minimize$[\pi^{(0)T} x(0) + \sum_{\theta \epsilon \Omega} q(\theta) p^{(1)}(\theta)^T x(1, \theta)]$.

The second player's problem is equivalent to the following:

Given $(x(0), \langle x(1, \theta) | \theta \epsilon \Omega \rangle)$ satisfying $A^{(0)} x(0) = b^{(0)}$, $x(0) \epsilon \mathbb{R}_+^{n_0}$, $A^{(1)}(\theta)\, x(0) + B^{(1)}(\theta) x(1, \theta) = b^{(1)}(\theta)$, $x(1, \theta) \epsilon \mathbb{R}_+^{n_1}$, choose $\pi \epsilon \Delta(\Omega)$ to Minimize $\sum_{\theta \epsilon \Omega} q(\theta) p^{(1)}(\theta)^T x(1, \theta)$.

Letting $|\Omega|$ denote the cardinality of Ω, let $m = m_0 + m_1|\Omega|$ and $n = n_0 + n_1|\Omega|$. We also assume some linear order (reflexive, connected/complete, transitive and anti-symmetric binary relation) on Ω according to which the occupancy of related coordinates in the required vectors and matrices are arranged.

Let $\pi = \begin{pmatrix} \pi^{(0)} \\ 0 \end{pmatrix}$ be an n-dimensional real valued column vector.

For each $\theta \epsilon \Omega$, let $p(\theta)$ be an n-dimensional real-valued column vector whose coordinates corresponding to θ are occupied by the column vector $p^{(1)}(\theta)$ and all other coordinates are zero. Thus, for all $\theta \epsilon \Omega$, the first n_0 coordinates of $p(\theta)$ are zero.

Let x denote an n-dimensional real valued column vector whose first n_0 coordinates are occupied by the n_0- dimensional real-valued column vector $x(0)$ and for $\theta \epsilon \Omega$, the corresponding coordinates of x are occupied by the column vector $x(1, \theta)$.

Let A be a real-valued matrix with m rows and n columns, whose first block along the diagonal is $A^{(0)}$ and for each $\theta \epsilon \Omega$, the corresponding block along the diagonal is occupied by the matrix $B^{(1)}(\theta)$. Further, for each $\theta \epsilon \Omega$, the first n_0 columns of A are occupied by the matrix $A^{(1)}(\theta)$. All other entries in the matrix A are zero.

Let b be the m-dimensional real-valued column vector whose first m_0 coordinates are occupied by $b^{(0)}$ and for each $\theta \epsilon \Omega$, the corresponding coordinates are occupied by $b^{(1)}(\theta)$.

Thus, the minimax stochastic linear programming game can be represented as an ASS-LP game $((A, b), \langle p(\theta)|\theta \epsilon \Omega \rangle, \pi, \rho)$ with $\rho(\theta) = 0$ for all $\theta \epsilon \Omega$.

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